where the $\log(\mu^2/s)$ term is completely governed by the above-given RGE for λ. This reflects that the observable effective 4-point coupling should not depend on the choice of reference scale μ.

The reference scale μ labels an entire equivalence class of parametrizations of the ϕ^4-theory and it should not matter which element of the class we choose for setting up the theory. These elements all lie on the same RGE trajectory.

When expressed in terms of the physical coupling λ_{ph}, the effective coupling $|\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)|^2$ is independent of the cutoff Λ, as expected for a correct observable! The cutoff dependence has been absorbed into a redefinition of the unobservable Lagrangian parameter (bare coupling) λ in terms of the observable physical parameter (effective coupling) λ_{ph}. In the literature this physical observable is usually referred to as the renormalized coupling λ_R, although this terminology is a bit strange bearing in mind that the original coupling was not normalized to begin with. This is an example of the concept of renormalization.

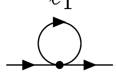
(10c) *Renormalization: express physically measurable quantities in terms of physically measurable quantities and not in terms of bare Lagrangian parameters.*

- For setting up a perturbative expansion, the bare Lagrangian parameters are in fact not the right parameters. Instead the physically measurable parameters should be used (cf. the discussion about m and m_{ph} in § 2.9.2).
- The occurrence of infinities in the loop integrals is linked to this. Our initial perturbative expansion consisted of taking $\Lambda \to \infty$ while keeping λ and m finite. From the renormalization group viewpoint, however, the set $(\mu = \Lambda = \infty, \lambda < \infty, m < \infty)$ does not belong to the equivalence class of the ϕ^4-theory!
- The convergence of the perturbative series can be further improved by using physical quantities at the "right scale", thereby avoiding large logarithmic factors like $\log(\mu^2/s)$ in the example above. This choice of scale has no consequence for all-order calculations, but it does if the series is truncated at a certain perturbative order.

To complete the story for the scalar ϕ^4-theory we consider the UV divergences that are present in the scalar self-energy. This time the mass parameter is essential and therefore should not be neglected.

Scalar self-energy at $\mathcal{O}(\lambda)$:

$$-i\Sigma(p^2) \;\overset{\mathcal{O}(\lambda)}{=\!=}\;$$



$$= \frac{-i\lambda}{2}\int \frac{d^4\ell_1}{(2\pi)^4}\,\frac{i}{\ell_1^2 - m^2 + i\epsilon}$$

$$\xrightarrow[\text{Wick rotation}]{\text{cutoff } \Lambda \gg m} \frac{-i\lambda}{32\pi^2}\int_0^{\Lambda^2} d\ell_E^2\,\frac{\ell_E^2}{\ell_E^2 + m^2 - i\epsilon} = \frac{-i\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\Big(\frac{\Lambda^2}{m^2}\Big)\right].$$

After Dyson summation the full propagator becomes

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \;\equiv\; \frac{iZ}{p^2 - m_{ph}^2} \;+\; \text{regular terms} \;.$$

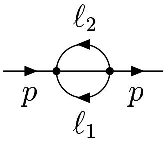
Since the 1-loop scalar self-energy does not depend on p^2, it is absorbed completely into the physical mass:

$$m_{ph}^2 \;=\; m^2 + \Sigma(m_{ph}^2) \;\stackrel{\mathcal{O}(\lambda)}{=\!=\!=}\; m^2 \;+\; \frac{\lambda}{32\pi^2}\Big[\Lambda^2 - m^2 \log\Big(\frac{\Lambda^2}{m^2}\Big)\Big] \;,$$

whereas the residue of the pole remains 1.

(10d) *Note the strong Λ^2 dependence of the scalar mass, which implies that this mass is very sensitive to high-scale quantum corrections. This is in fact a general feature of scalar particles, like the Higgs boson: intrinsically the quantum corrections to the mass of a scalar particle are dominated by the highest mass scale the scalar particle couples to!*

Scalar self-energy at $\mathcal{O}(\lambda^2)$: the residue of the pole is affected at 2-loop level by the contribution



$$= \frac{(-i\lambda)^2}{6}\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\int\frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\,\frac{i}{\ell_1^2 - m^2 + i\epsilon}\,\frac{i}{\ell_2^2 - m^2 + i\epsilon}\,\frac{i}{(\ell_1+\ell_2+p)^2 - m^2 + i\epsilon}$$

$$= \; a + bp^2 + cp^4 + \cdots \;.$$

To assess the UV behaviour of this diagram we perform naive power counting, which involves treating all loop momenta as being of the same order of magnitude. For $\ell_{1,2} \to \infty$ we obtain an integral of the order $\int \mathrm{d}^8\ell_E/\ell_E^6 \xrightarrow{\ell_E \,\leq\, \Lambda} \Lambda^{8-6} = \Lambda^2$.

- $a = \mathcal{O}(\Lambda^2)$ is obtained by setting $p = 0$;
- $b = \mathcal{O}(\log\Lambda)$ is obtained by taking $\frac{1}{2}\partial^2/\partial p_0^2$ and then setting $p = 0$. In naive power counting this logarithmically divergent term corresponds to integrals of order Λ^0.
- $c = \mathcal{O}(1)$ is obtained by taking $\frac{1}{4!}\partial^4/\partial p_0^4$ and then setting $p = 0$.

Adding all self-energy contributions and focussing on the diverging terms

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \quad\to\quad \frac{i}{p^2 - m^2 - A - Bp^2} \;\equiv\; \frac{iZ}{p^2 - m_{ph}^2} \;+\; \text{regular terms} \;,$$

$$Z \;=\; \frac{1}{1-B} \;=\; \mathcal{O}(\log\Lambda) \quad,\quad m_{ph}^2 \;=\; \frac{m^2 + A}{1-B} \;\equiv\; Zm^2 + \delta m^2 \quad,\quad \delta m^2 \;=\; \frac{A}{1-B} \;=\; \mathcal{O}(\Lambda^2) \;.$$

This leads to an $\mathcal{O}(\Lambda^2)$ shift in the mass and an $\mathcal{O}(\log\Lambda)$ contribution to the wave-function renormalization, which can be absorbed in the field ϕ itself.

So, UV divergent loop corrections in ϕ^4-theory are present in $\Sigma(p^2)$ and $\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)$, with

$$\Sigma(m_{ph}^2) \;=\; m_{ph}^2 - m^2 \;=\; (Z-1)m^2 + \delta m^2 \;\equiv\; m^2\delta_Z + \delta m^2 \quad , \quad \Sigma'(m_{ph}^2) \;=\; 1 - 1/Z$$

$$\text{and} \quad \mathcal{M}_{\phi\phi\to\phi\phi}(s=\mu^2,\pi/2) \;=\; -\,\lambda_{ph} \;\equiv\; -\,Z^2\lambda - \delta_\lambda \; .$$

The occurrence of the factor Z^2 in the last expression originates from the multiplicative factor $(\sqrt{Z}\,)^4$ that should be added according to the Feynman rules.

2.10.1 Physical perturbation theory (a.k.a. renormalized perturbation theory)

(10c) *The lowest-order ϕ^4-theory should have been written in terms of the experimentally measurable physical parameters m_{ph} and λ_{ph}, and perturbation theory should have been defined with respect to this lowest-order theory.*

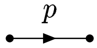
This is done as follows: take the original Lagrangian and write

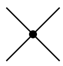
$$\phi \;=\; \phi_{_R}\sqrt{Z} \quad , \quad m^2 Z \;=\; m_{ph}^2 - \delta m^2 \quad , \quad \lambda Z^2 \;=\; \lambda_{ph} - \delta_\lambda \quad \text{and} \quad Z \;\equiv\; 1 + \delta_Z$$

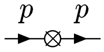
so that

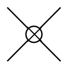
$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \;=\; \frac{1}{2}(\partial_\mu\phi_{_R})(\partial^\mu\phi_{_R}) - \frac{1}{2}m_{ph}^2\phi_{_R}^2 - \frac{\lambda_{ph}}{4!}\phi_{_R}^4$$
$$+\;\frac{1}{2}\delta_Z(\partial_\mu\phi_{_R})(\partial^\mu\phi_{_R}) + \frac{1}{2}\delta m^2\phi_{_R}^2 + \frac{\delta_\lambda}{4!}\phi_{_R}^4 \; .$$

We get back the original Lagrangian in terms of renormalized objects (first line) and we obtain extra interactions that are called counterterms (second line), since their purpose is to cancel the divergences in the theory. The Feynman rules for the propagators and vertices including counterterms are now given by

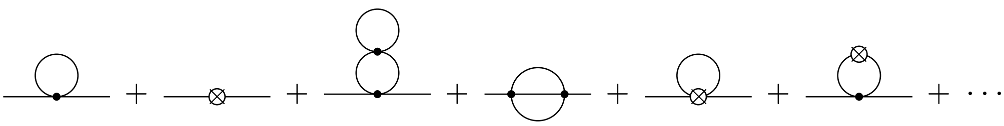


$$= \frac{i}{p^2 - m_{ph}^2 + i\epsilon} \quad ,$$

$$= -i\lambda_{ph} \; ,$$



$$= i(p^2\delta_Z + \delta m^2) \quad ,$$

$$= i\delta_\lambda \; .$$

Renormalization conditions: as an explicit example, the full propagator now reads $i/\big[p^2 - m_{ph}^2 - \Sigma_R(p^2)\big]$, with the renormalized self-energy given by

$$-\,i\Sigma_R(p^2) \;=$$


The parameters δ_Z and δm^2 can be fixed by imposing the renormalization conditions

$$\Sigma_R(m_{ph}^2) = 0 \quad \text{and} \quad \Sigma'_R(m_{ph}^2) = 0 \quad \Rightarrow \quad \text{full propagator} = \frac{i}{p^2 - m_{ph}^2} + \text{regular terms}\,.$$

The pole structure of the full propagator then resembles that of a free particle, so in that sense the physical 1-particle states have been re-normalized by this procedure. Adding one more renormalization condition based on $\mathcal{M}_{\phi\phi\to\phi\phi}$ in order to fix δ_λ, we have three conditions fixing three counterterm parameters. This will in fact be sufficient to make all observables of the ϕ^4-theory finite.

(10e) *The scalar ϕ^4-theory is called renormalizable: "the infinities of the theory can be absorbed into a finite number of parameters".*

2.10.2 What has happened?

The above procedure seems odd: we calculated something that turned out to be infinite, then subtracted infinity from our original mass and coupling in an arbitrary way and ended up with something finite. Moreover, we have added divergent terms to our Lagrangian and we have suddenly ended up with a scale-dependent coupling. Why would a procedure consisting of such ill-defined mathematical tricks be legitimate? To see what has really happened, let us closely examine the starting point of our calculation.

In general, we start with a Lagrangian containing all possible terms that are compatible with basic assumptions such as relativity, causality, locality, etc. It still contains a few parameters such as m and λ in the case of ϕ^4-theory. It is tempting to call them "mass" and "coupling", as they turn out to be just that in the classical (i.e. lowest-order) theory. However, up to this point they are just free parameters. In order to make the theory predictive, the parameters need to be fixed by a set of measurements: we should calculate a set of cross sections at a given order in perturbation theory, measure their values and then fit the parameters so that they reproduce the experimental data. After this procedure, the theory is completely determined and becomes predictive.

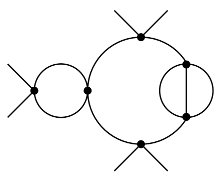
The bare parameters m and λ are only useful in intermediate calculations and will be replaced by physical (i.e. measured) quantities in the end anyway. So, we might as well parametrize the theory in terms of the latter. The renormalizability hypothesis is that this reparametrization of the theory is enough to turn the perturbation expansion into a well-defined expansion. The divergence problem then has nothing to do with the perturbation expansion itself: we have just chosen unsuitable parameters to perform it. Also, the fact that our physical coupling is scale-dependent should not surprise us. The physical reason for this "running" is the existence of quantum fluctuations, which were not there in the classical theory. These fluctuations correspond to intermediate particle states: at sufficiently high (i.e. relativistic) energies, new particles can be created and annihilated. As the available energy increases, more and more energetic particles can be created. This effectively changes the couplings.

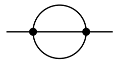
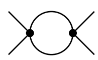
Having traded the bare parameters m and λ for renormalized parameters m_{ph} and λ_{ph}, let us take a closer look at the internal consistency of the renormalization procedure. We have introduced the physical coupling at a reference scale μ, but we could equally well have chosen an energy scale μ' with corresponding effective coupling λ'_{ph}. Physical processes should not depend on our choice of reference scale, hence the couplings should be related in such a way that for any observable O we have $O = O(m_{ph}, \mu, \lambda_{ph}) = O(m_{ph}, \mu', \lambda'_{ph})$. In other words, there should exist an equivalence class of parametrizations of the theory and it should not matter which element of the class we choose. This observation clarifies where the divergences came from: our initial perturbation expansion consisted of taking $\Lambda \to \infty$ while keeping m and λ finite. From the viewpoint of the renormalization group, however, the set $(\mu = \Lambda = \infty\,, m < \infty\,, \lambda < \infty)$ does not belong to any equivalence class of the ϕ^4-theory.

2.10.3 Superficial degree of divergence and renormalizability

(10e) *The statement at the end of §2.10.1 was a bit premature. In fact we still have to prove that amplitudes with more than four external particles do not introduce a new type of infinity that cannot be absorbed into the 2- and 4-point terms in the Lagrangian.*

A 6-point diagram like



will contain singular building blocks like  and  that should become finite once we perform the afore-mentioned renormalization procedure. The question that remains is whether the overall 6-point diagram can give rise to a new type of infinity. To assess this we perform naive power counting, i.e. we treat all loop momenta as being of the same large order of magnitude $\mathcal{O}(\Lambda)$. The outcome of this power counting is called the superficial degree of divergence D of the diagram, with $D = 0$ denoting logarithmic divergence.

Consider a 1PI amputated diagram with N external lines, P propagators and V vertices.

- In ϕ^4-theory four lines enter each vertex, each propagator counts twice towards the total number of lines entering vertices and each external line counts once. This results in the condition

$$4V = N + 2P \quad \Rightarrow \quad P = 2V - N/2 \qquad \text{and} \qquad N = \text{ even number}\,.$$

- The number of loop momenta is given by the number of propagators $-$ the number of four-momentum δ-functions $+\ 1$, since one of the δ-functions corresponds to the external momenta and will not fix an internal loop momentum (see page 53). This results in
$$L \ = \ P - V + 1 \ = \ V - N/2 + 1$$
independent *undetermined* loop momenta. So, loop diagrams require $V \geq N/2$.

Power counting: assume for argument's sake that the loop momenta are n-dimensional. That means that in the context of naive power counting each loop momentum contributes Λ^n and each propagator Λ^{-2}. The superficial degree of divergence of the diagram then reads

$$D \ = \ nL - 2P \ = \ n(V - N/2 + 1) - 2(2V - N/2) \ = \ n + V(n-4) + N(1 - n/2) \ ,$$

whereas the coupling λ has mass dimension $[\lambda] = 4 - n$ in n dimensions.

> *Superficially the diagram diverges like* Λ^D *if* $D > 0$ *and like* $\log(\Lambda)$ *if* $D = 0$, *provided it contains a loop. The diagram does not diverge superficially if* $D < 0$.

Let's now consider a few values for the dimensionality n of spacetime.

$\boldsymbol{n = 4}$: $D = 4 - N$ is independent of V and $[\lambda] = 0 \ \Rightarrow$ the theory is renormalizable. Divergences occur at all orders, but only a *finite number of amplitudes diverges superficially* (i.e. amplitudes with $N = 2$ or 4)! The theory keeps its predictive power in spite of the infinities that occur if we assume it to be valid at all energies.

$\boldsymbol{n = 3}$: $D = 3 - N/2 - V$ and $[\lambda] = 1 \ \Rightarrow$ the theory is superrenormalizable. At most a *finite number of diagrams diverges superficially* (i.e. the diagrams with $N = 2$ and $V = 1$ or $V = 2$), as the diagrams get less divergent if the loop order is increased!

$\boldsymbol{n = 5}$: $D = 5 - 3N/2 + V$ and $[\lambda] = -1 \ \Rightarrow$ the theory is nonrenormalizable. Now *all amplitudes will diverge superficially at a sufficiently high loop order!* An infinite amount of counterterms would be required to remove all divergences, which means that all predictive power is lost if we assume the theory to be valid at all energies!

10e

If we express the superficial degree of divergence in terms of V *and* N, *the coefficient in front of* V *determines whether the theory is superrenormalizable (negative coefficient), renormalizable (zero coefficient) or nonrenormalizable (positive coefficient)!*

In conclusion: for $n > 4$ the scalar ϕ^4-theory is nonrenormalizable and $[\lambda] < 0$, for $n = 4$ it is renormalizable and $[\lambda] = 0$, and for $n < 4$ it is superrenormalizable and $[\lambda] > 0$. These conclusions agree nicely with the general discussion on page 28 of these lecture notes.

3 The Dirac field

During the next three and a half lectures Chapter 3 of Peskin & Schroeder will be covered. We have seen various aspects of scalar theories, describing spin-0 particles. However, most particles in nature have spin $\neq 0$.

> (11a) *Question: how should we find Lorentz-invariant equations of motion for fields that do not transform as scalars?*

Consider to this end an n-component multiplet field $\Phi_a(x)$ with $a = 1, \cdots, n$, which has the following linear transformation characteristic under Lorentz transformations:

$$\Phi_a(x) \xrightarrow{\text{Lorentz transf.}} M_{ab}(\Lambda)\,\Phi_b(\Lambda^{-1}x)$$

with summation over the repeated index implied. A compact way of writing this is

$$\Phi(x) \xrightarrow{\text{Lorentz transf.}} M(\Lambda)\Phi(\Lambda^{-1}x) \ .$$

In the case of scalar fields the transformation matrix $M(\Lambda)$ was simply the identity matrix. In order to find different solutions, we make use of the fact that the Lorentz transformations form a group: $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu}$ is the unit element, $\Lambda^{-1} = \Lambda^T$ is the inverse, and for Λ_1 and Λ_2 being Lorentz transformations also $\Lambda_3 = \Lambda_2\Lambda_1$ is a Lorentz transformation. The transformation matrices $M(\Lambda)$ should reflect this group structure:

$$M(g) \ = \ I_n \quad , \quad M(\Lambda^{-1}) \ = \ M^{-1}(\Lambda) \qquad \text{and} \qquad M(\Lambda_2\Lambda_1) \ = \ M(\Lambda_2)M(\Lambda_1) \ ,$$

where I_n is the $n\times n$ identity matrix. To phrase it differently, the transformation matrices $M(\Lambda)$ should form an n-dimensional representation of the Lorentz group!

The continuous Lorentz group (rotations and boosts): transformations that lie infinitesimally close to the identity transformation define a vector space, called the Lie algebra of the group. The basis vectors for this vector space are called the generators of the Lie algebra. The Lorentz group has six generators $J^{\mu\nu} = -J^{\nu\mu}$, three for boosts and three for rotations. These generators are antisymmetric, as a result of $\Lambda^{-1} = \Lambda^T$, and they satisfy the following set of fundamental commutation relations:

$$\big[J^{\mu\nu}, J^{\rho\sigma}\big] \ = \ i\big(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\big) \ .$$

The three generators belonging to the boosts and the three generators belonging to the rotations are given by

$$K^j \ \equiv \ J^{0j} \quad \text{respectively} \quad J^j \ \equiv \ \frac{1}{2}\epsilon^{jkl}J^{kl} \ \Rightarrow \ J^{jk} \ = \ \epsilon^{jkl}J^l \qquad (j,k,l = 1,\cdots,3) \ ,$$

with summation over the repeated spatial indices implied. The latter generators, which span the Lie algebra of the rotation group, satisfy the fundamental commutation relations

$$\big[J^j, J^k\big] \ = \ i\epsilon^{jkl}J^l \ .$$

(11a) *In fact it is proven in Ex. 15 that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers (j_+, j_-), where both j_+ and j_- correspond to a representation of the rotation group. The sum $j_+ + j_-$ should be interpreted as the spin of the representation, since it corresponds to the actual rotations contained in the Lorentz group.*

A finite Lorentz transformation is then in general given by $\exp(-i\omega_{\mu\nu}J^{\mu\nu}/2)$, where the antisymmetric tensor $\omega_{\mu\nu} \in \mathbb{R}$ represents the Lorentz transformation. For instance:

$$\omega_{12} = -\omega_{21} = \delta\theta \quad , \quad \text{rest} = 0 \quad \Rightarrow \quad \omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta\theta & 0 \\ 0 & \delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for an infinitesimal rotation about the z-axis (see Ex. 14), and

$$\omega_{01} = -\omega_{10} = \delta v \quad , \quad \text{rest} = 0 \quad \Rightarrow \quad \omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \delta v & 0 & 0 \\ \delta v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for an infinitesimal boost along the x-direction (see Ex. 14).

The task at hand is now to find the matrix representations of the generators of the Lorentz group.

Examples: in Ex. 14 it is proven that

- $(J^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\mu}{}_{\beta}g^{\nu\alpha})$ are the six generators that describe Lorentz transformations of contravariant four-vectors:

  $$x^{\alpha} \xrightarrow{\text{Lorentz transf.}} \Lambda^{\alpha}{}_{\beta}\, x^{\beta} = \left[\exp(-i\omega_{\mu\nu}J^{\mu\nu}/2)\right]^{\alpha}{}_{\beta}\, x^{\beta} \approx \left[g^{\alpha}{}_{\beta} - \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}\right] x^{\beta} \ .$$

  This implies that $g^{\alpha}{}_{\beta} - \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta}$ represents the infinitesimal form of the Lorentz transformation matrix $\Lambda^{\alpha}{}_{\beta}$, as is indeed the case.

- $J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$ are the six generators in coordinate space, which describe the infinitesimal Lorentz transformations of scalar fields

  $$\phi(x) \xrightarrow{\text{Lorentz transf.}} \phi(\Lambda^{-1}x) \approx \phi(x) - \frac{1}{2}\,\omega_{\rho\sigma}[x^{\sigma}\partial^{\rho} - x^{\rho}\partial^{\sigma}]\,\phi(x) \ ,$$

  as derived on page 11.