3 The Dirac field

During the next three and a half lectures Chapter 3 of Peskin & Schroeder will be covered. We have seen various aspects of scalar theories, describing spin-0 particles. However, most particles in nature have spin $\neq 0$.

(11a) Question: how should we find Lorentz-invariant equations of motion for fields that do not transform as scalars?

Consider to this end an *n*-component multiplet field $\Phi_a(x)$ with $a = 1, \dots, n$, which has the following linear transformation characteristic under Lorentz transformations:

$$\Phi_a(x) \xrightarrow{\text{Lorentz transf.}} M_{ab}(\Lambda) \Phi_b(\Lambda^{-1}x)$$

with summation over the repeated index implied. A compact way of writing this is

$$\Phi(x) \xrightarrow{\text{Lorentz transf.}} M(\Lambda) \Phi(\Lambda^{-1}x)$$

In the case of scalar fields the transformation matrix $M(\Lambda)$ was simply the identity matrix. In order to find different solutions, we make use of the fact that the Lorentz transformations form a group: $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu}$ is the unit element, $\Lambda^{-1} = \Lambda^{T}$ is the inverse, and for Λ_{1} and Λ_{2} being Lorentz transformations also $\Lambda_{3} = \Lambda_{2}\Lambda_{1}$ is a Lorentz transformation. The transformation matrices $M(\Lambda)$ should reflect this group structure:

$$M(g) = I_n$$
, $M(\Lambda^{-1}) = M^{-1}(\Lambda)$ and $M(\Lambda_2\Lambda_1) = M(\Lambda_2)M(\Lambda_1)$,

where I_n is the $n \times n$ identity matrix. To phrase it differently, the transformation matrices $M(\Lambda)$ should form an n-dimensional representation of the Lorentz group!

The continuous Lorentz group (rotations and boosts): transformations that lie infinitesimally close to the identity transformation define a vector space, called the <u>Lie</u> <u>algebra</u> of the group. The <u>basis vectors</u> for this vector space are called the <u>generators</u> of the Lie algebra. The Lorentz group has six generators $J^{\mu\nu} = -J^{\nu\mu}$, three for boosts and three for rotations. These generators are antisymmetric, as a result of $\Lambda^{-1} = \Lambda^T$, and they satisfy the following set of fundamental commutation relations:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \,.$$

The three generators belonging to the boosts and the three generators belonging to the rotations are given by

$$K^j \equiv J^{0j}$$
 respectively $J^j \equiv \frac{1}{2} \epsilon^{jkl} J^{kl} \Rightarrow J^{jk} = \epsilon^{jkl} J^l$ $(j, k, l = 1, \dots, 3)$,

with summation over the repeated spatial indices implied. The latter generators, which span the Lie algebra of the rotation group, satisfy the fundamental commutation relations

$$\left[J^j, J^k\right] = i \epsilon^{jkl} J^l \; .$$

(11a) In fact it is proven in Ex. 15 that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers (j_+, j_-) , where both j_+ and j_- correspond to a representation of the rotation group. The sum $j_+ + j_-$ should be interpreted as the spin of the representation, since it corresponds to the actual rotations contained in the Lorentz group.

A finite Lorentz transformation is then in general given by $\exp(-i\omega_{\mu\nu}J^{\mu\nu}/2)$, where the antisymmetric tensor $\omega_{\mu\nu} \in \mathbb{R}$ represents the Lorentz transformation. For instance:

$$\omega_{12} = -\omega_{21} = \delta\theta \quad , \quad \text{rest} = 0 \quad \Rightarrow \quad \omega^{\mu}_{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta\theta & 0 \\ 0 & \delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for an infinitesimal rotation about the z-axis (see Ex. 14), and

for an infinitesimal boost along the x-direction (see Ex. 14).

The task at hand is now to find the matrix representations of the generators of the Lorentz group.

Examples: in Ex.14 it is proven that

• $(J^{\mu\nu})^{\alpha}_{\ \beta} = i(g^{\mu\alpha}g^{\nu}_{\ \beta} - g^{\mu}_{\ \beta}g^{\nu\alpha})$ are the six generators that describe Lorentz transformations of contravariant four-vectors:

$$x^{\alpha} \xrightarrow{\text{Lorentz transf.}} \Lambda^{\alpha}_{\ \beta} x^{\beta} = \left[\exp(-i\omega_{\mu\nu}J^{\mu\nu}/2) \right]^{\alpha}_{\ \beta} x^{\beta} \approx \left[g^{\alpha}_{\ \beta} - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}_{\ \beta} \right] x^{\beta} .$$

This implies that $g^{\alpha}_{\ \beta} - \frac{i}{2} \omega_{\mu\nu} (J^{\mu\nu})^{\alpha}_{\ \beta} = g^{\alpha}_{\ \beta} + \omega^{\alpha}_{\ \beta}$ represents the infinitesimal form of the Lorentz transformation matrix $\Lambda^{\alpha}_{\ \beta}$, as is indeed the case.

• $J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$ are the six generators in coordinate space, which describe the infinitesimal Lorentz transformations of scalar fields

$$\phi(x) \quad \frac{\text{Lorentz transf.}}{\longrightarrow} \quad \phi(\Lambda^{-1}x) \; \approx \; \phi(x) - \frac{1}{2} \, \omega_{\rho\sigma} [x^{\sigma} \partial^{\rho} - x^{\rho} \partial^{\sigma}] \, \phi(x) \; ,$$

as derived on page 11.

Dirac's trick: introduce four $n \times n$ matrices γ^{μ} that are referred to as the γ -matrices of Dirac, which satisfy the Dirac algebra (Clifford algebra)

(11b)

$$\left\{\gamma^{\mu},\gamma^{
u}
ight\} \equiv \gamma^{\mu}\gamma^{
u}+\gamma^{
u}\gamma^{\mu} = 2g^{\mu
u}I_n \; ,$$

with I_n the $n \times n$ identity matrix. In Ex. 14 it is proven that this implies that the $n \times n$ matrices $S^{\mu\nu} \equiv \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]$ form a representation of the generators $J^{\mu\nu}$ of the Lorentz group. In fact this is true for any spacetime dimensionality.

Four-dimensional solution to the Dirac algebra: since there are no solutions for n = 2 or 3, the first solution can be found for n = 4. Written in 2×2 block form in terms of the 2×2 identity matrix I_2 and the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

the solution reads

$$\gamma^{0} = \begin{pmatrix} 0 & I_{2} \\ I_{2} & 0 \end{pmatrix}$$
 and $\gamma^{j} = \begin{pmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{pmatrix}$ $(j = 1, 2, 3)$

in the <u>Weyl representation</u>, which is also known as the <u>chiral representation</u>. In fact there is an infinite number of such four-dimensional representations, since for any invertable 4×4 matrix V also $V\gamma^{\mu}V^{-1}$ is a solution. In the Weyl representation the generators of the Lorentz group have a block-diagonal form. The generators for boosts are given by

$$S^{0j} = \frac{i}{4} [\gamma^{0}, \gamma^{j}] = \frac{i}{2} \gamma^{0} \gamma^{j} = -\frac{i}{2} \begin{pmatrix} \sigma^{j} & 0 \\ 0 & -\sigma^{j} \end{pmatrix} \qquad (j = 1, 2, 3)$$

whereas the generators S^1 , S^2 and S^3 for rotations follow from

$$S^{jk} \stackrel{j \neq k}{=} \frac{i}{4} \begin{bmatrix} \gamma^{j}, \gamma^{k} \end{bmatrix} = -\frac{i}{4} \begin{pmatrix} \begin{bmatrix} \sigma^{j}, \sigma^{k} \end{bmatrix} & 0\\ 0 & \begin{bmatrix} \sigma^{j}, \sigma^{k} \end{bmatrix} \end{pmatrix} = \epsilon^{jkl} \begin{pmatrix} \frac{1}{2}\sigma^{l} & 0\\ 0 & \frac{1}{2}\sigma^{l} \end{pmatrix} \equiv \epsilon^{jkl}S^{l} \quad (j, k = 1, 2, 3)$$
$$\Rightarrow \quad S^{l} = \begin{pmatrix} \frac{1}{2}\sigma^{l} & 0\\ 0 & \frac{1}{2}\sigma^{l} \end{pmatrix} \equiv \frac{1}{2}\Sigma^{l} \qquad (l = 1, 2, 3) .$$

The generators for rotations look like twice replicated two-dimensional representations of the rotation group. We will come back to this point later on. As a result of the properties

$$(\gamma^0)^{\dagger} = \gamma^0$$
, $(\gamma^j)^{\dagger} = -\gamma^j$ $(j = 1, 2, 3) \Rightarrow (\gamma^{\mu})^{\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$,

the generators of the Lorentz group satisfy

$$(S^{\mu\nu})^{\dagger} = -\frac{i}{4} \left[(\gamma^{\nu})^{\dagger}, (\gamma^{\mu})^{\dagger} \right] = \frac{i}{4} \left[(\gamma^{\mu})^{\dagger}, (\gamma^{\nu})^{\dagger} \right] = \gamma^{0} S^{\mu\nu} \gamma^{0}$$

This means that the generators of rotations are hermitian, since $(S^{jk})^{\dagger} = S^{jk}$, indicating that rotations preserve normalization. On the other hand, the generators of boosts are non-hermitian, since $(S^{0j})^{\dagger} = -S^{0j}$, indicating that boosts do not preserve normalization owing to the Lorentz contraction of spatial volumes.

Dirac spinors and adjoint Dirac spinors: a four-component field $\psi(x)$ that Lorentz transforms according to this four-dimensional representation of the Lorentz group is called a Dirac spinor:

$$\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda_{1/2} \psi(\Lambda^{-1}x) \quad \text{with} \quad \Lambda_{1/2} = \exp(-i\omega_{\mu\nu}S^{\mu\nu}/2) .$$

The adjoint Dirac spinor $\overline{\psi}(x)$ is defined as

$$\bar{\psi}(x) \equiv \psi^{\dagger}(x) \gamma^{0}$$

and therefore transforms as

$$ar{\psi}(x) \ \ {{\rm Lorentz\ transf.}\over \longrightarrow} \ \ ar{\psi}(\Lambda^{-1}x) \, \Lambda^{-1}_{1/2} \ ,$$

 since

$$\gamma^{0} \Lambda^{\dagger}_{1/2} \gamma^{0} = \gamma^{0} \exp\left(i\omega_{\mu\nu} [S^{\mu\nu}]^{\dagger}/2\right) \gamma^{0} \xrightarrow{(S^{\mu\nu})^{\dagger} = \gamma^{0} S^{\mu\nu} \gamma^{0}} \exp(i\omega_{\mu\nu} S^{\mu\nu}/2) = \Lambda^{-1}_{1/2} .$$

Using the important γ -matrix property

$$\begin{bmatrix} \gamma^{\mu}, S^{\rho\sigma} \end{bmatrix} = \frac{i}{2} \begin{bmatrix} \gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma} \end{bmatrix} = \frac{i}{2} (\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}) = i(g^{\mu\rho}\gamma^{\sigma} - g^{\mu\sigma}\gamma^{\rho})$$
$$= i(g^{\rho\mu}g^{\sigma}{}_{\nu} - g^{\rho}{}_{\nu}g^{\sigma\mu})\gamma^{\nu} = (J^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} ,$$

the following infinitesimal Lorentz-transformation identity holds up to $\mathcal{O}(\omega)$:

$$\left(I_4 + \frac{i}{2}\,\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^{\mu}\left(I_4 - \frac{i}{2}\,\omega_{\alpha\beta}S^{\alpha\beta}\right) \approx \left[g^{\mu}_{\nu} - \frac{i}{2}\,\omega_{\rho\sigma}(J^{\rho\sigma})^{\mu}_{\nu}\right]\gamma^{\nu}.$$

This reflects that for finite transformations

$$\Lambda_{1/2}^{-1} \gamma^{\mu} \Lambda_{1/2} = \Lambda^{\mu}_{\ \nu} \gamma^{\nu} ,$$

which indicates that γ^{μ} transforms like a contravariant four-vector provided it is properly contracted with Dirac spinors and adjoint Dirac spinors.

(11d) Consequently, $\psi(x)$, $\gamma^{\mu}\partial_{\mu}\psi(x)$, $\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}\psi(x)$, \cdots are good building blocks for constructing a Lorentz-invariant wave equation for Dirac spinors, whereas $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x)$, \cdots are scalar building blocks for obtaining the corresponding Lagrangian.

3.1 Towards the Dirac equation $(\S 3.2 \text{ and } 3.4 \text{ in the book})$

(11d) **Dirac-field bilinears** (currents): the interesting objects in spinor space are of the form $\bar{\psi}\Gamma\psi$, with Γ a 4×4 matrix that consists of a sequence of γ -matrices. These objects are called <u>bilinears</u> or <u>currents</u>. They will be needed to construct Lagrangians that include interactions with other fields, like $\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)$ for interactions with a vector field and $\bar{\psi}(x)\gamma^{\mu}\gamma^{\nu}\psi(x)h_{\mu\nu}(x)$ for interactions with a tensor field. A basis for Γ that satisfies $\Gamma^{\dagger} = \gamma^{0}\Gamma\gamma^{0}$ is given by the following combinations of γ -matrices:

$$I_4$$
 , $\gamma^\mu, \, \sigma^{\mu
u} = rac{i}{2} \left[\gamma^\mu, \gamma^
u
ight]$, $\gamma^\mu\gamma^5, \, i\,\gamma^5$,

where

$$\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon^{\mu
u
ho\sigma} \gamma_\mu \gamma_
u \gamma_
ho \gamma_\sigma$$

in terms of the totally antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of (0123)} \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of (0123)} \\ 0 & \text{else} \end{cases}$$

Properties of γ^5 : the properties of the matrix γ^5 will prove important for the description of weak interactions. They read:

$$(\gamma^5)^{\dagger} = \gamma^5$$
, $(\gamma^5)^2 = I_4$ and $\{\gamma^5, \gamma^{\mu}\} = 0$ $(\mu = 0, \dots, 3)$
 $\Rightarrow [\gamma^5, S^{\mu\nu}] = 0 \Rightarrow [\gamma^5, \Lambda_{1/2}] = 0$.

This means that γ^5 is a "Lorentz scalar" if it is properly contracted with Dirac spinors and adjoint Dirac spinors. Since γ^5 commutes with the generators of Lorentz transformations in spinor space, eigenvectors of γ^5 corresponding to different eigenvalues transform independently (i.e. without mixing).

11c According to Schur's lemma this implies that the Dirac representation of the Lorentz group is reducible, i.e. we should be able to write it in terms of two independent lower-dimensional chiral representations.

In the Weyl representation of the γ -matrices, the matrix γ^5 has the following form in terms of 2×2 blocks:

$$\gamma^5 = \left(egin{array}{cc} -I_2 & 0 \ 0 & I_2 \end{array}
ight).$$

As a result,

$$P_R \equiv \frac{1}{2}(I_4 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}$$
 and $P_L \equiv \frac{1}{2}(I_4 - \gamma^5) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$

are (chiral) projection operators on 2-dimensional vectors ψ_R and ψ_L :

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow P_R \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \quad \text{and} \quad P_L \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix},$$

which are eigenvectors of γ^5 corresponding to the <u>chirality</u> eigenvalues +1 and -1. In terms of these <u>right-handed Weyl spinors</u> ψ_R and <u>left-handed Weyl spinors</u> ψ_L the infinitesimal Lorentz transformations of ψ can be rewritten as (cf. Ex. 15 and the generators that are given on page 91)

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{Lorentz transf.}} \begin{pmatrix} [I_2 - i\vec{\theta} \cdot \vec{\sigma}/2 - \vec{\beta} \cdot \vec{\sigma}/2] \psi_L \\ [I_2 - i\vec{\theta} \cdot \vec{\sigma}/2 + \vec{\beta} \cdot \vec{\sigma}/2] \psi_R \end{pmatrix}.$$

The real infinitesimal parameters $\vec{\theta}$ and $\vec{\beta}$ coincide with the parameters $\delta \vec{\alpha}$ and $\delta \vec{v}$ that were used in Ex. 14. We see that the Weyl spinors transform independently, which indeed implies that the four-dimensional Dirac representation of the Lorentz group is reducible and can be split into two two-dimensional representations. For later use we mention the following identity for the Pauli spin matrices:

$$\begin{split} \sigma^2 \vec{\sigma}^* &= -\vec{\sigma} \sigma^2 \quad \Rightarrow \quad \sigma^2 \psi_L^* \quad \xrightarrow{\text{Lorentz transf.}} \quad \sigma^2 \left[I_2 + i\vec{\theta} \cdot \vec{\sigma}^* / 2 - \vec{\beta} \cdot \vec{\sigma}^* / 2 \right] \psi_L^* \\ &= \left[I_2 - i\vec{\theta} \cdot \vec{\sigma} / 2 + \vec{\beta} \cdot \vec{\sigma} / 2 \right] \sigma^2 \psi_L^* \end{split}$$

which indicates that $\sigma^2 \psi_L^*$ transforms like a right-handed Weyl spinor.

Chirality and currents: from the 4×4 matrix basis on the previous page all possible hermitian currents can be obtained as $\bar{\psi}\Gamma\psi$, since $(\bar{\psi}\Gamma\psi)^{\dagger} = \psi^{\dagger}\Gamma^{\dagger}\gamma^{0}\psi \xrightarrow{\Gamma^{\dagger}=\gamma^{0}\Gamma\gamma^{0}} \bar{\psi}\Gamma\psi$. These currents and their associated continuous Lorentz transformations read:

$$\underline{scalar\ current}: \ j_{S}(x) \equiv \overline{\psi}(x)\psi(x) \xrightarrow{\text{Lorentz transf.}} j_{S}(\Lambda^{-1}x) ,$$

$$\underline{vector\ current}: \ j_{V}^{\mu}(x) \equiv \overline{\psi}(x)\gamma^{\mu}\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^{\mu}{}_{\alpha}j_{V}^{\alpha}(\Lambda^{-1}x) ,$$

$$\underline{tensor\ current}: \ j_{T}^{\mu\nu}(x) \equiv \overline{\psi}(x)\sigma^{\mu\nu}\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}j_{T}^{\alpha\beta}(\Lambda^{-1}x) ,$$

$$\underline{axial\ vector\ current}: \ j_{A}^{\mu}(x) \equiv \overline{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^{\mu}{}_{\alpha}j_{A}^{\alpha}(\Lambda^{-1}x) ,$$

$$\underline{pseudo\ scalar\ current}: \ j_{P}(x) \equiv i\overline{\psi}(x)\gamma^{5}\psi(x) \xrightarrow{\text{Lorentz transf.}} j_{P}(\Lambda^{-1}x) ,$$

making use of the fact that $\Lambda_{1/2}^{-1} \gamma^{\mu} \Lambda_{1/2} = \Lambda_{\ \nu}^{\mu} \gamma^{\nu}$ and $\Lambda_{1/2}^{-1} \gamma^{5} \Lambda_{1/2} = \gamma^{5}$.

Using the chiral projection operators $P_{L/R}$, the Dirac spinors can be decomposed into chiral components according to

$$P_{L/R}\,\psi(x) \equiv \psi_{L/R} \quad \Rightarrow \quad \bar{\psi}_{L/R} \equiv (\psi_{L/R})^{\dagger}\gamma^{0} = \psi^{\dagger}P_{L/R}\gamma^{0} = \psi^{\dagger}\gamma^{0}P_{R/L} = \bar{\psi}P_{R/L} \; .$$

This results in the following chiral decompositions of the currents.

• The scalar current mixes left- and right-handed Weyl spinors, since

$$\bar{\psi}\psi = \bar{\psi}(P_R + P_L)\psi = \bar{\psi}(P_R^2 + P_L^2)\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L .$$

This will prove important for the description of massive spin-1/2 particles.

• The vector current treats left- and right-handed Weyl spinors on equal footing, since

$$\bar{\psi}\gamma^{\mu}\psi = \bar{\psi}\gamma^{\mu}(P_R^2 + P_L^2)\psi = \bar{\psi}(P_L\gamma^{\mu}P_R + P_R\gamma^{\mu}P_L)\psi = \bar{\psi}_R\gamma^{\mu}\psi_R + \bar{\psi}_L\gamma^{\mu}\psi_L .$$

This will prove important for vector-like theories, describing for instance the electromagnetic and strong interactions.

• Similarly the tensor current mixes left- and right-handed Weyl spinors:

$$\bar{\psi}\sigma^{\mu\nu}\psi \ = \ \bar{\psi}_L\sigma^{\mu\nu}\psi_R + \bar{\psi}_R\sigma^{\mu\nu}\psi_L \ .$$

This is needed for describing Lorentz transformations, as we have seen already.

• The axial vector current treats left- and right-handed Weyl spinors in opposite ways:

$$\begin{split} \bar{\psi}\gamma^{\mu}\gamma^{5}\psi &= \bar{\psi}\gamma^{\mu}\gamma^{5}(P_{R}^{2}+P_{L}^{2})\psi = \bar{\psi}(P_{L}\gamma^{\mu}\gamma^{5}P_{R}+P_{R}\gamma^{\mu}\gamma^{5}P_{L})\psi \\ &= \bar{\psi}_{R}\gamma^{\mu}\gamma^{5}\psi_{R} + \bar{\psi}_{L}\gamma^{\mu}\gamma^{5}\psi_{L} \quad \underline{\gamma^{5}\psi_{R/L} = \pm\psi_{R/L}} \quad \bar{\psi}_{R}\gamma^{\mu}\psi_{R} - \bar{\psi}_{L}\gamma^{\mu}\psi_{L} \ . \end{split}$$

This will prove important for chiral theories, like the one that describes weak interactions.

• Similarly the pseudo scalar current decomposes according to

$$iar{\psi}\gamma^5\psi = iar{\psi}_L\gamma^5\psi_R + iar{\psi}_R\gamma^5\psi_L = iar{\psi}_L\psi_R - iar{\psi}_R\psi_L$$

This will prove important in describing interactions between spin-0 and spin-1/2 particles.

Handy combinations of such currents are given by the left/right-handed vector currents

$$j^{\mu}_{L/R}(x) \equiv \bar{\psi}(x)\gamma^{\mu}P_{L/R}\psi(x) = \bar{\psi}_{L/R}(x)\gamma^{\mu}\psi_{L/R}(x) ,$$

which will feature in the Standard Model of electroweak interactions.

(11e) **Dirac equation**: let's now try to construct a Lorentz-invariant wave equation that has the Klein-Gordon equation built in. The simplest candidate is the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu}-m)\psi(x) = 0$$
 .

This is a first order differential equation, whereas the Klein-Gordon equation was a second order equation. This is possible because γ^{μ} behaves like a vector without actually introducing a preferred direction, which is not possible in scalar theories!

Proof: first of all

$$0 = (i\gamma^{\nu}\partial_{\nu} + m)(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = -(\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu} + m^{2})\psi(x)$$
$$= -\left(\frac{1}{2}\left\{\gamma^{\mu},\gamma^{\nu}\right\}\partial_{\mu}\partial_{\nu} + m^{2}\right)\psi(x) \quad \underline{\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}I_{4}}{-(\Box + m^{2})\psi(x)}$$

so the Klein-Gordon equation is indeed built in! Secondly, under continuous Lorentz transformations a Dirac spinor transforms according to $\psi(x) \to \psi'(x) = \Lambda_{1/2} \psi(\Lambda^{-1}x)$. If $\psi(x)$ satisfies the Dirac equation then it follows that

$$\begin{aligned} &\forall x (i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \Rightarrow (i\gamma^{\mu}\partial_{\mu} - m)\Lambda_{1/2}\psi(\Lambda^{-1}x) = \Lambda_{1/2}(i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}\partial_{\mu} - m)\psi(\Lambda^{-1}x) \\ &= \Lambda_{1/2} \Big[i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}(\partial_{\nu}\psi)(\Lambda^{-1}x) - m\psi(\Lambda^{-1}x)\Big] \\ &= \Lambda_{1/2} \Big[i\gamma^{\nu}\partial_{\nu}\psi - m\psi\Big](\Lambda^{-1}x) = 0 \end{aligned}$$

$$\Rightarrow \quad (i\gamma^{\mu}\partial_{\mu} - m)\psi'(x) = 0 \; .$$

If the field $\psi(x)$ satisfies the Dirac equation then so does the Lorentz transformed field $\psi'(x)$, as required for having a Lorentz invariant wave equation.

In the Weyl representation the Dirac equation reads

$$0 = (i\gamma^{\mu}\partial_{\mu} - m)\psi = \begin{pmatrix} -mI_2 & i(I_2\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \\ i(I_2\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) & -mI_2 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \equiv \begin{pmatrix} -mI_2 & i\sigma^{\mu}\partial_{\mu} \\ i\bar{\sigma}^{\mu}\partial_{\mu} & -mI_2 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

using the compact notation

$$\sigma^{\mu} \equiv (I_2, \vec{\sigma}) \quad \text{and} \quad \bar{\sigma}^{\mu} \equiv (I_2, -\vec{\sigma}) \quad \Rightarrow \quad \gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$$

From this we conclude that

11e the two representations associated with ψ_L and ψ_R are mixed by the mass term in the Dirac equation! In the massless case the Dirac equation splits into two independent wave equations for ψ_L and ψ_R , the so-called Weyl equations

$$i ar{\sigma}^\mu \partial_\mu \psi_L(x) = 0$$
 and $i \sigma^\mu \partial_\mu \psi_R(x) = 0$.

The Dirac Lagrangian: the Lagrangian that corresponds to the Dirac equation reads

$$\mathcal{L}_{\text{Dirac}}(x) = \bar{\psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\psi(x)$$

Proof: the Euler-Lagrange equations for the $\bar{\psi}$ and ψ fields are given by

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = -(i\gamma^{\mu} \partial_{\mu} - m)\psi = 0 ,$$

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = \partial_{\mu} (i\bar{\psi}\gamma^{\mu}) + m\bar{\psi} = \bar{\psi} (i\overleftarrow{\partial}_{\mu}\gamma^{\mu} + m) = 0$$

which are indeed the Dirac equation and the corresponding adjoint equation

$$0 = \left[(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) \right]^{\dagger}\gamma^{0} = -i\left(\partial_{\mu}\psi^{\dagger}(x)\right)\gamma^{\mu}\gamma^{0} - m\psi^{\dagger}(x)\gamma^{0} = -\bar{\psi}(x)(i\overleftarrow{\partial}_{\mu}\gamma^{\mu} + m) .$$

(11e) Conserved currents: in preparation for the quantization of the free Dirac theory and the derivation of its particle interpretation, we have a closer look at the conserved currents for the solutions $\psi(x)$ of the Dirac equation.

1. The vector current $j_V^{\mu}(x)$ is conserved.

 $\underline{\underline{\operatorname{Proof 1}}}_{:}: \partial_{\mu} j_{V}^{\mu} = (\partial_{\mu} \bar{\psi}) \gamma^{\mu} \psi + \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi \xrightarrow{\underline{\operatorname{Dirac eqns.}}} im \bar{\psi} \psi - im \bar{\psi} \psi = 0 .$ $\underline{\underline{\operatorname{Proof 2}}}_{:}: \text{ in Ex. 17 an alternative proof is given based on global U(1) invariance.}$

2. The axial vector current $j^{\mu}_{A}(x)$ is conserved if $\mathbf{m} = 0$.

<u>Proof 1</u>: $\partial_{\mu} j_{A}^{\mu} = (\partial_{\mu} \bar{\psi}) \gamma^{\mu} \gamma^{5} \psi - \bar{\psi} \gamma^{5} \gamma^{\mu} \partial_{\mu} \psi$ <u>Dirac eqns.</u> $2im \bar{\psi} \gamma^{5} \psi = 0$ if m = 0. Proof 2: in Ex.17 an alternative proof is given based on global chiral invariance.

3. The energy-momentum tensor $T^{\mu\nu}$ is conserved.

Only the spacetime coordinates of $\bar{\psi}(x)$ and $\psi(x)$ transform under translations, i.e. the spinors themselves do not transform. Consequently, the energy-momentum tensor $T^{\mu\nu}$ derived on page 8 will be conserved. This gives rise to four conserved charges, the field energy

$$H = \int d\vec{x} \mathcal{H} = \int d\vec{x} \left[\pi_{\psi} \dot{\psi} + \dot{\psi} \pi_{\bar{\psi}} - \mathcal{L}_{\text{Dirac}} \right] = \int d\vec{x} \pi_{\psi} \dot{\psi}$$

and field momentum

$$\vec{P} = -\int \mathrm{d}\vec{x} \left[\pi_{\psi} \vec{\nabla} \psi + (\vec{\nabla} \bar{\psi}) \pi_{\bar{\psi}} \right] = -\int \mathrm{d}\vec{x} \ \pi_{\psi} \vec{\nabla} \psi.$$

Here we used that in these Noether charges $\psi(x)$ should satisfy the Dirac equation, and that $\pi_{\psi} = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_0 \psi)} = i \bar{\psi} \gamma^0 = i \psi^{\dagger}$ as well as $\pi_{\bar{\psi}} = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_0 \bar{\psi})} = 0$.

From these conjugate momenta we can read off that out of the eight real degrees of freedom of the Dirac spinor $\psi(x)$ in fact four belong to the conjugate momentum.

4. Under continuous Lorentz transformations a Dirac spinor transforms as

$$\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda_{1/2} \psi(\Lambda^{-1}x) \stackrel{\text{inf.}}{\approx} \left[I_4 - \frac{i}{2} \,\omega_{\rho\sigma} S^{\rho\sigma} \right] \psi(x) - \frac{1}{2} \,\omega_{\rho\sigma} \left[x^{\sigma} \partial^{\rho} - x^{\rho} \partial^{\sigma} \right] \psi(x) ,$$

where the first term is typical for Dirac spinors and the second term is the same as for scalar fields. Bearing in mind that the Dirac Lagrangian is a Lorentz scalar, we can generalize the derivation on page 11 to arrive at the following six conserved Noether currents:

$$J^{\mu\rho\sigma}(x) = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_{\mu}\psi)} \left[x^{\rho}\partial^{\sigma} - x^{\sigma}\partial^{\rho} - iS^{\rho\sigma} \right] \psi(x) + \left[g^{\mu\rho}x^{\sigma} - g^{\mu\sigma}x^{\rho} \right] \mathcal{L}_{\text{Dirac}}(x)$$
$$= T^{\mu\sigma}x^{\rho} - T^{\mu\rho}x^{\sigma} + \bar{\psi}(x)\gamma^{\mu}S^{\rho\sigma}\psi(x) .$$

(11e) The last term in these conserved Noether currents is specific for Dirac theories. After quantization of the Dirac theory this term will help us to determine the spin of the particles described by the (free) Dirac field theory.

3.2 Solutions of the free Dirac equation (\S 3.3 in the book)

11f) Since solutions of the (free) Dirac equation automatically satisfy the Klein-Gordon equation, we can use the standard plane-wave (Fourier) decomposition in order to decouple the degrees of freedom as much as possible.

The positive-energy case: according to this decomposition we introduce

$$\psi_p(x) \equiv u(p)e^{-ip \cdot x}$$
 with $p^2 = m^2$ and $p^0 > 0 \Rightarrow p^{\mu} = \left(\sqrt{\vec{p}^2 + m^2}, \vec{p}\right) \equiv (E_{\vec{p}}, \vec{p})$.

The spinor u(p) then has to satisfy the Dirac equation in momentum space:

$$(\gamma^{\mu}p_{\mu} - m)u(p) \equiv (\not p - m)u(p) = 0$$

using Feynman slash notation. The claim is now that u(p) can be written as

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \overline{\sigma}} \xi \end{pmatrix},$$

with ξ an arbitrary normalized 2-dimensional vector. <u>Proof</u>: using that

$$\sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})} = \sqrt{(p^0 I_2 - \vec{p} \cdot \vec{\sigma})(p^0 I_2 + \vec{p} \cdot \vec{\sigma})} \xrightarrow{\{\sigma^{j}, \sigma^{k}\} = 2\delta_{jk}I_2} I_2 \sqrt{p_0^2 - \vec{p}^2} = m I_2 ,$$

it easily follows that

$$(\not p - m)u(p) = \begin{pmatrix} -mI_2 & p \cdot \sigma \\ p \cdot \bar{\sigma} & -mI_2 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} = 0 .$$