Quantum Field Theory 2 : the Standard Model

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Conventions: in the following we will use so-called natural units ($\hbar = c = \mu_0 = \epsilon_0 = 1$) by absorbing these constants in the relevant fields/quantities
$\Rightarrow$ a single scale remains: mass.
For example: $E \to E * 1/c^2$ (cf $mc^2 \to m$),
$p \to p * 1/c$ (cf $mc \to m$),
$t \to t * c^2/\hbar$ (cf $\lambda_{compton}/c = \hbar/mc^2 \to 1/m$),
$r \to r * c/\hbar$ (cf $\lambda_{compton} = \hbar/mc \to 1/m$).
For the flat spacetime metric we will use the signature $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, with the Minkowski indices μ,ν running from 0 to 3. Repeated indices (of any kind) are implicitly summed over, unless stated otherwise.

Guided by the requirement of having local laws of nature, particles and their mutual interactions will be described by continuous (quantum) fields. These fields satisfy classical equations of motion, which can be best formulated in terms of Lagrangians for continuous systems. Such Lagrangians are particularly suitable for discussing symmetries, the cornerstones of relativistic quantum field theory. In order to have the relativity principle built in, we will be working with scalar Lagrangians to describe the physics that we observe.

set of fields $F_j(x)$, x = spacetime 4-vector

Generic: Lagrangian (density) $\mathcal{L}(\{F_j\},\{\partial_\mu F_j\})$, with $\partial_\mu F_j = \frac{\partial}{\partial x^\mu} F_j$
$\to$ equations of motion for F_j (Euler-Lagrange eqns.): $\boxed{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_j)}\right) = \frac{\partial\mathcal{L}}{\partial F_j}}$.

Examples: *) Real free Klein-Gordon field $\phi(x)$: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$
(1 d.o.f.) $\to$ $\Rightarrow$ ϕ satisfies the Klein-Gordon eqn. $\partial_\mu(\partial^\mu\phi) + m^2\phi = (\Box + m^2)\phi = 0$.
*) Complex free Klein-Gordon field $\phi(x)$: $\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*$
(2 d.o.f.) $\to$ $\Rightarrow$ ϕ and ϕ^* satisfy Klein-Gordon eqns. $(\Box + m^2)\phi = (\Box + m^2)\phi^* = 0$.
*) Free Dirac field $\psi(x)$: $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$, with γ^μ the γ-matrices of Dirac and $\bar\psi \equiv \psi^\dagger\gamma^0$ the Dirac-adjoint of ψ
(4 d.o.f.) $\to$ $\Rightarrow$ ψ and $\bar\psi$ satisfy the Dirac eqn. and adjoint Dirac eqn.
$(i\gamma^\mu\partial_\mu - m)\psi = 0$ and $\bar\psi(i\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$.
acting to the left

Noether's theorem for continuous symmetries: consider fields $F_j(x)$ that satisfy the Euler-Lagrange eqns. of $\mathcal{L}(\{F_j\},\{\partial_\mu F_j\})$ and apply the infinitesimal continuous transformations $F_j(x) \to F_j'(x) = F_j(x) + \vartheta \Delta F_j(x)$ (ϑ independent of x, infinitesimal). We speak of a symmetry under this transformation if $\mathcal{L}(x)$ changes by a 4-divergence, which leaves the eqn. of motion invariant. In that case

$$\vartheta \partial_\mu G^\mu \equiv \vartheta \Delta \mathcal{L} = \vartheta \left(\frac{\partial \mathcal{L}}{\partial F_j} \Delta F_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)} \underbrace{\Delta(\partial_\mu F_j)}_{\partial_\mu(\Delta F_j)} \right) = \vartheta \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)} \Delta F_j \right) + \vartheta \underbrace{\left[\frac{\partial \mathcal{L}}{\partial F_j} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)} \right) \right]}_{\text{E.-L. eqns.: } 0} \Delta F_j$$

$$\Rightarrow \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)} \Delta F_j - G^\mu \right) \equiv \partial_\mu j^\mu = 0},$$

"charge"

i.e. j^μ is a conserved current (Noether current) and $\int_V d\vec{x}\, j^0(x)$ is conserved locally since $\frac{d}{dt} \int_V d\vec{x}\, j^0 = - \int_V d\vec{x}\, \vec{\nabla}\cdot\vec{j} \overset{\text{Gauss}}{=} - \int_{S(V)} d\vec{S}\cdot\vec{j}$.

This forms the basis for the charge conservation laws that will feature prominently in a symmetry-based description of the electromagnetic, strong and weak interactions (and any interactions beyond the Standard Model).

Examples: *) Complex Klein-Gordon theory: $\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*$.

This Lagrangian is invariant under a continuous phase transformation

$$\phi \to \phi' = e^{i\vartheta}\phi \overset{\text{inf.}}{\approx} \phi + \vartheta \underbrace{(i\phi)}_{\Delta\phi}, \quad \phi^* \to \phi'^* = e^{-i\vartheta}\phi^* \overset{\text{inf.}}{\approx} \phi^* + \vartheta \underbrace{(-i\phi^*)}_{\Delta\phi^*}$$

$\Rightarrow \Delta\mathcal{L} = 0$, i.e. $G^\mu = 0$, and $j^\mu = i\phi\, \partial^\mu \phi^* - i\phi^* \partial^\mu \phi$ is conserved.

*) Free Dirac theory: $\mathcal{L}(\psi, \bar\psi, \partial_\mu \psi) = \bar\psi(i\gamma^\mu \partial_\mu - m)\psi$.

Also this Lagrangian is invariant under the transformation

$$\psi \to \psi' = e^{i\vartheta}\psi \overset{\text{inf.}}{\approx} \psi + \vartheta \underbrace{(i\psi)}_{\Delta\psi}, \quad \bar\psi \to \bar\psi' = e^{-i\vartheta}\bar\psi \overset{\text{inf.}}{\approx} \bar\psi + \vartheta \underbrace{(-i\bar\psi)}_{\Delta\bar\psi}$$

$\Rightarrow \Delta\mathcal{L} = 0$, i.e. $G^\mu = 0$, and $j_V^\mu = \bar\psi i \gamma^\mu (i\psi) + 0 = -\bar\psi \gamma^\mu \psi$ is conserved.

also: Abelian

These two phase transformations are referred to as "global $U(1)$ gauge transformations" [global: ϑ independent of x; $U(1)$: the same $e^{i\vartheta}$ is used for each component of the field]. Later on we will also encounter a generalization of this: global unitary transformations that mix different fields of the same type (such as 2 or 3 Dirac fields), referred to as non-Abelian gauge transf.

weak int.

strong int.

Conjugate momenta : $\pi_j(x) = \partial\mathcal{L}/\partial(\partial_0 F_j)$ conjugate momentum associated with F_j.

*) Free complex Klein-Gordon theory : $\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^* \equiv \pi$, $\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial_0\phi \equiv \pi^*$.

*) Free Dirac theory : $\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger \equiv \pi_\psi$, $\frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\psi})} = 0$

$\Rightarrow$ out of the 8 real d.o.f. of the Dirac spinor in fact 4 belong to the conjugate momentum !

Canonical quantization : the fields $F_j(x)$ and associated conjugate momenta $\pi_j(x)$ become operators that satisfy canonical commutation relations for bosonic fields F_j or anticommutation relations for fermionic fields F_j.

- Bosonic case, fields $\hat{\phi}_j(x)$: $[\hat{\phi}_j(t,\vec{x}), \hat{\pi}_k(t,\vec{y})] = i\delta_{jk}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  $[\hat{\phi}_j(t,\vec{x}), \hat{\phi}_k(t,\vec{y})] = [\hat{\pi}_j(t,\vec{x}), \hat{\pi}_k(t,\vec{y})] = 0$.
  equal-time commutation relations
- Fermionic case, fields $\hat{\psi}_{j,\alpha}(x)$: $\{\hat{\psi}_{j,\alpha}(t,\vec{x}), \hat{\pi}_{k,\beta}(t,\vec{y})\} = i\delta_{jk}\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  spinor label
  $\{\hat{\psi}_{j,\alpha}(t,\vec{x}), \hat{\psi}_{k,\beta}(t,\vec{y})\} = \{\hat{\pi}_{j,\alpha}(t,\vec{x}), \hat{\pi}_{k,\beta}(t,\vec{y})\} = 0$.
  equal-time anticommutation relations

This can also be formulated in terms of creation and annihilation operators, from which the particle interpretation of the fields follow: each field annihilates a certain type of "particle" and creates the corresponding type of "antiparticle".

Chapter 1 : Abelian and non-Abelian gauge theories

We will see that the Standard Model is built on the gauge principle, which was introduced in the course Quantum Field Theory in the context of Quantum Electrodynamics (QED).

§1.1 QED and the gauge principle

Consider the electromagnetic field in "vacuum" with charge density $\rho(x) = j_c^0(x) \in \mathbb{R}$ and current density $\vec{j}_c(x) \in \mathbb{R}^3$: $A^\mu(x) = (A^0(x), \vec{A}(x))$. vector potential

4-vector potential scalar potential

This e.m. field satisfies the electromagnetic wave equation $\boxed{\partial_\mu F^{\mu\nu}(x) = j_c^\nu(x)}$, with $\boxed{F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)}$ the antisymmetric electromagnetic field tensor. This wave equation summarizes the Maxwell equations in 4-vector language.

Gauge freedom: the physics described by the e.m. wave equation is unaffected by the following transformation of the e.m. field, $A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x)$, with $\chi(x) \in \mathbb{R}$ an arbitrary sufficiently smooth scalar function.

e.m. gauge transformation

Proof: $F^{\mu\nu}(x) \to F'^{\mu\nu}(x) = \partial^\mu A'^\nu(x) - \partial^\nu A'^\mu(x) = F^{\mu\nu}(x) + (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\chi(x) \stackrel{!}{=} F^{\mu\nu}(x)$.

The associated freedom to choose the e.m. field is called the gauge freedom.

Local charge conservation: $\partial_\nu j_c^\nu(x) = \partial_\nu \partial_\mu F^{\mu\nu}(x) = 0$, since $F^{\mu\nu} = -F^{\nu\mu}$. Hence, the current density $j_c^\mu(x)$ is a conserved current and the electric charge $\int_V d\vec{x}\, j_c^0(x) = \int_V d\vec{x}\, \rho(x)$ is conserved locally. Just as we have seen for Noether currents.

Electromagnetic Lagrangian: the Lagrangian density belonging to the e.m. wave equation is given by $\mathcal{L}_{e.m.}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - j_c^\mu(x) A_\mu(x)$.

Proof: $\frac{\partial \mathcal{L}_{e.m.}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} F_{\rho\sigma}\right) F^{\rho\sigma} = -F^{\mu\nu} \Rightarrow$ E.-L. eqn.: $-\partial_\mu F^{\mu\nu}(x) = -j_c^\nu(x)$.

We see that the e.m. field couples to the conserved current density caused by matter. The charged particles that matter is comprised of are all of the Dirac type (electrons, up quarks, down quarks). For a Dirac-type particle with charge q, the e.m. current density is given by the conserved current $j^\nu_{c,\,Dirac}(x) = q\bar{\psi}(x)\gamma^\nu\psi(x)$. In that case the complete Lagrangian density takes the form

$$\mathcal{L}_{QED}(x) = \overbrace{\bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x)}^{\text{free Dirac Lagr.}} \overbrace{- \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)}^{\text{free e.m. Lagr.}} \overbrace{- q\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)}^{\text{interaction}}$$

$$\equiv \boxed{\bar{\psi}(x)(iD_\mu\gamma^\mu - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \mathcal{L}_{QED}(x)},$$

with $D_\mu = \partial_\mu + iqA_\mu$. ← cf $-i\hat{p}_\mu + iqA_\mu = -i(\hat{p}_\mu - qA_\mu)$ in quantum mechanics

This last step we recognize as the concept of <u>minimal substitution</u>: the interaction between matter particles and e.m. fields is obtained by inserting $p^\mu \to p^\mu - qA^\mu$ in the free Lagrangian density.

<u>QED from a symmetry principle</u> (gauge invariance and the gauge principle): let's turn around the argument and alternatively start with the Dirac Lagrangian $\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$. As we have seen, this Lagrangian is invariant under the global $U(1)$ gauge transf. $\psi(x) \to e^{i\varphi}\psi(x)$, $\bar{\psi}(x) \to e^{-i\varphi}\bar{\psi}(x)$ ($\varphi \in \mathbb{R}$, independent of x). According to Noether's theorem this global gauge symmetry can be associated with a conserved current and charge, precisely what we need for e.m. interactions.

<u>Non-relativistic quantum mechanics</u>: this global gauge invariance of a free-fermion system simply underlines the unobservability of the absolute phase of a wave function (i.e. only relative phases are observable through interference).

<u>Relativistic quantum field theory</u> (postulate): demand this to hold locally, since relativistic quantum mechanics should be a local theory. As charges interact while being at non-zero distance, the local interactions should be mediated by force carriers.

$\Rightarrow$ <u>The gauge principle</u>: demand the gauge invariance to hold locally !

Local $U(1)$ gauge transf.: $\psi(x) \to e^{i\varphi(x)}\psi(x)$, $\bar{\psi}(x) \to e^{-i\varphi(x)}\bar{\psi}(x)$, with $\varphi(x)$ a real scalar field

$$\Rightarrow i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \to i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \underbrace{\bar{\psi}(x)\gamma^\mu\psi(x)\overbrace{[\partial_\mu\varphi(x)]}^{\text{covariant vector}}}_{\neq 0 \text{ in general } \Rightarrow \text{ not locally invariant}}.$$

Remedy: replace the ordinary derivative ∂_μ by a gauge covariant derivative (or short: <u>covariant derivative</u>) D_μ such that

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = e^{i\vartheta(x)} D_\mu \psi(x),$$

causing $D_\mu\psi(x)$ and $\psi(x)$ to transform similarly under local gauge transformations! This can be achieved by

gauge field

=0 in global case

$$D_\mu \equiv \partial_\mu + ig A_\mu(x), \quad \text{with } A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \vartheta(x).$$

~~cf minimal substitution~~ gauge coupling cf e.m. gauge transf. with $\chi(x) = -\frac{\vartheta(x)}{g}$

we are exploiting the gauge freedom here!

Proof: $D'_\mu \psi'(x) = (\partial_\mu + ig[A_\mu(x) - \frac{1}{g}\partial_\mu \vartheta(x)])e^{i\vartheta(x)}\psi(x)$
$= e^{i\vartheta(x)}(\partial_\mu + [i\partial_\mu \vartheta(x)] + igA_\mu(x) - [i\partial_\mu\vartheta(x)])\psi(x) = e^{i\vartheta(x)} D_\mu \psi(x).$

The Dirac Lagrangian density with $\partial_\mu \to D_\mu$ is now invariant under the collective local "U(1) gauge transformations" of $\psi(x)$, $\bar\psi(x)$ and $A_\mu(x)$. However, a gauge-invariant kinetic term for a dynamical gauge field still has to be added. To this end we define the gauge-invariant field tensor

$$ig F_{\mu\nu}(x) \equiv [D_\mu, D_\nu] = (\partial_\mu + igA_\mu(x))(\partial_\nu + igA_\nu(x)) - (\mu \leftrightarrow \nu)$$
$$= ig\,(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))$$

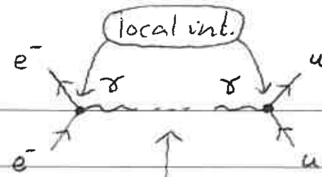
and add the term $-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$ to the Lagrangian, resulting in the same Lagrangian density as obtained from electromagnetism and minimal substitution. This locally invariant theory describes the fundamental e.m. interactions between matter fermions as being mediated by photons (gauge bosons):

$$\mathcal{L}^{int.}_{e.m.} = -g\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x). \leftarrow \text{this is called a gauge interaction}$$

Scaling the coupling strength: different matter particles can (and do) have different charges and therefore should have a different coupling strength to the e.m. field. This can be taken into account by scaling $\vartheta(x) \to Q\vartheta(x)$ and $g \to Qg$:

$\Rightarrow$ • the transformation property of $A_\mu(x)$ is unaltered (because of $\vartheta(x)/g$). so one and the same gauge field can couple to particles of different charge and can therefore mediate the interaction.

Graphically:



with the photon coupling to conserved currents!

a single gauge field mediates the finite-distance interaction between (differently) charged particles

- the covariant derivative changes to $D_\mu \to \partial_\mu + iQgA_\mu(x)$, thereby changing the interaction strength. Setting $g = |e|$ (unit charge) and $q = Q|e|$ (particle charge) we retrieve the e.m. interaction given on p. 4

<u>Massless gauge fields</u>: a massive gauge field would require a mass term $+\frac{1}{2}m_A^2 A_\mu(x)A^\mu(x)$ in the Lagrangian, which is not gauge invariant $\Rightarrow$ <u>without an additional trick local gauge invariance implies $m_A = 0$</u>!
This is fine for the e.m. field, which we know to be massless, but it will come back to haunt us once we discuss the Standard Model.

§1.2 Non-Abelian gauge theories

Motivated by the success of describing QED through the gauge principle, turning a global gauge symmetry (phase symmetry) into a local one, we now generalize this idea to other types of gauge transformations in an attempt to describe other fundamental interactions in nature. As in non-relativistic quantum mechanics we will use continuous unitary transformations to describe such extended symmetries, belonging to the class of <u>Lie groups</u>.

<u>Some relevant aspects of Lie groups</u>:

*) Transformations that lie infinitesimally close to the identity element,

$$g(\vartheta^1, \ldots, \vartheta^n) = 1 + i\vartheta^a T^a + \mathcal{O}(\vartheta^2) \qquad (\vartheta^1, \ldots, \vartheta^n \in \mathbb{R} \text{ infinitesimal};\ T^a \text{ hermitian}),$$

define a vector space called the <u>Lie algebra</u> of the group. The basis label the independent transformations

vectors $T^1, \ldots, T^n$ for this vector space are called the generators of the Lie algebra. These generators satisfy a characteristic set of fundamental commutation relations: $[T^a, T^b] = i C^{abc} T^c$. ← reflects the group structure

↑ structure constants

*) A finite group element is obtained by the repeated action of infinitesimal elements:

$$g(\varphi^1, \ldots, \varphi^n) = [g(\varphi^1/N, \ldots, \varphi^n/N)]^N \xrightarrow{N\to\infty} e^{i\varphi^a T^a},$$

using that $\lim_{N\to\infty} (1 + x/N)^N = \lim_{N\to\infty} (1 + N\frac{x}{N} + \frac{N(N-1)}{2!}\frac{x^2}{N^2} + \ldots) = e^x$.

$\vec{e}_K^{\,2} = 1$

Example: finite rotations of a spin-½ system about the $\vec{e}_K$-axis,

$\boxed{\text{Ex. 2}}\rightarrow$ $$g(\underbrace{-\theta^1, -\theta^2, -\theta^3}_{\equiv -\theta\vec{e}_K}) = e^{-i\theta\vec{e}_K\cdot\vec{\sigma}/2} = I_2 \cos(\theta/2) - i(\vec{e}_K\cdot\vec{\sigma}) \sin(\theta/2),$$

with $\sigma^1, \sigma^2, \sigma^3$ the Pauli spin matrices which satisfy the normalization condition $\{\sigma^a/2, \sigma^b/2\} = \frac{1}{2}\delta^{ab} I_2$ and the SU(2) commutation relations $[\sigma^a/2, \sigma^b/2] = i\epsilon^{abc}\sigma^c/2$. Here

SU(2) generators

$$\epsilon^{abc} = \begin{cases} +1 & \text{even perm. of } (123) \\ -1 & \text{odd perm. of } (123) \\ 0 & \text{else} \end{cases}$$ are the SU(2) structure constants.

*) Most important classes of gauge groups used in particle physics:

- rotation group in N dimensions SO(N).

  Properties: real N×N matrices with $OO^T = 1$, $\det O = 1$

  $\boxed{\text{Ex. 1}}\rightarrow$ ⇒ • $\frac{1}{2}N(N-1)$ generators = # planes in N dimensions,

  • purely imaginary generators with $\mathrm{Tr}(T^a) = 0$.

- unitary transformations of N-dimensional vectors: contains overall phase transf. [Abelian U(1) subgroup] and SU(N) transformations.

  ↑ cf QED gauge group

  Properties of SU(N≥2): complex N×N matrices with $UU^\dagger = 1$, $\det U = 1$

  ⇒ • $N^2 - 1$ generators,

  • $\mathrm{Tr}(T^a) = 0$,

  $\boxed{\text{Ex. 1}}\rightarrow$ • $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ (normalization),

  • $[T^a, T^b] = i f^{abc} T^c$, $f^{abc} \in \mathbb{R}$ totally antisymmetric.

  ≠ 0 : non-Abelian gauge group

These groups have been specified in their natural, defining representation. This is called the fundamental representation: it acts on N-dimensional vectors, with the generators represented by $N\times N$ matrices. In an $SU(N)$ gauge theory the fermionic matter fields live in the fundamental representation, i.e. they form

N-component multiplets of Dirac fields $\psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix}$, $\bar\psi = (\bar\psi_1, \dots, \bar\psi_N)$.

For instance (see later): $SU(2)$ doublets such as $\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L$,

$SU(3)$ quark colour triplets $\begin{pmatrix}q_R\\ q_G\\ q_B\end{pmatrix}$.

Another representation of the Lie algebra that will feature prominently is the adjoint representation: it acts on (N^2-1)-dimensional vectors, with the generators $(T^a)^{bc} = -i f^{abc}$ being $(N^2-1)\times(N^2-1)$ matrices.

For instance for $SU(2)$: $(T^a)^{bc} = -i\epsilon^{abc}$ are 3×3 matrices in the adjoint representation. Whereas the fundamental representation involves doublets, the adjoint representation deals with triplets.

The gauge fields live in the adjoint representation, e.g. $W_\mu^{1,2,3}$ triplet in $SU(2)$ or gluon octet $G_\mu^{1,\dots,8}$ in $SU(3)$.

Non-Abelian gauge theories: consider the Lagrangian

$$\mathcal{L}(x) = i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - m\bar\psi(x)\psi(x), \qquad \psi(x) = \begin{pmatrix}\psi_1(x)\\ \vdots \\ \psi_N(x)\end{pmatrix}$$

for an N-component multiplet of Dirac fields. This Lagrangian is invariant under global continuous $SU(N)$ transformations

$$\psi(x) \to \psi'(x) = U\psi(x), \qquad \bar\psi(x) \to \bar\psi'(x) = \bar\psi(x)U^\dagger$$

with $U = e^{i\theta^a T^a}$ $(\theta^1, \dots, \theta^{N^2-1} \in \mathbb{R}$, independent of x). fund. repr.

Generalized gauge principle: demand the $SU(N)$ invariance to also hold locally [i.e. $\theta^a \to \theta^a(x)$] in order to describe the interaction between matter and gauge bosons. Here N refers to the number of components in the multiplets linked by the gauge interactions.

experiments have to guide us towards the correct multiplet description !

Replace to this end the derivative ∂_μ by the covariant derivative
$\boxed{D_\mu \equiv \partial_\mu + i g W_\mu^a T^a \equiv \partial_\mu + i g W_\mu}$, acting on ψ in the fundamental repr.
Here g is the SU(N) gauge coupling (like $|e|$ in QED) and the gauge field $W_\mu = W_\mu^1 T^1 + \dots + W_\mu^{N^2-1} T^{N^2-1}$ lives in the SU(N) Lie algebra

$\uparrow$ N^2-1 independent transf. $\Rightarrow$ N^2-1 gauge-field components

$\Rightarrow$ $W_\mu^1, \dots, W_\mu^{N^2-1}$ form a multiplet in the adjoint representation!

To derive the transformation rule for W_μ, consider the local SU(N) transformation $\psi(x) \to \psi'(x) = U(x)\psi(x)$, $\bar\psi(x) \to \bar\psi'(x) = \bar\psi(x) U^\dagger(x)$, with $U(x) = e^{i\theta^a(x) T^a}$ and impose that

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = U(x) D_\mu \psi(x) \ \ \forall_\psi \ \Rightarrow \ D'_\mu U \psi = U D_\mu \psi \ \ \forall_\psi \ \Rightarrow \ \boxed{D'_\mu = U D_\mu U^{-1}}.$$

This implies that $(\partial_\mu + i g W'_\mu) = U(\partial_\mu + i g W_\mu) U^{-1} = \partial_\mu + U(\partial_\mu U^{-1}) + i g U W_\mu U^{-1}$

$$\Rightarrow \boxed{W'_\mu = U W_\mu U^{-1} - \frac{i}{g} U (\partial_\mu U^{-1})}.$$

Infinitesimally: $(1 + i\theta^b T^b) W_\mu^a T^a (1 - i\theta^{b'} T^{b'}) - \frac{i}{g}(1 + i\theta^b T^b)(-i\partial_\mu \theta^a T^a)(1 - i\theta^{b'} T^{b'})$

$$\overset{\text{inf.}}{\approx} W_\mu^a T^a - i\theta^b W_\mu^a \overbrace{[T^a, T^b]}^{i f^{abc} T^c} - \frac{1}{g}\partial_\mu \theta^a T^a = \Big(\underbrace{W_\mu^a - \frac{1}{g}\partial_\mu \theta^a}_{\text{QED style}} - \underbrace{f^{abc}\theta^b W_\mu^c}_{\text{non-Abelian}}\Big) T^a = W_\mu'^a T^a.$$

relabel: $a \leftrightarrow c$

This transformation characteristic of the gauge field contains $W_\mu^a - f^{abc}\theta^b W_\mu^c$ $= [\delta^{ac} + i (T^b)^{ac} \theta^b] W_\mu^c$, corresponding to the infinitesimal SU(N) transformation $e^{i\theta^b(x) T^b}$ acting on the adjoint representation.

Next we generalize the "kinetic term" for the gauge fields. To this end we define the SU(N) field tensor $W_{\mu\nu}$:

$$\underbrace{i g W_{\mu\nu} \equiv [D_\mu, D_\nu]}_{\text{Ricci identity}} \Rightarrow W_{\mu\nu} = -\frac{i}{g}(\partial_\mu + i g W_\mu)(\partial_\nu + i g W_\nu) + \frac{i}{g}(\mu \leftrightarrow \nu)$$

$$= i g \overbrace{[W_\mu, W_\nu]}^{\text{involves } [T^b, T^c] \neq 0} + \partial_\mu W_\nu - \partial_\nu W_\mu$$

$$= \Big(\underbrace{\partial_\mu W_\nu^a - \partial_\nu W_\mu^a}_{\text{cf. QED}} - \underbrace{g f^{abc} W_\mu^b W_\nu^c}_{\text{non-Abelian}}\Big) T^a \equiv W_{\mu\nu}^a T^a.$$

Transformation property: $ig W'_{\mu\nu} = [D'_\mu, D'_\nu] = [U D_\mu U^{-1}, U D_\nu U^{-1}] = U [D_\mu, D_\nu] U^{-1}$

$$= ig U W_{\mu\nu} U^{-1} \Rightarrow \boxed{W'_{\mu\nu} = U W_{\mu\nu} U^{-1}} \text{ (just like } D_\mu)$$

The so-called <u>Yang-Mills Lagrangian</u>

scalar, no open labels

$$\mathcal{L}_{YM} = -\tfrac{1}{2} \mathrm{Tr}(W_{\mu\nu} W^{\mu\nu}) = -\tfrac{1}{2} W^a_{\mu\nu} W^{b,\mu\nu} \mathrm{Tr}(T^a T^b) = -\tfrac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} \leftarrow \text{cf QED}$$

is then gauge invariant.

Proof: $\mathrm{Tr}(W'_{\mu\nu} W'^{\mu\nu}) = \mathrm{Tr}(U W_{\mu\nu} \underbrace{U^{-1} U}_{1} W^{\mu\nu} U^{-1}) \overset{\text{cyclic}}{=} \mathrm{Tr}(\underbrace{U^{-1} U}_{1} W_{\mu\nu} W^{\mu\nu}) = \mathrm{Tr}(W_{\mu\nu} W^{\mu\nu})$.

This results in the following Lagrangian that is invariant under local SU(N) gauge transformations:

$$\boxed{\mathcal{L}_{SU(N)} = i \bar\psi(x) \gamma^\mu D_\mu \psi(x) - m \bar\psi(x) \psi(x) - \tfrac{1}{2} \mathrm{Tr}(W_{\mu\nu}(x) W^{\mu\nu}(x)),}$$

$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$: N-plet in fundamental repr., $\psi_1 \cdots \psi_N$ are N Dirac fields,

W^a_μ $(a = 1, \ldots, N^2-1)$: N^2-1 gauge fields, which form a multiplet in the adjoint repr.,

T^a $(a = 1, \ldots, N^2-1)$: N^2-1 generators, acting on ψ in the fundamental repr. and on the gauge fields in the adjoint repr.

<u>Interactions</u>: *) <u>gauge interactions between matter fermions and gauge bosons</u> (like in QED): the term $i\bar\psi \gamma^\mu D_\mu \psi$ contains the interaction $-g \bar\psi(x) \gamma^\mu T^a \psi(x) W^a_\mu(x) = -g\, j^{a,\mu}(x) W^a_\mu(x)$, which involves N^2-1 gauge fields coupled to N^2-1 currents. These currents

Ex. 3 → are not conserved as the gauge bosons also interact amongst themselves. Remark: in the case of QED we had the freedom to change the interaction strength by scaling. The non-Abelian case is more restrictive: in order to leave the transformation property of the gauge bosons unaltered, also the <u>interactions have to remain unchanged under scaling</u>!

*) Yang-Mills interactions between three or four gauge bosons: the kinetic gauge-boson term $-\frac{1}{2}Tr(W_{\mu\nu}W^{\mu\nu}) = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}$ contains three distinct terms involving two, three and four gauge fields $-\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$,

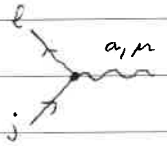
just like in QED

triple gauge couplings (TGC) $\to + g(\partial_\mu W^a_\nu)W^{b,\mu}W^{c,\nu}f^{abc}$,

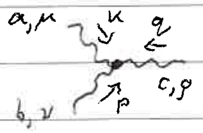
absent in QED: $Q_\gamma = 0$

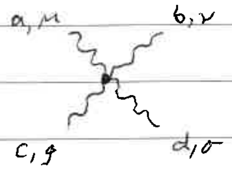
quartic gauge couplings (QGC) $\to -\frac{g^2}{4}f^{abe}f^{cde}W^a_\mu W^b_\nu W^{c,\mu}W^{d,\nu}$

Feynman rules for interaction vertices: —— = fermion , ∼∼ = gauge boson

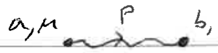

$= -ig\gamma^\mu(T^a)_{\ell j}$ (ℓ, j = multiplet label ; a = gauge-boson label),

used : $\partial_\lambda \to -i *$(incoming momentum)$_\lambda$


$= \underbrace{-gf^{abc}}_{-ig(T^a)^{bc}}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$,


$= -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$

$+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]$,


$= \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}\delta_{\ell j}$,

$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}\delta_{ab}$.

*) $\underline{SU(2)}$: matter doublets and gauge triplets , $T^a = \sigma^a/2$ $(a = 1,2,3)$

Pauli spin matrices

$$\Rightarrow W_\mu = \frac{1}{2}W^a_\mu\sigma^a \equiv \frac{1}{2}\vec{W}_\mu\cdot\vec{\sigma} = \frac{1}{2}\begin{pmatrix} W^3_\mu & \overbrace{W^1_\mu - iW^2_\mu}^{\equiv\sqrt{2}W^+_\mu} \\ \underbrace{W^1_\mu + iW^2_\mu}_{\equiv\sqrt{2}W^-_\mu} & -W^3_\mu \end{pmatrix} = \underbrace{T^3W^3_\mu}_{\text{diagonal}} + \frac{\overbrace{T^+W^+_\mu}^{\text{raising}} + \overbrace{T^-W^-_\mu}^{\text{lowering}}}{\sqrt{2}},$$

$$U(x) = e^{\frac{i}{2}\varphi^a(x)\sigma^a} \equiv e^{\frac{i}{2}\vec{\varphi}(x)\cdot\vec{\sigma}} = \cos(\varphi(x)/2)\,I_2 + i\frac{\vec{\varphi}(x)\cdot\vec{\sigma}}{\varphi(x)}\sin(\varphi(x)/2),$$

with $\varphi(x) \equiv |\vec{\varphi}(x)| = \sqrt{\varphi^a(x)\varphi^a(x)}$.

This gauge group will be needed for the gauge-theory description of the $\underline{\text{weak interactions}}$ and the associated $\underline{W^\pm \text{ and } Z \text{ gauge bosons}}$ [although only part of the Z gauge boson belongs to SU(2)].

*) <u>SU(3)</u> : matter triplets and gauge octets , $T^a = \lambda^a/2$ $(a = 1, \dots, 8)$
in terms of the Gell-Mann λ-matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

This gauge group is used in the gauge-theory description of the <u>strong interactions</u>, called <u>Quantum Chromodynamics (QCD)</u>. The associated gauge bosons are called <u>gluons</u>.

(indicated by)

<u>Chapter 2</u> : <u>Building towards the Standard Model</u>

In the next step we discuss the experimental input that has led to the gauge-group ingredients of the Standard Model, which is a gauge-theory description of the strong, weak and hypercharge interactions based on the gauge group $\underbrace{SU(3)_c}_{\text{strong int.}} \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{electroweak int.}}$. The electromagnetic interactions are automatically contained in this description, hidden in the electroweak $SU(2)_L \times U(1)_Y$ sector.

<u>$SU(3)_c$, c = colour</u> : matter triplets (quarks) and gauge octets (gluons)

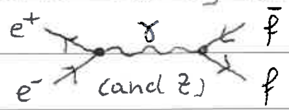
(Quantum Chromodynamics (<u>QCD</u>))

$\begin{pmatrix} \psi_R \\ \psi_G \\ \psi_B \end{pmatrix}$ $\qquad$ $(G_\mu^a \;\; (a = 1, \dots, 8))$

Gauge coupling : g_s = strong interaction strength.

Why? : *) spin-½ <u>quarks</u> confined inside hadrons have an <u>intrinsic property</u> (quantum number) called <u>colour</u> ⇒ <u>colour multiplets</u> !
This follows from the existence of the Δ^{++} baryon (= uuu bound state with L = 0 and S = 3/2) $\xrightarrow{\text{fermions}}$ at least three colours !
(spatial + spin symmetric)

*) Consider the reaction $e^+e^- \to f\bar{f}$ $(f\bar{f} = \mu^-\mu^+, \tau^-\tau^+, q\bar{q})$ for unpolarized e^+e^- beams, which is dominated by virtual photon exchange at $\mathcal{O}$(GeV) energies : (quarks)



.

$$\sigma^{\text{unpol.}}_{e^+e^-\to f\bar f} \underset{\text{QFT}}{\overset{\text{Ex. 22}}{=}} \frac{\pi \alpha^2 Q_f^2}{3E^2}\sqrt{1-m_f^2/E^2}\left[1+\frac{m_f^2}{2E^2}\right]\Theta(E-m_f) \xrightarrow{E \gg m_f} \frac{\pi\alpha^2}{3E^2}Q_f^2$$

(energy of e^+ and e^- in CM)

for each type of fermion f. production threshold

$$\text{Hence,}\quad R = \frac{\sigma^{\text{unpol.}}_{e^+e^-\to \text{hadrons}}}{\sigma^{\text{unpol.}}_{e^+e^-\to\mu^+\mu^-}} \xrightarrow[Q_\mu=-1]{E\gg m_\mu} \sum_{\substack{\text{quarks}\\ q}} Q_q^2\sqrt{1-m_q^2/E^2}\left[1+\frac{m_q^2}{2E^2}\right]\Theta(E-m_q).$$

threshold for different flavours of quarks

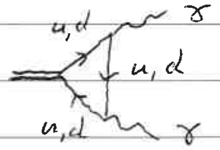
Given that each quark flavour $q = d, u, s, c, b, t$ (u, c, t: $Q_q = +2/3$ charge; d, s, b: $Q_q = -1/3$ charge)

comes in N_c colours, R will display resonances followed by steps given by $R \to N_c \sum\limits_{\substack{\text{active}\\ \text{flavours}}} Q_q^2$ involving all (active) flavours for which $m_q \ll E$.

$E > 1.5$ GeV: d, u, s, c active flavours $\Rightarrow$ R approaches $\frac{10}{9}N_c$ in the limit that $E \gg m_c$.

$E > 4.7$ GeV: d, u, s, c, b active flavours $\Rightarrow$ R approaches $\frac{11}{9}N_c$ in the limit that $E \gg m_b$.

see figure $\rightarrow$ Data suggest that $N_c = 3$. Also the decay $\pi^0 \to 2\gamma$ supports ($\frac{1}{\sqrt{2}}(u\bar u - d\bar d)$ bound state)

this: π^0  + reversed loop, with $\Gamma_{\pi^0\to 2\gamma} \propto N_c^2$.

colour neutral $\Rightarrow$ no preference for colour d.o.f.

The weak interactions: the matter particles that are subject to weak interactions are grouped into isospin doublets and the associated interactions are governed by an SU(2) gauge theory. However, each Dirac fermion has one spin-related mode that is subject to weak interactions (left-handed part) and one mode that is not (right-handed part). The latter modes are therefore supposed to not transform under SU(2) gauge transformations, i.e. they form singlets under SU(2).

e.g. $\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix}\begin{matrix}\leftarrow I_3=+1/2\\ \leftarrow I_3=-1/2\end{matrix}$: isospin $I=1/2$

$SU(2)_L$, L = left-handed chirality: matter doublets (left-handed quarks/leptons), matter singlets (right-handed quarks/leptons) and gauge-boson triplets.

isospin

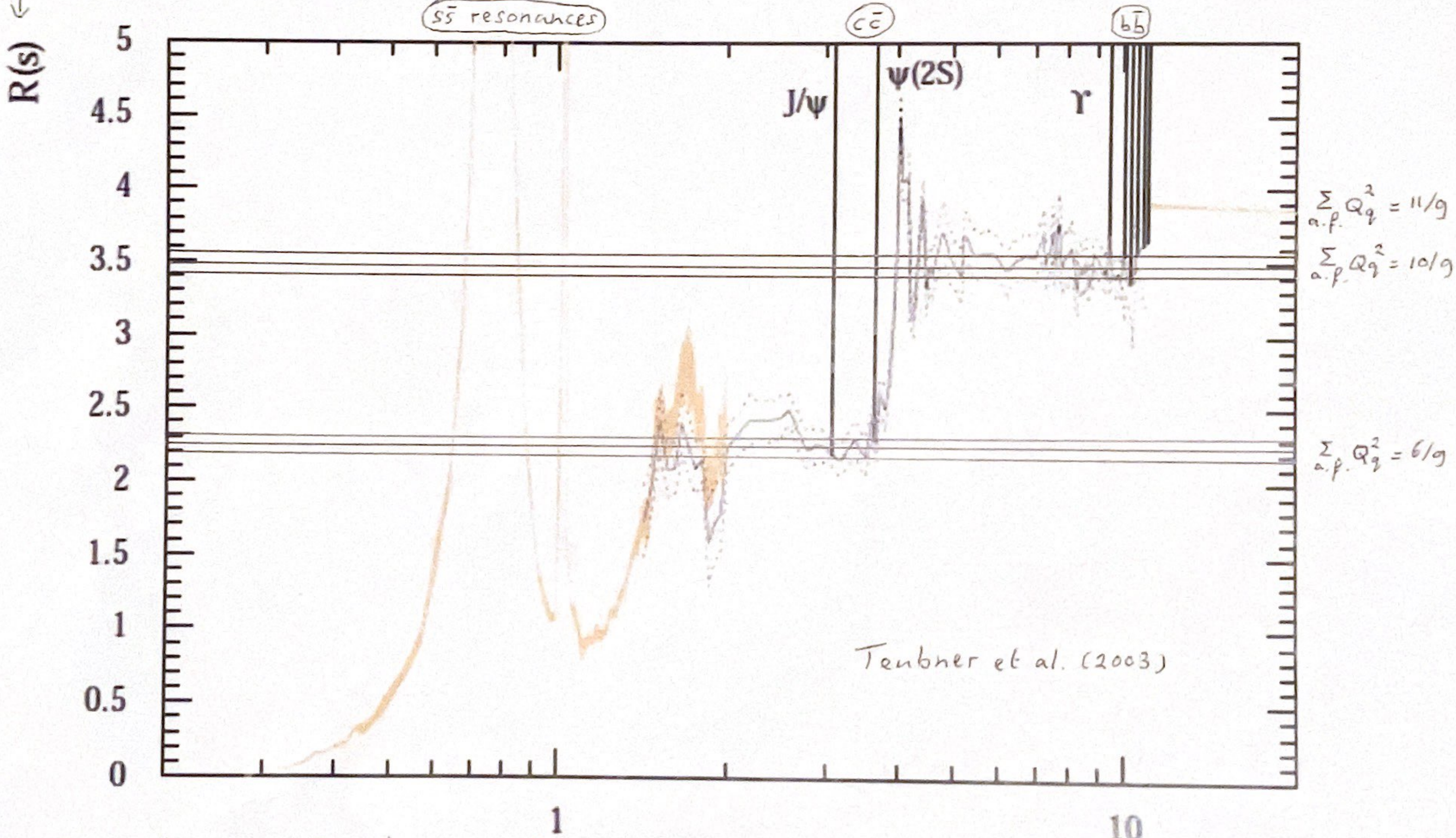
e.g. ν_{eR}, e_R: isospin $I=0$

W_μ^a $(a=1,2,3)$

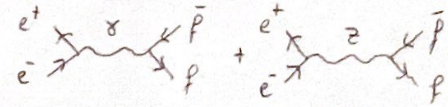
Gauge coupling: g = weak interaction strength.

using that $R(s) = N_c \sum_{a.f.} Q_q^2 \left\{ 1 + \frac{\alpha_s(s)}{\pi} + \cdots \right\}$ from perturbative QCD

active flavours for which $m_q \ll E$



$$R \equiv \frac{\sigma^{unpol.}_{e^+e^- \to hadrons}}{\sigma^{unpol.}_{e^+e^- \to \mu^+\mu^-}} :$$



+ higher orders,

$$\sigma^{unpol.}_{e^+e^- \to f\bar{f}} \approx \frac{\pi \alpha^2 Q_f^2}{3E^2} \sqrt{1 - \frac{m_f^2}{E^2}} \left[1 + \frac{m_f^2}{2E^2} \right] \Theta(E - m_f) \xrightarrow{E \gg m_f} \frac{\pi \alpha^2}{3E^2} Q_f^2$$ for each type of fermion f

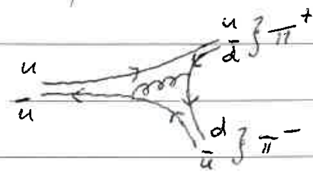
$E \ll M_Z$

Let's go through the experimental evidence:

*) The existence of <u>slow decays</u> like $\pi^- \to \mu^- \bar{\nu}_\mu$, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ and β-decay $n \to p e^- \bar{\nu}_e$ with lifetimes $\tau > 10^{-8}$ sec hints at a new type of interaction, bearing in mind that strong-interaction decays have $\tau \sim 10^{-23}$ sec and electromagnetic decays have $\tau \sim 10^{-16}$ sec $\Rightarrow$ <u>weak interactions</u> (much weaker than the electromagnetic interactions at low energies)!

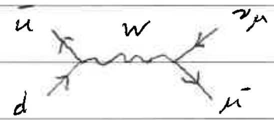
e.m. decay $\pi^0 \to 2\gamma$ (see before) $\longrightarrow \tau = 8.4 \times 10^{-17}$ sec

strong decay $\rho^0 \to \pi^+ \pi^-$:



$\longrightarrow \tau = 4.5 \times 10^{-24}$ sec

weak decays $\pi^- \to \mu^- \bar{\nu}_\mu$:

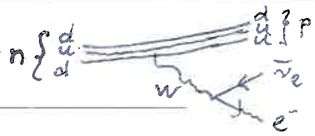


$\longrightarrow \tau = 2.6 \times 10^{-8}$ sec

$\mu^- \to e^- \bar{\nu}_e \nu_\mu$:

$\longrightarrow \tau = 2.2 \times 10^{-6}$ sec

$n \to p e^- \bar{\nu}_e$:



$\longrightarrow \tau = 15$ min

*) Most weak interactions appear to be <u>charge-current interactions</u>, which couple fermions of non-opposite charge (unlike e.m. interactions)

$(e^- \bar{\nu}_e, \mu^- \bar{\nu}_\mu, n \bar{p}, \ldots)$

in fact: $d\bar{u}$

$\Rightarrow$ <u>hints at doublet structure</u>!

*) Fixed interaction strength G_F at low energies between 4 fermions:

– <u>short-range interactions</u> (unlike e.m. and gluon-mediated int.)!

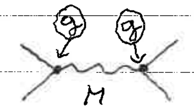
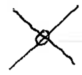
Concept: if $E = 1/\lambda_{dB} \ll 1/a$ (a = range int.), only the net effect of

$1/$mass force carrier

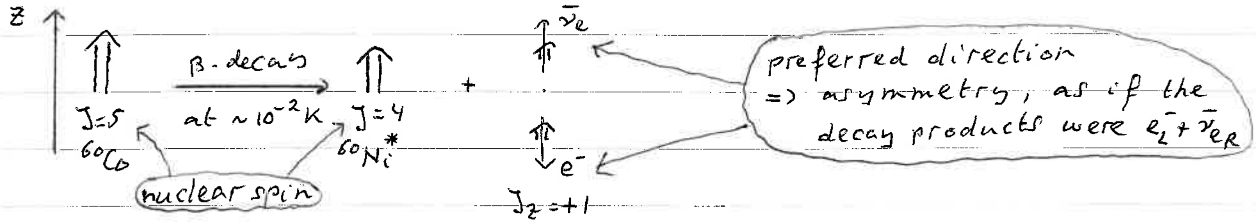
the interaction potential is felt (i.e. not enough energy for resolving details of the interaction potential)

$\Rightarrow$ effectively constant interaction strength [unlike QED where $a \to \infty$ in view of $m_\gamma = 0$ $\Rightarrow$ in that case there will always be sensitivity to the detailed form of the interaction potential].

- Fermi-coupling $G_F \approx 10^{-5}\,\text{GeV}^{-2} = \mathcal{O}(e^2/M^2)$ with $M = \mathcal{O}(100\,\text{GeV})$ the mass of the force carrier (gauge boson)!

Concept:  $\propto \frac{ig^2}{q^2 - M^2} \xrightarrow{\sqrt{q^2} \ll M} \frac{-ig^2}{M^2} =$  effective coupling with mass dimension -2.

*) The decay ${}^{60}Co \to {}^{60}Ni^{*} + e^- + \bar{\nu}_e$ (excited nucleus) displays a preferred decay direction in the presence of an external magnetic field along the z-axis



If the interaction current would be like in QED/QCD [current $\propto \bar{\psi}\gamma^\mu\psi$], then we would expect an equal number of e^- to be emitted parallel and antiparallel to the magnetic field

similar experiments ⇒ in charged-current weak interactions only $\bar{\nu}_{e_R}$, e^-_L and ν_{e_L}, e^+_R feature [corresponding to Dirac fields with left-handed chirality: $\psi_L = \frac{1}{2}(I_4 - \gamma^5)\psi$, $\bar{\psi}_L = (\psi^\dagger \frac{1}{2}(I_4 - \gamma^5))\gamma^0 = \bar{\psi}\frac{1}{2}(I_4 + \gamma^5)$], and no ν_{e_R} or $\bar{\nu}_{e_L}$ are "observed"!

⇒ hints at a so-called chiral gauge theory, which treats L/R modes differently! This implies the following for weak interactions:

- parity is violated, since $L/R \xrightarrow{P} R/L$.
  Indeed: $\Gamma(\pi^+ \to \mu^+_R \nu_{\mu_L}) \neq \Gamma(\pi^+ \to \mu^+_L \nu_{\mu_R}) = 0$ (P violated maximally).
- charge-conjugation invariance is violated, since $\nu_L \xrightarrow{C} \bar{\nu}_L$.
  Indeed: $\Gamma(\pi^+ \to \mu^+_R \nu_{\mu_L}) \neq \Gamma(\pi^- \to \mu^-_R \bar{\nu}_{\mu_L}) = 0$ (C violated maximally)
- CP could still be conserved.
  Indeed: $\Gamma(\pi^+ \to \mu^+_R \nu_{\mu_L}) = \Gamma(\pi^- \to \mu^-_L \bar{\nu}_{\mu_R})$ (CP conserved).

Difference between vector (QED/QCD-like) theories and chiral theories:

*) vector theories are democratic w.r.t. the two chiral modes and therefore conserve parity (which interchanges chiral modes).

Proof: $\underbrace{P_L \equiv \frac{1}{2}(I_4 - \gamma^5) = P_L^2 \;,\; P_R \equiv \frac{1}{2}(I_4+\gamma^5) = P_R^2 \;,\; P_L + P_R = I_4}$ with $(\gamma^5)^2 = I_4$, $\{\gamma^\mu, \gamma^5\} = 0$ and $(\gamma^5)^\dagger = \gamma^5$

$P_{L/R}$ project on L/R chiral modes:

$\psi_{L/R} = P_{L/R}\,\psi \;,\; \bar{\psi}_{L/R} = (P_{L/R}\psi)^\dagger \gamma^0 = \psi^\dagger P_{L/R}\gamma^0 = \bar{\psi} P_{R/L}$

$\Rightarrow \bar{\psi}\gamma_\mu\psi = \bar{\psi}\gamma_\mu (P_L^2 + P_R^2)\psi = \bar{\psi} P_R \gamma_\mu P_L \psi + \bar{\psi} P_L \gamma_\mu P_R \psi$

vector current

$= \bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R$ (conserving parity).

under the corresponding local gauge transformations chiral states transform in the same way, i.e. $\psi_{L/R} \to \psi_{L/R}' = U(x)\,\psi_{L/R}$.

*) chiral theories treat different chiral modes differently and thereby implicitly violate parity. Under the corresponding local gauge transformations chiral states transform differently.

Example: under $SU(2)_L$, $\psi_L \to \psi_L' = U(x)\,\psi_L$ and $\psi_R \to \psi_R' = \psi_R$.

A chiral $SU(2)_L$ gauge theory for the weak interactions requires three massive gauge bosons, two responsible for the charge-current (CC) weak interactions [$W^\pm$ corresponding to the generators $T^\pm = T^1 \pm i T^2 = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$] and one predicted to be responsible for the neutral-current (NC) weak interactions [W^3 corresponding to the generator $T^3 = \sigma^3/2$]. The existence of this third massive vector boson was confirmed experimentally by the

not photon, since $SU(2)_L$ is chiral and $U(1)_{e.m.}$ is not!

*) observation of weak NC effects in 1973 @ CERN:

e.g. $\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$, $\nu_\mu N \to \nu_\mu X$, $\bar{\nu}_\mu N \to \bar{\nu}_\mu X$,

nucleon anything

neutral!

*) discovery of the massive gauge bosons $W^\pm$ and Z in 1983 at the CERN $p\bar{p}$ experiments UA1 and UA2.

Including electromagnetic interactions: the most economic gauge group for the electroweak sector (= weak + e.m. interactions) would therefore be $SU(2)_L \times U(1)_Y$. The generator T_Y belonging to this U(1) group has to

gauge coupling g   gauge coupling g' with $g \neq g'$ ⇒ not a true unification!

commute with the SU(2) generators T^+, T^- and T^3 ⇒ $[T_Y, T^\pm] = 0$, i.e. the Y quantum number is not affected by raising/lowering ⇒ each $SU(2)_L$ multiplet has one Y quantum number, i.e. $U(1)_Y$ does not refer to the electromagnetic gauge group as the neutrino and electron have different charges!

$y(f_L) \neq y(f_R)$: another sign that the SM is a chiral theory !

$U(1)_y$, y = hypercharge: QED-like (Abelian), one gauge boson B_μ.

Gauge coupling: g', which can be scaled by a quantum number $y/2$ that specifies the int. strength.

$\begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}, \begin{pmatrix}\nu_{\mu L}\\ \mu_L^-\end{pmatrix}, \begin{pmatrix}\nu_{\tau L}\\ \tau_L^-\end{pmatrix}$ $\quad e_R^-, \mu_R^-, \tau_R^-$ $\quad \nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$

y(lepton doublet L) $= -1$, $y(\ell_R^-) = -2$, $y(\nu_R) = 0$, !

pure singlet (no strong/electroweak int.)

y(quark doublet Q) $= 1/3$, y(up-type q_R) $= 4/3$, y(down-type q_R) $= -2/3$.

$\begin{pmatrix}u_L\\ d_L\end{pmatrix}, \begin{pmatrix}c_L\\ s_L\end{pmatrix}, \begin{pmatrix}t_L\\ b_L\end{pmatrix}$ $\quad u_R, c_R, t_R$ $\quad d_R, s_R, b_R$

Relation between I_3 (3rd comp. isospin), y and Q (e.m. charge) (Gell-Mann – Nishijima relation): $\boxed{Q = I_3 + y/2}$ †,
with $I_3 = \pm 1/2$ for left-handed matter and $I_3 = 0$ for right-handed matter.
since photon = lin. combination of W^3 and B !

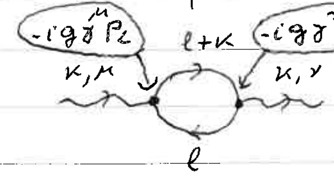
The fact that $N_C = 3$ and that the leptons and quarks come in a family structure with at least three generations (such as ν_e, e; u, d) will turn out to be a crucial ingredient of the Standard Model.

Additional theoretical input: the issue of anomaly cancellation.

What happens to the gauge symmetry (Ward-Takahashi identities) in the presence of chiral fermions in the loop corrections?

Consider to this end a chiral $U(1)_L$ theory with massless fermions: not essential

*) The impact on the gauge-boson self-energy:



$$= -g^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{Tr}\,(\not\ell\,\gamma^\nu P_L\,(\not k+\not\ell)\,\gamma^\mu P_L)}{\ell^2\,(\ell+k)^2}$$

$\not\ell\gamma^\nu(\not k+\not\ell)\gamma^\mu P_L$ $\qquad -4i\epsilon^{\sigma\nu\beta\mu}\,\ell_\sigma(k+\ell)_\beta$ (totally antisymmetric)

$$= \frac{1}{2} * \text{QED self-energy} + \underbrace{\frac{g^2}{2}\int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{Tr}\,(\not\ell\gamma^\nu(\not k+\not\ell)\gamma^\mu\gamma^5)}{\ell^2(\ell+k)^2}}$$

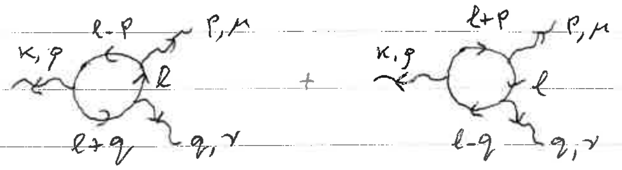
0, since $\int d^4\ell\, \frac{\ell_\sigma k_\beta}{\ell^2(\ell+k)^2} \propto k_\sigma k_\beta$!

due to Lorentz covariance

So, the contribution from chiral fermions equals half the QED self-energy, which is consistent with $\psi = \psi_L + \psi_R$. Nothing spectacular is happening at this point, since the QED self-energy respects the Ward-Takahashi identity.

† $W^{1,2,3}$ do not satisfy this, but $\underbrace{W^+, W^3, W^-}_{I=1 \text{ isospin triplet}}$ do since $y=0$ and $Q = I_3 = +1, 0, -1$

*) The impact on the triple gauge-boson couplings: yet again we obtain half the QED vertex correction (which happens to be 0) and an axial-vector contribution



$\xrightarrow[\text{vector}]{\text{axial}} g^3 V_5^{\mu\nu\rho}(p,q,k)$,

$l \to -l$: same integral with all matrices in reversed order = same integral

with $$V_5^{\mu\nu\rho}(p,q,k) = \frac{1}{2}\int\frac{d^4l}{(2\pi)^4}\,\frac{Tr([\not l-\not p]\gamma^\mu\not l\gamma^\nu[\not l+\not q]\gamma^\rho\gamma^5)}{(l-p)^2\,l^2\,(l+q)^2} + (p,\mu \leftrightarrow q,\nu)$$

$$= \int\frac{d^4l}{(2\pi)^4}\,\frac{Tr([\not l-\not p]\gamma^\mu\not l\gamma^\nu[\not l+\not q]\gamma^\rho\gamma^5)}{(l-p)^2\,l^2\,(l+q)^2}$$

Ward identities: $k_\rho V_5^{\mu\nu\rho}(p,q,k)$, $q_\nu V_5^{\mu\nu\rho}(p,q,k)$ and $p_\mu V_5^{\mu\nu\rho}(p,q,k)$ should all vanish, otherwise gauge invariance and renormalizability are lost.

$$k_\rho V_5^{\mu\nu\rho}(p,q,k) = \int\frac{d^4l}{(2\pi)^4}\,\frac{Tr([\not l-\not p]\gamma^\mu\not l\gamma^\nu[\not l+\not q]\not k\gamma^5)}{(l-p)^2\,l^2\,(l+q)^2}$$ and use that

$$\not k = -\not p-\not q = (\not l-\not p)-(\not l+\not q) \Rightarrow \not k\gamma^5 = -\gamma^5(\not l-\not p) - (\not l+\not q)\gamma^5$$

and $$Tr([\not l-\not p]\gamma^\mu\not l\gamma^\nu[\not l+\not q]\not k\gamma^5) = -(l-p)^2\underbrace{Tr(\gamma^\mu\not l\gamma^\nu[\not l+\not q]\gamma^5)}_{-4i\epsilon^{\mu\sigma\nu\beta}l_\sigma q_\beta} - (l+q)^2\underbrace{Tr([\not l-\not p]\gamma^\mu\not l\gamma^\nu\gamma^5)}_{-4i\epsilon^{\mu\sigma\nu\beta}l_\sigma p_\beta}$$

Hence, $$k_\rho V_5^{\mu\nu\rho}(p,q,k) = 4i\epsilon^{\mu\sigma\nu\beta}\int\frac{d^4l}{(2\pi)^4}\left[\underbrace{\frac{l_\sigma q_\beta}{l^2(l+q)^2}}_{\to\,\propto\, q_\sigma q_\beta} + \underbrace{\frac{l_\sigma p_\beta}{l^2(l-p)^2}}_{\to\,\propto\, p_\sigma p_\beta}\right] \to 0$$ (as before).

So far so good, but what about the 3rd Ward identity?

$$p_\mu V_5^{\mu\nu\rho}(p,q,k) = \int\frac{d^4l}{(2\pi)^4}\,\frac{Tr(\overbrace{[\not l-\not p]\not p\not l}^{\to\, l^2(\not l-\not p)-(l-p)^2\not l}\gamma^\nu[\not l+\not q]\gamma^\rho\gamma^5)}{(l-p)^2\,l^2\,(l+q)^2}$$

$$= 4i\epsilon^{\sigma\nu\beta\rho}\int\frac{d^4l}{(2\pi)^4}\left[\underbrace{\frac{l_\sigma(q+\cancel{l})_\beta}{l^2(l+q)^2}}_{\to\, q_\sigma q_\beta,\ \text{which vanishes}} - \underbrace{\frac{(l-p)_\sigma(p+q+\cancel{l-p})_\beta}{(l-p)^2(l+q)^2}}_{\text{would also vanish if the shift } l\to l+p \text{ is allowed in the integral}}\right].$$

So, the Ward identity $p_\mu V_5^{\mu\nu\rho}(p,q,k) = 0$ is satisfied, provided that we are allowed to shift the integration momentum by a constant 4-vector without paying a price. The same remark applies to the remaining Ward identity $q_\nu V_5^{\mu\nu\rho}(p,q,k) = 0$.

We would be allowed to perform the indicated shift if the integral would be convergent. In fact this is not the case. Since the integrand behaves as $|l|^{-3}$ for $|l| \to \infty$, we are dealing with a linearly divergent integral!

n-dim. Euclidean vectors

Consider the Euclidean n-dimensional integral $I(\vec{y}) = \int d^n\vec{x}\,[f(\vec{x}+\vec{y}) - f(\vec{x})]$, with $f(x) = \mathcal{O}(|\vec{x}|^{1-n})$ for $|\vec{x}| \to \infty$. The behaviour at $|\vec{x}| \to \infty$ will determine whether this integral will vanish or not $\Rightarrow$ perform a Taylor expansion:

$$I(y) = \lim_{R\to\infty} \int_{|\vec{x}| \le R} d^n x \Big(\underbrace{y^i \frac{\partial f}{\partial x^i}}_{\vec{\nabla}\cdot(f\vec{y})} + \underbrace{\frac{1}{2} y^i y^j \frac{\partial^2 f}{\partial x^i \partial x^j}}_{\vec{\nabla}\cdot(\frac{1}{2} y^j \frac{\partial f}{\partial x^j}\vec{y})} + \dots \Big) \overset{\text{Gauss}}{=} \lim_{R\to\infty} \int_{S_n(R)} d\sigma \frac{x^i}{R} [f(x) y^i]$$

$d\vec{\sigma}$ pointing outward

area $\propto R^{n-1}$

+ negligible terms.

In the case of a linearly divergent integral the surface term survives in general and $\int d^n x\, f(x)$ depends on the reference point of the integration!!! Using Wick rotation to turn the actual Minkowskian integral into a Euclidean one, it is indeed found that

- not all three Ward identities can be satisfied simultaneously, not even by redefining the reference point of the loop integral. This is a problem since we couple the axial current to a gauge boson;
- the Ward identities are violated by terms $\propto \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta$;
- no further Ward-identity violating terms occur at higher loop order or in the corrections to vertices involving four or more gauge bosons. $\leftarrow$ typical for $U(1)$

Consequence: renormalizability spoiled, predictive power lost!

This phenomenon of quantum (loop) effects invalidating the gauge invariance of a gauge theory is called a gauge anomaly.

What about a chiral $SU(N)_L$ theory?

linked to the momenta p, q, k

In that case the 1st vertex correction on p. 19 gets a factor $Tr(T^a T^b T^c)$ and the 2nd one a factor $Tr(T^b T^a T^c) \Rightarrow$ overall factor $\frac{1}{2} Tr(\{T^a, T^b\} T^c)$ in front of $g^3 V_5^{\mu\nu\rho}(p,q,k)$, which vanishes for $SU(2)_L$ since $\{\sigma^a, \sigma^b\} = 2\delta^{ab} I_2$ and $Tr(\sigma^c) = 0$.

Hence, in order to make the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge theory anomaly free, the following two theoretical constraints should be imposed:

1) $\boxed{\sum_{f_L} Y(f_L) = 0}$ for one $U(1)_Y$ boson and two $SU(2)_L$ bosons,

$\rightarrow 2\underbrace{Y(L_L)}_{-1} + 2N_c \underbrace{Y(Q_L)}_{+1/3} = -2 + \frac{2}{3}N_c \stackrel{N_c=3}{=\!=} 0$ for each fermion generation!

2) $\boxed{\sum_f [Y^3(f_L) - Y^3(f_R)] = 0}$ for three $U(1)_Y$ bosons.

$\gamma^5 \to -\gamma^5$

$\rightarrow 2\underbrace{Y^3(L_L)}_{-2} - \underbrace{Y^3(e_R)}_{-8} + N_c[2\underbrace{Y^3(Q_L)}_{2/27} - \underbrace{Y^3(u_R)}_{64/27} - \underbrace{Y^3(d_R)}_{-8/27}] = 6 - 2N_c \stackrel{N_c=3}{=\!=} 0$

for each fermion generation!

Therefore, $N_c = 3$ and the lepton-quark family structure are crucial for making the Standard Model anomaly free $\Rightarrow$ predictive power restored!

The mass issue: the above-constructed gauge theory only alows massless particles. A gauge-boson mass term $+\frac{1}{2}m^2 W_\mu^a W^{a,\mu}$ would clearly break gauge invariance and result in problems with the high-energy behaviour and renormalizability of the theory (see Ex. 5)! Actually the same holds for the matter fermions of the theory. Gauge theories based on vector currents, such as QED and QCD, are democratic w.r.t. chiral states and the chiral states transform in the same way under the corresponding local gauge transformations (see p. 17). In that case a mass term $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$ would be gauge invariant. However, in a chiral gauge theory, such as the $SU(2)_L \times U(1)_Y$ gauge theory, different chiral states transform differently and a mass term $m\bar{\psi}\psi$ would break gauge invariance!

Chapter 3: Particle masses in the Standard Model

In order to have non-zero particle masses in the Standard Model we have to break gauge invariance without actually breaking it!

Way out: spontaneous symmetry breaking. The Lagrangian is invariant under local gauge transformations, but the dynamics is such that the ground state ('vacuum') is NOT a singlet of the symmetry group, i.e. the symmetry is not apparent in the ground state $\Rightarrow$ "choosing" one of the degenerate set of ground states as the physical vacuum breaks the symmetry!

Quantum mechanical concept: if there is symmetry under a group of transformations with elements $\hat{u}$, then $[\hat{H}, \hat{u}] = 0$ for the Hamilton operator. For the ground state $|0\rangle$, with $\hat{H}|0\rangle = E_0|0\rangle$, this implies that $\hat{H}\hat{u}|0\rangle = \hat{u}\hat{H}|0\rangle = E_0\hat{u}|0\rangle \Rightarrow \{\hat{u}|0\rangle\}$ forms a degenerate set of ground states if $\exists_{u}\ \hat{u}|0\rangle \neq |0\rangle$, which is equivalent to $|0\rangle$ not being a singlet under the transformation group.

Higgs toy model !

see Ex. 7

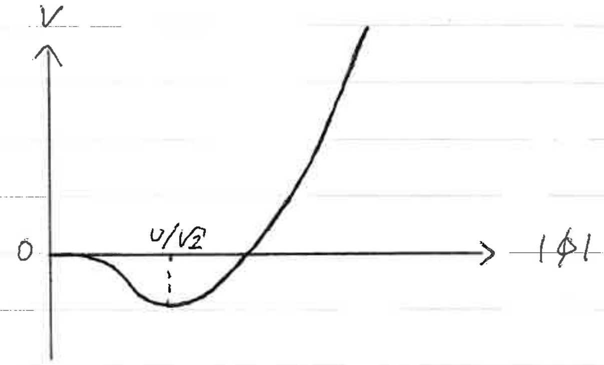
Let's start with a U(1) example ... the so-called relativistic superconductor;

wrong sign mass term

potential bounded from below !

$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi)$, $V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4$ $(\mu^2 < 0,\ \lambda > 0)$ and

ϕ = complex scalar field.



V has a maximum at $|\phi| = 0$ and a ring of minima at $|\phi|^2 = -\mu^2/2\lambda \equiv v^2/2 > 0$ linked by global U(1) phase transformations.

Consequences: $\phi = 0$ is not suitable as ground state, around which the quantum fluctuations (particle interpretation) should be considered. One of the degenerate ring of minima should be used for that purpose!

vacuum expectation value (vev) $\langle \phi \rangle_0$

Choose the ground state such that $\langle 0|\hat{\phi}|0\rangle = v/\sqrt{2} \Rightarrow$ global U(1) phase symmetry broken! Reparametrize $\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))\, e^{i w(x)/v}$, with $h(x)$ and $w(x)$ real scalar fields, in order to separate classical and quantum fluctuation parts

$$\Rightarrow -V(\phi) = \overset{\frac{1}{2}\lambda v^2}{-\tfrac{1}{2}\mu^2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 = \frac{1}{4}\lambda v^4 \overset{+\mu^2 h^2}{\underbrace{-\lambda v^2 h^2}_{\text{mass term} \,\equiv\, -\frac{1}{2} m_h^2 h^2}} \overbrace{-\lambda v h^3 - \frac{\lambda}{4} h^4}^{\text{interactions}},$$

$$(\partial_\mu \phi)^\dagger (\partial^\mu \phi) = \frac{1}{2}\Big[\partial_\mu h - i(v+h)\frac{\partial_\mu w}{v}\Big] e^{-iw/v} \Big[\partial^\mu h + i(v+h)\frac{\partial^\mu w}{v}\Big] e^{+iw/v}$$
$$= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}\underbrace{(1+h/v)^2}_{\text{interactions}}(\partial_\mu w)(\partial^\mu w).$$

Combined: - the boson described by the h-field has acquired a mass $m_h = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2}$

↳ massive d.o.f. describing physical oscillations

– the w-field corresponds to a massless boson (Goldstone boson)

↑ massless d.o.f. describing the symmetry of the true ground states (flat direction along the ring of minima);

going beyond U(1) toy model !

→ for each generator belonging to a broken symmetry there will be one such Goldstone boson.

generator that does not annihilate the vacuum

Next we turn this into a local U(1) gauge theory and local symmetry breaking:

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi)\,, \quad \text{with } D_\mu\phi = (\partial_\mu + igA_\mu)\phi\,,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

This Lagrangian is invariant under local gauge transformations

$$\phi(x) \to \phi'(x) = e^{i\varphi(x)}\phi(x)\,, \qquad A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \tfrac{1}{g}\partial_\mu\varphi(x)$$

using reparametrization of ϕ

$$\Rightarrow (D_\mu\phi)^\dagger(D^\mu\phi) = \tfrac{1}{2}[\partial_\mu h - i(v+h)(gA_\mu + \partial_\mu w/v)]e^{-iw/v}[\partial^\mu h + i(v+h)(gA^\mu + \partial^\mu w/v)]e^{+iw/v}$$

$$= \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{2}(1+h/v)^2[(\partial_\mu w)(\partial^\mu w) + g^2v^2A_\mu A^\mu + 2vgA^\mu\partial_\mu w]$$

interactions

bilinear A-w mixing $\Rightarrow$ be careful in interpreting $\mathcal{L}$ and its particle content!

In fact $w(x)$ can be removed by means of a specific gauge transformation (switching to the unitary gauge): $\phi^{(u)}(x) = e^{-iw(x)/v}\phi(x) = \frac{1}{\sqrt{2}}(v+h(x))$,

$$A^{(u)}_\mu(x) = A_\mu(x) + \frac{1}{gv}\partial_\mu w(x)$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}[\partial_\mu h - igA^{(u)}_\mu(v+h)][\partial^\mu h + igA^{(u),\mu}(v+h)] - \tfrac{1}{4}F^{(u)}_{\mu\nu}F^{(u),\mu\nu} - \tfrac{1}{2}\mu^2(v+h)^2 - \tfrac{\lambda}{4}(v+h)^4$$

$$= \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \mu^2h^2 - \tfrac{1}{4}F^{(u)}_{\mu\nu}F^{(u),\mu\nu} + \tfrac{1}{2}A^{(u)}_\mu A^{(u),\mu}[g^2v^2 + 2g^2vh + g^2h^2] - \lambda vh^3 - \tfrac{\lambda}{4}h^4 + \tfrac{1}{4}v^4\lambda .$$

interactions

Hence: – the gauge boson has acquired a mass $m_A = gv$

$\Rightarrow$ this gauge boson has acquired an extra (longitudinal) d.o.f.

[massless: 2 d.o.f. $\to$ massive: 3 d.o.f.],

– the Goldstone boson w has supplied this extra d.o.f., as the unitary-gauge expression has revealed (by gauging away w).

This is called the Higgs mechanism ! ← invented for describing the Meissner effect in superconductors

Subsequently we adapt this to the Standard Model electroweak sector.

to ensure that the ground state is Lorentz invariant

$SU(2)_L \times U(1)_Y$: we need a set of scalar fields Φ that develops a $U(1)_{e.m.}$ symmetric vacuum expectation value $\langle\Phi\rangle_0$ so that $SU(2)_L \times U(1)_Y \xrightarrow{\langle\Phi\rangle_0} U(1)_{e.m.}$

$\Rightarrow$ three out of four $SU(2)_L \times U(1)_Y$ gauge bosons acquire mass (3 broken generators), the photon remains massless by taking the chosen ground state to be a singlet under $U(1)_{e.m.}$ (i.e. $\langle\Phi\rangle_0$ is chosen to be neutral)!

meaning : $I_3 + Y/2 = 0$!

Most economic solution : $\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ is a complex scalar doublet of Higgs fields, with weak isospin $I = 1/2$ (doublet) and hypercharge $Y(\Phi) = 1 \Rightarrow I_3 = +1/2$ component has positive charge and $I_3 = -1/2$ component is neutral

added by hand !

needed for $\langle\Phi\rangle_0$

$$\Rightarrow \boxed{\mathcal{L}_H = (D_\mu\Phi)^\dagger(D^\mu\Phi) - \underbrace{\mu^2}_{<0}(\Phi^\dagger\Phi) - \underbrace{\lambda}_{>0}(\Phi^\dagger\Phi)^2 \quad , \quad \Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 - i\phi_2\\ \sigma + i\phi_3\end{pmatrix}}$$

$$D_\mu\Phi = (\partial_\mu + \underbrace{\tfrac{i}{2} g\,\vec{\sigma}.\vec{W}_\mu}_{SU(2)_L} + \underbrace{\tfrac{i}{2} g' Y B_\mu}_{U(1)_Y})\Phi \overset{Y=1}{=\!=} (\partial_\mu + \tfrac{i}{2}[g\,\vec{\sigma}.\vec{W}_\mu + g' B_\mu])\Phi \,.$$

Transformation property : $\Phi(x) \to \Phi'(x) = e^{\frac{i}{2}\vec{\sigma}.\vec{\varphi}(x)}\, e^{\frac{i}{2}\varphi_Y(x)}\,\Phi(x)$.

choose $\langle\Phi\rangle_0 = \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}$, $v^2 = -\mu^2/\lambda$, reparametrize $\Phi(x) = \underbrace{e^{i\vec{\sigma}.\vec{\omega}(x)/v}}_{\Sigma(x)\in SU(2)}\begin{pmatrix}0\\ [v+h(x)]/\sqrt{2}\end{pmatrix}$ and

switch to the unitary gauge $\Phi(x) \to \Phi^{(u)}(x) = \Sigma^{-1}(x)\,\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h(x)\end{pmatrix}$

$\Rightarrow D_\mu\Phi^{(u)} = (\partial_\mu + \frac{i}{2}[g\,\vec{\sigma}.\vec{W}_\mu^{(u)} + g' B_\mu^{(u)}])\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h\end{pmatrix}$, h = physical Higgs-boson field,

$-V(\Phi^{(u)}) = \mu^2 h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 + \frac{1}{4}\lambda v^4$ (as on p.22).

Consequence : the three would-be Goldstone bosons represented by $\vec{\omega}(x)$ are "eaten" by the gauge bosons to form three massive mass eigenstates

starting from unitary gauge

$$\boxed{\begin{array}{l} Z_\mu = c_W W_\mu^3 - s_W B_\mu \quad (Z \text{ boson}) \ , \quad A_\mu = s_W W_\mu^3 + c_W B_\mu \quad (\text{photon}), \\ \qquad\qquad \tan\theta_W = g'/g \\ W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2) \quad (W^\pm \text{ bosons}) \ , \text{ with } c_W = \cos\theta_W \ , \ s_W = \sin\theta_W \\ \qquad\qquad \text{and } \theta_W = \text{weak mixing angle}, \\ \mathcal{L}_{mass} = m_W^2 W_\mu^+ W^{-,\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu + 0 . A_\mu A^\mu , \\ m_W^2 = \frac{1}{4}v^2 g^2 \ , \ m_Z^2 = \frac{1}{4}v^2(g^2 + g'^2) = m_W^2/c_W^2 \ , \ m_A^2 = m_\gamma^2 = 0 \ , \ v = 246 \text{ GeV}. \end{array}}$$

see Ex.8

§3.1 Giving mass to fermions and intergenerational mixing

As mentioned on p.21, a mass term $-m\bar\psi\psi = -m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)$ would break gauge invariance in the Standard Model. We will therefore use the Higgs mechanism to also give mass to fermions. To this end we introduce $SU(2)_L \times U(1)_Y$ invariant interactions between the fermions and scalar doublets, which compensate the $SU(2)_L$ transformations of $\bar\psi_L$. Moreover, $U(1)_Y$ gauge invariance requires Higgs doublets with hypercharge $-y(\bar\psi_L\psi_R) = y(\psi_L) - y(\psi_R) = 2(Q - I_3) - 2Q = -2I_3$

$\Rightarrow$ use Φ, $y(\Phi) = -y(\bar\psi_L\psi_R) = +1$ to give mass to fermions with $I_3 = -1/2$,
use $\tilde\Phi$, $y(\tilde\Phi) = -y(\bar\psi_L\psi_R) = -1$ to give mass to fermions with $I_3 = +1/2$.

For one generation of fermions (ν_e, e, u, d) these so-called Yukawa interactions read

$$\mathcal{L}_{Yu} = -f_e \underbrace{\bar L_{1_L}}_{(\bar\psi_{\nu_{eL}},\, \bar\psi_{e_L})} \Phi \overbrace{e_R}^{\psi_{e_R}} - f_d \underbrace{\bar Q_{1_L}}_{(\bar\psi_{u_L},\, \bar\psi_{d_L})} \Phi \overbrace{d_R}^{\psi_{d_R}} - f_{\nu_e} \bar L_{1_L} \tilde\Phi \overbrace{\nu_{e_R}}^{\psi_{\nu_{eR}}} - f_u \bar Q_{1_L} \tilde\Phi \overbrace{u_R}^{\psi_{u_R}} + h.c.,$$

with $\tilde\Phi \equiv i\sigma^2\Phi^* = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi^* = \begin{pmatrix}\phi^{0*}\\ -\phi^{+*}\end{pmatrix} \equiv \begin{pmatrix}\phi^{0*}\\ -\phi^{-}\end{pmatrix} \xrightarrow[\text{gauge}]{\text{unitary}} \frac{1}{\sqrt2}\begin{pmatrix}v+h\\ 0\end{pmatrix}$,

$I(\tilde\Phi) = 1/2$, $y(\tilde\Phi) = -1$ $\Rightarrow$ $\tilde\Phi$ has the same $SU(2)_L$ and opposite $U(1)_Y$ transformation properties as Φ (see Ex.2),

$f_e, f_d, f_{\nu_e}, f_u \in \mathbb{R}$ (called "Yukawa couplings").

f_{ν_e} = 0 in the "standard" version of the Standard Model

The fermion mass terms now follow directly from the non-zero vev's $\langle\Phi\rangle_0 = \begin{pmatrix}0\\ v/\sqrt2\end{pmatrix}$ and $\langle\tilde\Phi\rangle_0 = \begin{pmatrix}v/\sqrt2\\ 0\end{pmatrix}$ $\Rightarrow$ $m_i = v f_i/\sqrt2$ for $i = e, d, \nu_e, u$.
↳ mass lower doublet comp. ⟶ mass upper doublet comp.

The mass generation for the other fermion species proceeds in a similar way.

Gauge eigenstates need not be identical to mass eigenstates: eigenstates with the same quantum numbers can mix, as observed in the weak decays

$\underset{(udd)}{n} \to \underset{(uud)}{p}\, e^-\bar\nu_e$ as well as $\underset{(uds)}{\Lambda} \to \underset{(uud)}{p}\, e^-\bar\nu_e$!

$d \to u$ transition $\qquad$ $s \to u$ transition

Hence, we need a more general version of the Yukawa sector in the actual Standard Model.

Introduce therefore several generations of leptons and quarks, and indicate the gauge eigenstates (which feature in/are produced by weak interactions) as

generation label

$$\nu'_{A_R} = (\nu'_{e_R}, \nu'_{\mu_R}, \ldots),\quad e'_{A_R} = (e'_R, \mu'_R, \ldots),\quad u'_{A_R} = (u'_R, c'_R, \ldots),\quad d'_{A_R} = (d'_R, s'_R, \ldots),$$

gauge singlets

$$L'_{A_L} = \begin{pmatrix} \nu'_{A_L} \\ e'_{A_L} \end{pmatrix},\quad Q'_{A_L} = \begin{pmatrix} u'_{A_L} \\ d'_{A_L} \end{pmatrix}$$ ← gauge doublets

The matter part of the Lagrangian, incl. electroweak covariant derivatives, then reads

$$\mathcal{L}_F = \bar{L}'_{A_L} i\gamma^\mu [\partial_\mu + \tfrac{i}{2} g \vec{\sigma}\cdot\vec{W}_\mu - \tfrac{i}{2} g' B_\mu] L'_{A_L} + \bar{e}'_{A_R} i\gamma^\mu [\partial_\mu - i g' B_\mu] e'_{A_R} + \bar{\nu}'_{A_R} i\gamma^\mu \partial_\mu \nu'_{A_R} + \ldots .$$

Furthermore we allow for Yukawa interactions with non-diagonal complex couplings:

$$\mathcal{L}_{Yu} = -(f_e)_{AB} \bar{L}'_{A_L} \Phi e'_{B_R} - (f_d)_{AB} \bar{Q}'_{A_L} \Phi d'_{B_R} - (f_{\nu_e})_{AB} \bar{L}'_{A_L} \tilde{\Phi} \nu'_{B_R} - (f_u)_{AB} \bar{Q}'_{A_L} \tilde{\Phi} u'_{B_R} + h.c.,$$

with $(f_i)_{AB}$ $(i = e, d, \nu_e, u)$ complex $n\times n$ matrices in family space with n generations of leptons and quarks. This results in mass terms

$$-\frac{v}{\sqrt{2}} \left[(f_e)_{AB} \bar{e}'_{A_L} e'_{B_R} + (f_d)_{AB} \bar{d}'_{A_L} d'_{B_R} + (f_{\nu_e})_{AB} \bar{\nu}'_{A_L} \nu'_{B_R} + (f_u)_{AB} \bar{u}'_{A_L} u'_{B_R} + h.c. \right],$$

involving mass matrices $\boxed{M^{(i)}_{AB} = \frac{v}{\sqrt{2}} (f_i)_{AB} \quad (i = e, d, \nu_e, u)}$ for the gauge eigenstates!

Next we want to bring this into diagonal form with positive entries on the diagonal, thereby switching to the mass eigenstates. For an arbitrary complex matrix $M^{(i)}$ we know from linear algebra that

$$\underset{S_i, T_i \text{ unitary}}{\exists} \quad S_i^\dagger M^{(i)} T_i = M^{(i)}_{diag} = \text{diagonal matrix with } (M^{(i)}_{diag})_{AA} \geq 0 .$$

↑ no summation

Consequence: gauge eigenstates contain a mixture of mass eigenstates

$$\left.\begin{array}{ll} u'_{A_L} = (S_u)_{AB}\, u_{B_L} & \text{etc.} \\ u'_{A_R} = (T_u)_{AB}\, u_{B_R} & \text{etc.} \end{array}\right\} \Rightarrow \bar{u}'_{A_L} M^{(u)}_{AB} u'_{B_R} = \bar{u}_{A_L} (S_u^\dagger M^{(u)} T_u)_{AB} u_{B_R} = \bar{u}_{A_L} (M^{(u)}_{diag})_{AB} u_{B_R} \text{ etc.}$$

mass eigenstates   properly diagonalized

Next we check whether the electroweak interactions can mix the generations (flavours).

Flavour-changing mixing effects: we start at lowest-order level

- No lowest order flavour-changing neutral currents (FCNC):

$$\bar{u}'_{AL}\gamma_\mu u'_{AL} = \bar{u}_{AL}\gamma_\mu \underbrace{(S_u^\dagger S_u)}_{I_n}{}_{AB}\, u_{BL} = \bar{u}_{AL}\gamma_\mu u_{AL}$$ etc., and similarly for R modes.

- Mixing does occur in the charged-current (CC) interactions:

$$\bar{u}'_{AL}\gamma_\mu d'_{AL} = \bar{u}_{AL}\gamma_\mu (S_u^\dagger S_d)_{AB}\, d_{BL} \equiv \bar{u}_{AL}\gamma_\mu (V_{CKM})_{AB}\, d_{BL} \,,$$

V_{CKM} = unitary quark-mixing matrix (Cabibbo-Kobayashi-Maskawa, 1973)
↳ Nobel Prize 2008

$$\equiv \begin{pmatrix} V_{ud} & V_{us} & \cdots \\ V_{cd} & V_{cs} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \longleftarrow$$ can be assigned to the down-type quarks, ... but there is no strong need for that

A similar thing can be done in the lepton sector, although that is not seen as a standard procedure in many texts on the Standard Model. We will come back to this.

How many independent parameters will determine V_{CKM}, bearing in mind that some phases can be absorbed into a redefinition of the quark fields?

Concept: $\psi \to e^{i\varphi}\psi \Rightarrow \pi_\psi = i\psi^\dagger \to e^{-i\varphi}\pi_\psi$ (independent of x) and therefore all fundamental anticommutation relations stay the same ⇒ physics unaltered!

In expressions like $\bar{u}_{AL}\gamma_\mu (V_{CKM})_{AB}\, d_{BL}$ one phase can be absorbed for each quark flavour, with the exception of an overall phase. Assuming V_{CKM} to be a complex $n\times n$ matrix, the d.o.f. counting goes as follows:

$2n^2$ d.o.f. $\xrightarrow{V^\dagger V = I_n}$ $2n^2 - n^2 = n^2$ d.o.f. $\xrightarrow[\text{quark phases}]{\text{absorb unphysical}}$ $n^2 - (2n-1) = (n-1)^2$ d.o.f., out of which $\binom{n}{2}$ d.o.f. correspond to real n-dim. rotations (i.e. angles) and $(n-1)^2 - \binom{n}{2} = \frac{1}{2}(n-1)(n-2)$ d.o.f. correspond to independent physical phases.

Hence:
- for $n=2$ one rotation (Cabibbo) angle determines V_{CKM}, which is effectively real since all phases are absorbable;
- for $n=3$ three rotation angles and one unabsorbable phase determine V_{CKM}, opening up the possibility that $V_{CKM} \notin \mathbb{R}$

⇒ at least three generations are needed for having $V_{CKM} \notin \mathbb{R}$ in the CC interactions!

The experimental results on V_{CKM} can be summarized in the Wolfenstein parametrization:

$$V_{CKM} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix},$$

good approximation

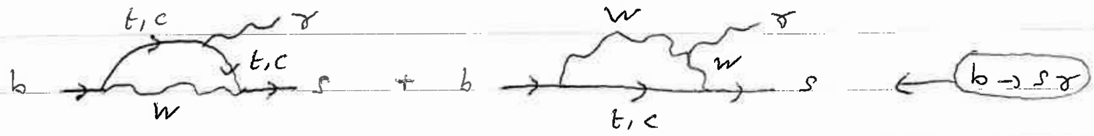
in fact: $|V^{exp.}_{CKM}|_{AB} \approx \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.009 & 0.04 & 1.00 \end{pmatrix}$

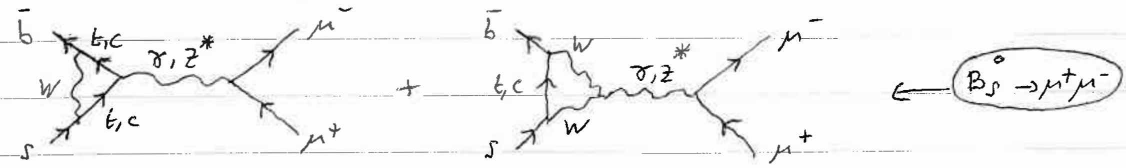
with $\lambda = 0.226 \pm 0.001$, $A = 0.81 \pm 0.02$, $\rho = 0.14 \pm 0.02$, $\eta = 0.35 \pm 0.02$.

↳ pronounced hierarchy among components !

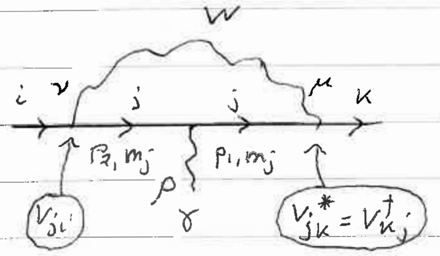
The mixing in the quark sector gives rise to some interesting phenomena.

*) FCNC occur in higher loop order, inducing rare decays such as $b \to s\gamma$ and $B^0_s \to \mu^+\mu^-$:





The associated amplitudes are small due to the GIM mechanism (Glashow–Iliopoulos–Maiani, 1970):



shorthand for V_{CKM}

spinors

$$\propto \sum_j V^\dagger_{kj} V_{ji}\, \bar{u}_k \gamma_\mu (I_4 - \gamma^5) \frac{\not{p}_1 + m_j}{p_1^2 - m_j^2} \gamma_\rho \frac{\not{p}_2 + m_j}{p_2^2 - m_j^2} \gamma_\nu (I_4 - \gamma^5) u_i$$

$$= \sum_j \frac{2 V^\dagger_{kj} V_{ji}}{(p_1^2 - m_j^2)(p_2^2 - m_j^2)}\, \bar{u}_k \gamma_\mu \left[\not{p}_1 \gamma_\rho \not{p}_2 + m_j^2 \gamma_\rho \right] \gamma_\nu (I_4 - \gamma^5) u_i$$

needed: additional j dependence in order to avoid that $\sum_j V^\dagger_{kj} V_{ji} = \delta_{ki}$ (which would mean no FCNC)

$\Rightarrow$ suppression factors $V^\dagger_{kj} V_{ji}$ for $k \neq i$,
$g^2 m_j^2 / m_W^2$ if j light, and $g^2 m_W^2 / m_j^2$ if $j = t$.

Such FCNC processes put severe constraints on physics beyond the SM: models beyond the SM might give rise to additional generational mixing, involving new (non-SM) particles that feature in the FCNC loop diagrams

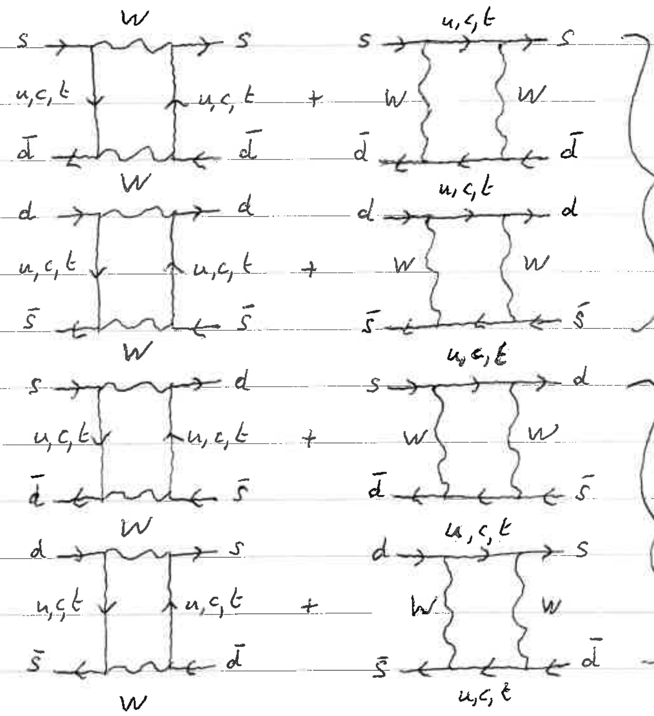
$\Rightarrow$ constraints on the associated mixing matrices and/or the masses of the new particles of the model, in order not to exceed the amount of FCNC effects that we observe in experiment !

*) Meson mixing effects + CP violation: let's consider the mixing effects in the $\bar{K}^0$-K^0 (kaon) system, which give rise to energy eigenstates $K_{S/L}$.
$(s\bar{d})$ $(d\bar{s})$ — short/long lived

under a CP transformation $|K^0\rangle \overset{\hat{CP}}{\longleftrightarrow} -|\bar{K}^0\rangle$ (kaons, pions are P-odd). Imposing CP symmetry would imply $[\hat{CP}, \hat{H}] = 0 \Rightarrow \exists$ simultaneous set of eigenfunctions of $\hat{H}$ and $\hat{CP}$.

Consequence: K_S = CP-even $\Rightarrow$ can decay into CP-even $\pi^0\pi^0$, $\pi^+\pi^-$ final states,
K_L = CP-odd $\Rightarrow$ cannot decay into CP-even $\pi^0\pi^0$, $\pi^+\pi^-$ final states,
the CP-odd $\pi^0\pi^0\pi^0$, $\pi^0\pi^+\pi^-$ final states are possible!

$\boxed{\bar{K}^0 \to \bar{K}^0}$:
$\boxed{K^0 \to K^0}$:
$\boxed{\bar{K}^0 \to K^0}$:
$\boxed{K^0 \to \bar{K}^0}$:



matrix elements (e.g. $|V_{us}V_{ud}^*|^2$) are real and equal to each other (e.g. $|V_{ud}V_{us}^*|^2$)

matrix elements (e.g. $(V_{us}V_{ud}^*)^2$) are potentially non-real due to V_{td} and c.c. of each other (e.g. $(V_{ud}V_{us}^*)^2$)

The mixing matrix $\begin{pmatrix} A & B \\ B^* & A \end{pmatrix} = \begin{pmatrix} A & |B|e^{i\varphi_B} \\ |B|e^{-i\varphi_B} & A \end{pmatrix}$ ($A \in \mathbb{R}$) has eigenvalues $A \pm |B|$ and

eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm e^{-i\varphi_B} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\bar{K}^0\rangle \pm e^{-i\varphi_B}|K^0\rangle\right)$.

Scenario 1: $B \in \mathbb{R}$, $\varphi_B = 0 \Rightarrow$ eigenstates $K_L = \underbrace{\frac{1}{\sqrt{2}}(|\bar{K}^0\rangle + |K^0\rangle)}_{\text{CP-odd}} \equiv K_+$, $K_S = \underbrace{\frac{1}{\sqrt{2}}(|\bar{K}^0\rangle - |K^0\rangle)}_{\text{CP-even}} \equiv K_-$
$\Rightarrow$ CP symmetry, $K_L \not\to \pi^0\pi^0, \pi^+\pi^-$.

Scenario 2: $B \notin \mathbb{R}$, $\varphi_B \neq 0$ (small) $\Rightarrow$ eigenstates $K_L \approx (1 - \frac{i}{2}\varphi_B)K_+ + \frac{i}{2}\varphi_B K_-$, (almost CP-odd)
$K_S \approx \frac{i}{2}\varphi_B K_+ + (1 - \frac{i}{2}\varphi_B)K_-$ (almost CP-even)
$\Rightarrow$ no CP symmetry, suppressed $K_L \to \pi^0\pi^0, \pi^+\pi^-$ allowed.

Scenario 2 observed in 1964! Consequence: $V_{CKM} \notin \mathbb{R} \overset{\text{p.27}}{\Longrightarrow}$ the SM should involve at least three generations of quarks!

§3.2 The lepton sector: neutrino masses and mixing

Standard-Model style neutrino masses: as already indicated on p. 25-27, an explicit Yukawa interaction $-(f_{\nu_e})_{AB}\,\bar{L}'_{A_L}\tilde{\Phi}\,\nu'_{B_R} + h.c.$ can be added to the Standard Model in order to obtain neutrino mass terms $-\frac{v}{\sqrt{2}}(f_{\nu_e})_{AB}\,\bar{\nu}'_{A_L}\nu'_{B_R} + h.c.$. Here pure singlet right-handed neutrino modes have been added, which are not subject to electroweak or strong interactions. In analogy with the quark sector, gauge and mass eigenstates need not coincide in the lepton sector as well. This gives rise to lepton mixing in the charged-current interactions (Pontecorvo-Maki-Nakagawa-Sakata [PMNS, 1962]):

$$\bar{\nu}'_{A_L}\gamma_\mu e'_{A_L} = \bar{\nu}_{A_L}\gamma_\mu (S_\nu^\dagger S_e)_{AB}\, e_{B_L} \equiv \bar{\nu}_{A_L}\gamma_\mu (U^\dagger_{PMNS})_{AB}\, e_{B_L},$$

and for three generations of fermions

$$U_{PMNS} \equiv \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

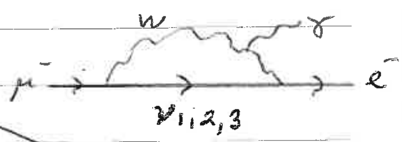
with $s_{ij} \equiv \sin(\theta_{ij})$, $c_{ij} \equiv \cos(\theta_{ij})$.

This unitary PMNS-matrix describes lepton mixing. It involves 3 mixing angles and 1 CP-violating phase, just as argued for the CKM-matrix.

(may be important for matter-antimatter asymmetry)

*) If all neutrino masses would be equal (e.g. all 0), then mixing would be irrelevant as each linear combination of mass eigenstates would be a mass eigenstate ⇒ gauge eigenstate = mass eigenstate!

*) Lepton mixing induces rare decays such as $\mu^- \to e^-\gamma$:



(extremely rare, due to a more pronounced GIM mechanism)

The similarities with the quark sector stop here, in fact there are also clear differences:

*) $|U_{PMNS}|^{\text{exp.}}_{AB} \approx \begin{pmatrix} 0.8 & 0.5 & 0.1 \\ 0.4 & 0.6 & 0.7 \\ 0.4 & 0.6 & 0.7 \end{pmatrix}$ ⇒ contrary to V_{CKM}, U_{PMNS} effectively shows no sign of a component hierarchy!

*) Neutrinos can only be produced and detected through the weak interactions. Actually the neutrinos are detected by observing the associated lepton mass eigenstates! In most cases also a particular lepton mass eigenstate is involved in the production of neutrinos through weak CC interactions, such as in proton-proton fusion reactions in the Sun ($pp \to {}^2_1D + e^+ + \nu_e$), charged-meson decays (e.g. $\pi^+ \to \mu^+ \nu_\mu$, $\pi^- \to \mu^- \bar{\nu}_\mu$) or muon decays ($\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$). Therefore U_{PMNS} is usually assigned to the neutrino mass eigenstates to form so-called neutrino flavour eigenstates:

$$\bar{\nu}_{A_L} \equiv \bar{\nu}_{q_L} (U_{PMNS})_{qA} \Rightarrow \bar{\nu}_{A_L} \gamma_\mu (U^\dagger_{PMNS})_{AB}\, e_{B_L} = \bar{\nu}_{q_L} \gamma_\mu \underbrace{(U_{PMNS} U^\dagger_{PMNS})}_{I_3}{}_{qB}\, e_{B_L}$$

$$= \bar{\nu}_{q_L} \gamma_\mu e_{q_L} \quad [\text{with } q = e, \mu, \tau \text{ labeling the flavours}],$$

which reflects the intimate link between neutrino flavours and specific lepton mass eigenstates!

*) A non-unit lepton mixing matrix U_{PMNS} implies that the $\nu_{e,\mu,\tau}$ neutrinos, which are produced in weak-interaction processes, are linear superpositions of the neutrino mass eigenstates $\nu_{1,2,3}$. Quantum mechanically these three mass-eigenstate components evolve differently with time. As a result, the $\nu_{e,\mu,\tau}$ neutrinos can oscillate into each other while traveling a certain distance!

See Ex.10 (neutrinos are effectively stable)

This can be observed by measuring the neutrinos as flavour eigenstates (through weak interactions in the detector). Neutrino oscillations were observed in 1998 @ Superkamiokande and without realizing it could have been inferred from the solar ν_e deficit at the Homestake experiment (1960's) ← solar neutrino problem

$\Rightarrow (\nu_{e,\mu,\tau}) \neq (\nu_{1,2,3}) \Rightarrow$ massive neutrinos exist!

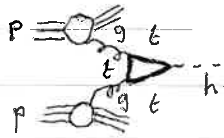
Question still to be answered: is the mass hierarchy the same as in the quark/charged-lepton sector? Reason: no direct access to $m_{\nu_{1,2,3}}$ in oscillation experiments.

Opening the door to physics beyond the Standard Model: alternative mass terms.

A generic mass term for fermions takes the form $-m(\overline{\psi}_{1L}\psi_{2R} + \overline{\psi}_{1R}\psi_{2L} + h.c.)$ for any combination of two spinor fields $\psi_{1,2}$. Since electroweak symmetry breaking leaves the $U(1)_{e.m.}$ symmetry intact, in order to guarantee that the photon remains massless, the fields $\psi_{1,2}$ are required to describe particles with the same electrostatic charge! We exploited this fact in the discussion of lepton mixing on p.30. However, neutrinos are neutral and therefore do not possess quantum numbers that would allow us to differentiate between particles and antiparticles experimentally. We only know that a neutral fermion field should be combined with a charged lepton field to form an $SU(2)_L$ doublet under the weak interactions. The experimental evidence for this construction was based solely on helicity (chirality) considerations --- at no point did we have to demand that the doublet partners of the charged leptons should refer to particles rather than antiparticles. This opens up a new avenue of possibilities: ψ_{2R} could either be a pure singlet right-handed neutrino mode (as on p.30) or a charge-conjugated version of the left-handed neutrino mode that features in an $SU(2)_L$ doublet*.

↳ the latter option goes beyond the realm of the Standard Model
$\Rightarrow$ in various texts on the Standard Model non-zero neutrino masses are interpreted as a sign of beyond the Standard Model physics (although that is not strictly necessary)!

§3.3 No fourth Standard-Model style fermion generation

Higgs production @ LHC :  followed by subsequent Higgs-boson decay.

The corresponding matrix element becomes independent of m_t for $2m_t \gg m_h$ (heavy-mass limit)
$\Rightarrow$ adding heavier quarks from a fourth fermion generation would give rise to a three times larger matrix element and therefore a nine times larger Higgs production cross section, which is not observed experimentally!

* Typically one could add alternative mass terms $-\frac{m_L}{2}\big[\overbrace{(\overline{\nu_L})^c\,\nu_L}^{\text{part of doublet}} + h.c.\big] - \frac{m_R}{2}\big[\overbrace{(\overline{\nu_R})^c\,\nu_R}^{\text{pure singlet}} + h.c.\big]$, involving charge-conjugated $(\overline{\nu}_{L/R})^c = \overline{\nu^c}P_{L/R}$ fields. For $m_L \neq 0$ and/or $m_R \neq 0$ lepton number is violated, since ν and ν^c have opposite lepton number (just like ν and $\bar{\nu}$) $\Rightarrow$ lepton-number violating processes would be a smoking gun for this!

Epilogue : status of the Standard Model

Over the past three decades the Standard Model has proven extremely successful, passing all high-precision tests that collider experiments (such as LEP1, SLC, LEP2, Tevatron, LHC) have thrown at it. It has truly established itself as the standard theoretical framework for describing strong-interaction and electroweak physics. Also the non-Abelian nature of the strong and weak interactions has been confirmed conclusively (e.g. the existence of the $W^+W^-\gamma$ / W^+W^-Z TGC in $e^+e^- \to W^+W^-$ @ LEP2). With the discovery of the Higgs boson in 2012 at the LHC, the Standard Model's full predicted particle content has been confirmed!

SM strengths →
(The good)
- passed all high-precision collider tests;
- complete particle content confirmed;
- minimal particle, gauge and interaction content;
- one Higgs doublet does all;
- all properties of the Higgs particle are fixed ... except its mass (λ).

SM weaknesses ... what it lacks →
(The bad)
- gravity and a reason why $\underset{\mathcal{O}(10^{19}\,\text{GeV})}{\Lambda_{Planck}} \gg \underset{\mathcal{O}(100\,\text{GeV})}{\Lambda_{ew}}$;
- a dark-matter candidate;
- an explanation for the matter-antimatter asymmetry in the universe;
- full-fledged massive ν's (including all options).

SM weaknesses ... what came out of the blue →
(The ugly)
- the family structure, the magic quantum numbers, the parameters, the mass hierarchy;
- charge quantization;
- the three gauge couplings;
- the Higgs potential;
- the left-right asymmetry.