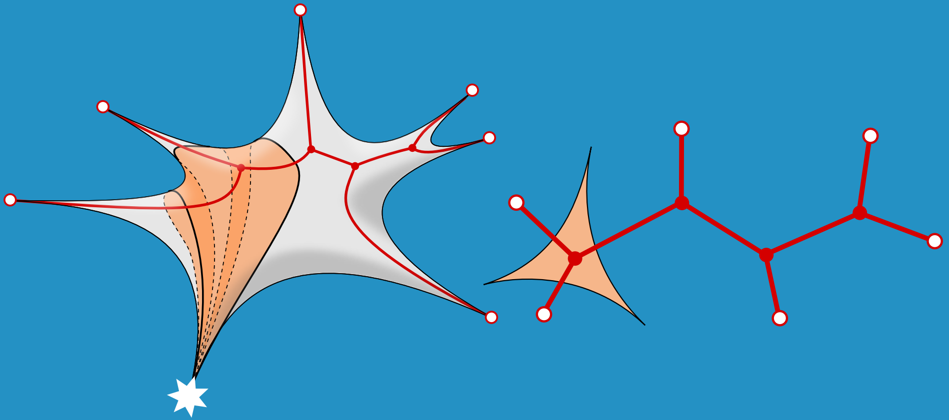
01-04-2020, Virtual Quantum Gravity Seminar @ Radboud University

Tree bijections for Riemann surfaces

Timothy Budd



Radboud University

T.Budd@science.ru.nl
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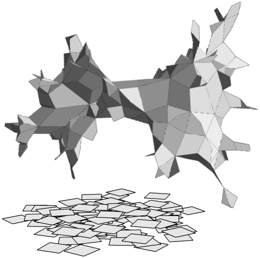
What is a natural random metric on S^2?

Random planar map



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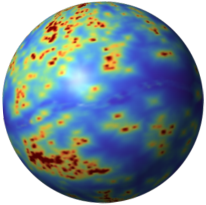
Liouville quantum gravity



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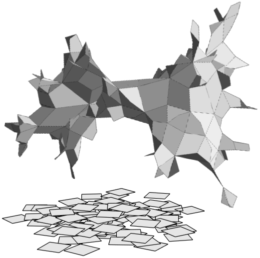
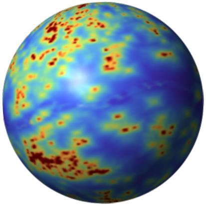
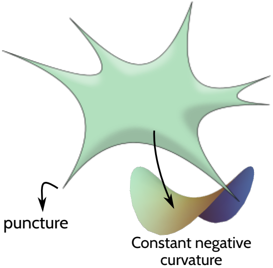


Liouville quantum gravity



$$\text{"}\mathrm{d}s^2 = e^{\gamma h(x,y)}(\mathrm{d}x^2 + \mathrm{d}y^2)\text{"}$$

Random Riemann surface



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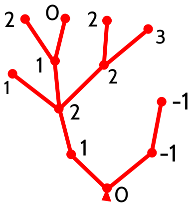
[Cori, Vauquelin, Schaeffer, Bouttier, Di Francesco, Guitter, ...]



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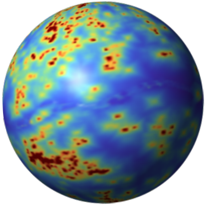

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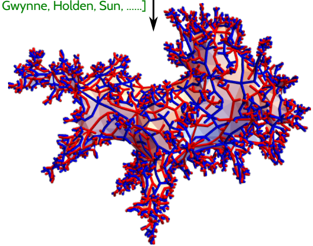
Mating of trees

[Miller, Sheffield, Duplantier, Gwynne, Holden, Sun, ...]



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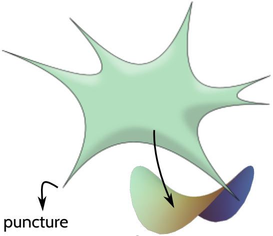

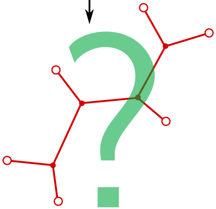

Trees all over the place....

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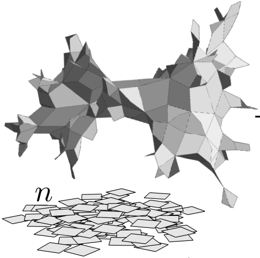
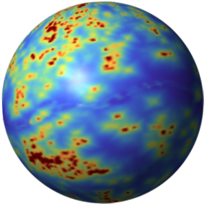


$n \to \infty$

Liouville quantum gravity



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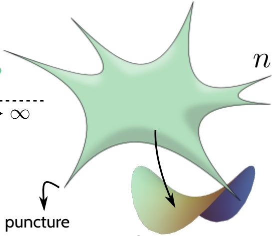

Constant negative curvature

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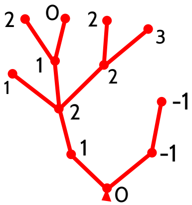

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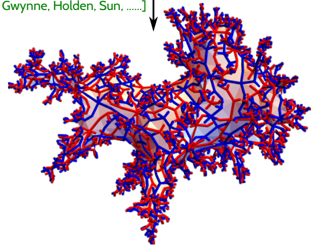

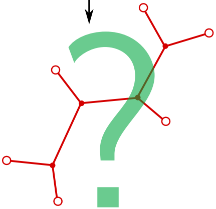

Trees all over the place....

Quadrangulations and labeled trees

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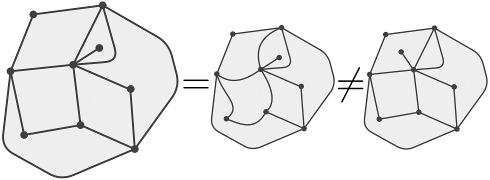

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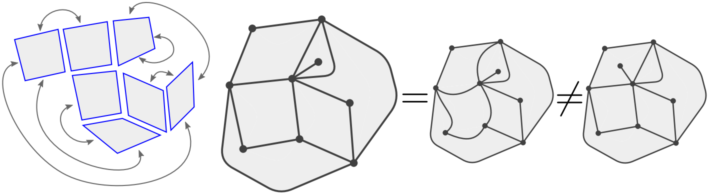

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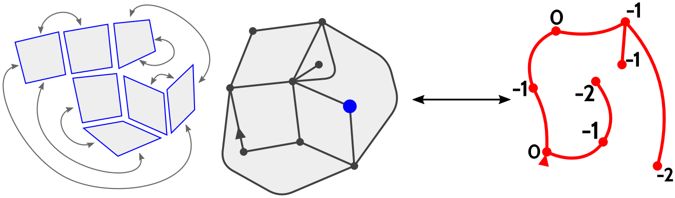

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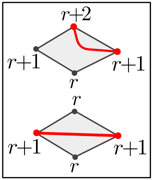

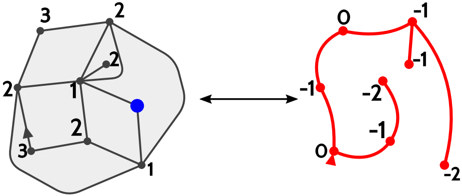

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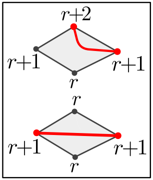

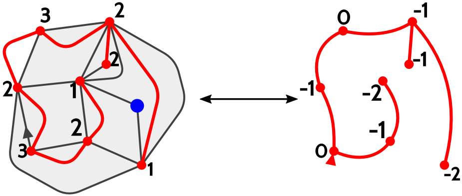

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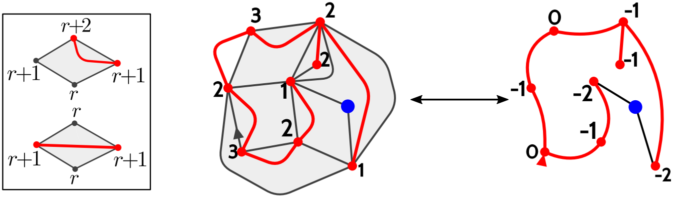

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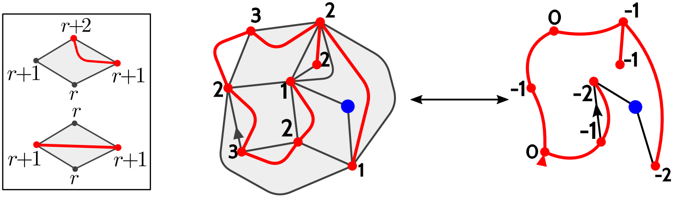

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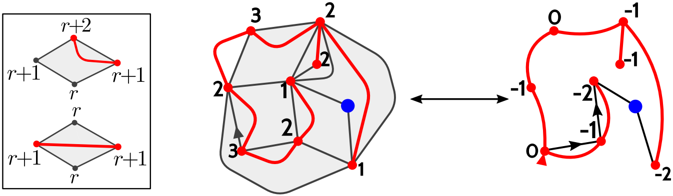

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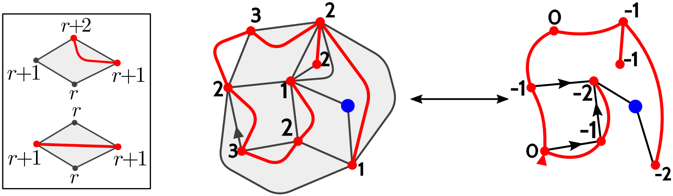

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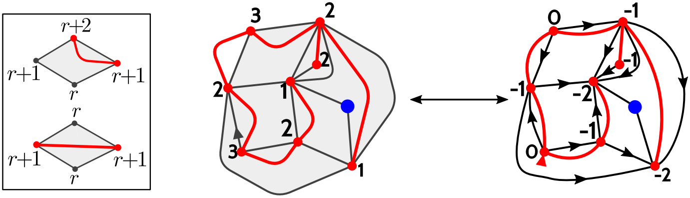

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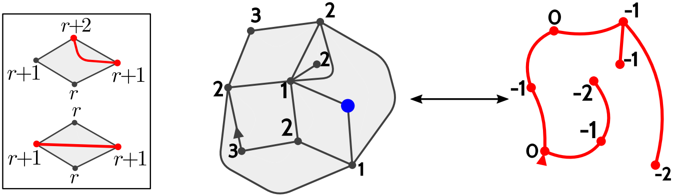
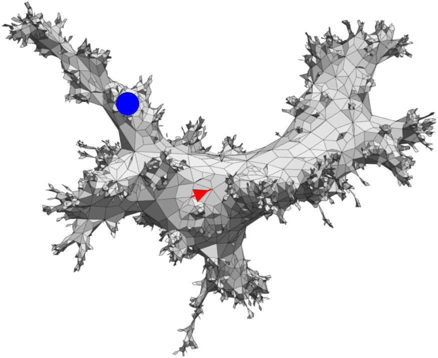
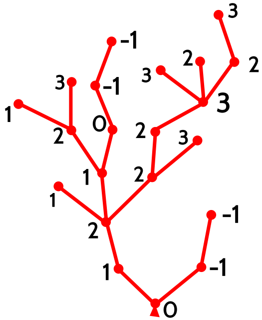
- uniform random quadrangulations with n faces $\longleftrightarrow$ uniform plane tree with n edges with a uniform label assignment.



Distance statistics in large quadrangulations

- The trees provide an easy way to deduce the fractal dimension:
  geodesic distance $\sim$ volume$^{1/4}$ [Ambjørn, Watabiki, '95]





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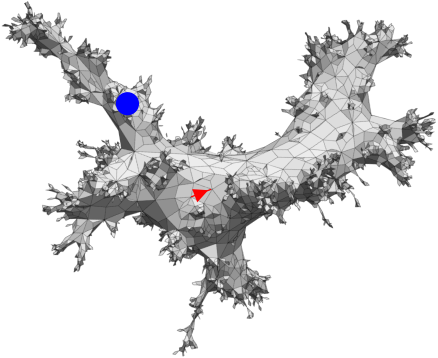

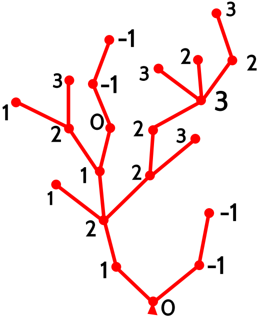

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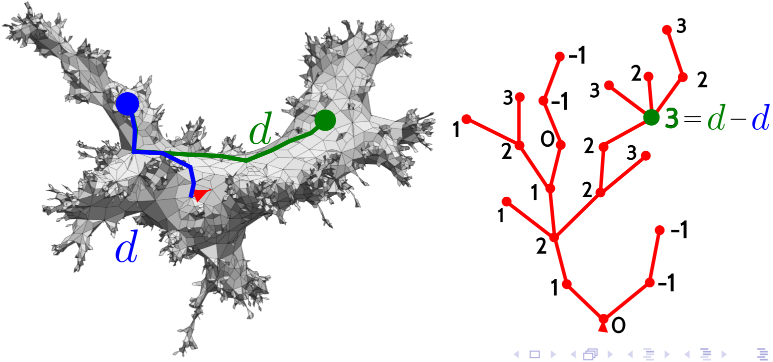

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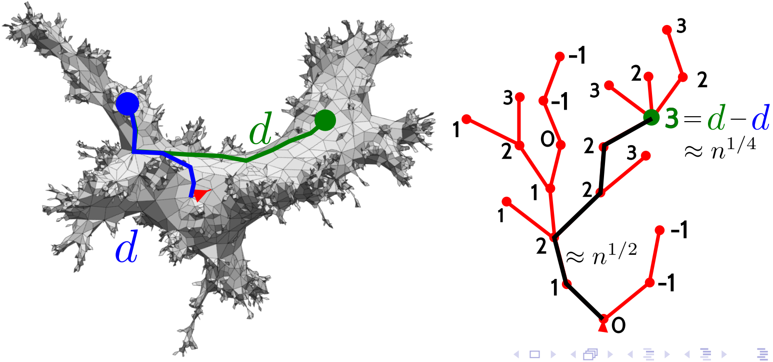

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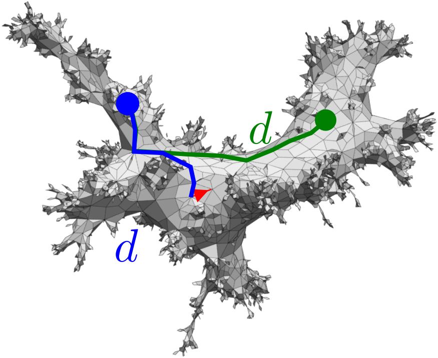
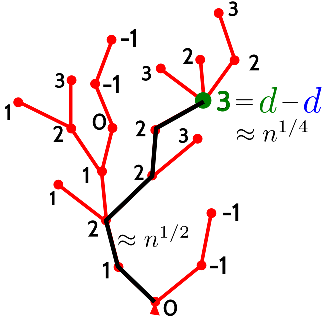
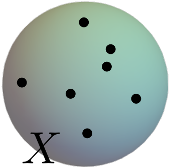
- Scaling limit: $(\{\text{vertices}\}, n^{-1/4} d(\cdot,\cdot)) \xrightarrow[n\to\infty]{(\mathrm{d})}$ Brownian sphere
  [Le Gall, '10][Miermont, '10]





Riemann surfaces

- Consider a punctured Riemann surface X: sphere with $n \geq 3$ labeled points removed and equipped with a complex structure.



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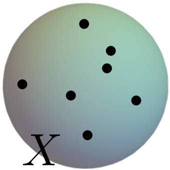

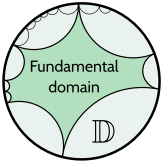

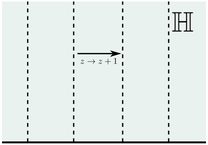

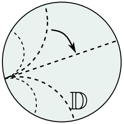

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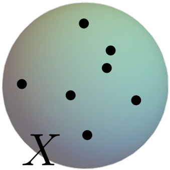
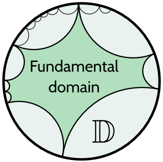
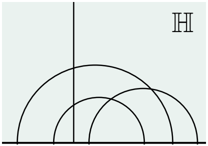
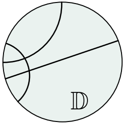
- $\mathbb{H}$ comes with a natural Poincaré metric: $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$.









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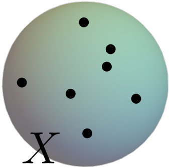
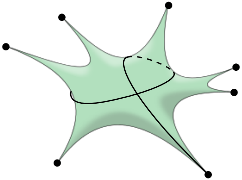
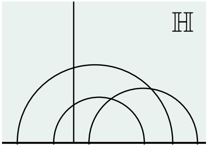
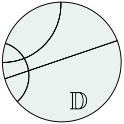
- $\mathrm{Aut}(\mathbb{H})$ acts by isometries $\longrightarrow$ X comes naturally equipped with complete metric of curvature -1: hyperbolic surface with cusps.





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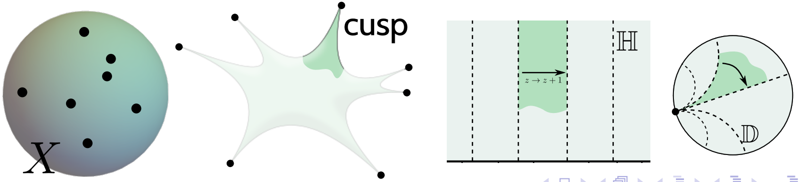

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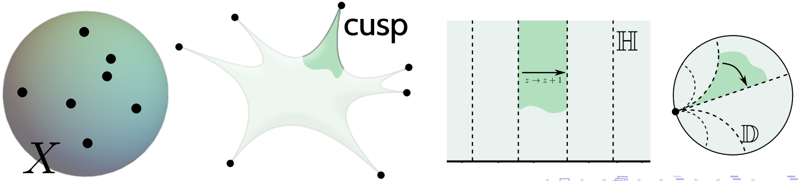

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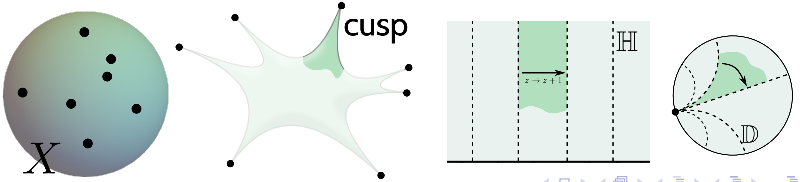

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- Several results on random Riemann surfaces for large genus g:
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    $\langle$ Diameter$\rangle \approx \langle$length shortest separating curve$\rangle \approx \log g$ as $g \to \infty$.
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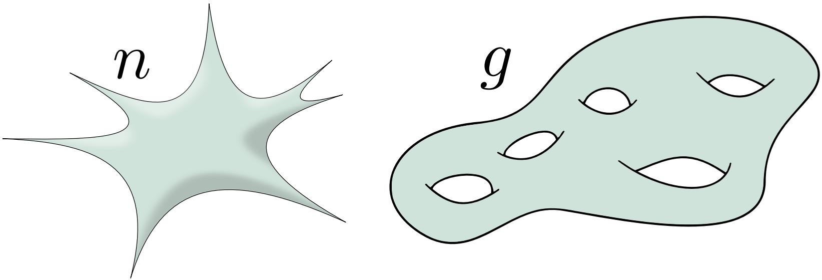

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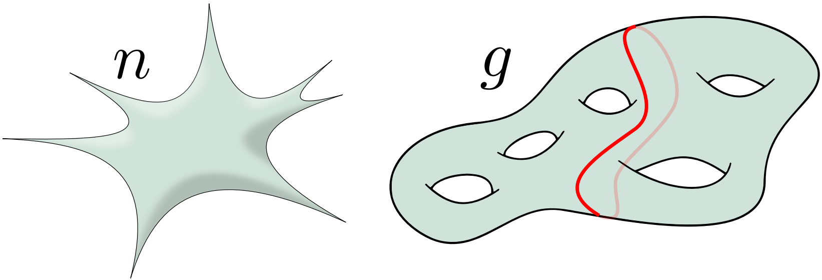

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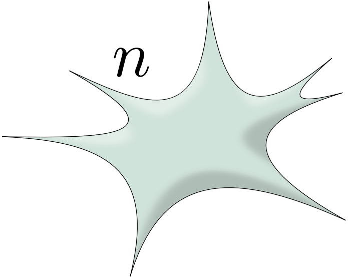

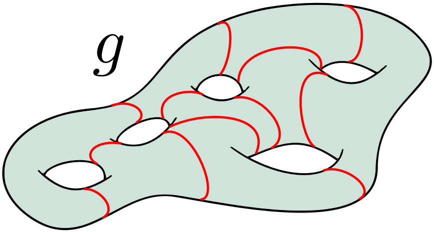

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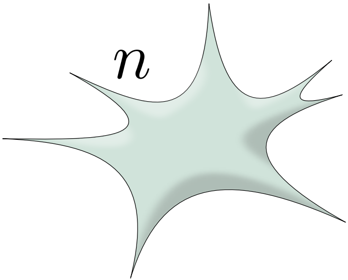

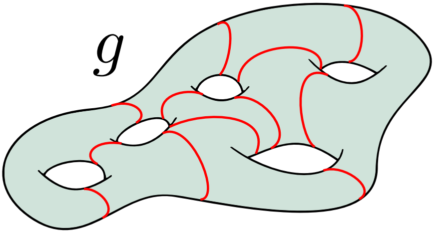

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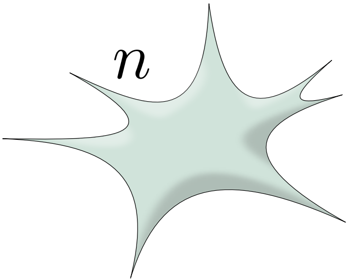
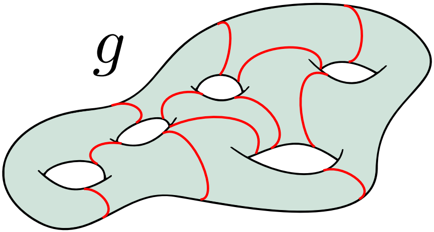
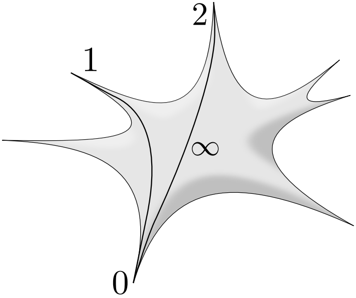
- Problem: until now no good algorithm to sample random Riemann surfaces.





Distance statistics?

- Distance between punctures is infinite.



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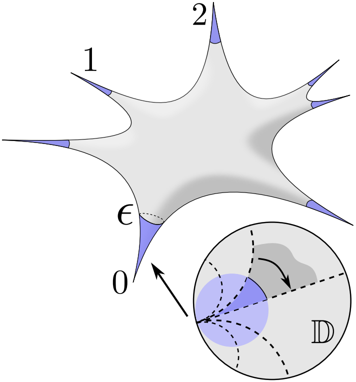

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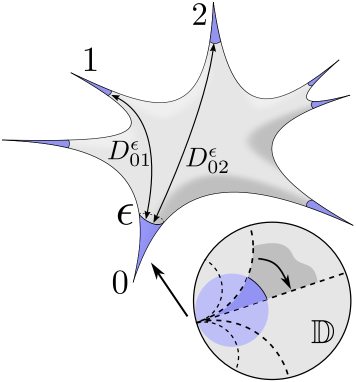

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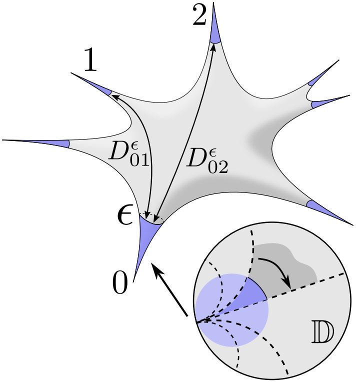

Theorem (TB, '19+)

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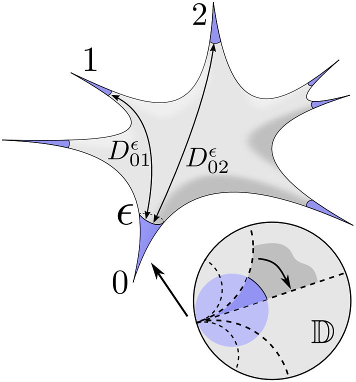

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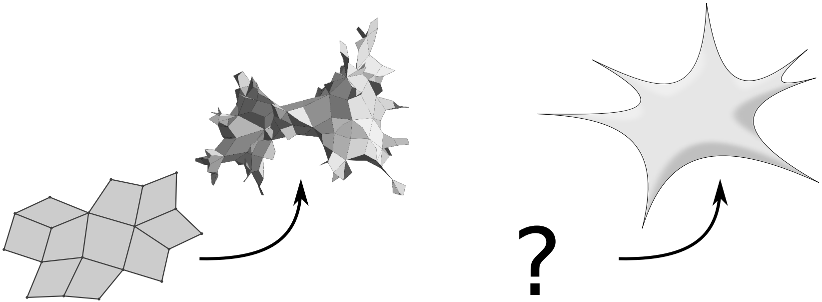
- Open problem: $(\{1,\ldots,n\}, n^{-1/4}(D^\epsilon_{ij})_{i,j}) \xrightarrow[n\to\infty]{(d)}$ Brownian sphere?

Building blocks?

- Quadrangulations: gluing of identical squares.
  Riemann surfaces: gluing of ???.



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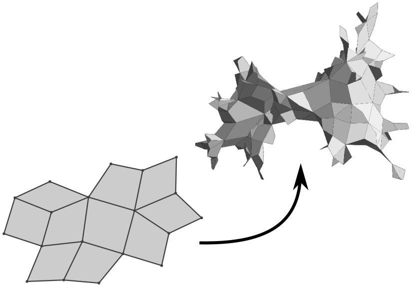

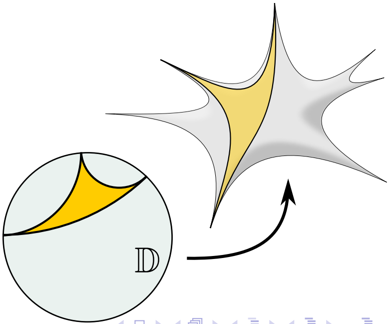

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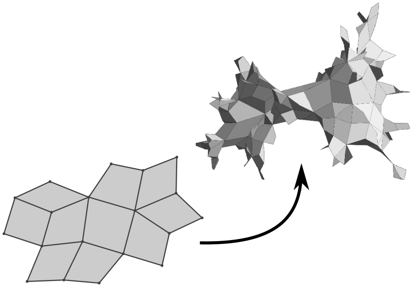

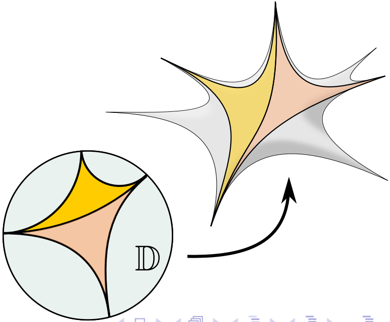

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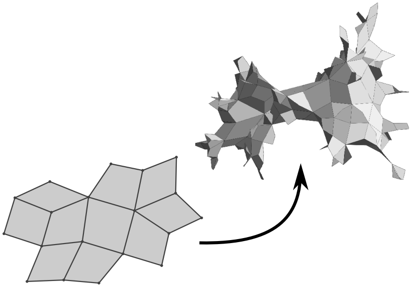

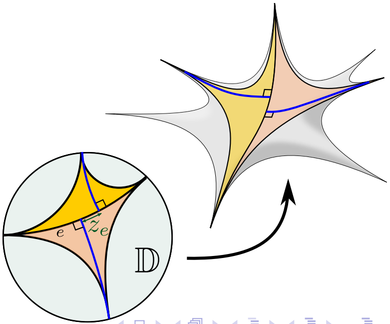

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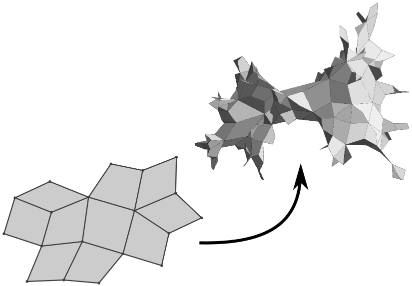

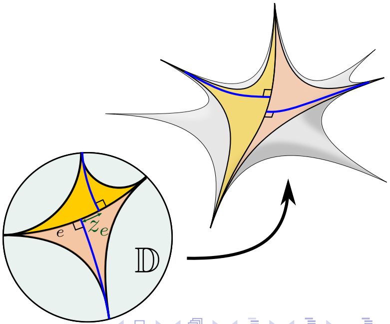

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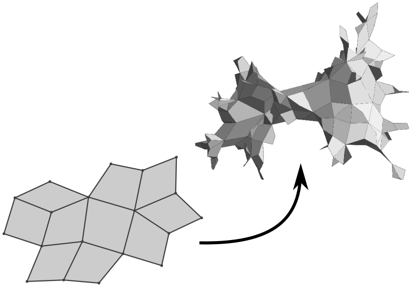

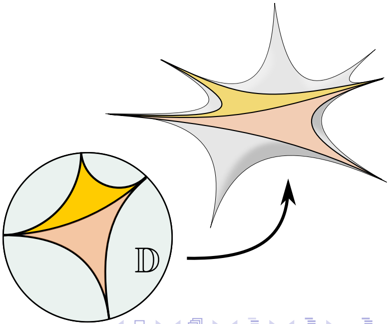

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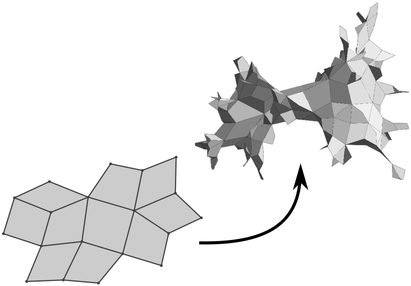
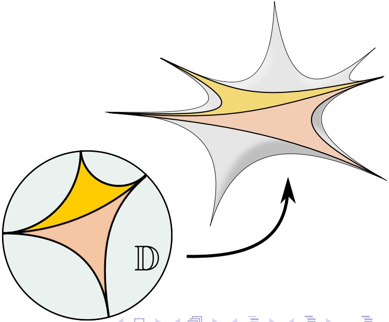
- Determine a canonical triangulation!





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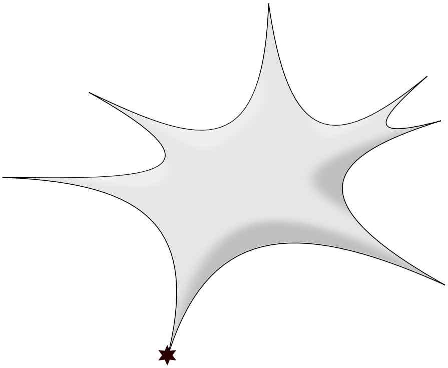

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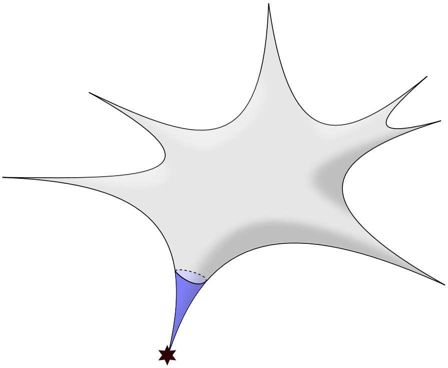

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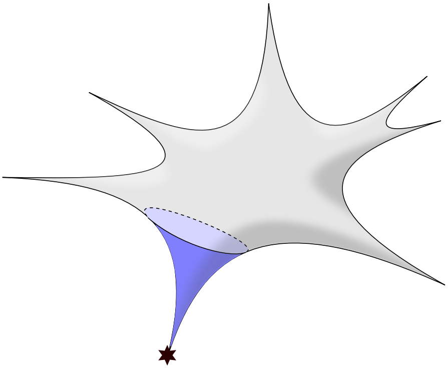

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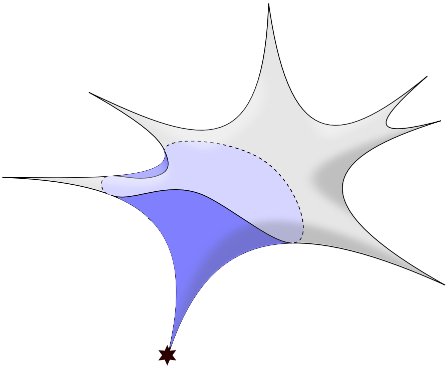

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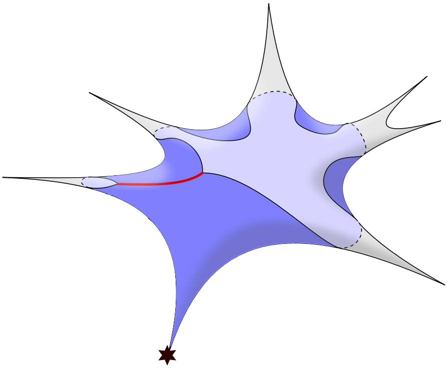

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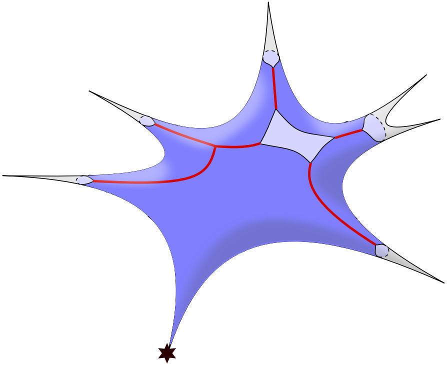

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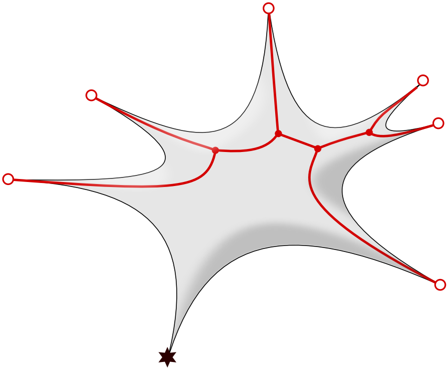

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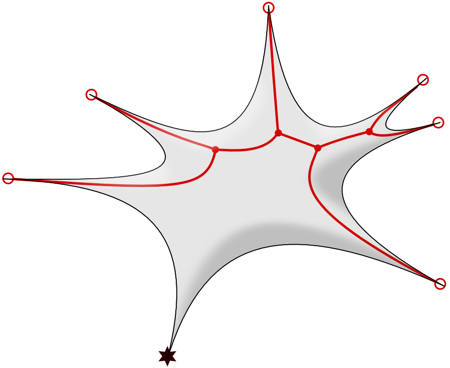

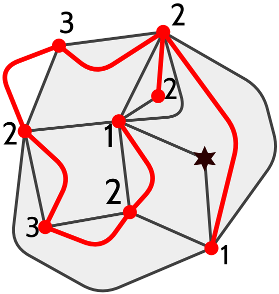

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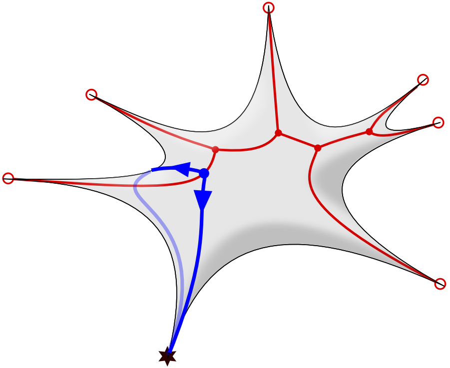

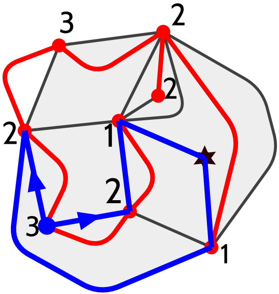

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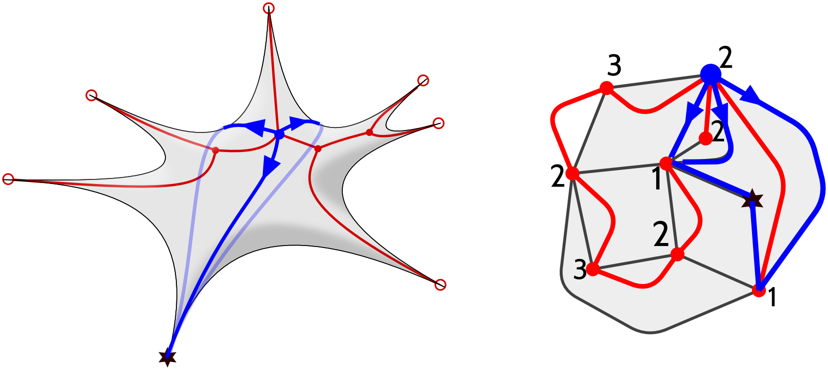

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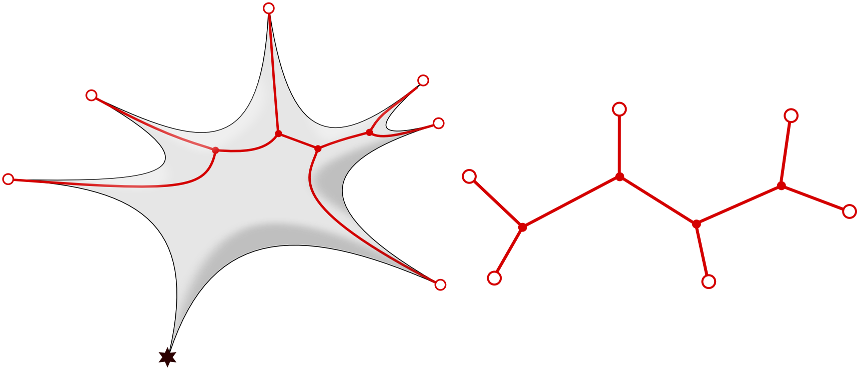

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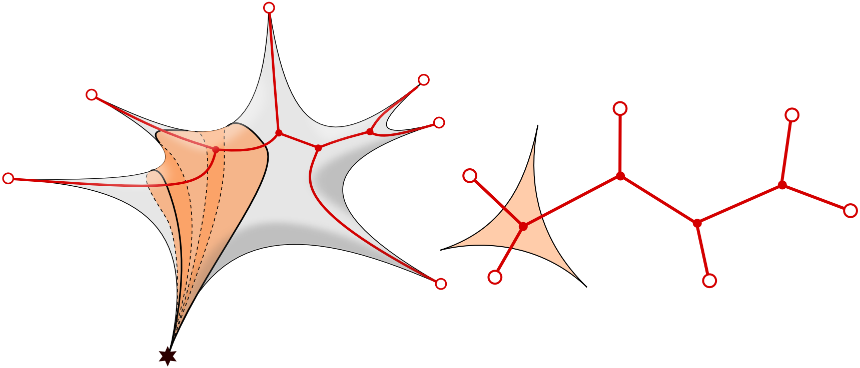

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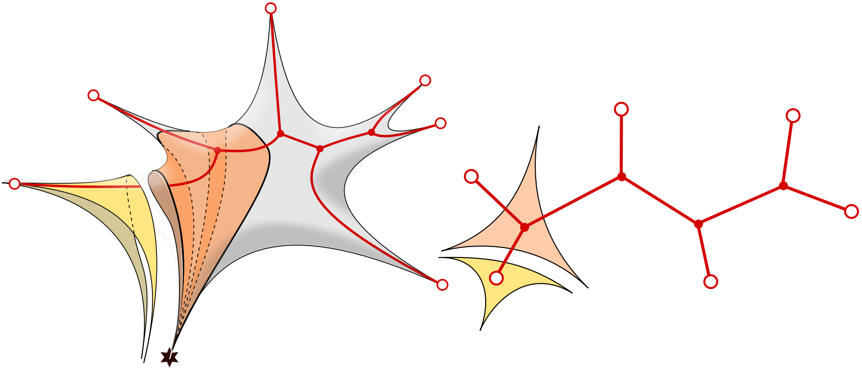

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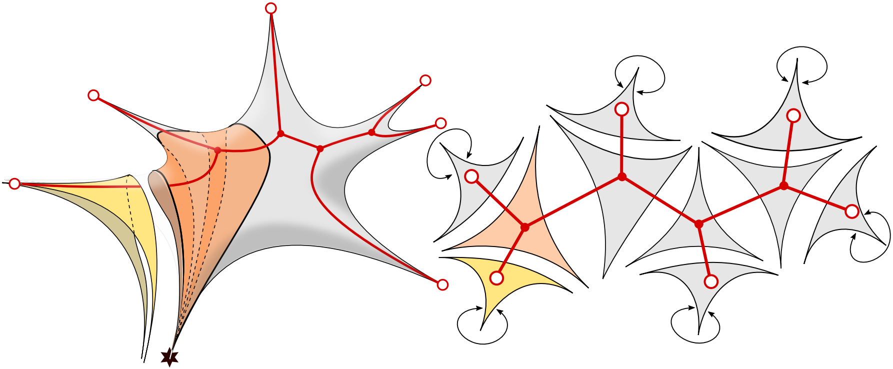

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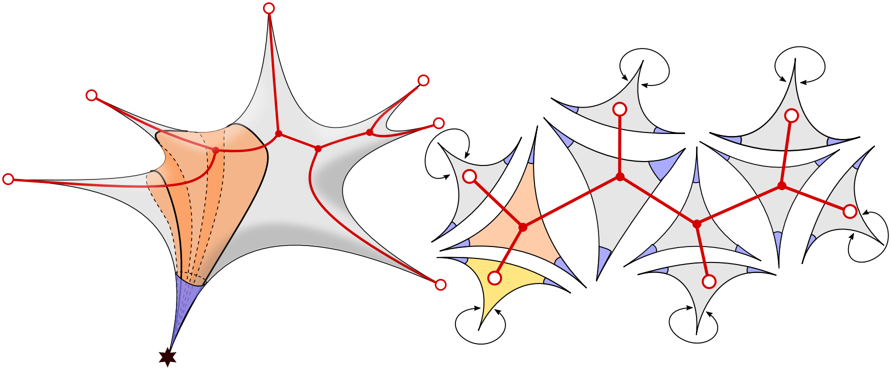

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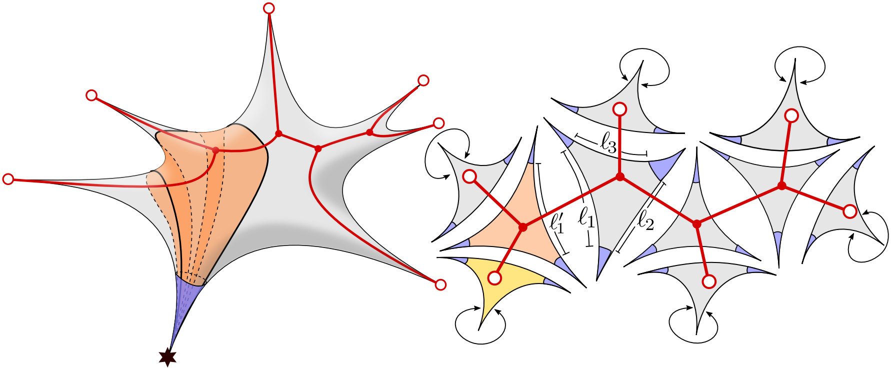

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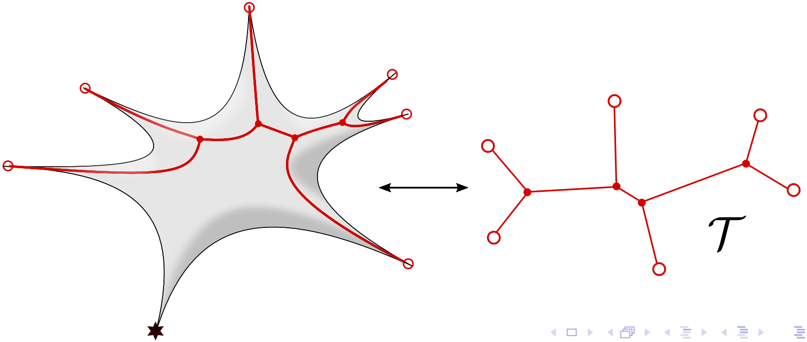

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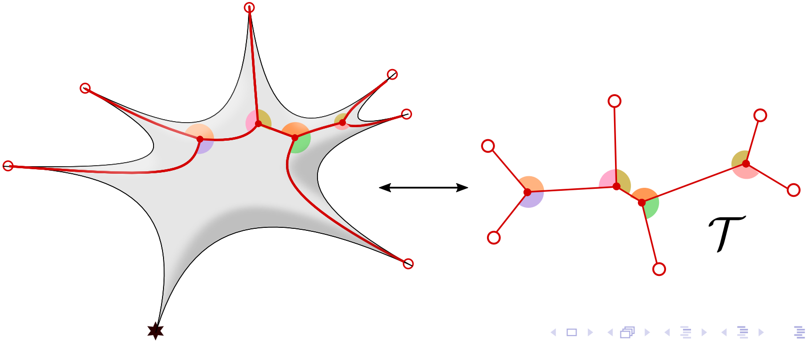
$$\mathcal{A}_{\mathcal{T}} = \{(\alpha_1, \ldots, \alpha_{2n-6}) : \text{sum of opposing angles} > \pi\} \subset (0, \pi)^{2n-6}.$$



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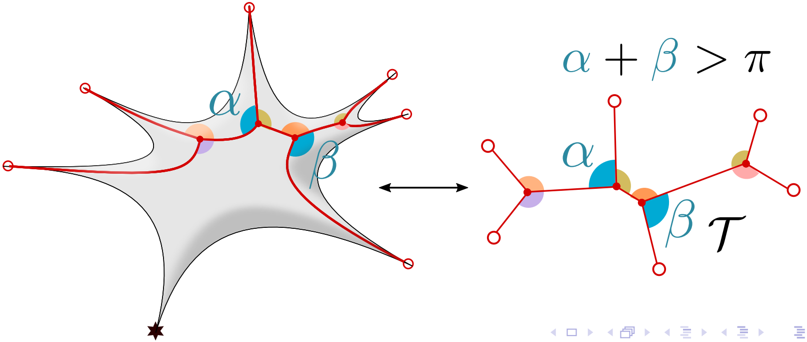

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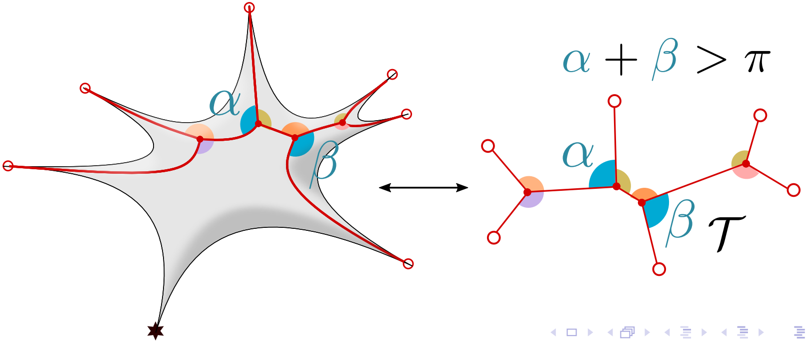

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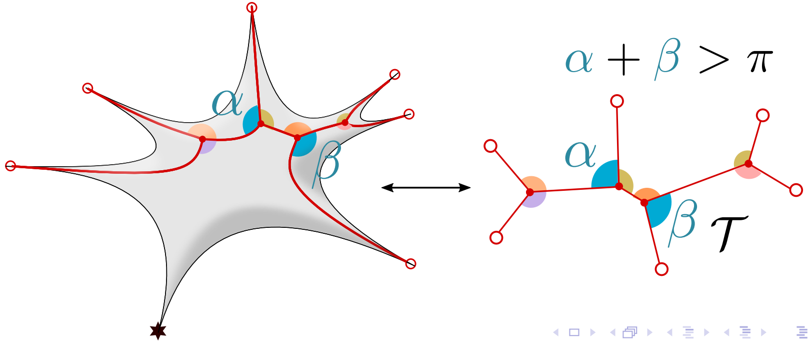

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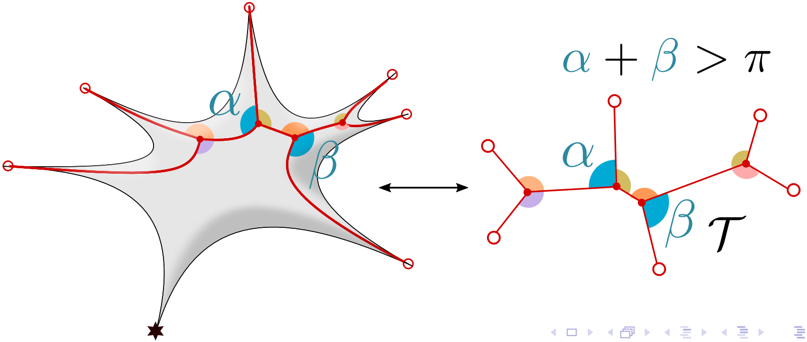
- Corollary: random balanced tree $\longleftrightarrow$ random Riemann surface.

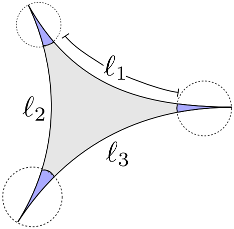


Why Lebesgue measure ($\mathrm{d}^{2n-6}\alpha$) on angles?

- Bijection:

$$\left\{ \begin{array}{c} \text{ideal triangles with horocycles} \\ \text{that admit equidistant point} \end{array} \right\} \leftrightarrow \left\{ \text{Euclidean triangles} \right\}$$

preserving angles between bisectors. [Penner, '95], [Bobenko, Pinkall, Springborn, '10]



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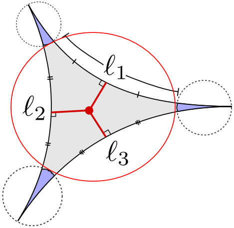

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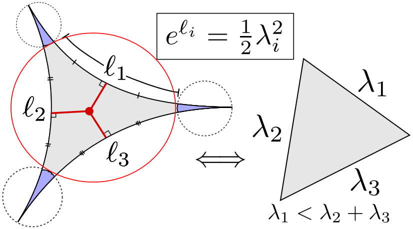

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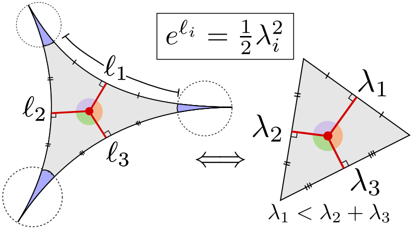

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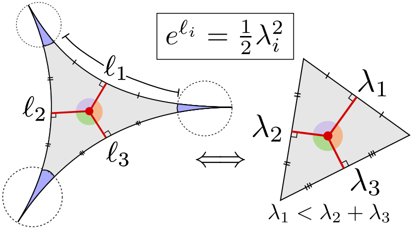

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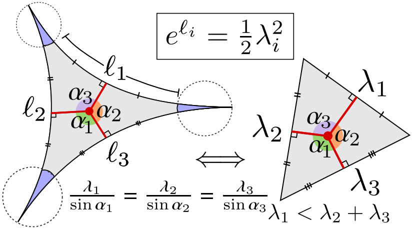

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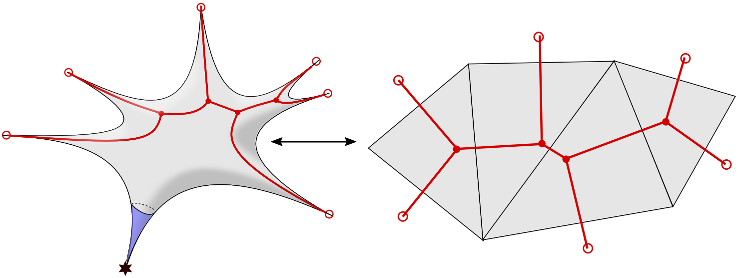

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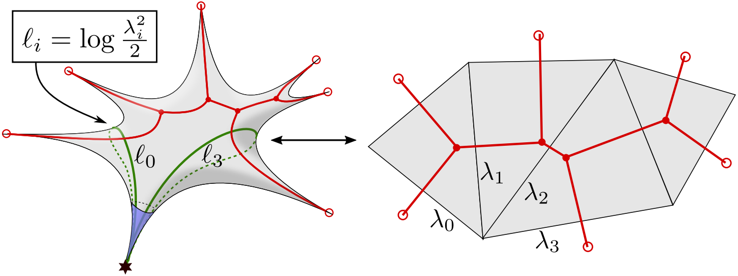

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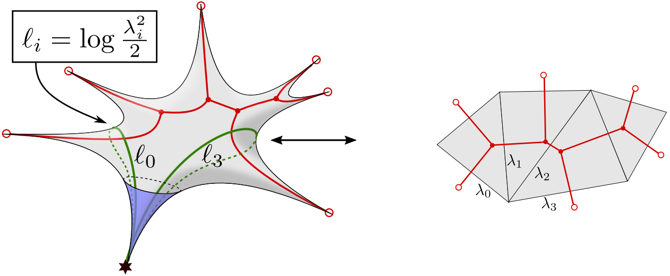
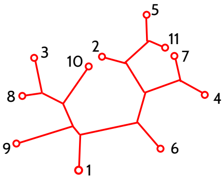


Reproducing Weil-Petersson volumes

Zograf's formula: $\mathcal{Z}(x) = \sum_{n\geq 3} V_n \frac{x^{n-2}}{(n-2)!}, \quad \frac{\sqrt{\mathcal{Z}(x)}}{\pi} J_1(2\pi\sqrt{\mathcal{Z}(x)}) \stackrel{?}{=} x$



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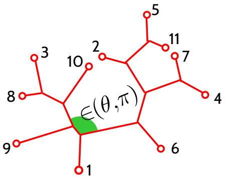

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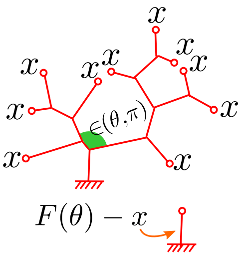

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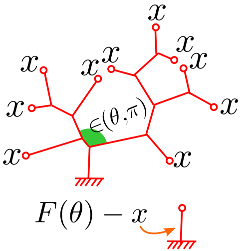

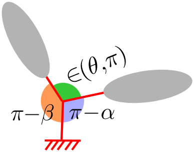

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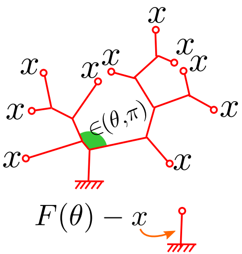

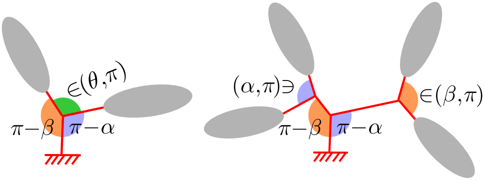

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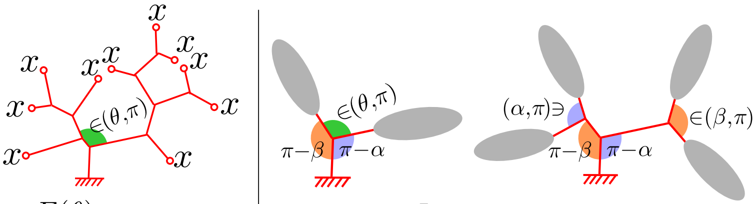

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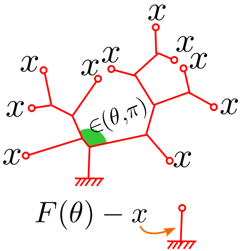
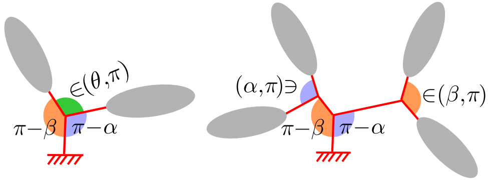
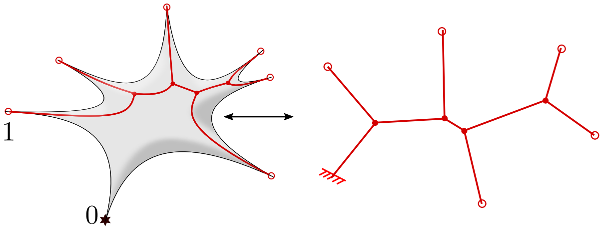




$$F(\theta) - x = \int_0^\pi \mathrm{d}\alpha \mathrm{d}\beta\, F(\alpha)F(\beta)\mathbf{1}_{\{\theta<\alpha+\beta<\pi\}}$$

- General solution: $F(\theta) = \frac{\sqrt{F(0)}}{\theta} J_1(2\theta\sqrt{F(0)})$.
- $F(0) = \mathcal{Z}(x)$, $F(\pi) = x \quad \Longrightarrow \quad$ Zograf's formula!

- Statistics in grand-canonical ensemble with three marked cusps.



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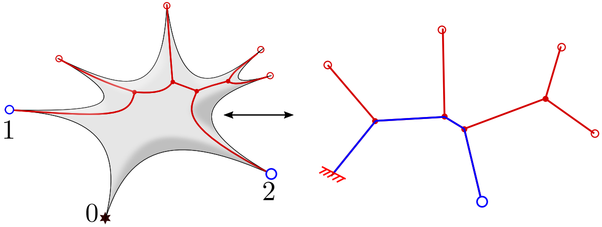

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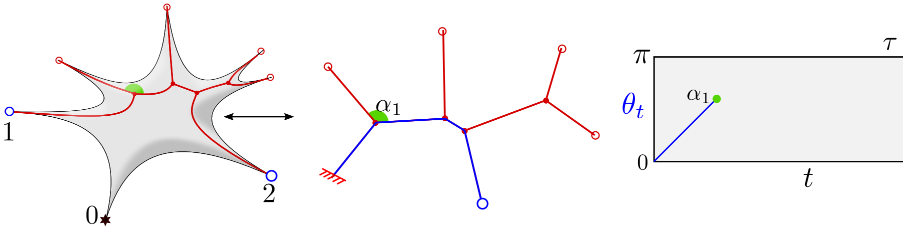

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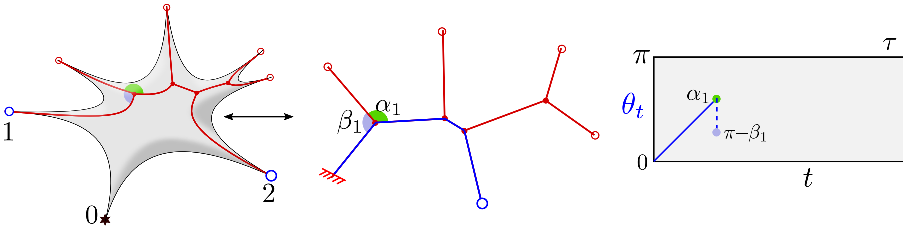

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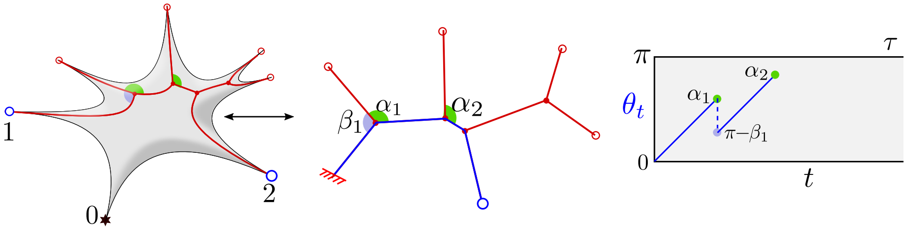

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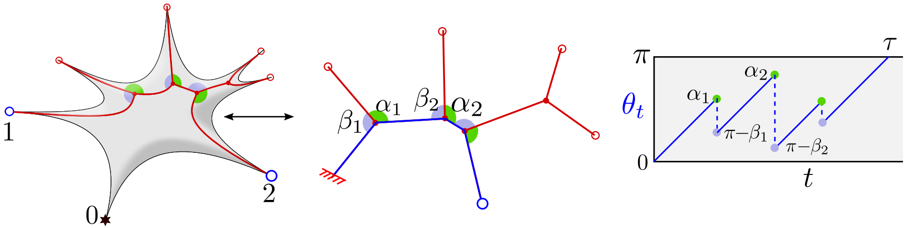

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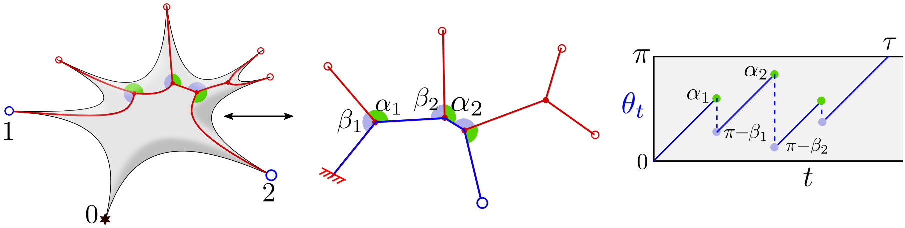

Proposition

In the grand-canonical ensemble with $x \in (0, x_*)$,

- *(θ_t) has the law of a Markov process with slope 1 and downward jumps $\theta \to \beta$ at rate $2F(\theta-\beta)\frac{F_\bullet(\beta)}{F_\bullet(\theta)}$;*

$$F(\theta) = \frac{\sqrt{\mathcal{Z}(x)}}{\theta} J_1(2\theta\sqrt{\mathcal{Z}(x)}), \qquad F_\bullet(\theta) = \tfrac{1}{\mathcal{Z}'(x)} \partial_x F(\theta) = J_0(2\theta\sqrt{\mathcal{Z}(x)})$$

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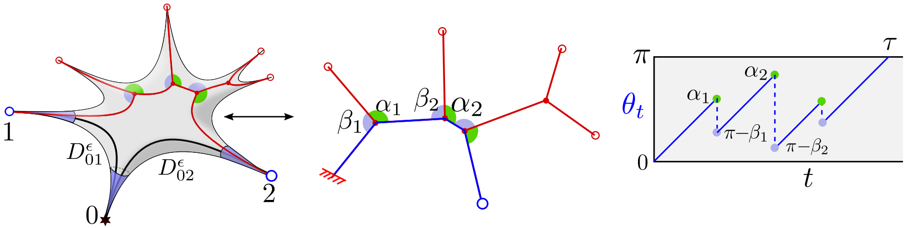

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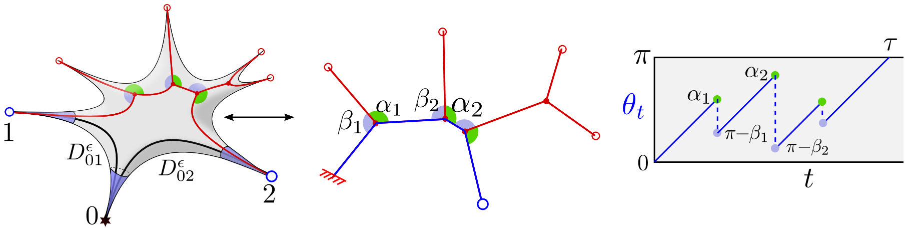

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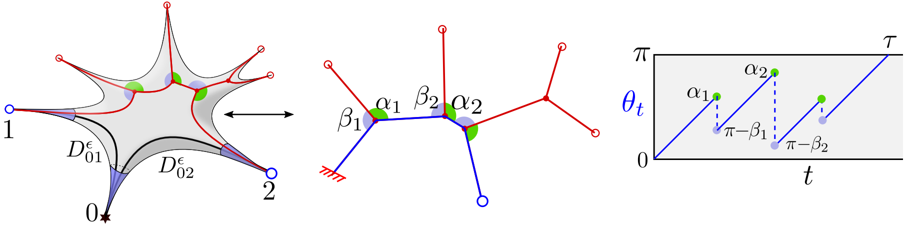
$$\mathrm{Var}_x[D^\epsilon_{01} - D^\epsilon_{02}] = -\frac{2}{F_\bullet(\pi)} \int_0^\pi \mathrm{d}\alpha \int_0^\alpha \mathrm{d}\beta \cot\alpha \cot\beta \, F_\bullet(\pi-\alpha) F_\bullet(\alpha-\beta) F_\bullet(\beta)$$



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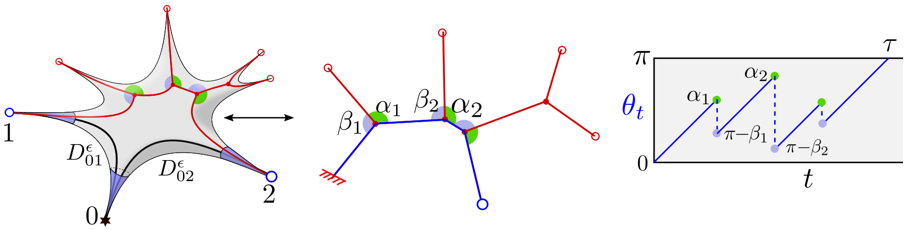
- Singularity analysis:
$\mathrm{Var}_n[D^\epsilon_{01} - D^\epsilon_{02}] = \frac{[x^n]F_\bullet(\pi)\mathrm{Var}^x[D^\epsilon_{01}-D^\epsilon_{02}]}{[x^n]F_\bullet(\pi)} \overset{n\to\infty}{\sim} 3.429137077\ldots \cdot \sqrt{n}$

- Where does $D_{01}^{\epsilon} - D_{02}^{\epsilon} = \int_0^{\tau} \cot\theta_t \,\mathrm{d}t$ come from?



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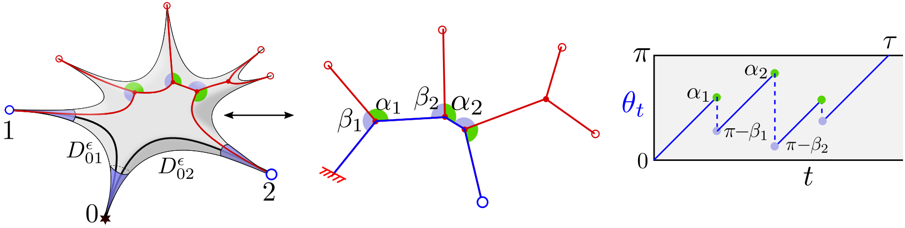

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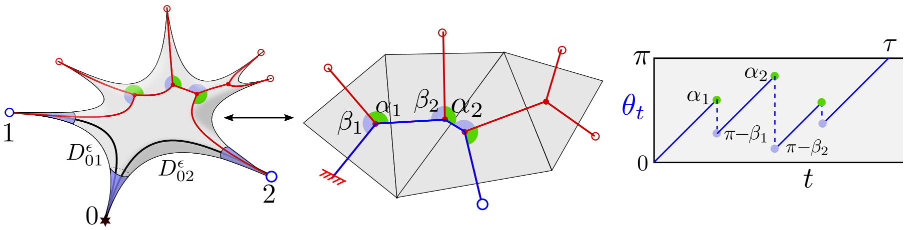

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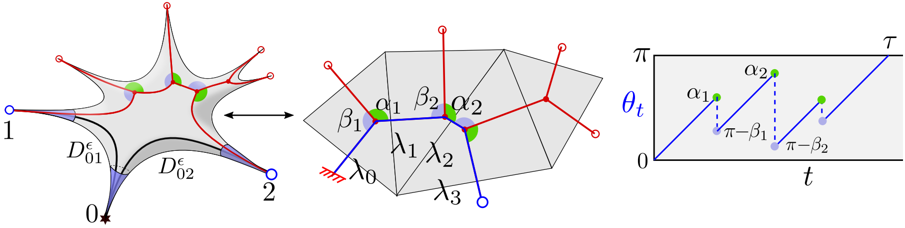

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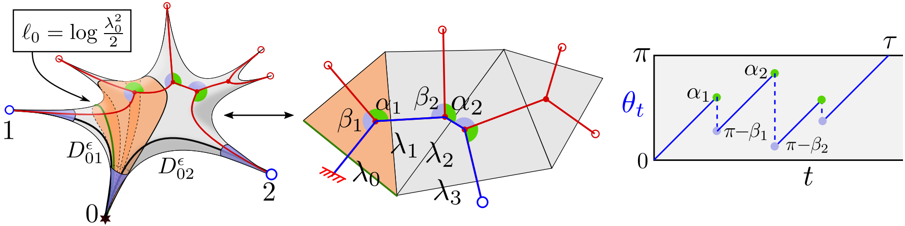

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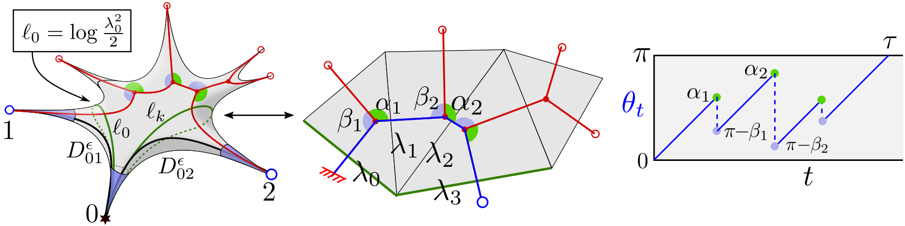

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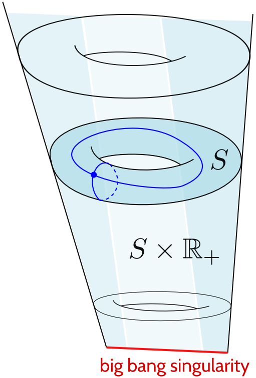
Importance for (2+1)-dimensional (quantum) gravity?

- Einstein eqs in 2+1D with $\Lambda = 0 \implies$ locally Minkowski space.



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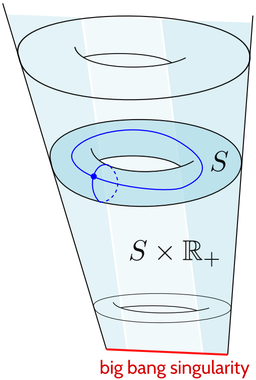
Phase space: $\tilde{\mathcal{P}}_{2+1} = \mathrm{Hom}(\pi_1(S), \mathrm{ISO}(2,1)) \,/\, \mathrm{ISO}(2,1), \quad \mathcal{P}_{2+1} = \tilde{\mathcal{P}}_{2+1}/\mathrm{MCG}.$



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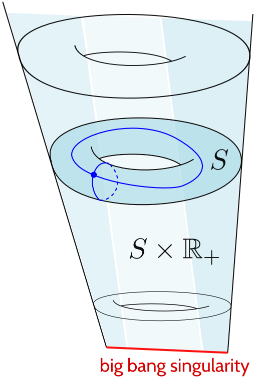
$\tilde{\mathcal{P}}_{2+1} = T\mathcal{T}_g$, $\quad$ Teichmüller space $\mathcal{T}_g = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) \,/\, \mathrm{PSL}(2,\mathbb{R})$



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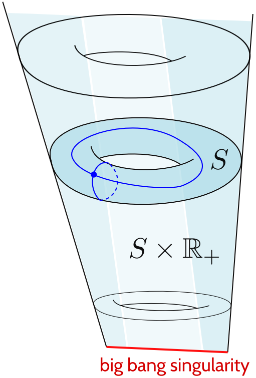

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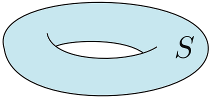
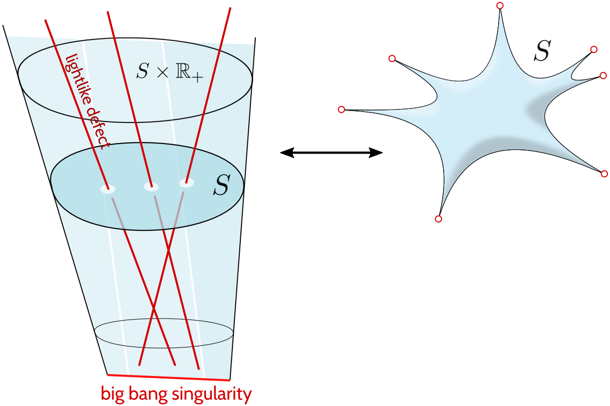


The Weil-Petersson symplectic structure on $\mathcal{M}_g$ induces correct Poisson bracket on $\mathcal{P}_{2+1}$

[TB, Loll, '09]

Importance for (2+1)-dimensional (quantum) gravity?

- S = n-punctured sphere $\longrightarrow S \times \mathbb{R}_+$ = universe with n massless particles



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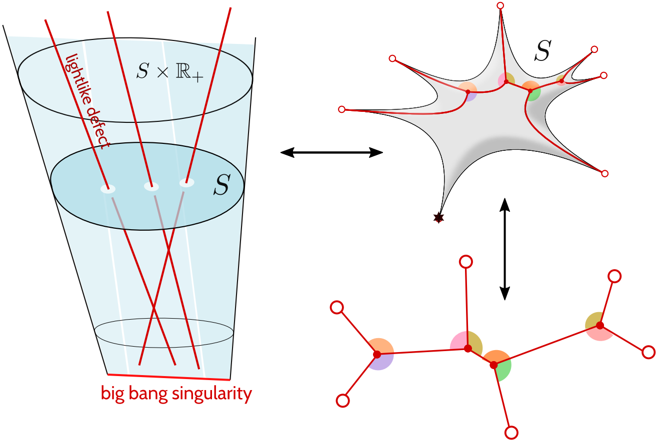

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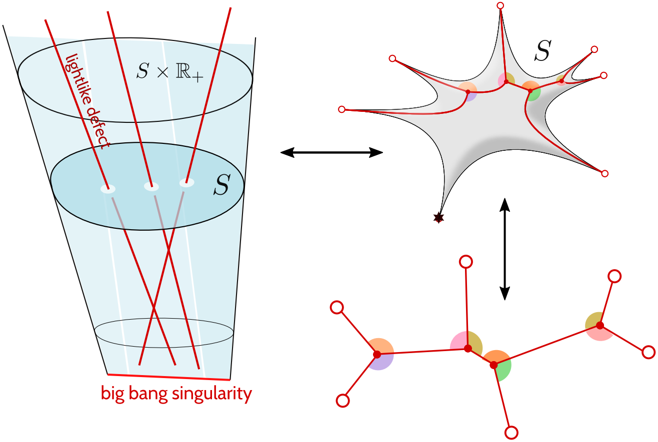

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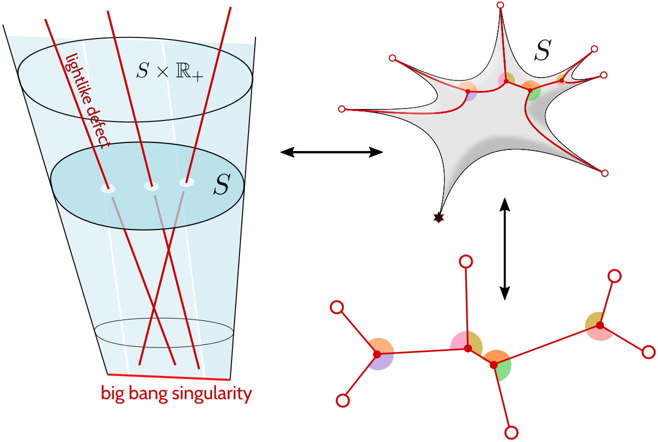
- Quantizing 2+1D gravity $\longleftrightarrow$ quantizing balanced trees.



Conclusion

- The bijection between genus-0 punctured Riemann surfaces and balanced trees provides
  - a convenient way to compute Weil-Petersson volumes;
  - detailed information on global distance statistics;
  - a potential avenue to tree bijections in higher dimensions;
  - efficient simulation of random Riemann surfaces via Boltzmann sampling of balanced trees!