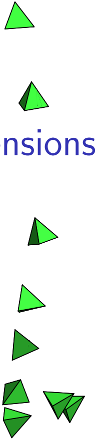
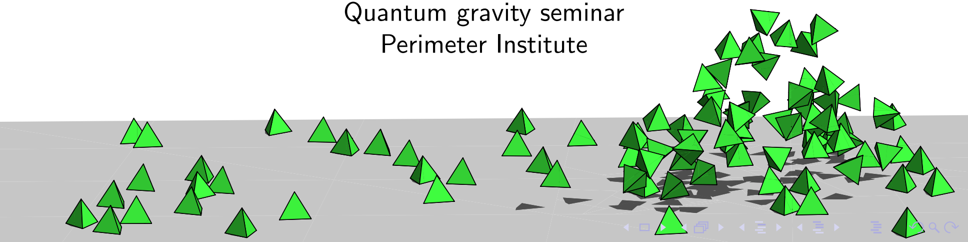
Effective dynamics of CDT in 2+1 dimensions



Timothy Budd

Institute for Theoretical Physics, Utrecht University

December 14, 2011
Quantum gravity seminar
Perimeter Institute



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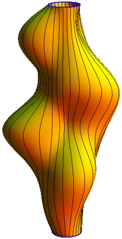

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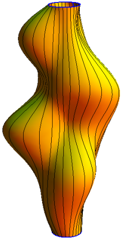

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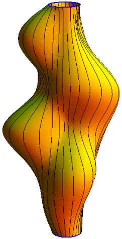

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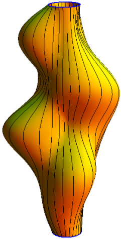

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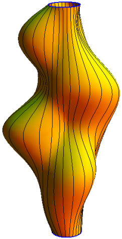

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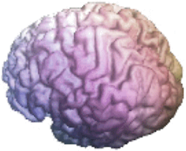
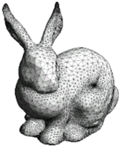
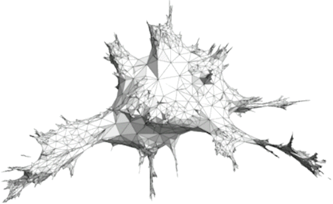
- Succes story in CDT: spatial volumes as observables.
- Notoriously difficult to go beyond this, but necessary since we believe that the true degrees of freedom in gravity have to do with the shape of space.

Challenges

- Construct observables that measure macroscopic shape in a meaningful way.
  - This is perhaps the hardest part. In fact: whole journals are dedicated to shape recognition in medical imaging, computer graphics, etc. However the random geometries in CDT are much wilder.







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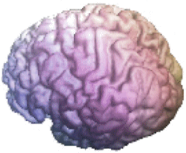
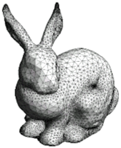
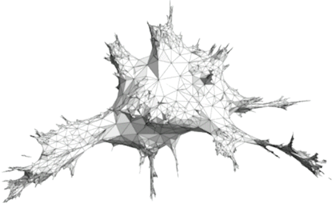

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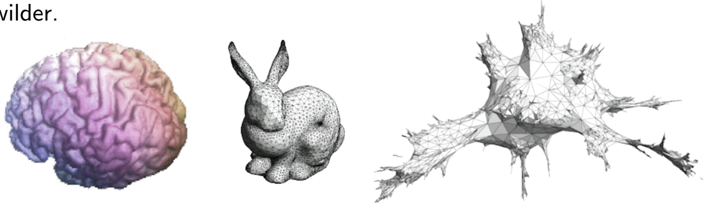

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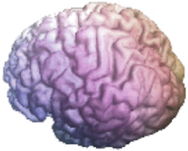
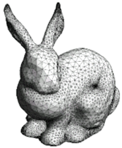
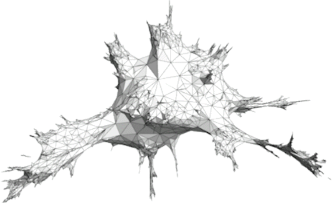

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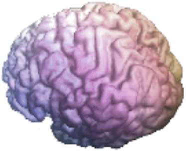
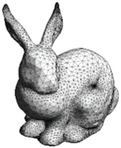
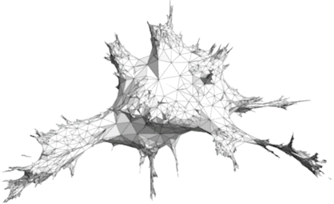

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I will show how to overcome these challenges for a particular observable in the case of CDT in 2+1 dimensions on the torus.

Outline

- Introduction to CDT in 2+1 dimensions
  - Previous results for spherical topology
- Effective actions for CDT
  - Conformal mode problem
  - Alternative ansatz a la Hořava–Lifshitz
- Moduli in CDT with torus topology
  - Introduce moduli as observables
  - Boundary conditions
  - Comparison with ansatz
- Summary and outlook

CDT in 2+1 dimensions

- Causal Dynamical Triangulation is a regularization of the Euclidean path integral over geometries

$$Z = \int \frac{\mathcal{D}g}{Diff} e^{-S_{EH}[g]} \quad \rightarrow \quad Z_{CDT} = \sum_{\text{triangulations}\, T} \frac{1}{C_T} e^{-S_{CDT}[T]}.$$

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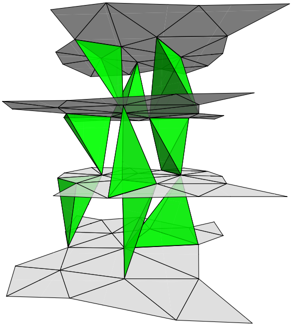

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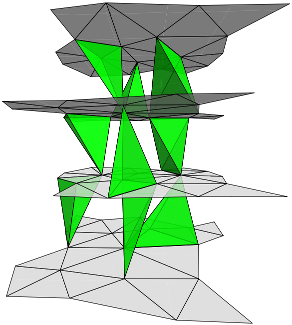
- "Causal" in CDT means that we only allow triangulations that are foliated by 2D triangulations with constant topology.
- The Euclidean Einstein–Hilbert action $S[g] = -\kappa \int d^3x \sqrt{g}(R - 2\Lambda)$ evaluated on the piecewise linear geometry leads to

$$S_{CDT}[T] = k_3 N_3 - k_0 N_0.$$



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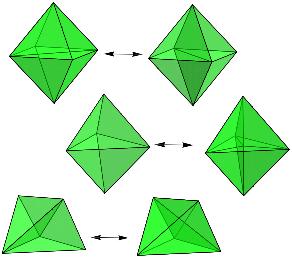

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- We use Monte Carlo methods to approximate these:

$$\langle \mathcal{O} \rangle_{N_3} \approx \frac{1}{n} \sum_{i=1}^{n} \mathcal{O}(T_i),$$

where the $\{T_i\}$ is a large set of random triangulations generated by applying a large number of random update moves satisfying a detailed balance condition.

Effective actions for CDT

- Given a set of observables $f_i : \{\text{CDT triangulations}\} \to \mathbb{R}$, $i = 1, \ldots, k$, measuring large scale geometry, we can write

$$Z_{CDT}(N_3) = \sum_T \frac{1}{C_T} e^{-k_0 N_0} = \int df_1 \int df_2 \cdots \int df_k \, e^{-S_{eff}[f_i]}, \quad (1)$$

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- We can learn about $S_{eff}[f_i]$ by measuring expectation values $\langle f_i \rangle$ and correlations $\langle f_i f_j \rangle$.
- Simplest set of observables in CDT: $\{V(t)\}_t$, spatial volume $V(t)$ at time t.

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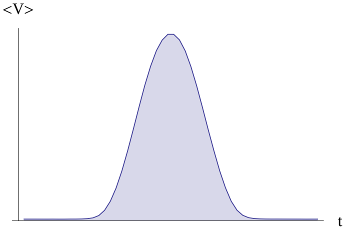

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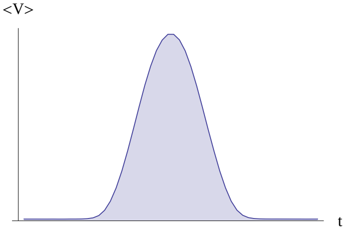

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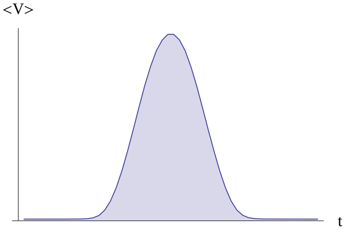

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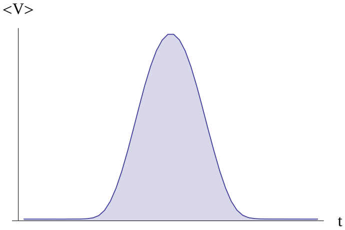

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- They differ by an overall minus sign! S_{eff} is bounded below (for fixed 3-volume), S_{EH} is not.

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where $\mathcal{G}^{abcd}$ is the Wheeler-DeWitt metric,

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- Indefinite metric! Positive definite on traceless directions, negative definite on trace/conformal directions in superspace.
- CDT is a (well-defined) statistical system, therefore it better be described by a bounded effective action, e.g. in

$$Z_{CDT}(N_3) = \int dV(1) \int dV(2) \cdots \int dV(T) e^{-S_{eff}[V(t)]}. \quad (5)$$

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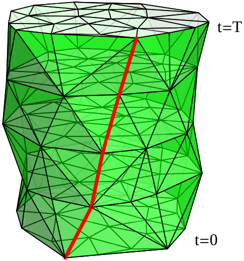

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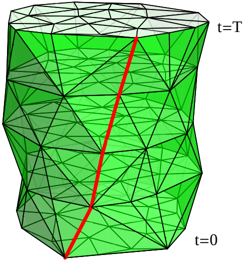

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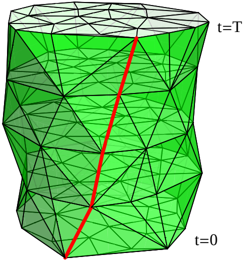

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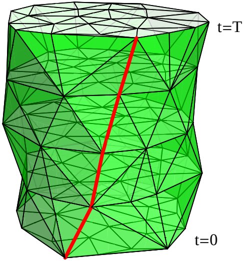

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CDT with spatial topology of the torus

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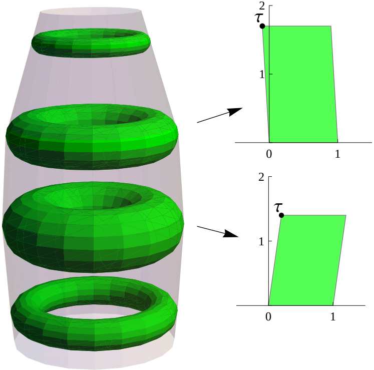

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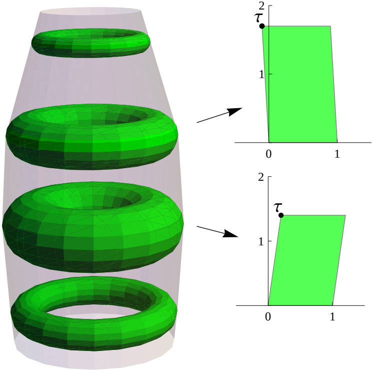
- What can we learn about the effective action by measuring $V(t)$, $\tau_1(t)$ and $\tau_2(t)$?
- But, first of all, how do we measure τ in CDT?



Measurement of τ in the continuum

- Any metric g_{ab} on the torus is conformally flat and up to diffeomorphisms the flat unit-volume metrics are given by
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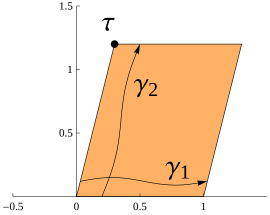

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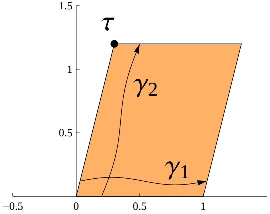

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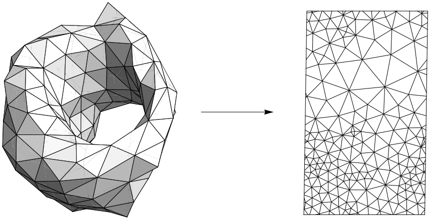
- Once we have these ingredients we can construct discrete conformal maps:



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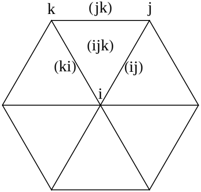

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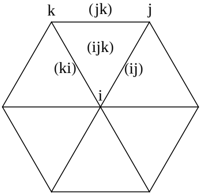

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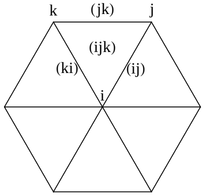

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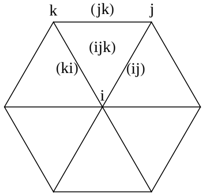

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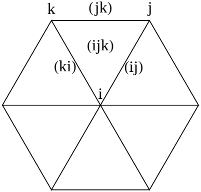

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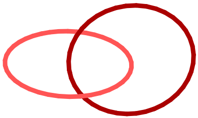

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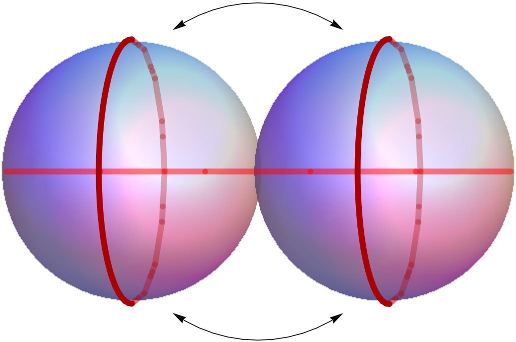

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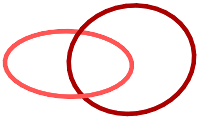

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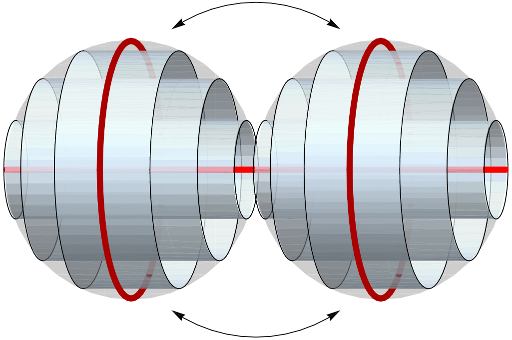

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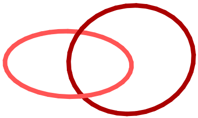

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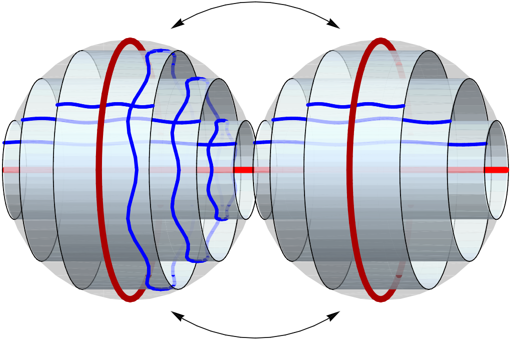
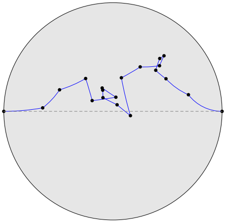


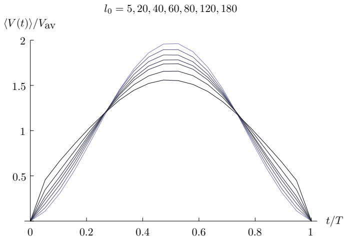
- The initial and final (degenerate) geometry correspond to moduli $\tau = 0$ and $\tau = i\infty$ respectively.
- A CDT configuration determines a sequence of moduli in the upper-half plane. Map to Poincaré disk for convenience:

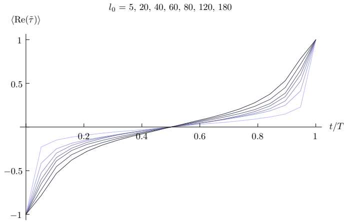
$$\tilde{\tau} = \frac{\tau - i}{\tau + i}$$

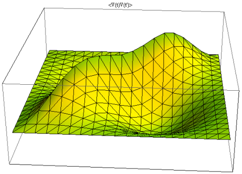


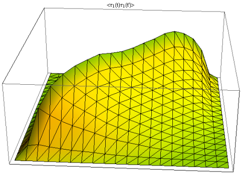
- We can measure expectation values $\langle \tilde{\tau}_i(t) \rangle$ and correlations $\langle \tilde{\tau}_i(t)\tilde{\tau}_j(t') \rangle$ in addition to the spatial volume $\langle V(t) \rangle$ and $\langle V(t)V(t') \rangle$.

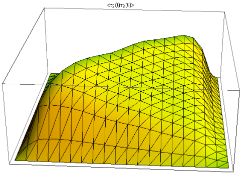
Data











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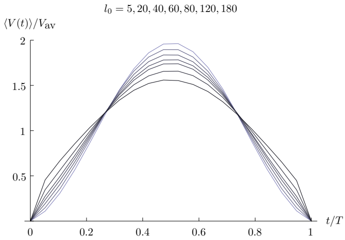

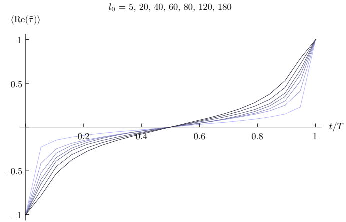

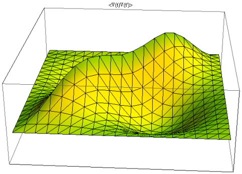

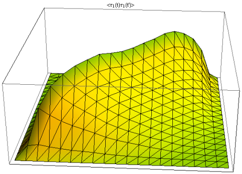

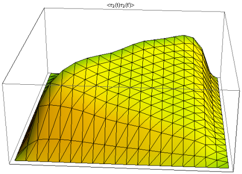

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- The “cusp” on the diagonal of the correlators tells us about the effective kinetic term!

Comparison to ansatz

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- We can do better: this prefactor is related to the change of τ under a normalized metric deformation δg_{ab} w.r.t. $\mathcal{G}_\lambda$.

$$S[V,\tau] = \kappa \int dt \Big((\frac{1}{2} - \lambda) \frac{\dot{V}^2}{V} + \frac{1}{2A[g]} \frac{\dot{\tau}_1^2 + \dot{\tau}_2^2}{\tau_2^2} - U(V,\tau) \Big).$$

with

$$A[g] = \frac{\delta_{ij}}{4\tau_2^2} \int d^2x \sqrt{g} \frac{\delta \tau_i}{\delta g_{ab}} \mathcal{G}^\lambda_{abcd} \frac{\delta \tau_j}{\delta g_{cd}}.$$

- Writing the metric in conformal gauge $ds^2 = \Omega(x)^2 \hat{g}_{ab} dx^a dx^b$ we find

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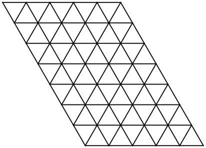
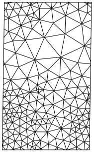
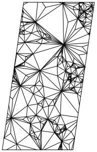
- A natural discretization of $A[g]$ to triangulations T is $A[T] = \sum_{\sigma \in T} \mathrm{area}(\sigma)^2$, where $\mathrm{area}(\sigma)$ is the area of the triangle σ in the conformal embedding of T in the plane.



$VA[T] = 1$

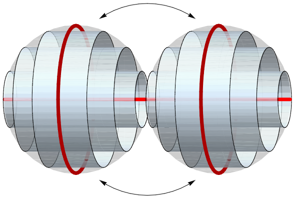


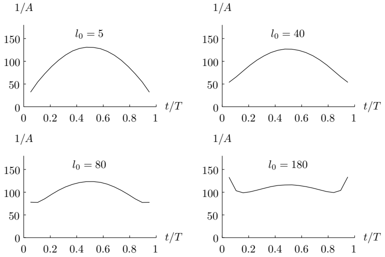
$VA[T] \approx 1.9$



$VA[T] \approx 4.9$

- Measure $\langle A[g(t)]\rangle$ in CDT configurations: plots for $k_0 = 2.5$, $V = 60000$ and different singularity lengths l_0.





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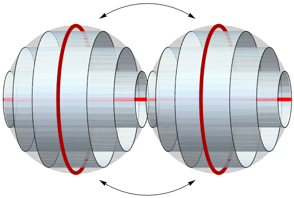

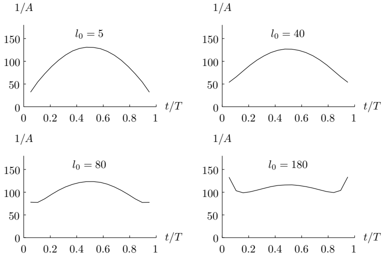

- Now we can start comparing! Semi-classically the correlation matrix $\langle \Delta\tau_i(t)\Delta\tau_j(t')\rangle$ is proportional to the inverse of $\delta^2 S_{eff}/\delta\tau_i(t)\delta\tau_j(t')$.

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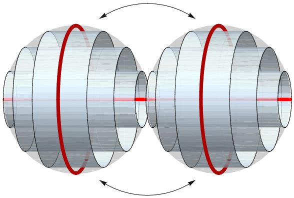

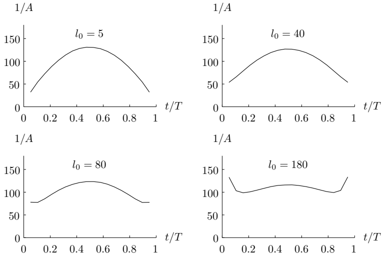

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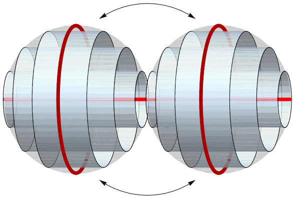

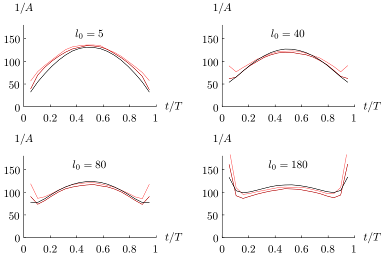

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- Fit to measured $A[g]$ gives $\lambda \approx 0.18$.

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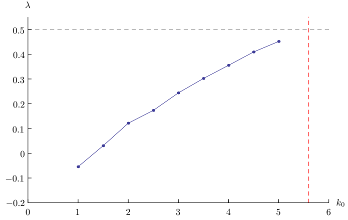

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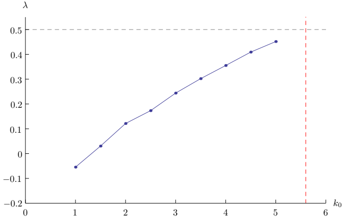

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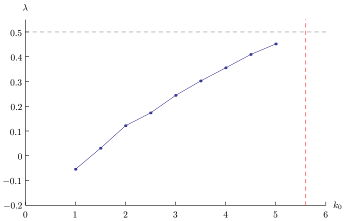

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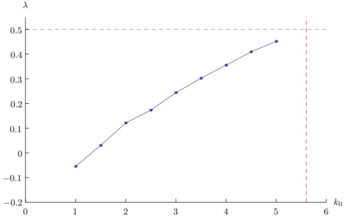

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- Indication that spatial conformal symmetry is implemented at/near the phase transition. Relation to Shape Dynamics in 2+1 dimensions? [TB, T. Koslowski, arXiv:1107.1287]
  The weak coupling between consecutive spatial geometries may ensure that the conformal properties of 2d gravity are maintained.

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- CDT and Hořava–Lifshitz gravity seem to be living in the same theory space and indeed our data is well-described by a kinetic term of the type appearing in HL.

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Thanks for your attention!