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Relating discrete and continuum 2d quantum gravity

Timothy Budd

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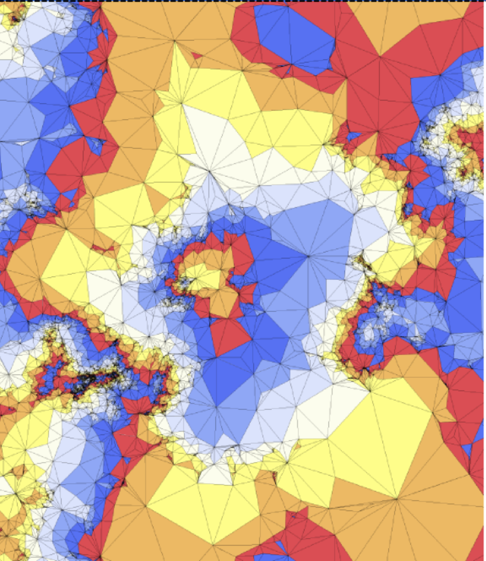
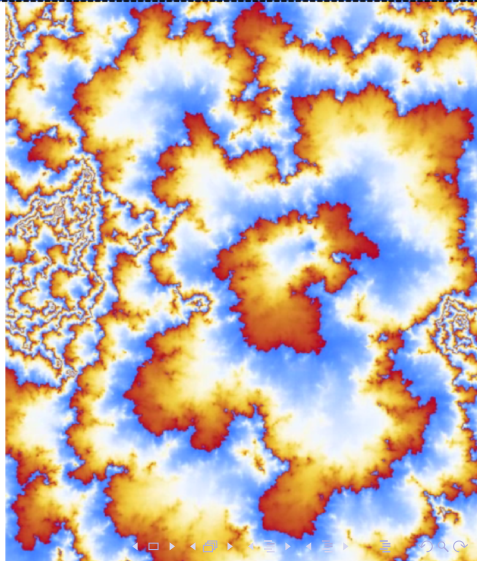


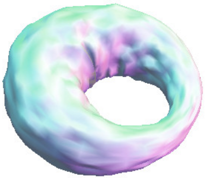


2D quantum gravity

- Formally 2d gravity is a statistical system of random metrics on a surface of fixed topology with partition function

$$Z = \int [\mathcal{D}g][\mathcal{D}X] \exp(-\lambda V[g] - S_m[g, X]) \,,$$

possibly coupled to some matter fields X with action $S_m[g, X]$.



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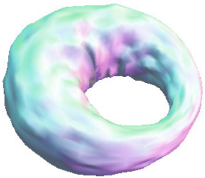

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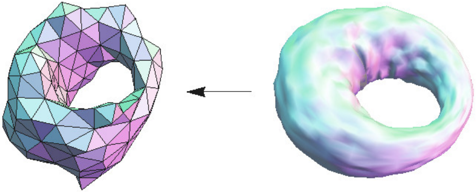

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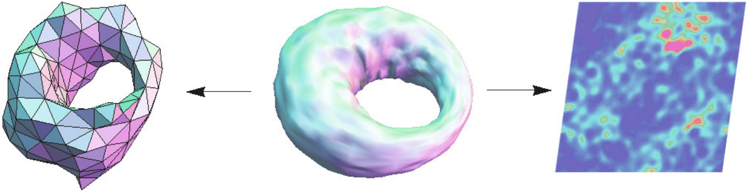
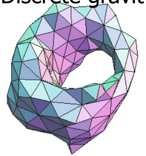
- Roughly two strategies to make sense of this path-integral:
  - Combinatorially: $Z = \sum_T e^{-\lambda N_T} Z_m(T)$
  - Liouville path integral: gauge fix $g_{ab} = e^{\gamma \phi} \hat{g}_{ab}(\tau)$.



Discrete gravity



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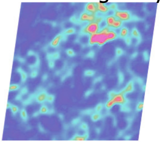
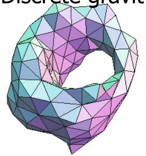


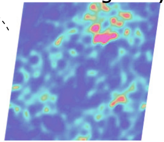
Outline

Discrete gravity



Matrix models

Liouville gravity



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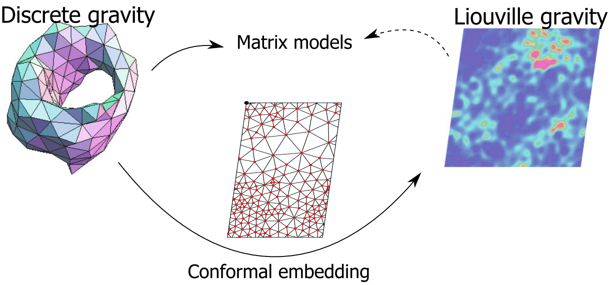
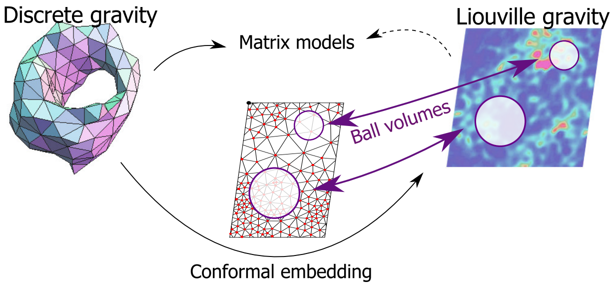


Outline

1. Conformal embedding



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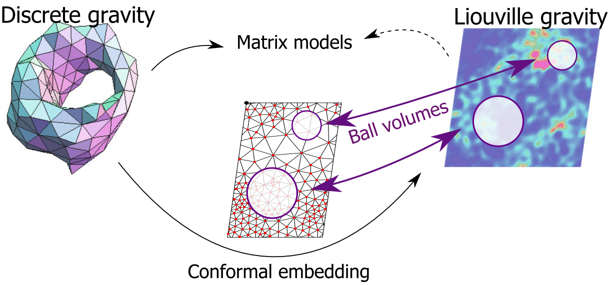
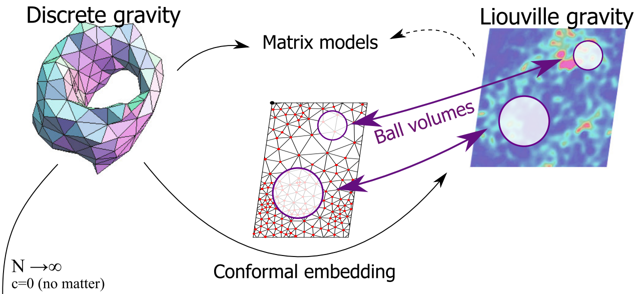
2. Ball volumes



Hausdorff dimension

Outline

1. Conformal embedding
2. Ball volumes
3. Hausdorff dimension



Hausdorff dimension

Brownian map

$d_h = 4$

Outline

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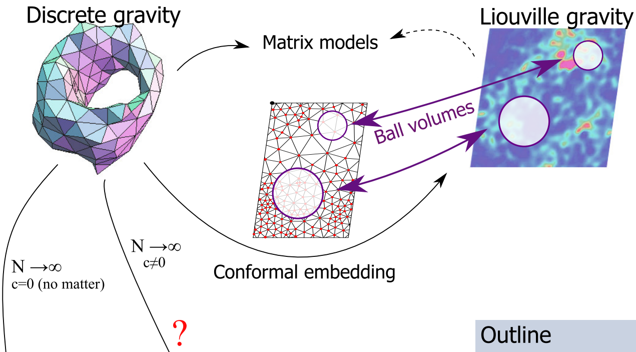
2. Ball volumes
3. Hausdorff dimension
   - in DT



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Numerically
d_h (c=-2)= 3.56

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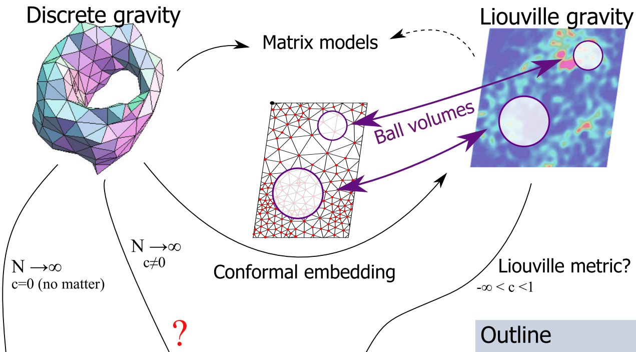

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Watabiki's formula:

$$d_h = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}$$

Check numerically for a Gaussian free field.

Outline

1. Conformal embedding
2. Ball volumes
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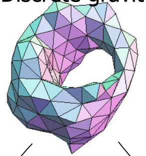
   - in DT
   - in Liouville

Discrete gravity



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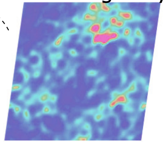

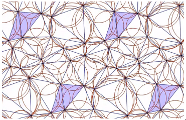

Circle patterns

$N \to \infty$
c=0 (no matter)

$N \to \infty$
$c \neq 0$

?

Liouville metric?
$-\infty < c < 1$

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Check numerically for a Gaussian free field.

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2. Ball volumes
3. Hausdorff dimension
   - in DT
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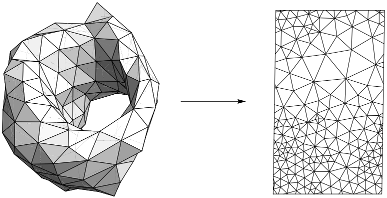
Mapping a triangulation to the plane

- Given a triangulation of the torus, there is a natural way to associate a harmonic embedding in $\mathbb{R}^2$ and a Teichmüller parameter τ.



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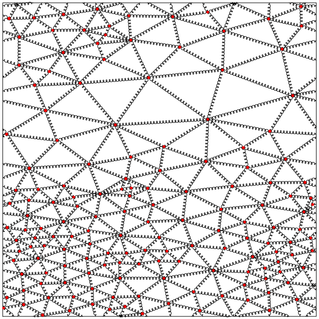

Mapping a triangulation to the plane

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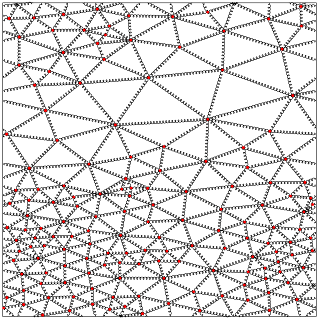
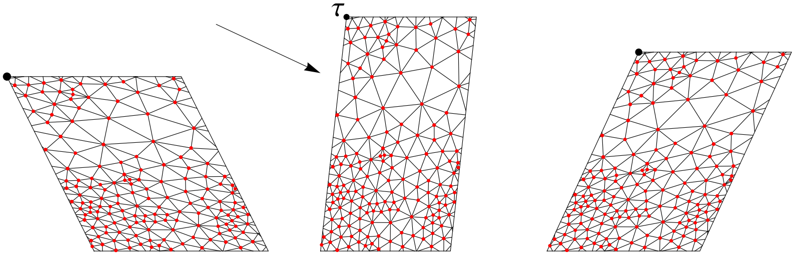
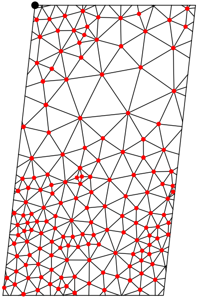
- Find linear transformation that minimizes energy while fixing the volume.





- Two pieces of information: modulus τ and periodic discrete measure on $\mathbb{R}^2$.



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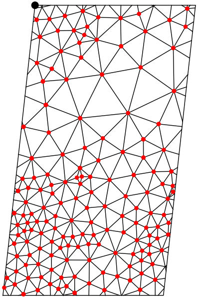

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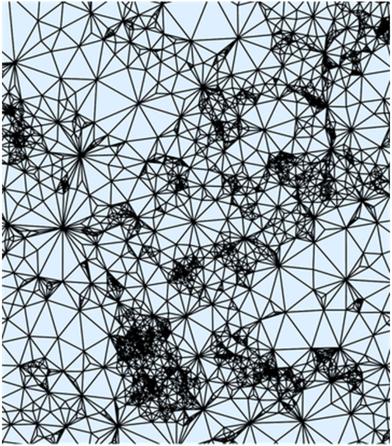

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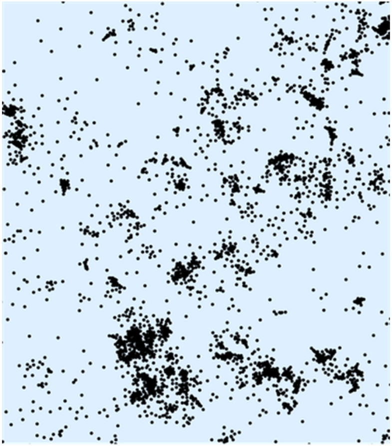

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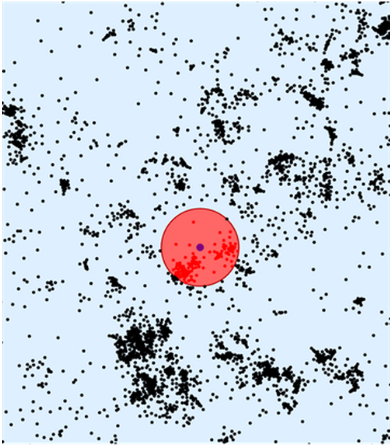
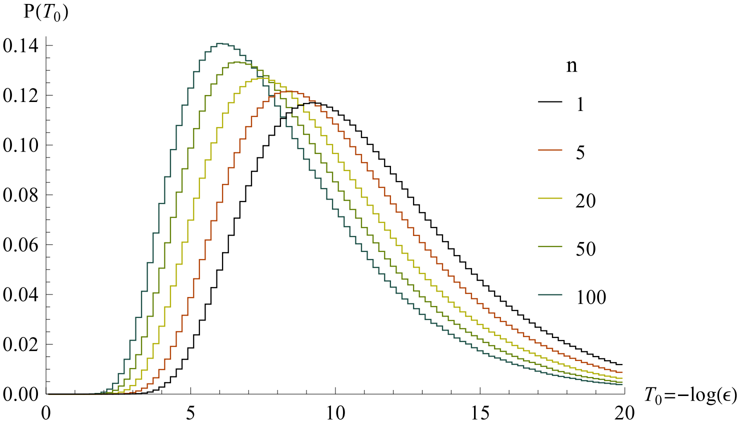
- What is the distance ϵ_n to the n'th nearest neighbour of a randomly chosen vertex?
- ϵ_n can be interpreted as the radius of a Euclidean disk with "quantum volume" $\delta = n/N$.



- Can measure the distribution $P_{N,n}(T_0 = -log(\epsilon))$ in Dynamical Triangulations. See plot for $N = 400k$ and $c = -2$ and $n = 1, \ldots, 100$.



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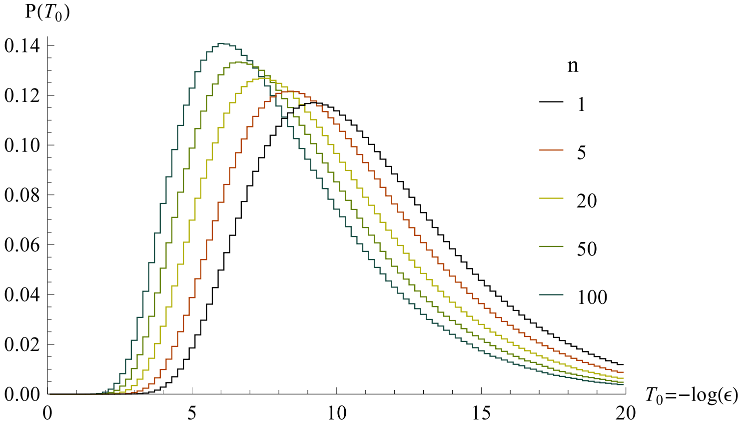

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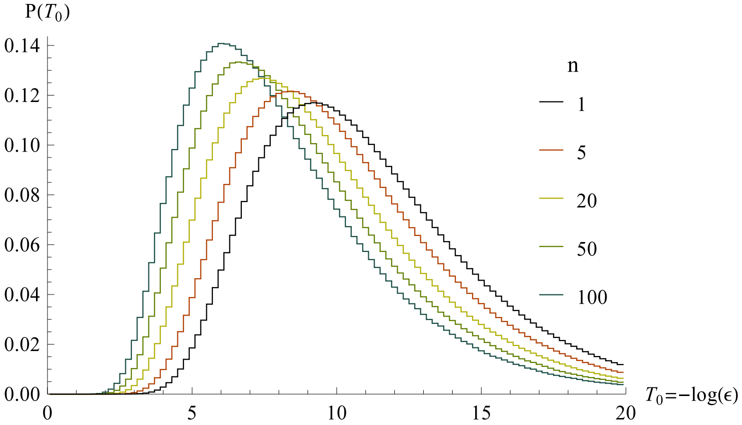
- As we will see, Liouville theory explains why.



Quantum Liouville gravity [David, '88] [Distler, Kawai, '89]

- Consider 2d gravity coupled to c scalar fields, i.e. the Polyakov string in c dimensions,

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- Conformal bootstrap: assuming Z to be of the form

$$Z = \int d\tau [\mathcal{D}_{\hat{g}}\phi][\mathcal{D}_{\hat{g}}X] \exp\left(-S_L[\hat{g},\phi] - S_m[X,\hat{g}]\right)$$

with the Liouville action

$$S_L[\hat{g},\phi] = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} (\hat{g}^{ab} \partial_a\phi \partial_b\phi + Q\hat{R}\phi + \mu e^{\gamma\phi})$$

and requiring invariance w.r.t. $\hat{g}_{ab}$ fixes the constants Q and γ:

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} = \sqrt{\frac{25-c}{6}}$$

- If we ignore τ-integral and set $\hat{g}_{ab} = \delta_{ab}$ flat and $\mu = 0$,

$$Z = \int [\mathcal{D}\phi] \exp\left(-\frac{1}{4\pi} \int d^2x\, \partial^a\phi\, \partial_a\phi\right),$$

i.e. simple Gaussian Free Field (GFF)!

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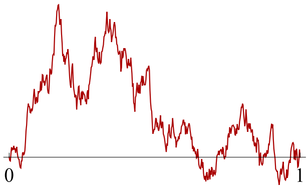
  and does it agree with DT?
- Care required: $e^{\gamma\phi}\delta_{ab}$ is almost surely not a Riemannian metric! Need to take into account the fractal properties of the geometry and regularize appropriately.

Gaussian free field basics

- Gaussian free field in 1d is a.s. a continuous function: Brownian motion.



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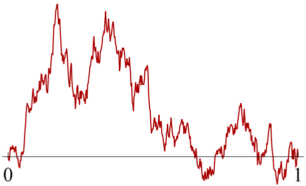

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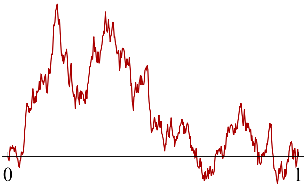

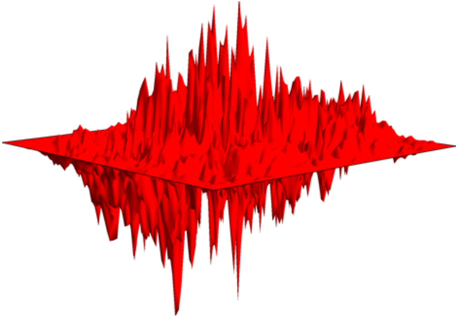

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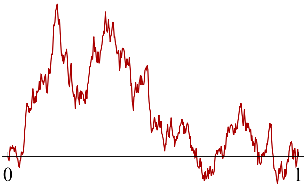
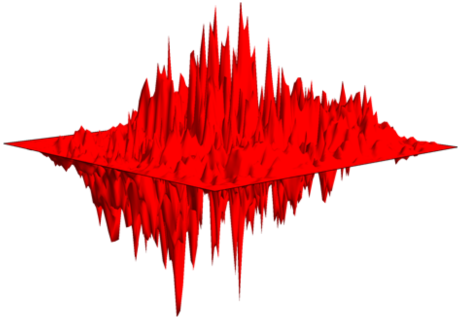
- How do we make sense of the measure $e^{\gamma\phi}$?





Regularization [Sheffield, Duplantier]

- The integral $(f, \phi) = \int d^2x\, f(x)\phi(x)$ has finite variance.
- In particular, for circle average $\phi_\epsilon(x) := \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \phi(x + \epsilon e^{i\theta})$,

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- Define regularized measure $d\mu_\epsilon = \epsilon^{\gamma^2/2} e^{\gamma\phi_\epsilon(x)} d^2x$.
- $d\mu_\epsilon$ converges almost surely to a well-defined random measure $d\mu_\gamma$ as $\epsilon \to 0$. [Sheffield, Duplantier]

Regularization [Sheffield, Duplantier]

- The integral $(f, \phi) = \int d^2x\, f(x)\phi(x)$ has finite variance.
- In particular, for circle average $\phi_\epsilon(x) := \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \phi(x + \epsilon e^{i\theta})$,

$$\langle \phi_\epsilon(x)^2 \rangle = -\log \epsilon - \tilde{G}(x, x).$$

- Therefore,

$$\left\langle e^{\gamma\phi_\epsilon(x)} \right\rangle = e^{\langle(\gamma\phi_\epsilon)^2\rangle/2} = \left(\frac{\tilde{G}(x,x)}{\epsilon}\right)^{\gamma^2/2}.$$

- Define regularized measure $d\mu_\epsilon = \epsilon^{\gamma^2/2} e^{\gamma\phi_\epsilon(x)} d^2x$.
- $d\mu_\epsilon$ converges almost surely to a well-defined random measure $d\mu_\gamma$ as $\epsilon \to 0$. [Sheffield, Duplantier]
- Alternatively, one can use a momentum cut-off. Given an orthonormal basis $\Delta_E f_i = \lambda_i f_i$,

$$\phi_p := \sum_{\lambda_i \le p^2} (f_i, \phi) f_i, \quad d\mu_p = p^{-\gamma^2/2} e^{\gamma\phi_p(x)} d^2x$$

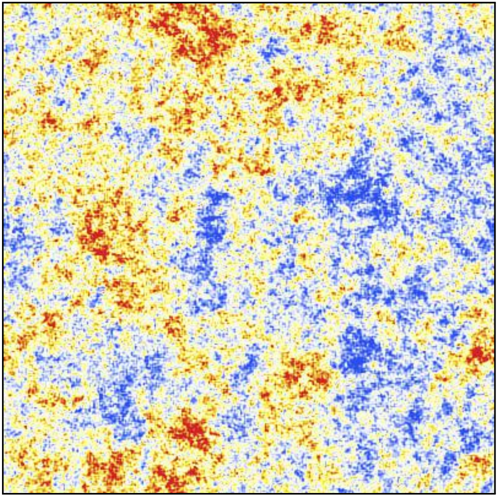
On the lattice

- We can easily put a Gaussian free field on a lattice, say, $L \times L$ with periodic boundary conditions.

```
RandomField[L_] :=
  Re@Fourier[RandomVariate[NormalDistribution[], {L, L, 2}].{1, i}
     Table[If[i == j == 1, 0, (2/π Sin[π (i - 1) / L]^2 + 2/π Sin[π (j - 1) / L]^2)^(-1/2)],
      {i, L}, {j, L}]];
```

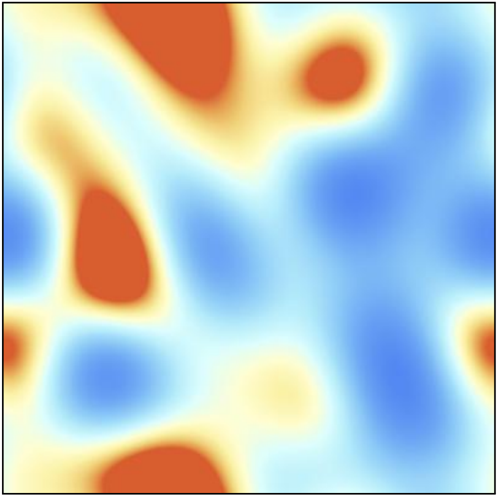
On the lattice

- $L \times L$ with periodic boundary conditions.



On the lattice

- $L \times L$ with periodic boundary conditions.
- Consider $d\mu_p = p^{-\gamma^2/2} e^{\gamma \phi_p(x)} d^2x$ with $p \ll L$.



$\gamma = 0.6, p = 10$

On the lattice

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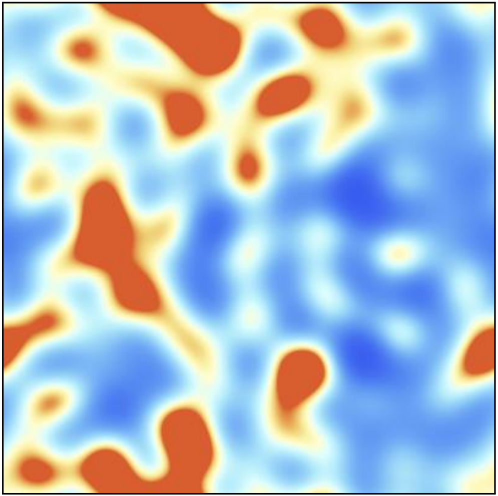

$\gamma = 0.6, p = 20$

On the lattice

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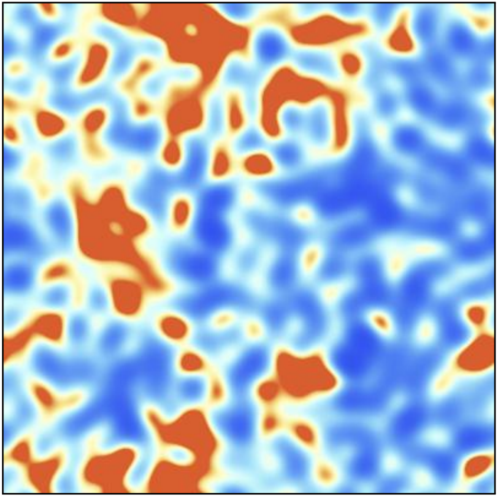

$\gamma = 0.6, p = 40$

On the lattice

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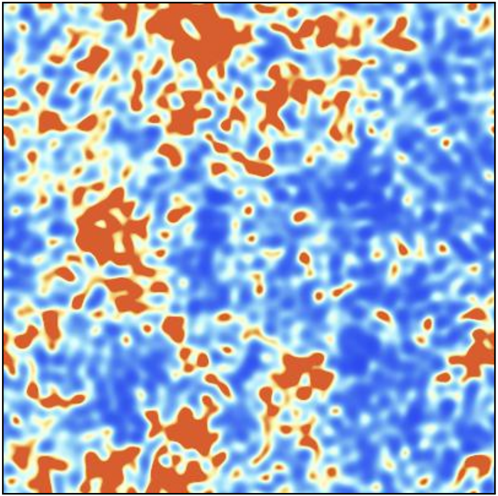
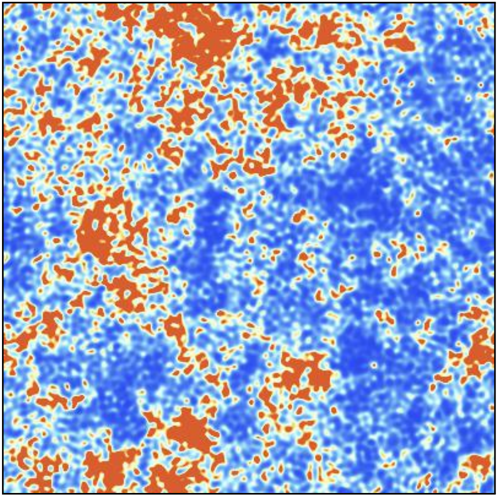


$\gamma = 0.6, p = 80$

On the lattice

- $L \times L$ with periodic boundary conditions.
- Consider $d\mu_p = p^{-\gamma^2/2} e^{\gamma \phi_p(x)} d^2x$ with $p \ll L$.



$\gamma = 0.6, p = 160$

On the lattice

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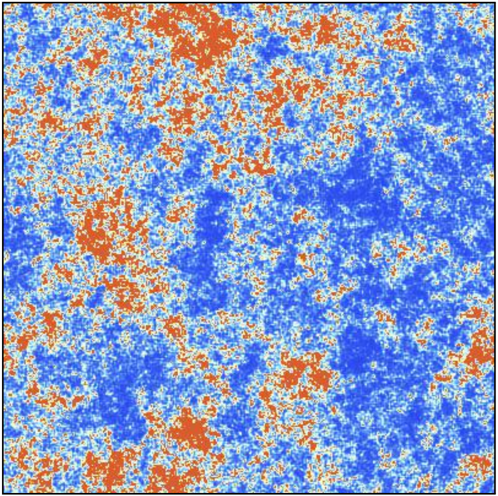
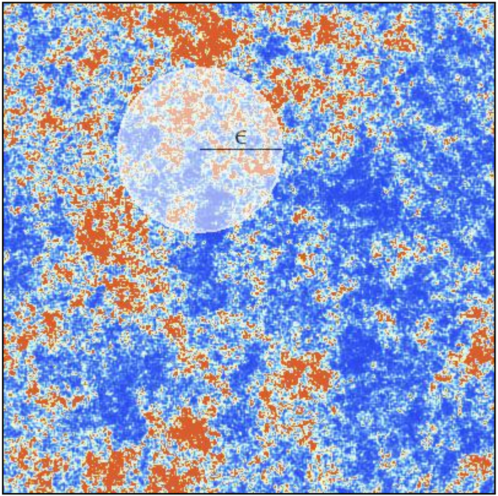


$\gamma = 0.6, p = 320$

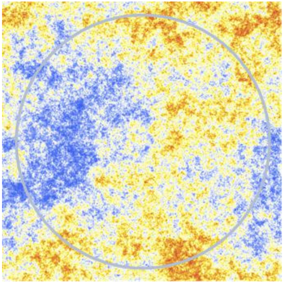
On the lattice

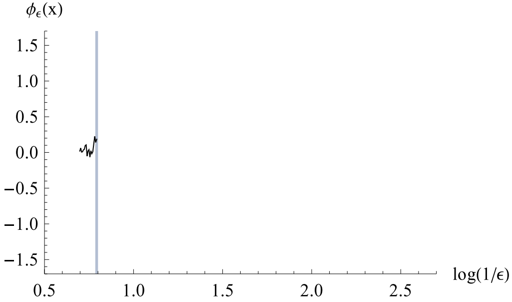
- $L \times L$ with periodic boundary conditions.
- Consider $d\mu_p = p^{-\gamma^2/2} e^{\gamma\phi_p(x)} d^2x$ with $p \ll L$.
- Can we understand the relation between $\delta = \mu(B_\epsilon(x))$ and ϵ?



$\gamma = 0.6, p = 320$

- Look at the circle average $\phi_\epsilon(x)$ as function of ϵ.





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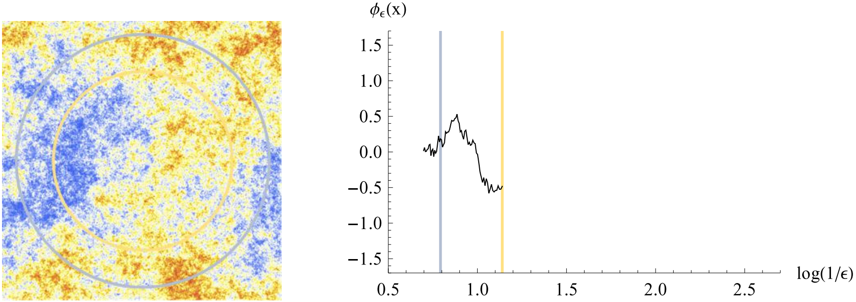

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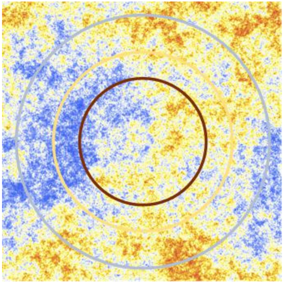

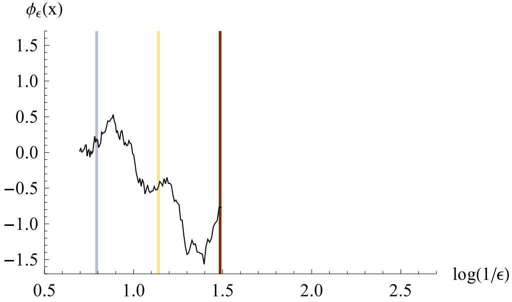

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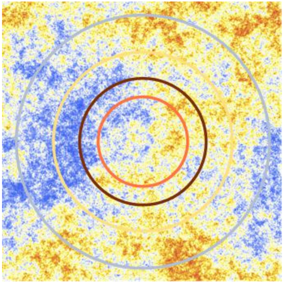

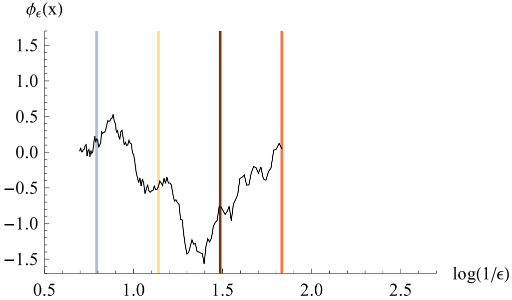

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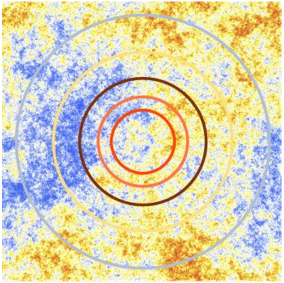
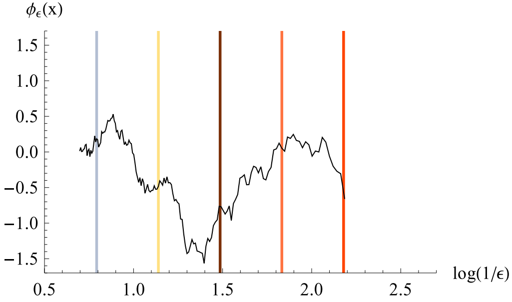
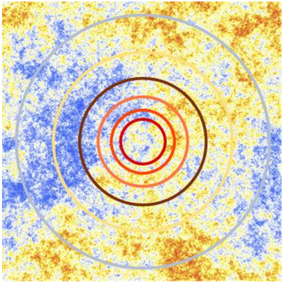
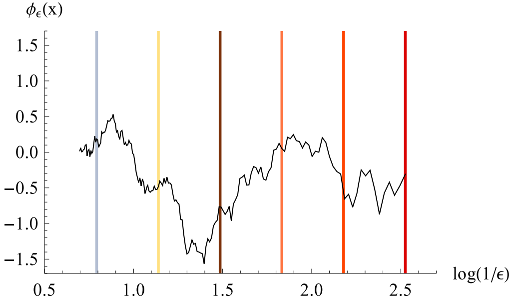




- Look at the circle average $\phi_\epsilon(x)$ as function of ϵ.
- $\langle \phi_\epsilon(x)\phi_{\epsilon'}(x)\rangle = -\log \frac{\max(\epsilon,\epsilon')}{\epsilon_0} = \min(t,t'), \quad t = -\log(\frac{\epsilon}{\epsilon_0})$





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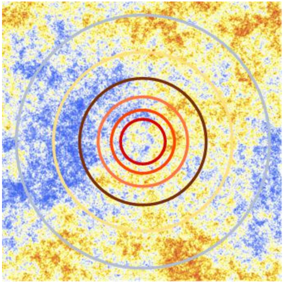

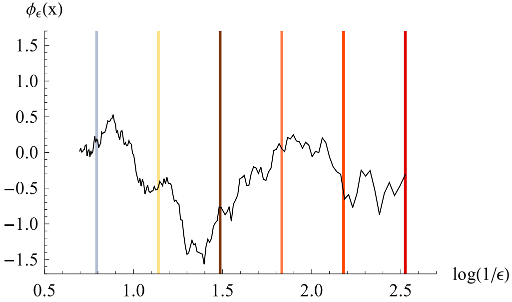

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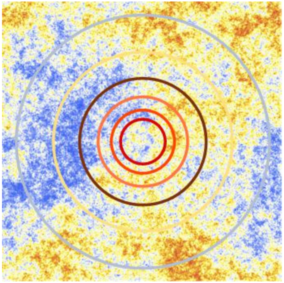

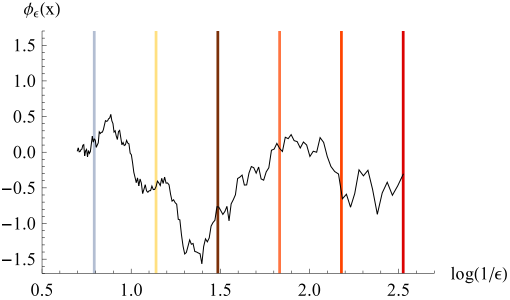

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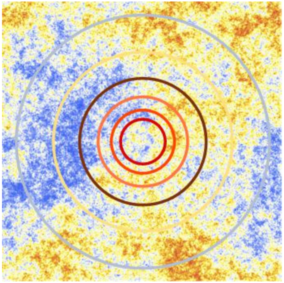
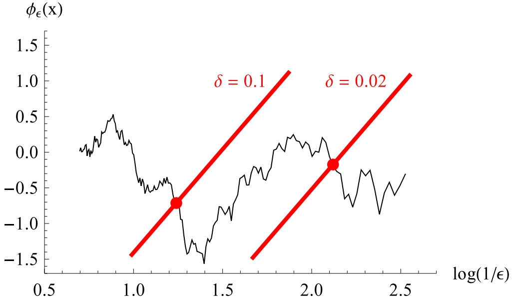
- Hence $\epsilon(\delta)$ is found by solving

$$\delta = \pi\epsilon^2\epsilon^{\gamma^2/2}e^{\gamma\phi_\epsilon(x)} = \pi\epsilon^{\gamma Q}e^{\gamma\phi_\epsilon(x)}$$





- $\epsilon(\delta) = \epsilon_0 e^{-T}$, where T is the first time a Brownian motion with drift Q reaches level $A := \frac{\log(\pi/\delta)}{\gamma} + Q \log \epsilon_0$.

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- It follows that

$$\left\langle \epsilon(\delta)^{2\Delta_0 - 2} \right\rangle = \int dT\, e^{-(2\Delta_0 - 2)T} P_\delta(T) \;\propto\; \delta^{\frac{1}{\gamma}(\sqrt{Q^2 + 4\Delta_0 - 4} - Q)} = \delta^{\Delta - 1}$$

where Δ satisfies the famous KPZ relation [Knizhnik, Polyakov, Zamolodchikov, '88][Duplantier, Sheffield, '10]

$$\Delta_0 = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta,$$

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which relates the conformal weight Δ_0 of an operator in CFT to its scaling dimension Δ when coupled to quantum gravity.
- If (1) holds in DT, then KPZ follows!

- Detail: should not choose x uniformly, but w.r.t. Liouville measure.

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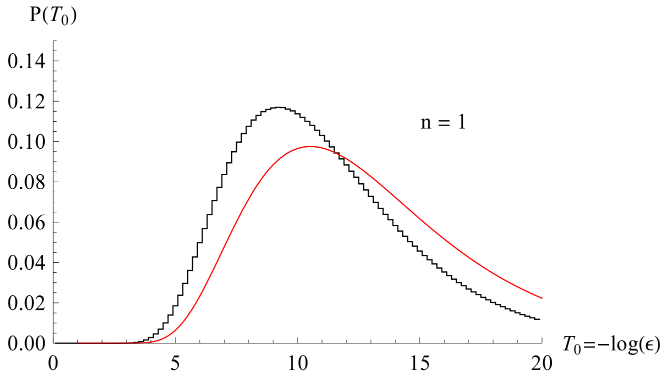

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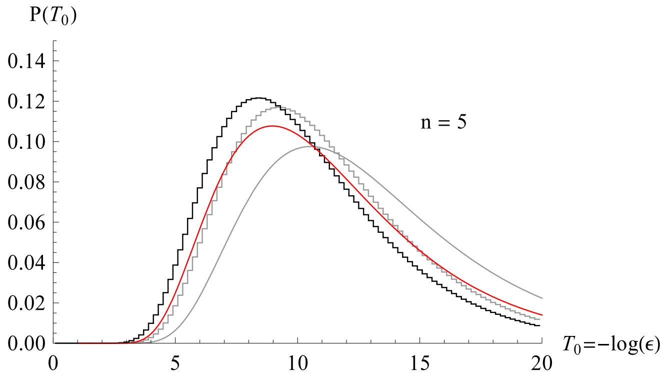

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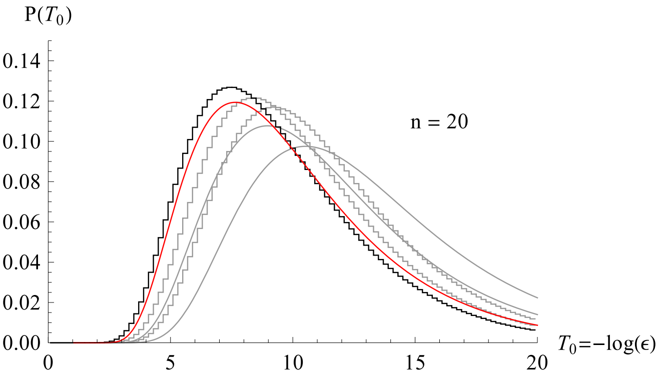

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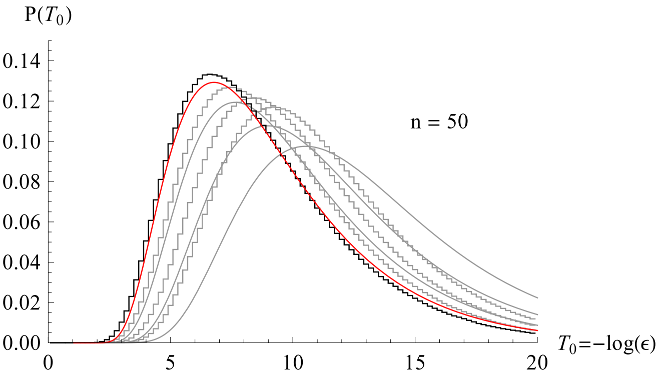

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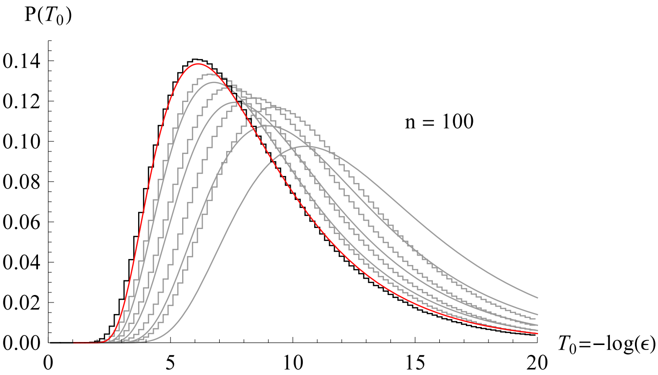

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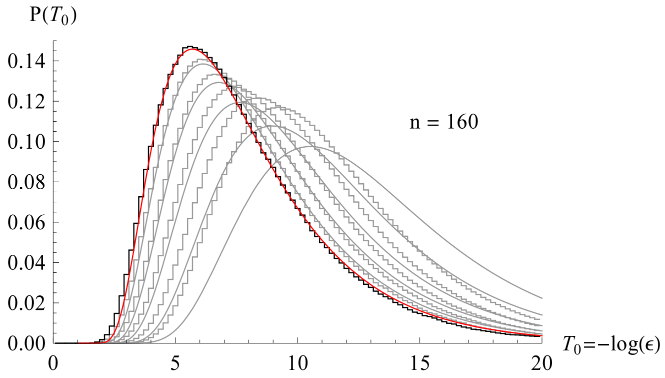
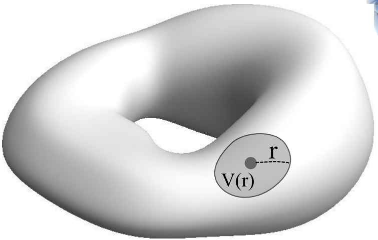


Hausdorff dimension

- The Hausdorff dimension d_h measures the relative scaling of geodesic distance and volume.

$$V(r) \sim r^{d_h}, \quad d_h = \lim_{r \to 0} \frac{\log V(r)}{\log r}$$



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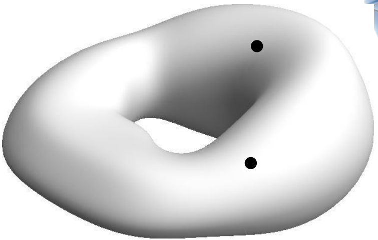

- In terms of 2-point function

$$G(r) = \int d^2x \int d^2y \sqrt{g(x)}\sqrt{g(y)}\, \delta(d_g(x,y) - r),$$

$$G(r) \sim r^{d_h - 1}, \quad d_h - 1 = \lim_{r \to 0} \frac{\log G(r)}{\log r}$$

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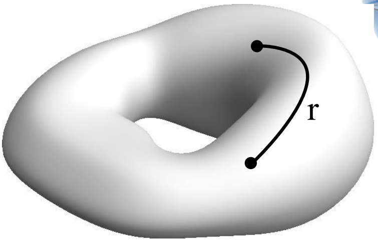
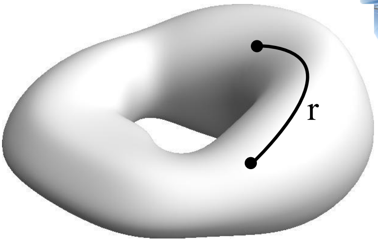


- In terms of 2-point function

$$G(r) = \int d^2x \int d^2y \sqrt{g(x)} \sqrt{g(y)}\, \delta(d_g(x,y) - r),$$

$$G(r) \sim r^{d_h - 1}, \quad d_h - 1 = \lim_{r \to 0} \frac{\log G(r)}{\log r}$$

Hausdorff dimension



- The Hausdorff dimension d_h measures the relative scaling of geodesic distance and volume.

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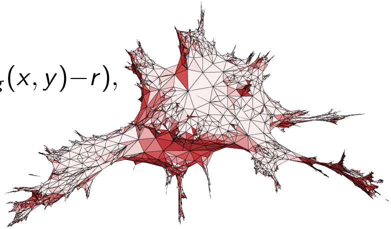
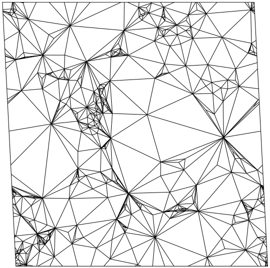


- For Riemannian surfaces $d_h = 2$ but in random metrics we may find $d_h > 2$. In fact, a typical geometry in pure 2d quantum gravity has $d_h = 4$.

Hausdorff dimension from shortest cycles [Ambjørn, TB, '13]

- A shortest non-contractible loop is automatically a geodesic and therefore we expect its length to scale with the volume V as $L \sim V^{\frac{1}{d_h}}$.



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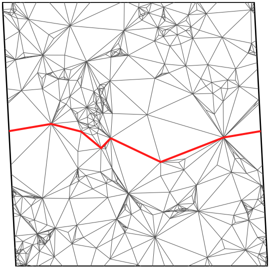

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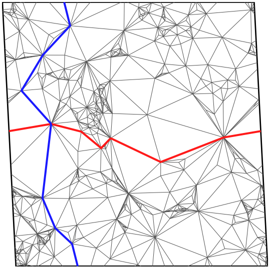

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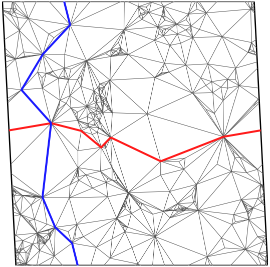

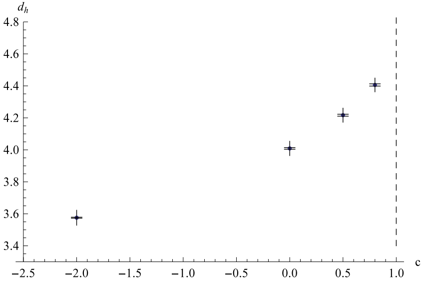

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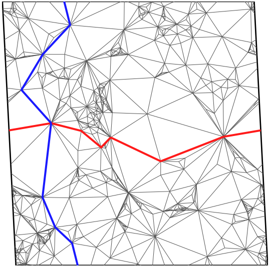
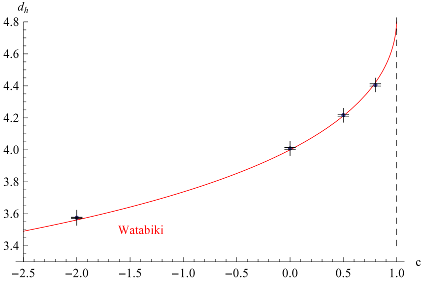
- Data agrees well with Watabiki's formula: $d_h = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}$





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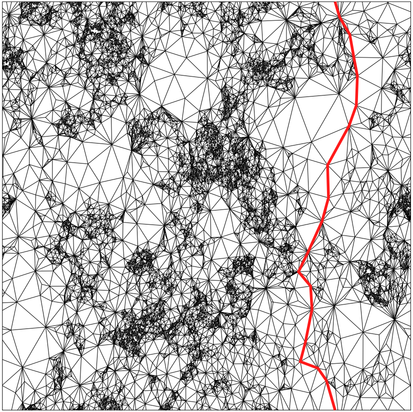
- Try numerically!

Triangulations versus Liouville

- The harmonic embedding of a random triangulation represents roughly a piecewise constant field ϕ^δ: $e^{\gamma\phi^\delta(x)}|_{x\in\triangle} = 1/(N\,a_\triangle)$



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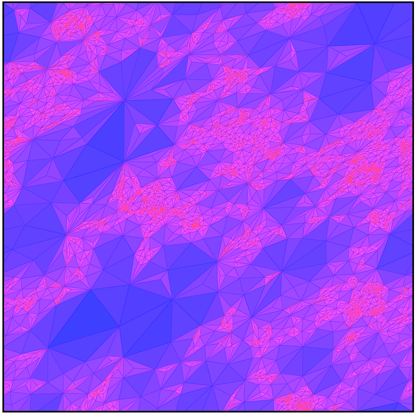

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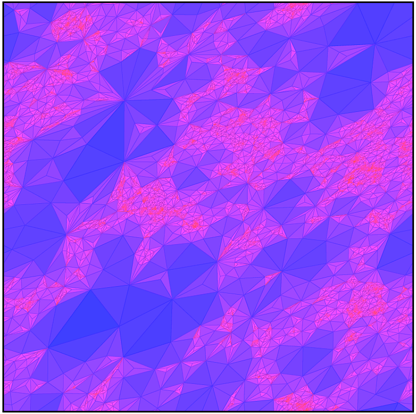
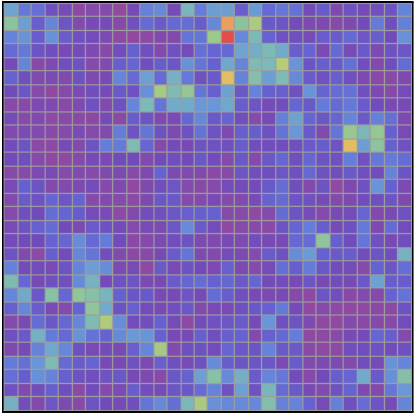
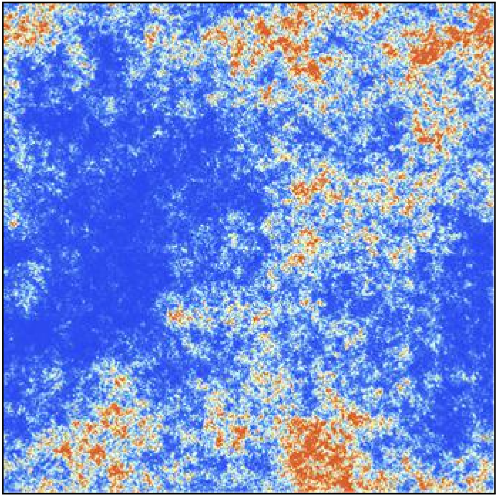


Covariant: lattice sites contain equal volume



Non-covariant: lattice site contains volume $\propto e^{\gamma\phi}$

- Mimic a covariant cutoff.



$\gamma = 0.6$

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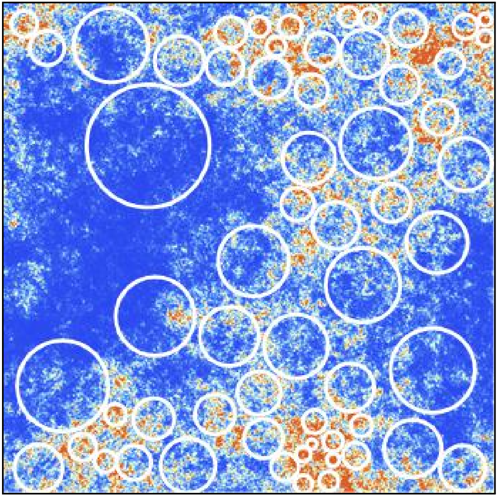
- For $\delta > 0$, find the ball $B_{\epsilon(\delta)}(x)$ around x with volume $\mu(B_{\epsilon(\delta)}) = \delta$.
- Replace the measure with the average measure over the ball.



$\gamma = 0.6, \delta = 0.01$

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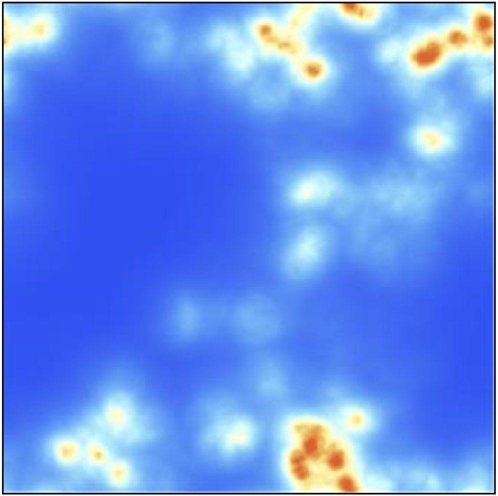

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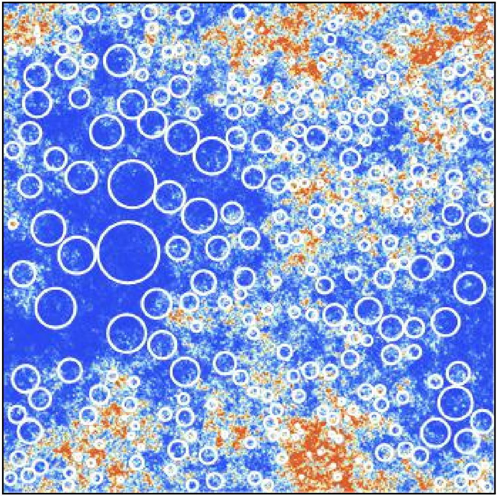

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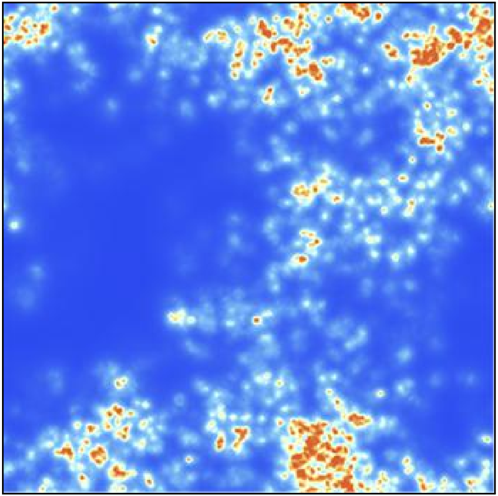
- Compare to DT: $\delta \sim 1/N$



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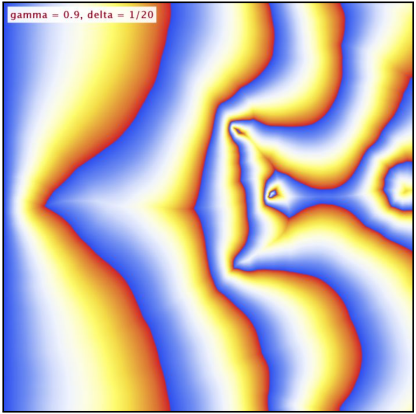
Measure distance w.r.t. $g_{ab} = e^{\gamma\phi^\delta}\delta_{ab}$

$$d_\delta(x,y) = \inf_\Gamma \left\{ \int_\Gamma ds\, e^{\frac{\gamma}{2}\phi^\delta(x(s))} \right\}$$



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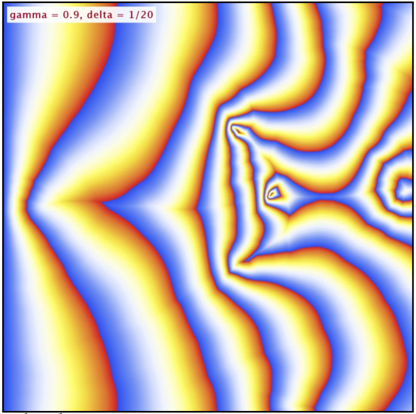

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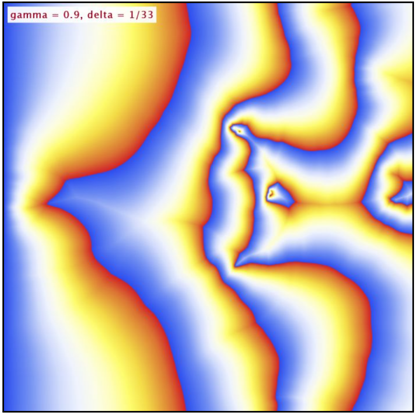

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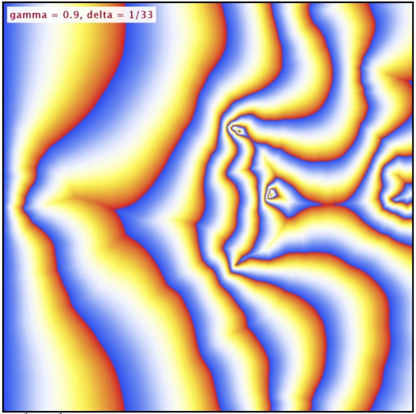

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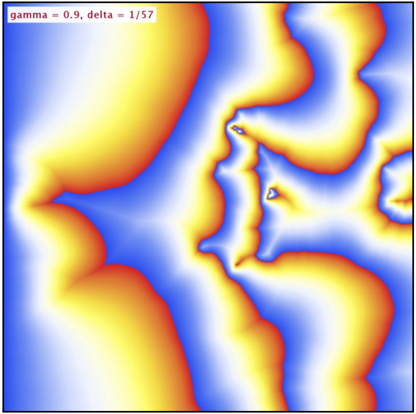

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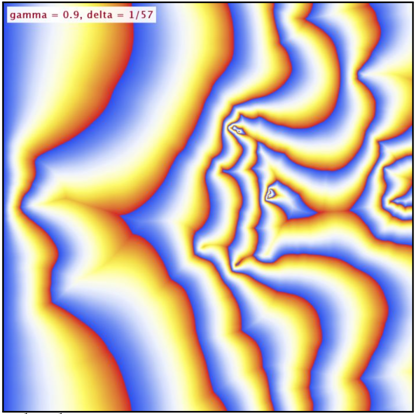

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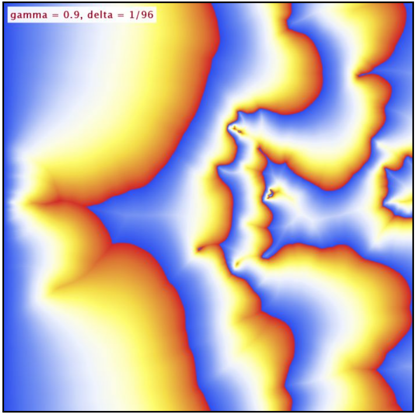

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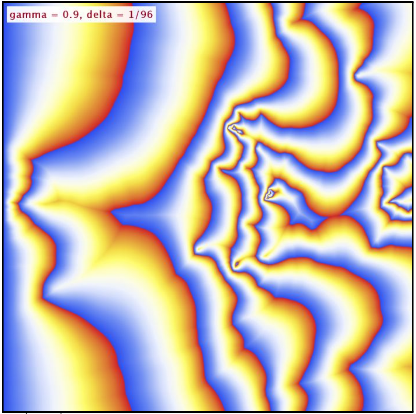

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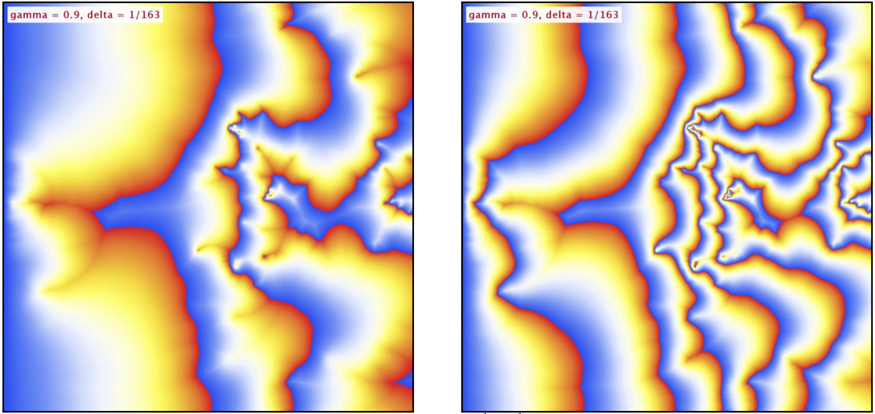

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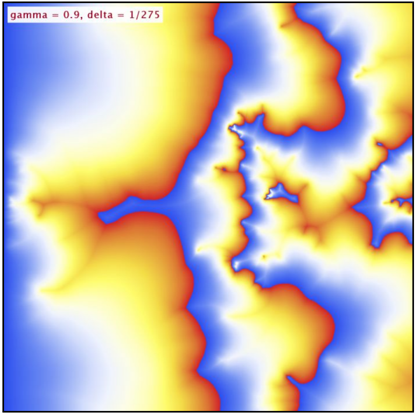

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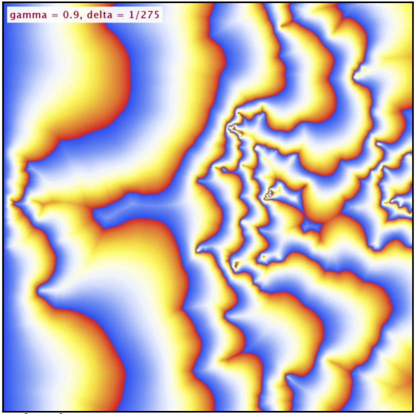

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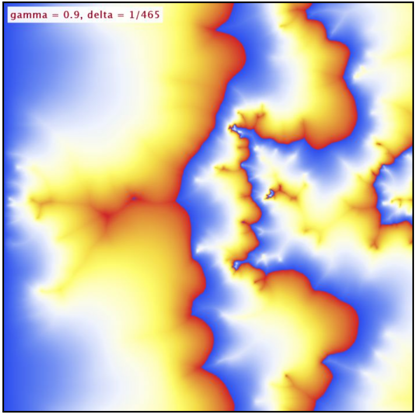

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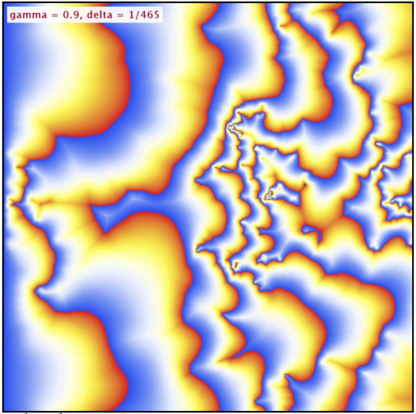

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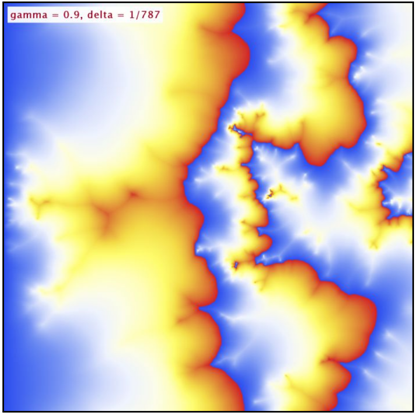

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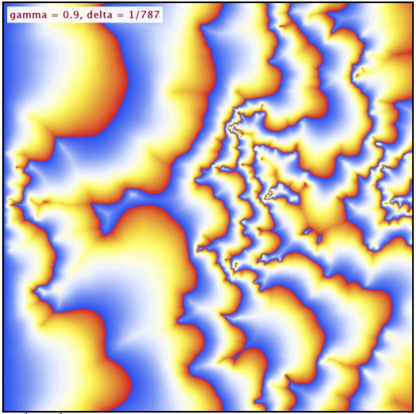

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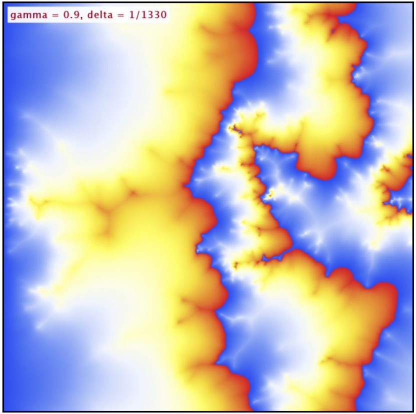

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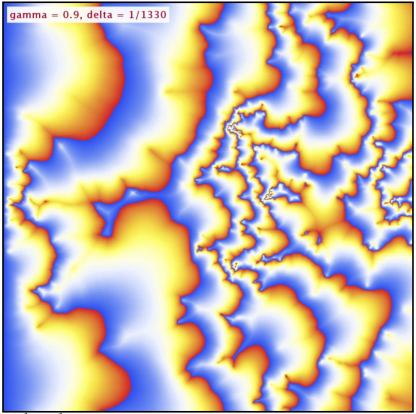

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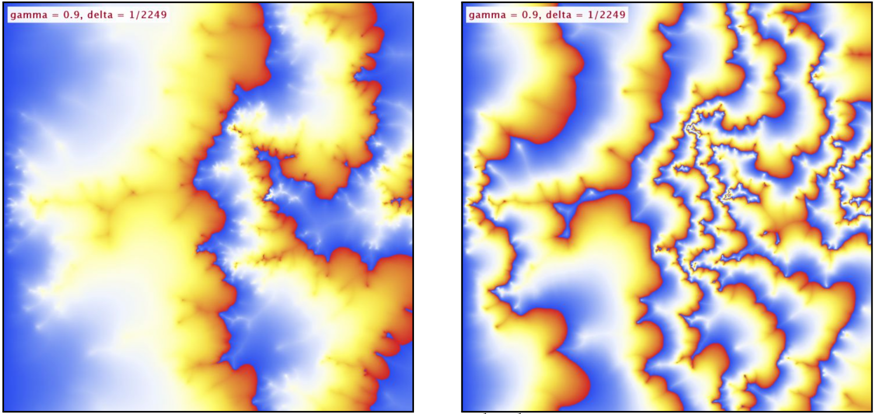

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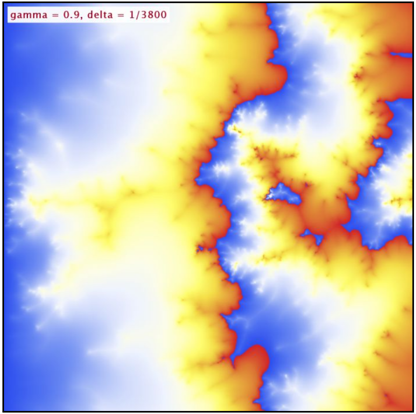

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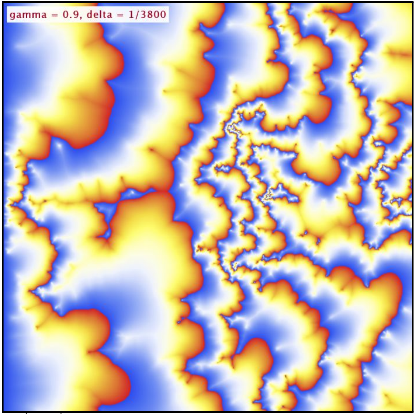

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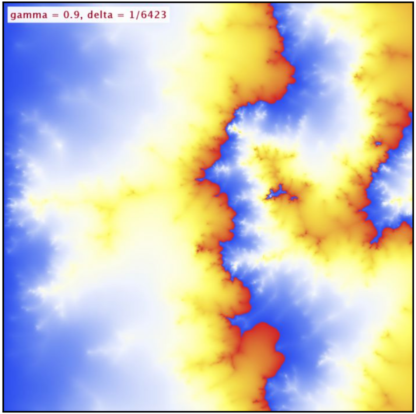
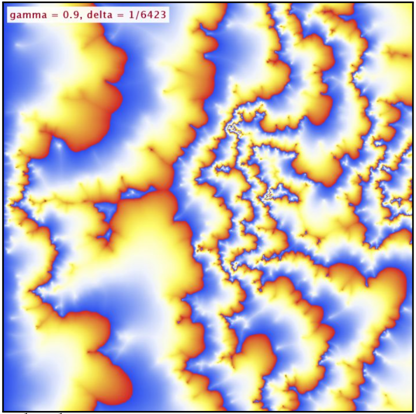
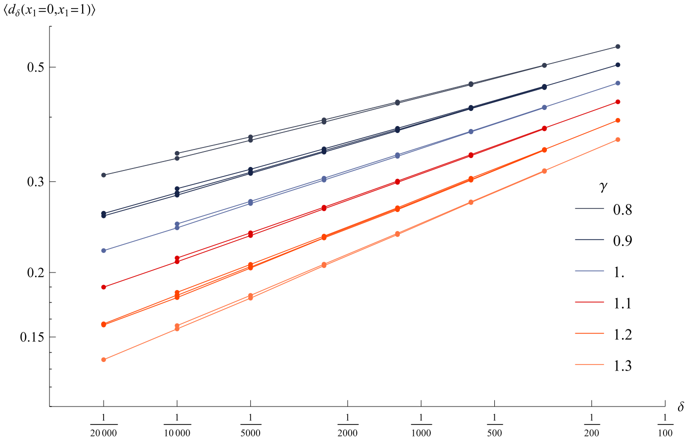


$d_\delta(x, \{x_1 = 0\})$



$\delta^{\frac{1}{d_h} - \frac{1}{2}} d_\delta(x, \{x_1 = 0\}), \; d_h \approx 2.70$

- To extract $d_h(\gamma)$, measure the expectation value $\langle d_\delta(\{x_1 = 0\}, \{x_1 = 1\})\rangle$ of the distance between left and right border as function of δ.



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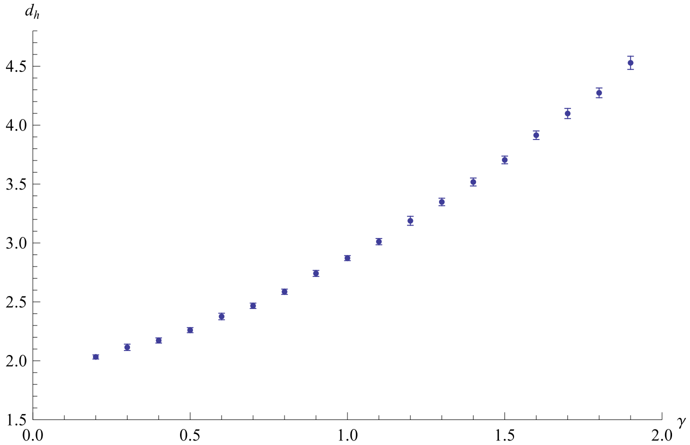

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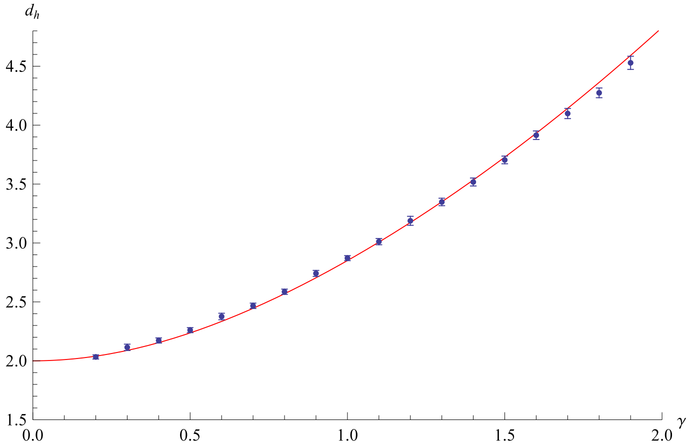
- Compare with Watabiki's formula, $d_h = 1 + \frac{\gamma^2}{4} + \sqrt{1 + \frac{3}{2}\gamma^2 + \frac{1}{16}\gamma^4}$.



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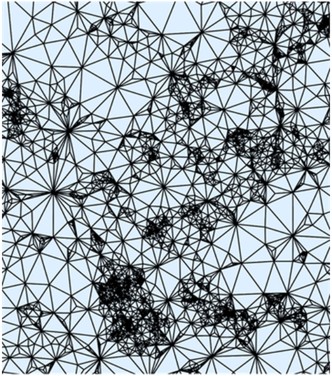

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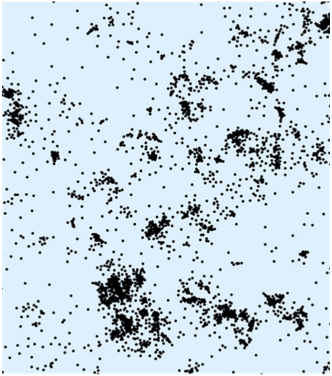

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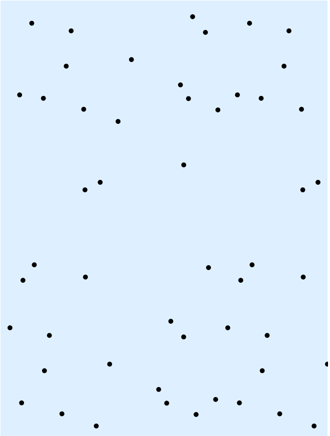

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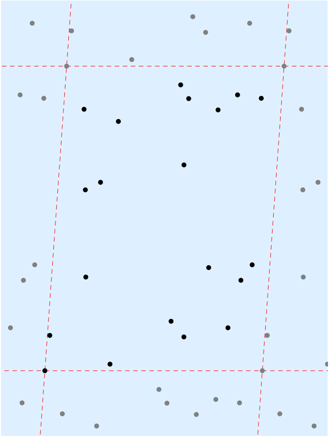

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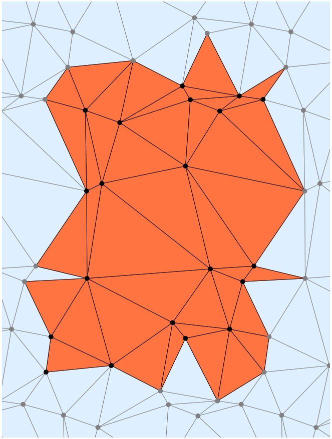

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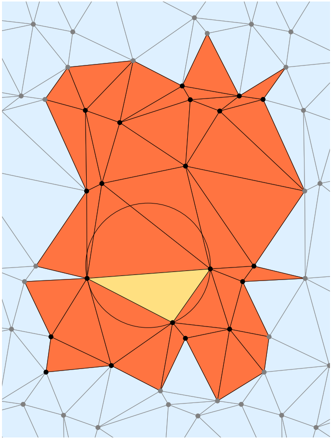

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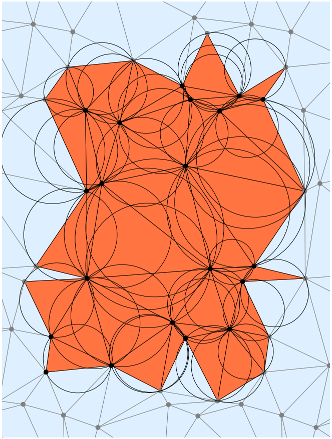

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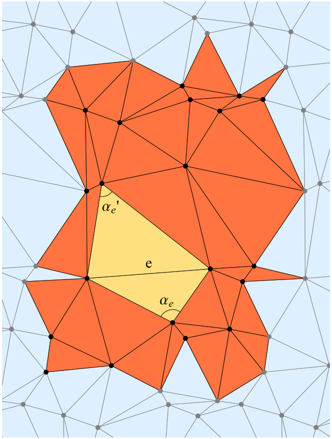

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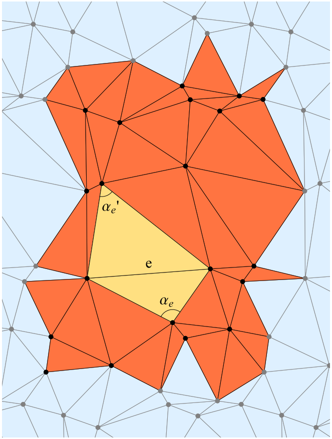
- Circle pattern theorem [Rivin, '94]: the embedding of the abstract triangulation is uniquely determined by the values $\{\theta_e\}$.



- To be precise, there exists a bijection

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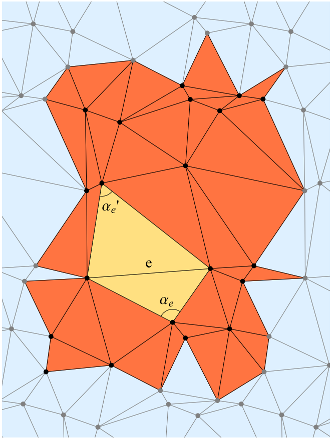

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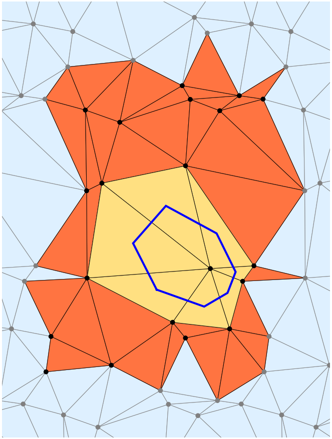

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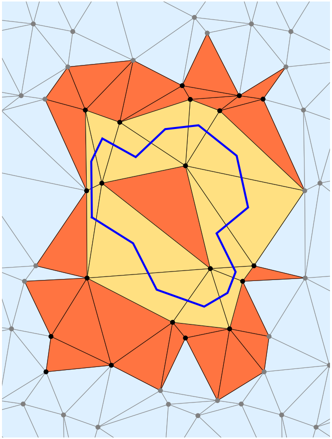

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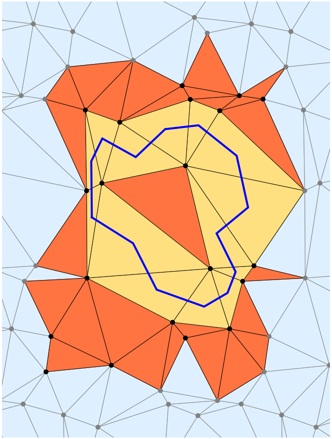

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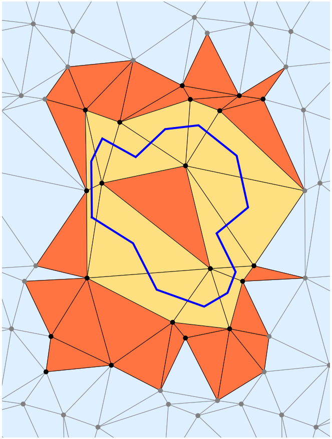

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- If true: $Z(g,n) \propto n^{-\frac{7}{2}+\frac{5}{2}g}C^n(1+\mathcal{O}(n^{-1}))$, $\quad C \approx 15.6$

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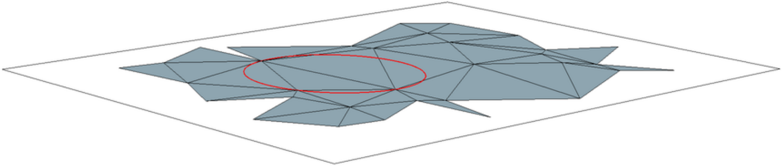

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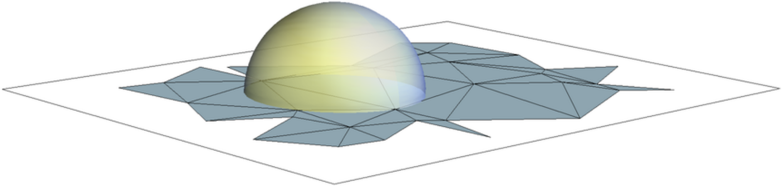

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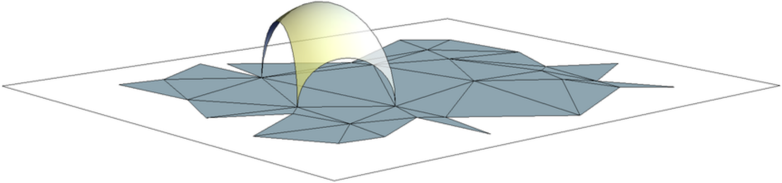

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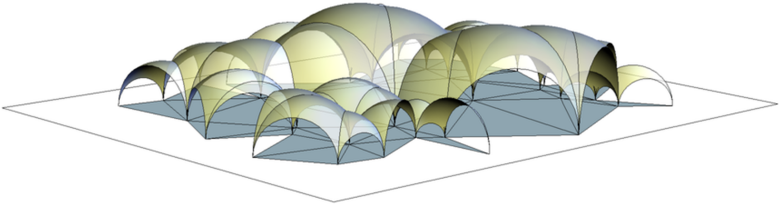

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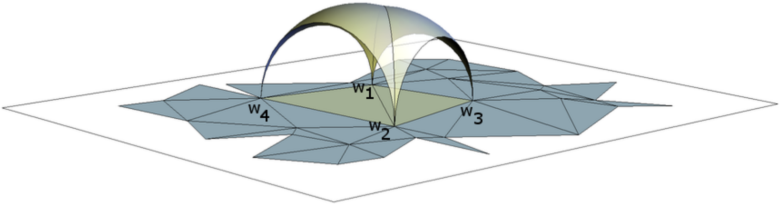

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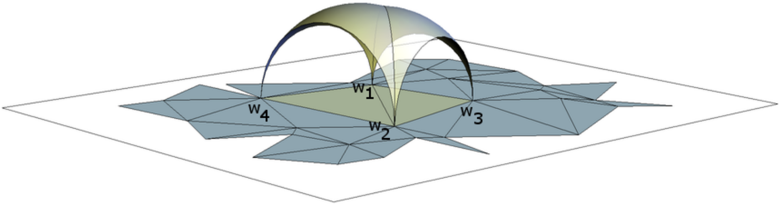

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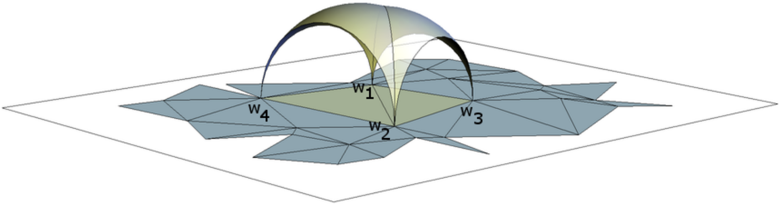

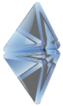

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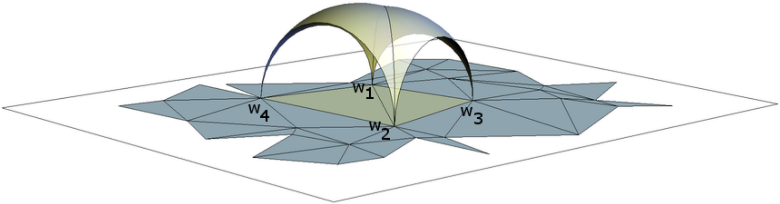

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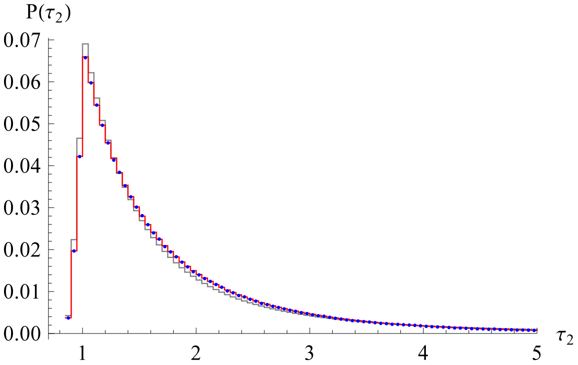
- Example: distribution of the modulus τ for genus 1 with 25 vertices.



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  - Circle patterns give a more precise conformal mapping between triangulations and discrete measures and reveal a close connection with the well-studied Weil-Petersson geometry of Riemann surfaces.
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  - Conversely, one can assign a geometric interpretation to a Liouville measure by implementing a covariant cut-off. This is used to measure the Hausdorff dimension, which agrees well with Watabiki's formula.
  - Circle patterns give a more precise conformal mapping between triangulations and discrete measures and reveal a close connection with the well-studied Weil-Petersson geometry of Riemann surfaces.
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Thanks! Questions? Slides available at `http://www.nbi.dk/~budd/`