Approaches to Quantum Gravity, Clermont-Ferrand, Jan. 6, 2014

Fractal dimensions of 2d quantum gravity

Timothy Budd

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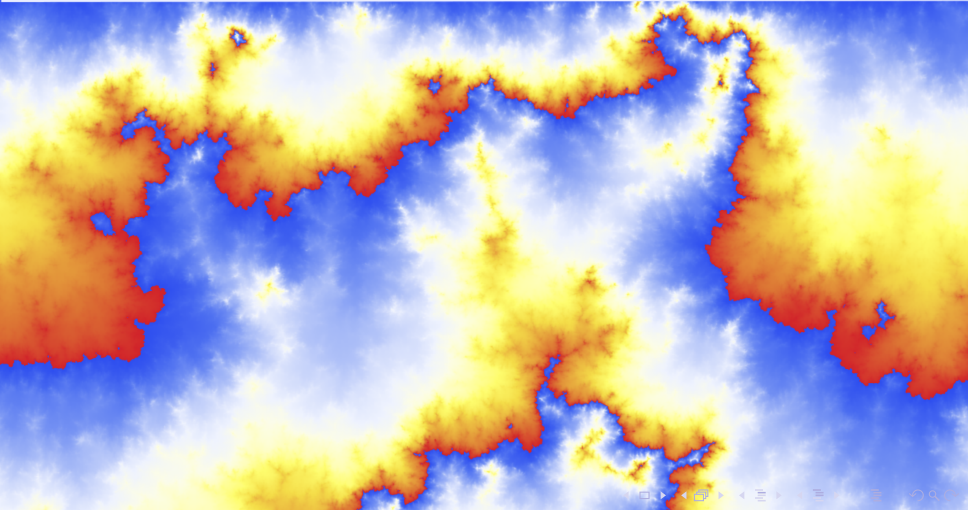


Outline

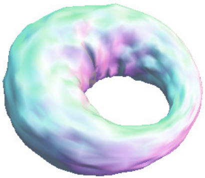
- Introduction to 2d gravity
- Fractal dimensions
  - Hausdorff dimension d_h
  - "Teichmüller deformation dimension" d_{TD}
- Hausdorff dimension in dynamical triangulations
  - Overview of results in the literature
  - Recent numerical results via shortest cycles
- Quantum Liouville gravity
  - Gaussian free field basics
  - Distance in the Liouville metric
  - Measurement of d_h
  - Derivation of d_{TD}
- Summary & outlook

2D quantum gravity

- Formally 2d gravity is a statistical system of random metrics on a surface of fixed topology with partition function

$$Z = \int [\mathcal{D}g][\mathcal{D}X] \exp(-\lambda V[g] - S_m[g,X])\,,$$

possibly coupled to some matter fields X with action $S_m[g,X]$.



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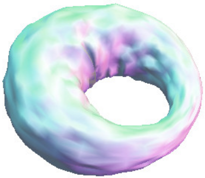

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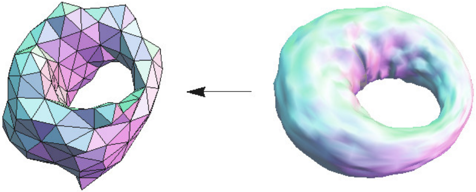

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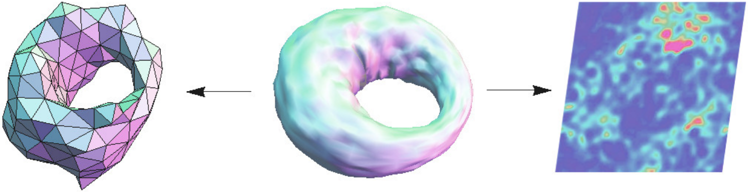
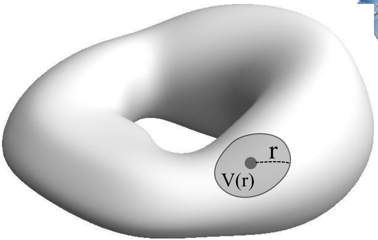
  - Liouville path integral: gauge fix $g_{ab} = e^{\gamma\phi}\hat{g}_{ab}(\tau)$.



Hausdorff dimension

- The Hausdorff dimension d_h measures the relative scaling of geodesic distance and volume.

$$V(r) \sim r^{d_h}, \quad d_h = \lim_{r \to 0} \frac{\log V(r)}{\log r}$$



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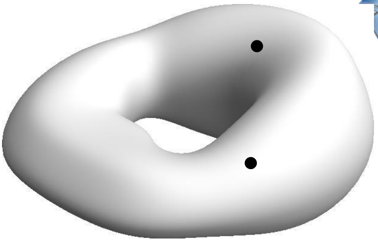

- In terms of 2-point function

$$G(r) = \int d^2x \int d^2y \sqrt{g(x)}\sqrt{g(y)}\,\delta(d_g(x,y) - r),$$

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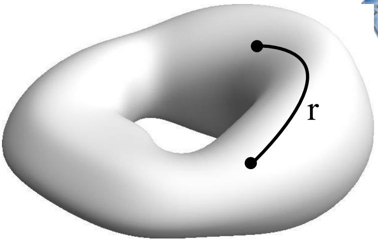

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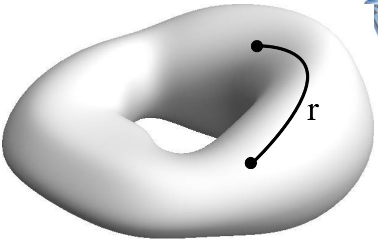

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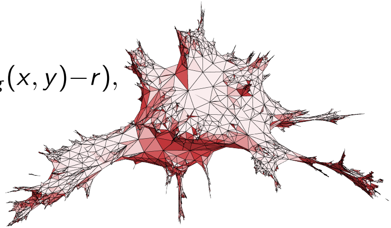

- For Riemannian surfaces $d_h = 2$ but in random metrics we may find $d_h > 2$. In fact, a typical geometry in pure 2d quantum gravity has $d_h = 4$.

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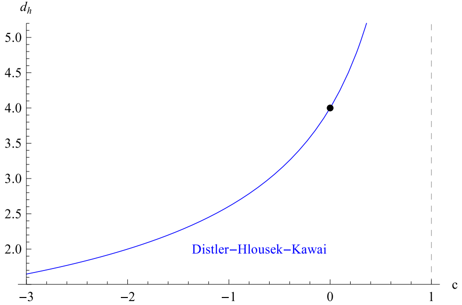

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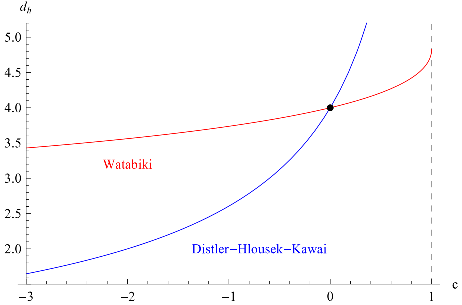

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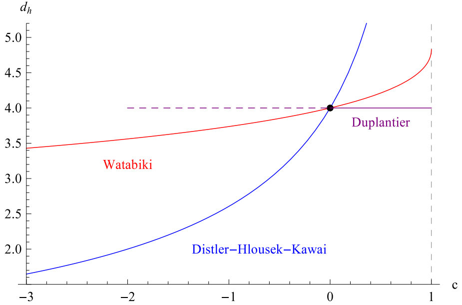
$$d_h = 4 \qquad \text{[Duplantier,'11]}$$



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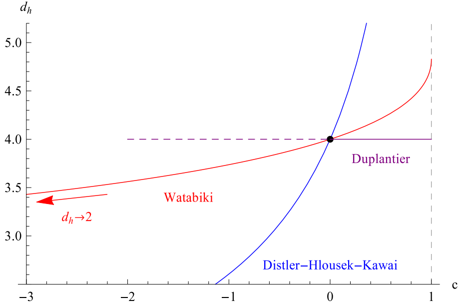

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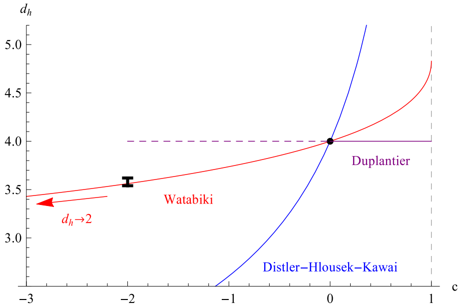

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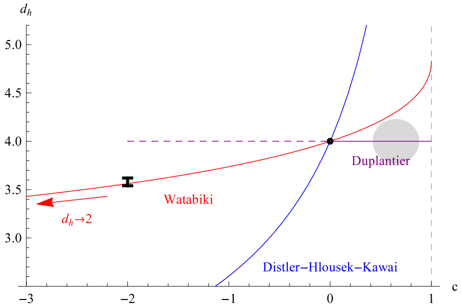
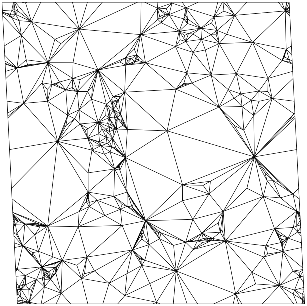
- First numerical results for $c = -2$ from measuring 2-point function: $d_h = 3.58 \pm 0.04$ [Ambjørn, Anagnostopoulos, ..., '95]
- Measurements for Ising model ($c = 1/2$) and 3-states Potts ($c = 4/5$) are inconclusive: various values between $d_h \approx 3.8$ and $d_h \approx 4.3$ are obtained, but $d_h = 4$ seems to be preferred. [Catterall et al, '95] [Ambjørn, Anagnostopoulos, '97]



Hausdorff dimension from shortest cycles

- A shortest non-contractible loop is automatically a geodesic and therefore we expect its length to scale with the volume V as $L \sim V^{\frac{1}{d_h}}$.



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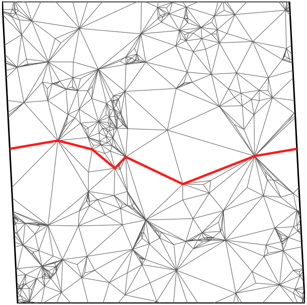

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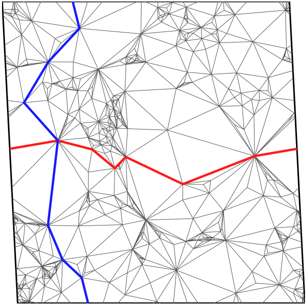
- Especially for $c > 0$ these loops are really short, so we also measure second shortest cycles.



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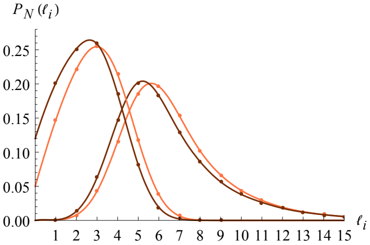

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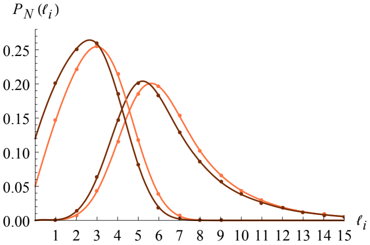

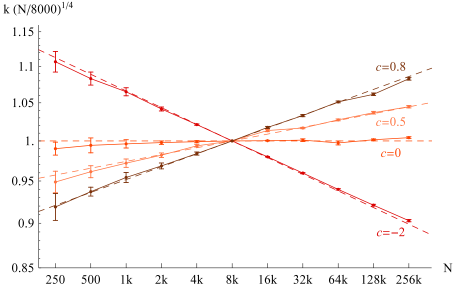

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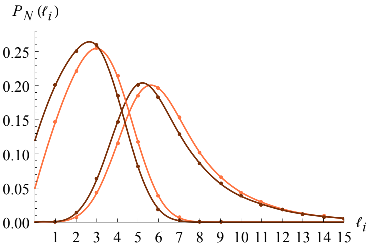

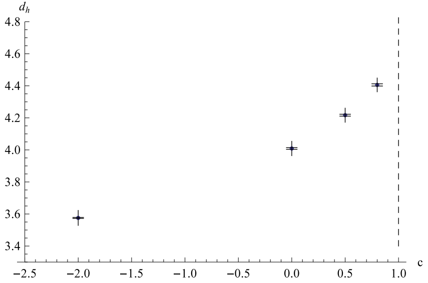

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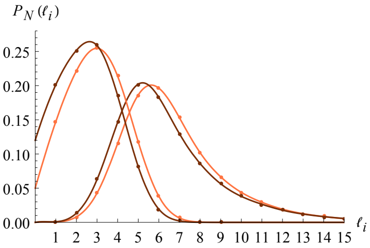

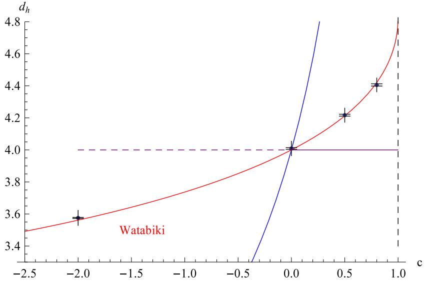

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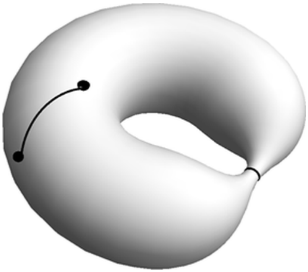
- Shortest cycle is not a generic geodesic, it is the shortest in its homotopy class.
- The “real” Hausdorff dimension corresponds to distances between typical points.
- If the shortest cycle scales with larger dimensions than d_h, then in the continuum limit the geometry becomes pinched.



Teichmüller deformation dimension d_{TD}

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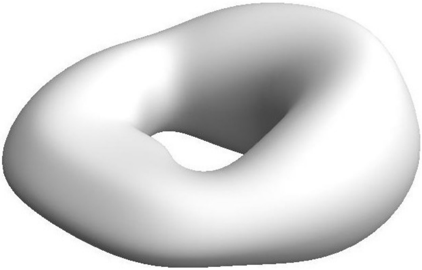

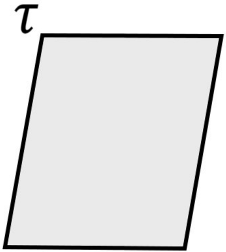

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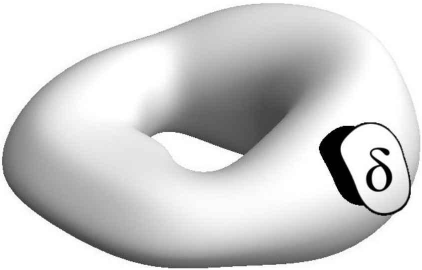

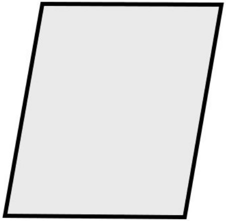

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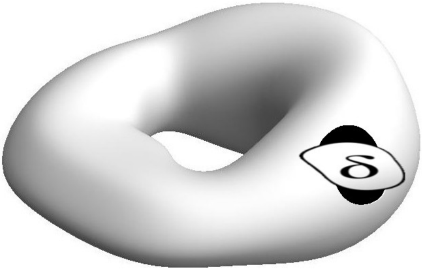

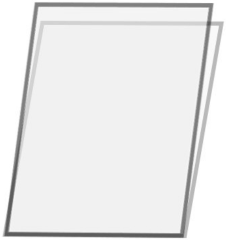

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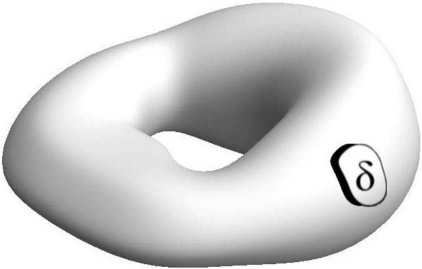

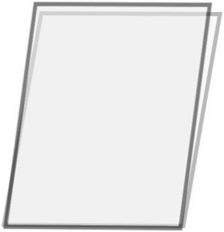

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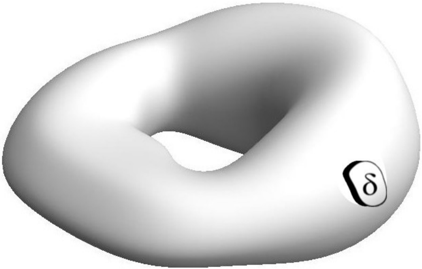

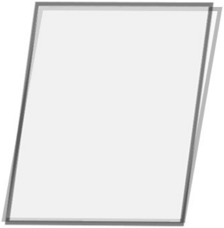

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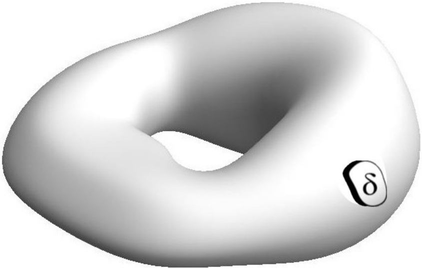
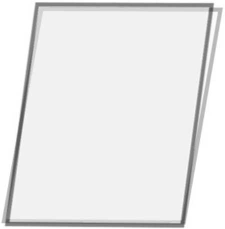
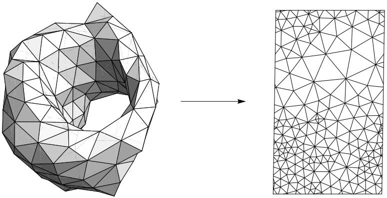
- More generally, define d_{TD} by $\langle d^2_{\text{Poincaré}}(\tau, \tau + \Delta\tau)\rangle \sim \delta^{1+\frac{2}{d_{TD}}}$.





Teichmüller deformation for a triangulation

- Given a triangulation of the torus, there is a natural way to associate a harmonic embedding in $\mathbb{R}^2$ and a Teichmüller parameter τ.



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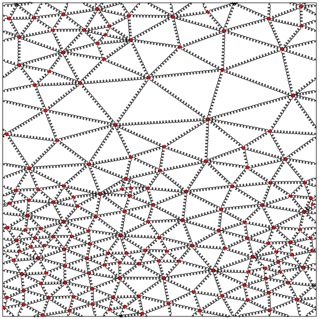

Teichmüller deformation for a triangulation

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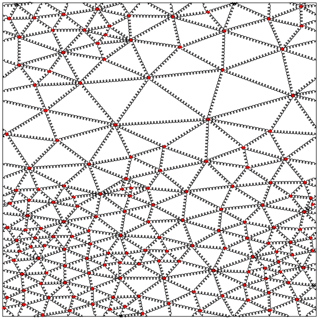
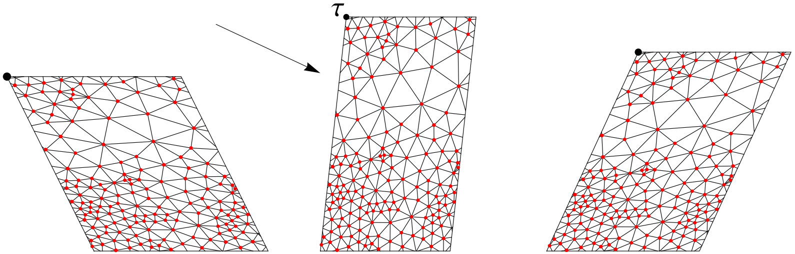
- Replace edges by ideal springs and find equilibrium.
- Find linear transformation that minimizes energy while fixing the volume.





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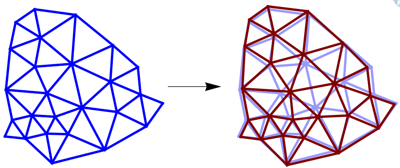

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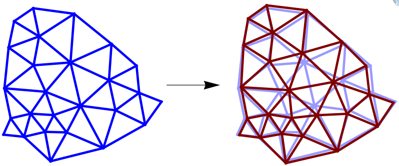

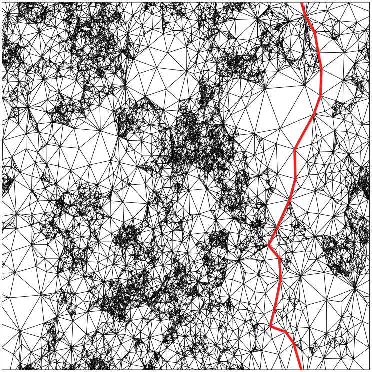

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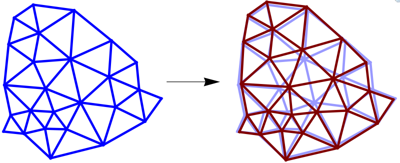
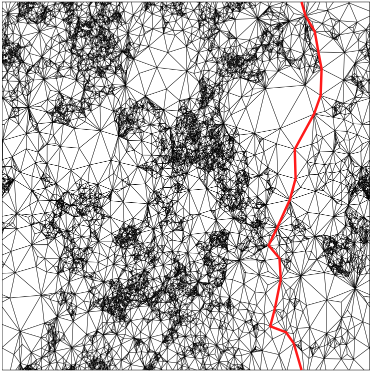
- $d^2_{\text{Poincaré}}(\tau, \tau + \Delta\tau)$ scales for large N like the square $a^2_\triangle$ of the areas $a_\triangle$ of the triangles involved. [Budd,'12]
- Since $\delta \approx 1/N$,

$$\langle d^2_{\text{P.}}(\tau, \tau{+}\Delta\tau)\rangle \approx \frac{1}{N}\sum_\triangle a^2_\triangle \sim \delta^{1+\frac{2}{d_{TD}}}$$

is equivalent to

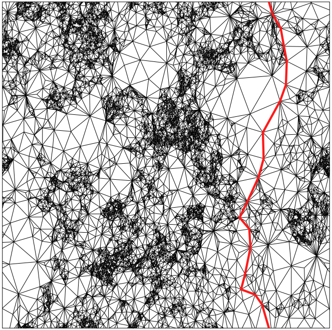
$$\sum_\triangle a^2_\triangle \sim N^{-\frac{2}{d_{TD}}}$$





Relation between d_h and d_{TD}?

- Conjecture: $d_{TD} = d_h$. Why?



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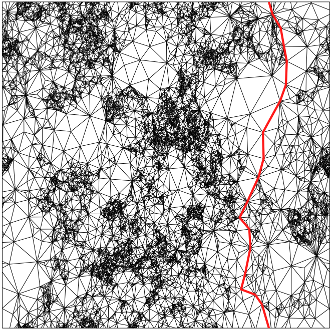

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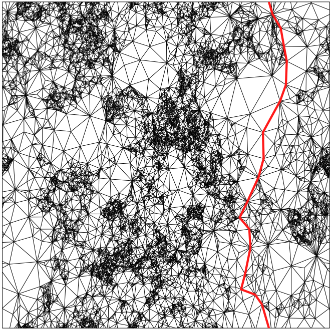

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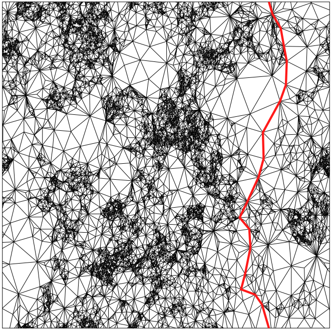

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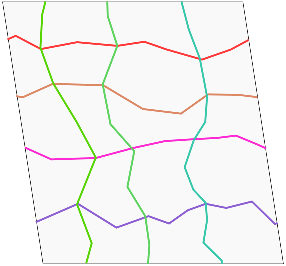

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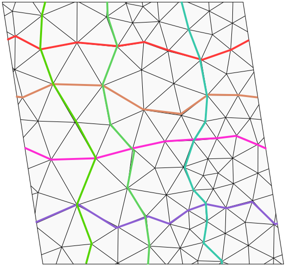

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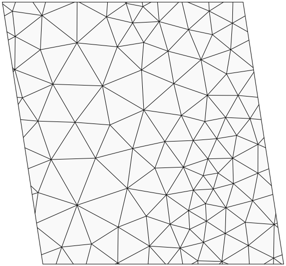

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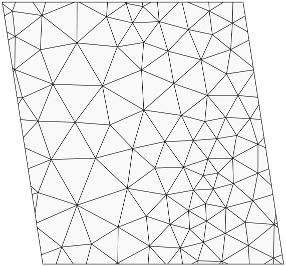

- Hence

$$\sum_\triangle a_\triangle^2 \sim N^{\frac{2}{d_h}} (N^{\frac{2}{d_h}})^{-2} = N^{-\frac{2}{d_h}} =: N^{-\frac{2}{d_{TD}}}$$

Quantum Liouville gravity [David, '88] [Distler, Kawai, '89]

- Consider 2d gravity coupled to c scalar fields, i.e. the Polyakov string in c dimensions,

$$Z = \int [\mathcal{D}g][\mathcal{D}X] \exp\left(-\lambda V[g] - \int d^2x \sqrt{g} g^{ab} \partial_a X^i \partial_b X^j \delta_{ij}\right), \quad X \in \mathbb{R}^c.$$

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- Conformal bootstrap: assuming Z to be of the form

$$Z = \int d\tau [\mathcal{D}_{\hat{g}}\phi][\mathcal{D}_{\hat{g}}X] \exp\left(-S_L[\hat{g},\phi] - S_m[X,\hat{g}]\right)$$

with the Liouville action

$$S_L[\hat{g},\phi] = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} (\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q\hat{R}\phi + \mu e^{\gamma\phi})$$

and requiring invariance w.r.t. $\hat{g}_{ab}$ fixes the constants Q and γ:

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} = \sqrt{\frac{25-c}{6}}$$

- If we ignore τ-integral and set $\hat{g}_{ab} = \delta_{ab}$ flat and $\mu = 0$,

$$Z = \int [\mathcal{D}\phi] \exp\left(-\frac{1}{4\pi} \int d^2x \, \partial^a \phi \, \partial_a \phi \right) ,$$

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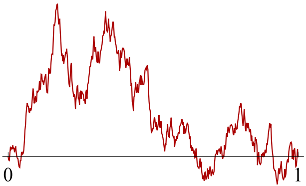
  and does it agree with DT?
- Care required: $e^{\gamma\phi}\delta_{ab}$ is almost surely not a Riemannian metric! Need to take into account the fractal properties of the geometry and regularize appropriately.

Gaussian free field basics

- Gaussian free field in 1d is a.s. a continuous function: Brownian motion.



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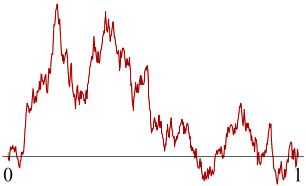
$$\langle \phi(x)\phi(y) \rangle = G(x,y) = -\log|x-y| + \tilde{G}(x,y).$$



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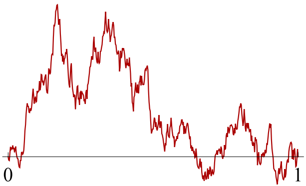

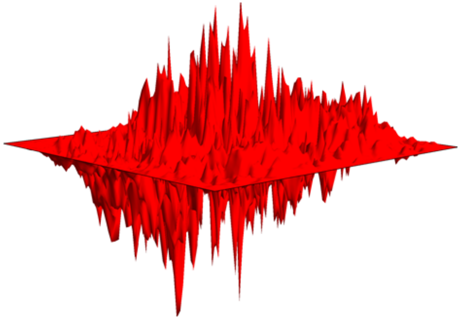

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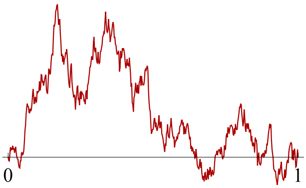
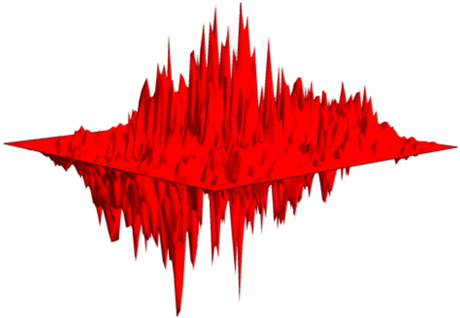
- How do we make sense of the measure $e^{\gamma\phi}$?





Regularization [Sheffield, Duplantier]

- The integral $(f, \phi) = \int d^2x\, f(x)\phi(x)$ has finite variance.
- In particular, for circle average $\phi_\epsilon(x) := \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \phi(x + \epsilon e^{i\theta})$,

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- $d\mu_\epsilon$ converges almost surely to a well-defined random measure $d\mu_\gamma$ as $\epsilon \to 0$. [Sheffield, Duplantier]
- Alternatively, one can use a momentum cut-off. Given an orthonormal basis $\Delta_E f_i = \lambda_i f_i$,

$$\phi_p := \sum_{\lambda_i \le p^2} (f_i, \phi) f_i, \quad d\mu_p = p^{-\gamma^2/2} e^{\gamma\phi_p(x)} d^2x$$

On the lattice

- We can easily put a Gaussian free field on a lattice, say, $L \times L$ with periodic boundary conditions.

```
RandomField[L_] :=
  Re@Fourier[RandomVariate[NormalDistribution[], {L, L, 2}].{1, i}
    Table[If[i == j == 1, 0, (2/π Sin[π (i - 1) / L]^2 + 2/π Sin[π (j - 1) / L]^2)^(-1/2)],
     {i, L}, {j, L}]];
```

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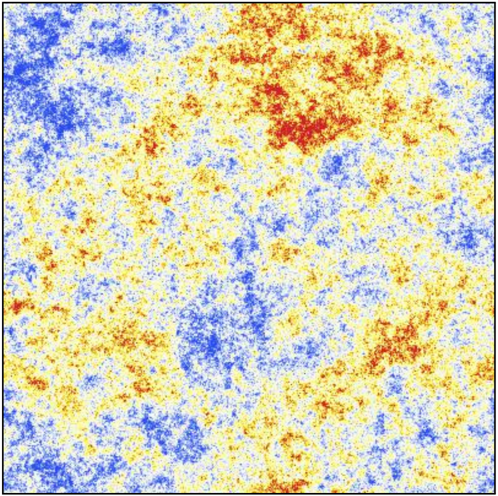

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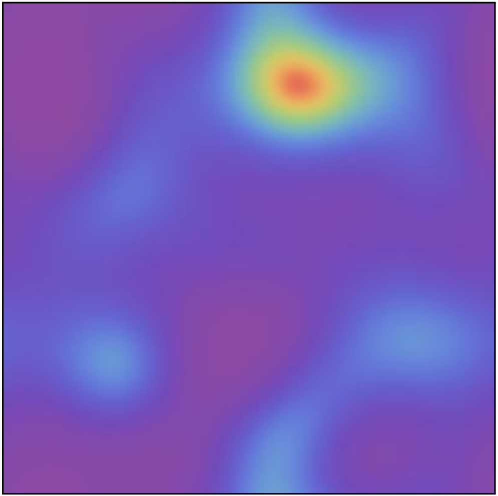

$\gamma = 0.6, p = 10$

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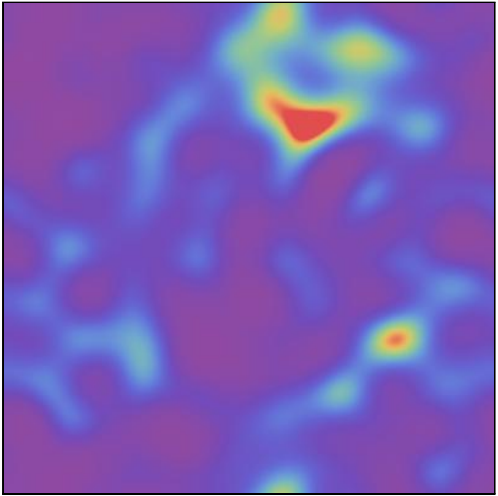

$\gamma = 0.6$, $p = 20$

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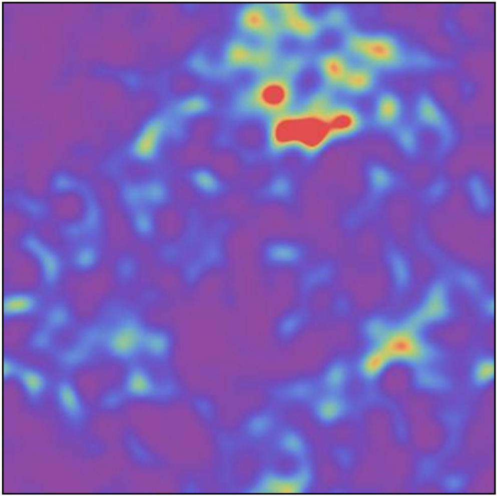

$\gamma = 0.6, p = 40$

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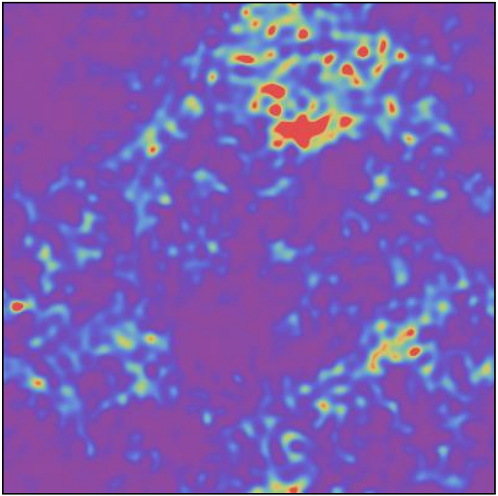

$\gamma = 0.6, p = 80$

On the lattice

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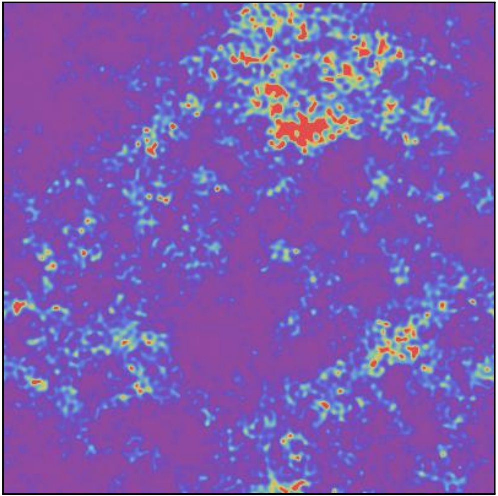

$\gamma = 0.6, p = 160$

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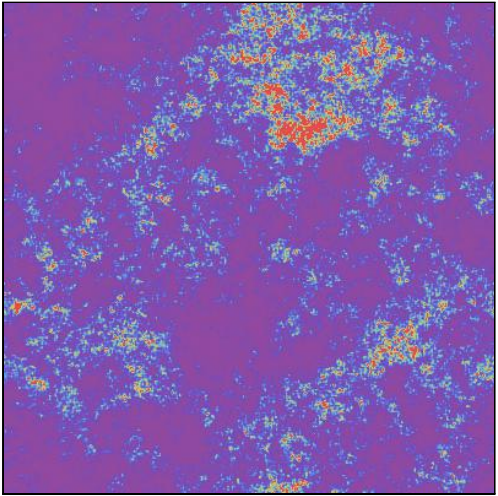
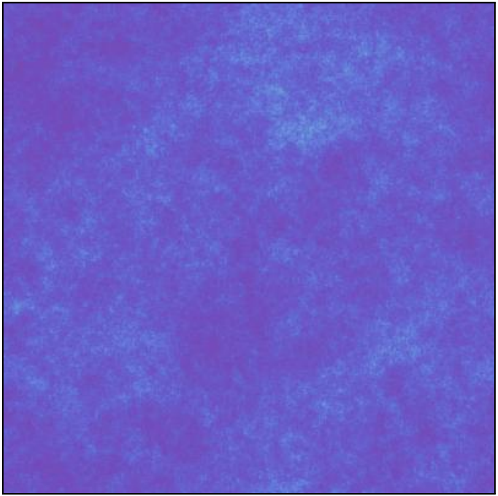


$\gamma = 0.6, p = 320$

On the lattice

- $L \times L$ with periodic boundary conditions.
- Consider $d\mu_p = p^{-\gamma^2/2} e^{\gamma \phi_p(x)} d^2 x$ with $p \ll L$.
- Almost all volume contained in *thick points*, a subset of dimensions $2 - \gamma^2/2$. [Hu, Miller, Peres, '10]



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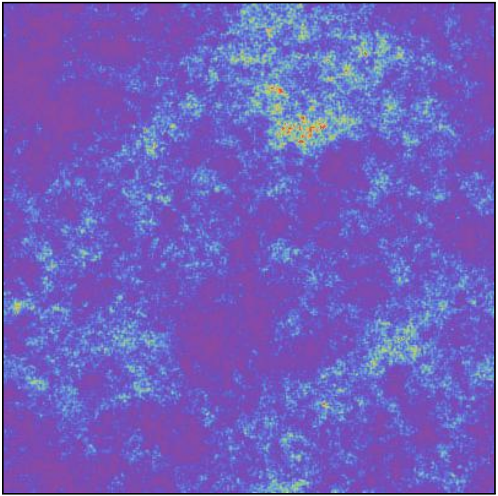

$\gamma = 0.3, p = 320$

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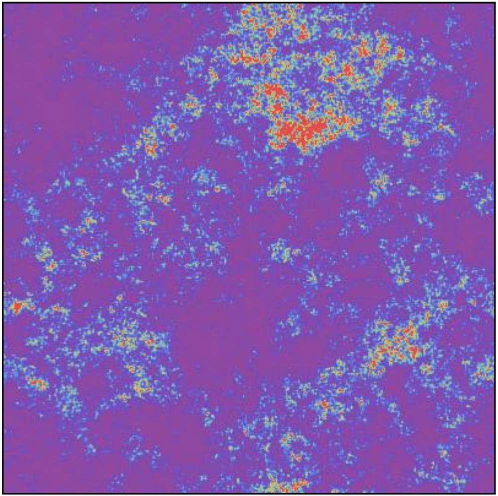
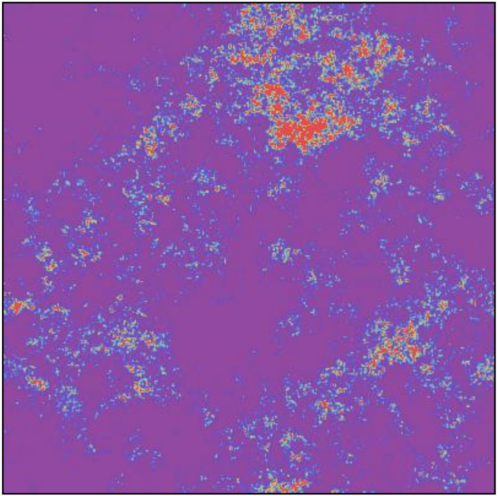


$\gamma = 0.6, p = 320$

On the lattice

- $L \times L$ with periodic boundary conditions.
- Consider $d\mu_p = p^{-\gamma^2/2} e^{\gamma \phi_p(x)} d^2x$ with $p \ll L$.
- Almost all volume contained in *thick points*, a subset of dimensions $2 - \gamma^2/2$. [Hu, Miller, Peres, '10]



$\gamma = 0.9, p = 320$

Geodesic distance?

- For each $\epsilon \approx 1/p$, $g^{\epsilon}_{ab} = \epsilon^{\gamma^2/2} e^{\gamma\phi_\epsilon(x)} \delta_{ab}$ defines a Riemannian metric with associated geodesic distance

$$d_\epsilon(x,y) = \inf_\Gamma \left\{ \int_\Gamma ds\, \epsilon^{\gamma^2/4} e^{\frac{\gamma}{2}\phi_\epsilon(x(s))} \right\}$$

- Does $\epsilon^\sigma d_\epsilon(x,y)$ converge to a continuous $d(x,y)$ for some value $\sigma = \sigma(\gamma)$?

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  - Looking at cycle length we would like to compare to total volume, but what total volume? $\langle \int d^2x \sqrt{g^{\epsilon}} \rangle \sim 1$
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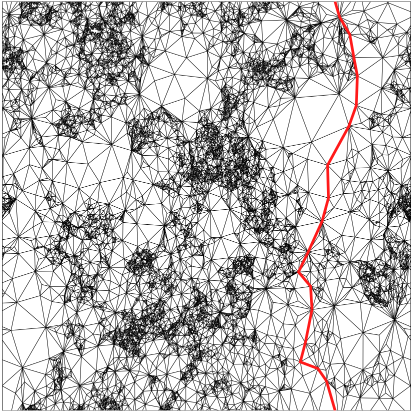
  - When looking at the short-distance behavior of the 2-point function, one is zooming in on the "thick points", where the lattice is "always" too coarse.
- To get closer to DT: use *covariant* cut-off!

- The harmonic embedding of a random triangulation represents roughly a piecewise constant field ϕ^δ: $e^{\gamma\phi^\delta(x)}|_{x\in\triangle} = 1/(N\, a_\triangle)$



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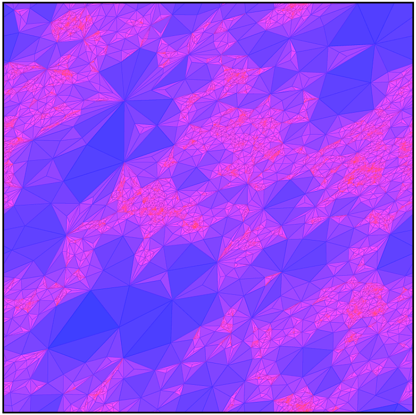

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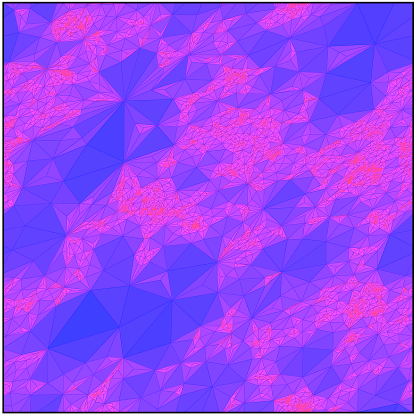
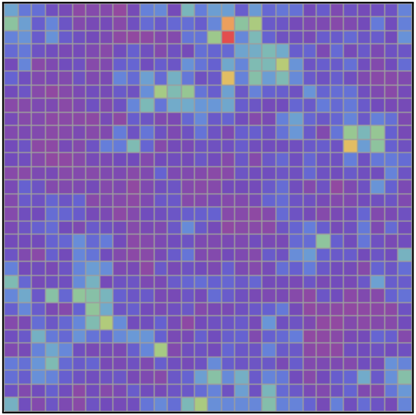
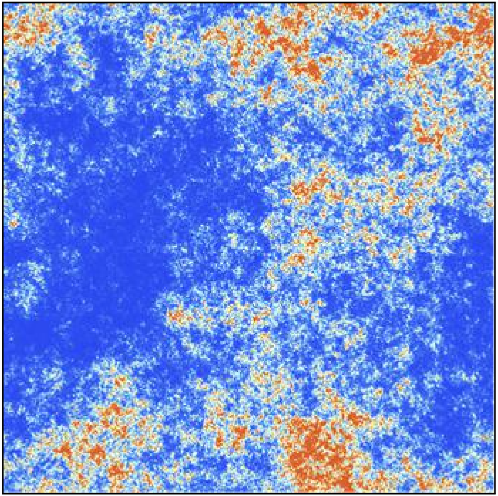


Covariant: lattice sites contain equal volume



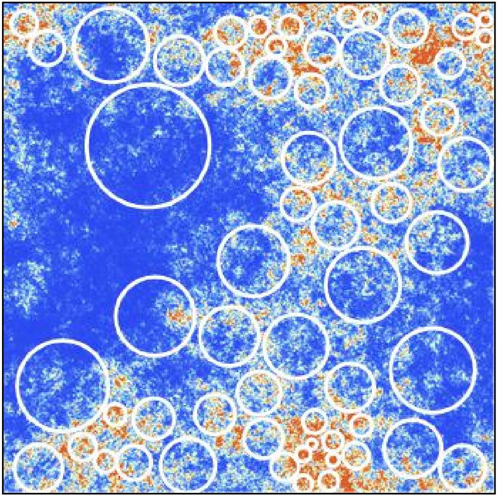
Non-covariant: lattice site contains volume $\propto e^{\gamma\phi}$

- Mimic a covariant cutoff.



$\gamma = 0.6$

- Mimic a covariant cutoff.
- For $\delta > 0$, find the ball $B_{\epsilon(\delta)}(x)$ around x with volume $\mu(B_{\epsilon(\delta)}) = \delta$.
- Replace the measure with the average measure over the ball.



$\gamma = 0.6, \delta = 0.01$

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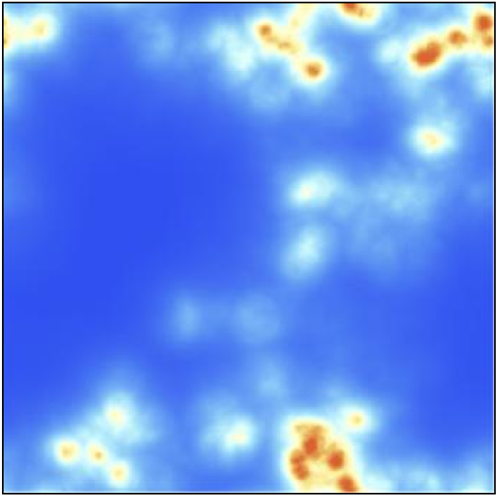
- Define $e^{\gamma\phi^\delta(x)} := \frac{\delta}{\pi\epsilon(\delta)^2}$.



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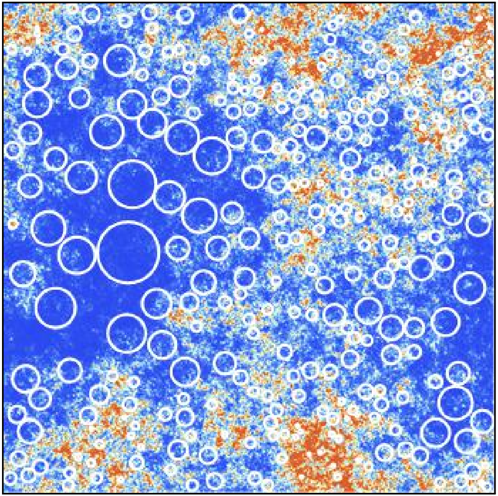
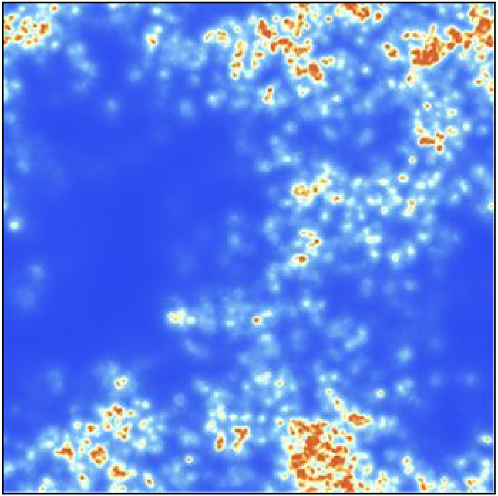


$\gamma = 0.6, \delta = 0.0005$

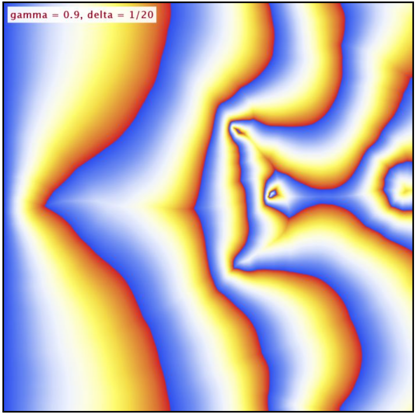
- Mimic a covariant cutoff.
- For $\delta > 0$, find the ball $B_{\epsilon(\delta)}(x)$ around x with volume $\mu(B_{\epsilon(\delta)}) = \delta$.
- Replace the measure with the average measure over the ball.
- Define $e^{\gamma\phi^{\delta}(x)} := \frac{\delta}{\pi\epsilon(\delta)^2}$.
- Compare to DT: $\delta \sim 1/N$



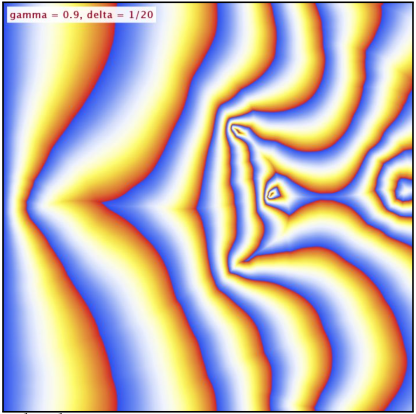
$\gamma = 0.6, \delta = 0.0005$

Measure distance w.r.t. $g_{ab} = e^{\gamma\phi^\delta}\delta_{ab}$

$$d_\delta(x,y) = \inf_\Gamma \left\{ \int_\Gamma ds\, e^{\frac{\gamma}{2}\phi^\delta(x(s))} \right\}$$



$d_\delta(x, \{x_1 = 0\})$



$\delta^{\frac{1}{d_h}-\frac{1}{2}} d_\delta(x, \{x_1 = 0\}),\ d_h \approx 2.70$

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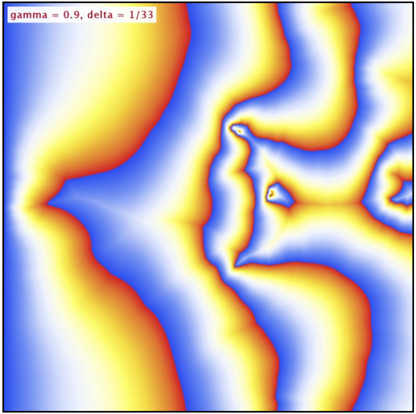

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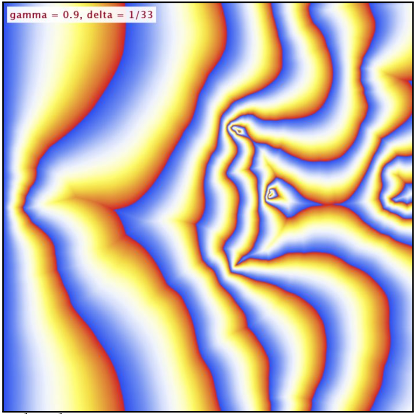

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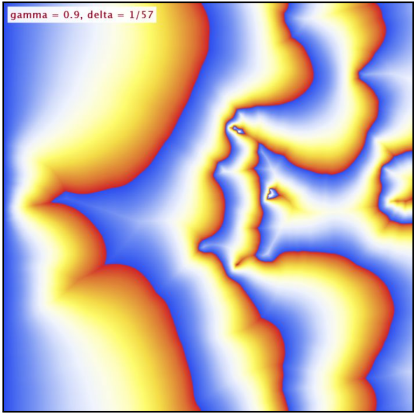

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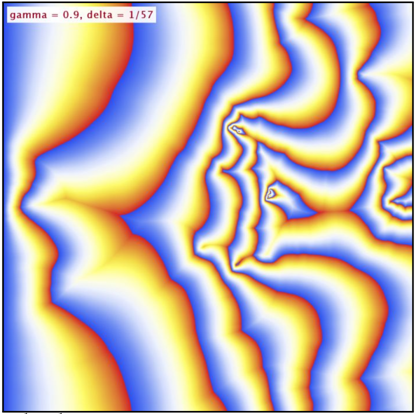

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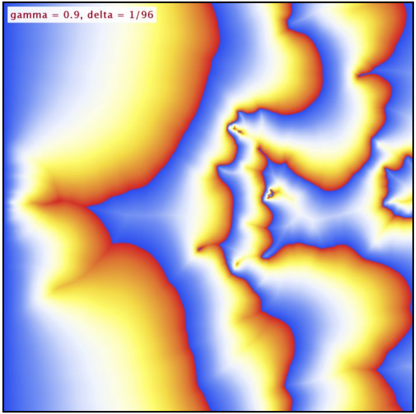

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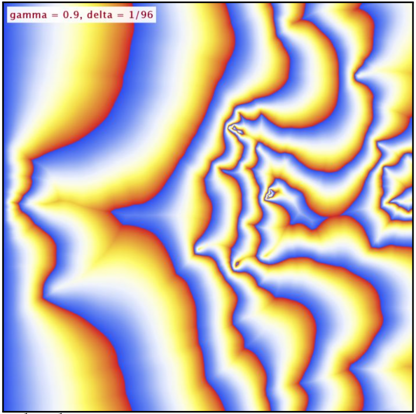

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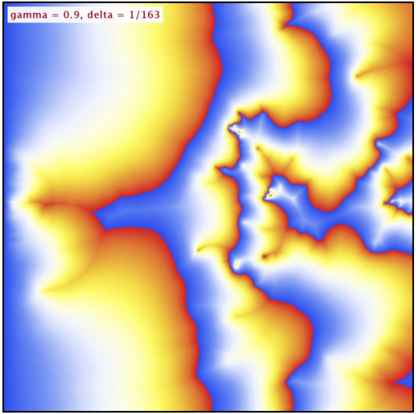

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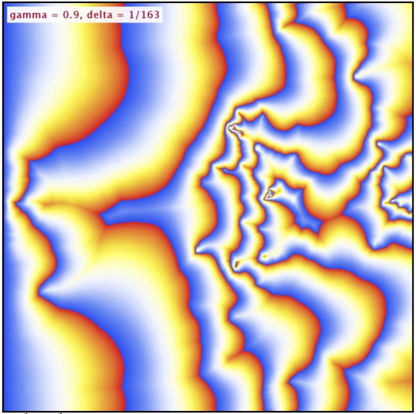

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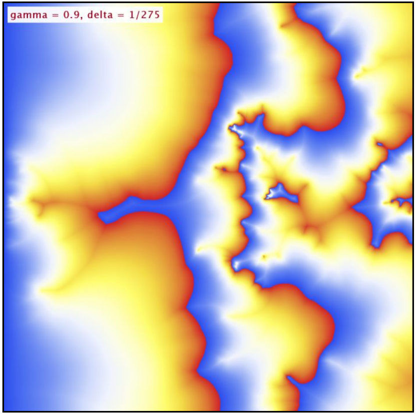

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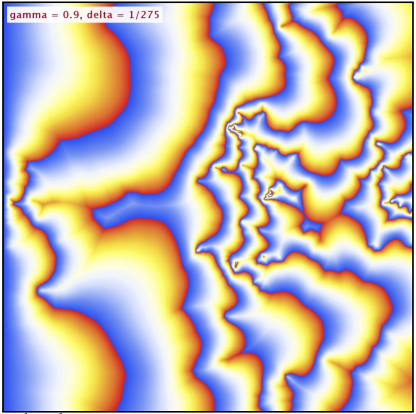

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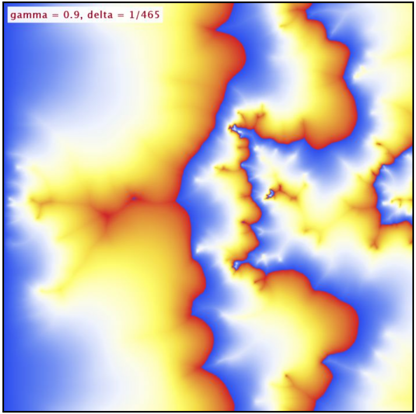

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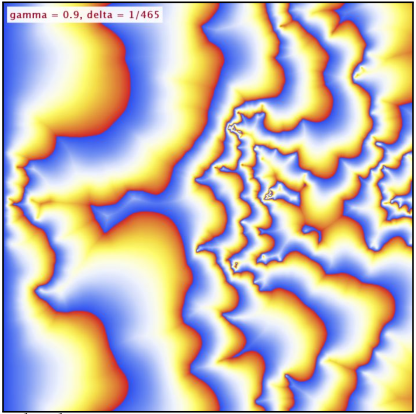

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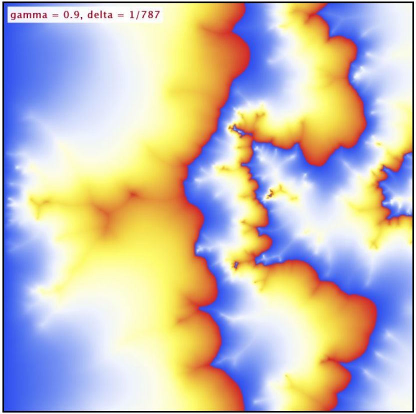

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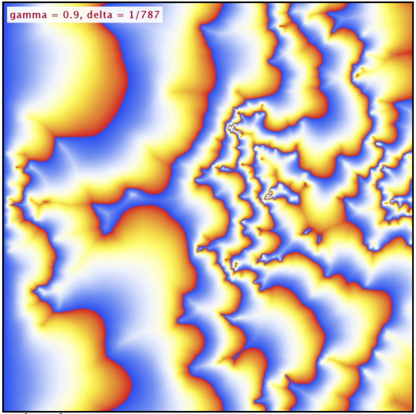

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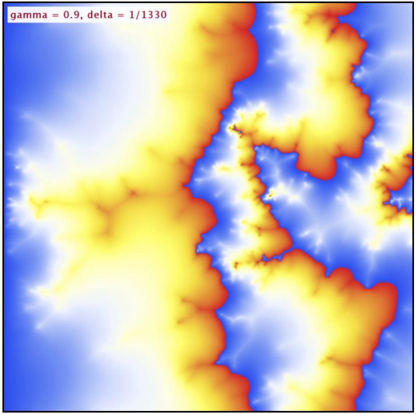

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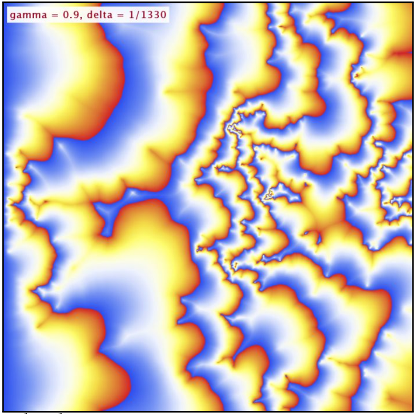

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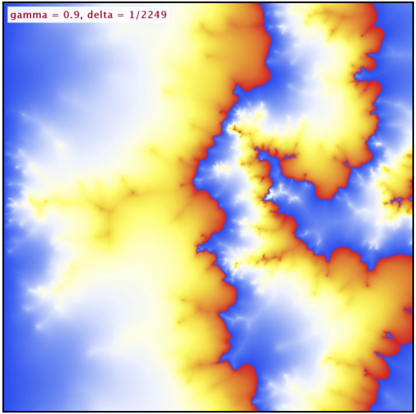

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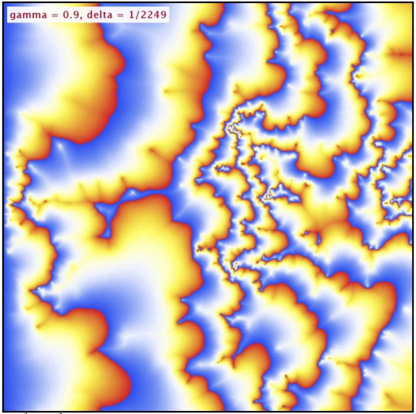

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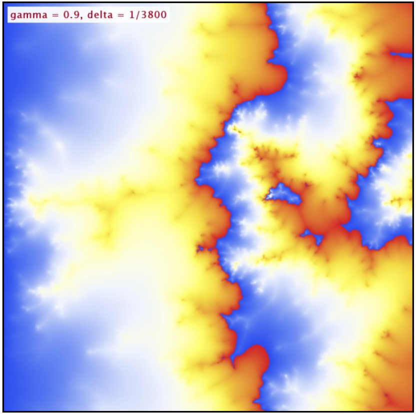

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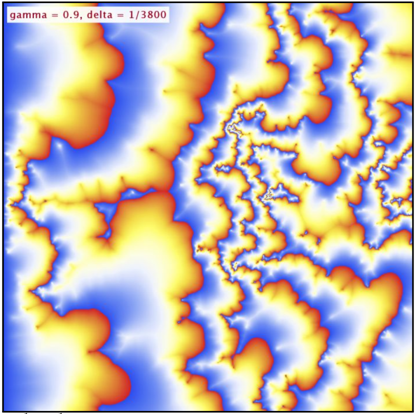

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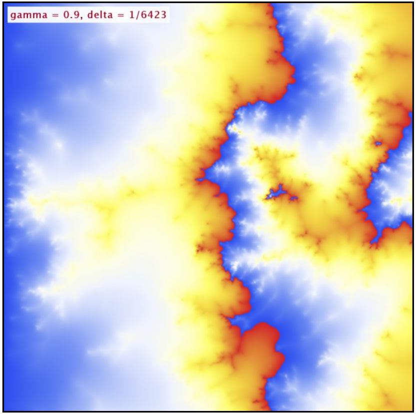

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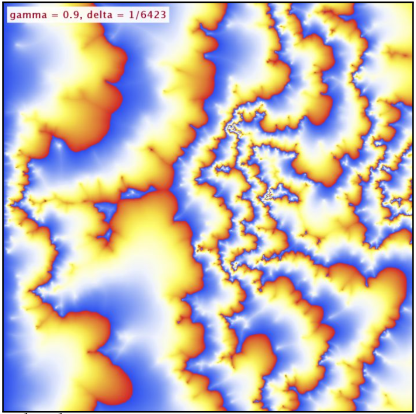
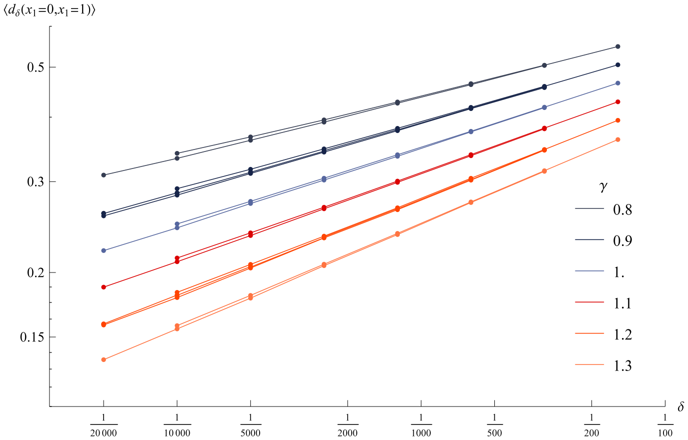
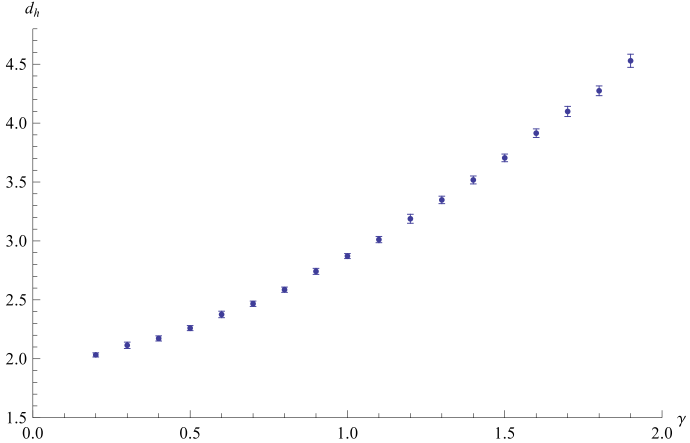


$\delta^{\frac{1}{d_h}-\frac{1}{2}} d_\delta(x, \{x_1 = 0\}),\ \ d_h \approx 2.70$

- To extract $d_h(\gamma)$, measure the expectation value $\langle d_\delta(\{x_1 = 0\}, \{x_1 = 1\})\rangle$ of the distance between left and right border as function of δ.



- The slopes of the curves, $\langle d_\delta(\{x_1 = 0\}, \{x_1 = 1\})\rangle \propto \delta^{\frac{1}{2} - \frac{1}{d_h}}$, lead to the following estimate of the Hausdorff dimension.



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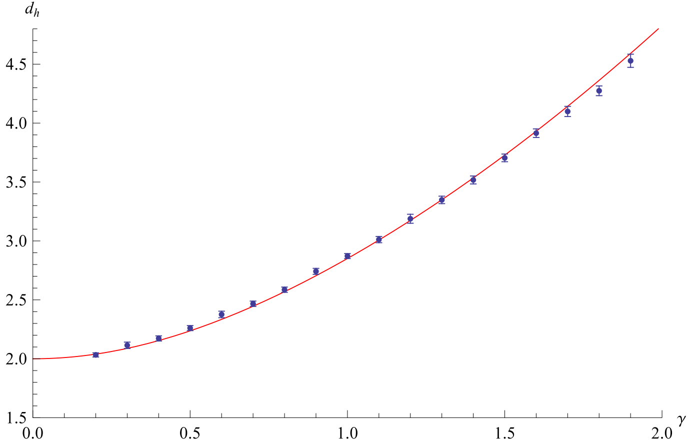
- Compare with Watabiki's formula, $d_h = 1+\frac{\gamma^2}{4}+\sqrt{1+\frac{3}{2}\gamma^2+\frac{1}{16}\gamma^4}$.



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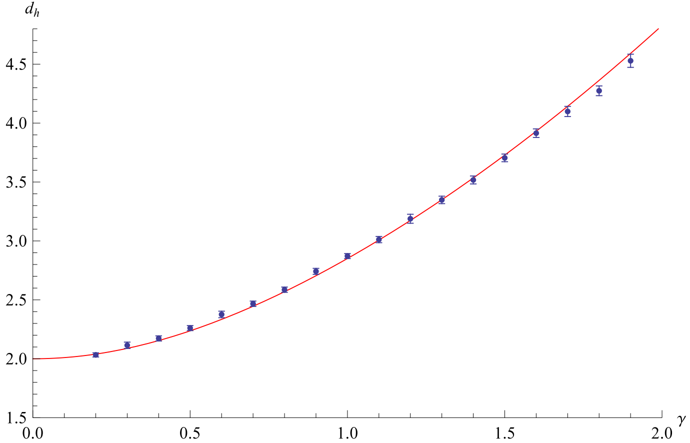
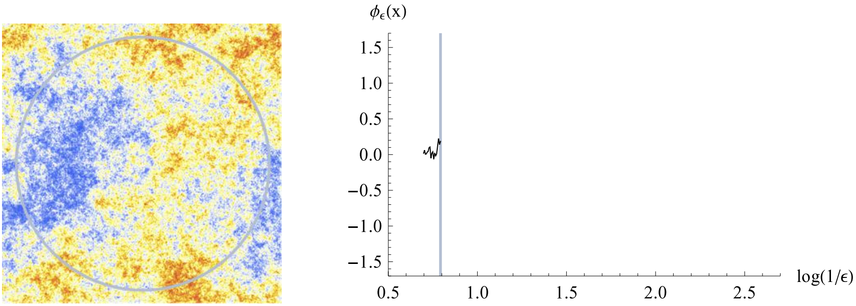
- Compare with Watabiki's formula, $d_h = 1 + \frac{\gamma^2}{4} + \sqrt{1 + \frac{3}{2}\gamma^2 + \frac{1}{16}\gamma^4}$.
- Can we understand where this formula comes from?



- Need to understand the relation $\epsilon(\delta)$. Back to the circle average $\phi_\epsilon(x)$!



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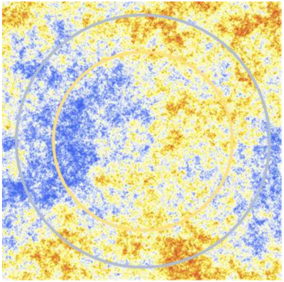

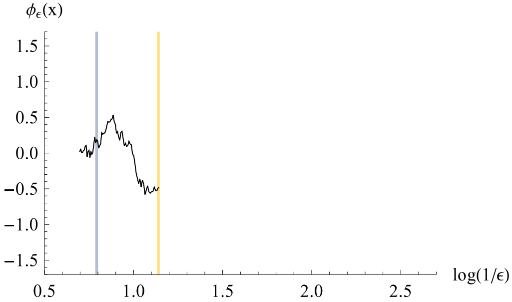

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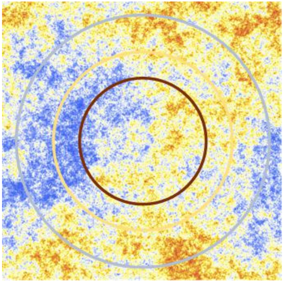

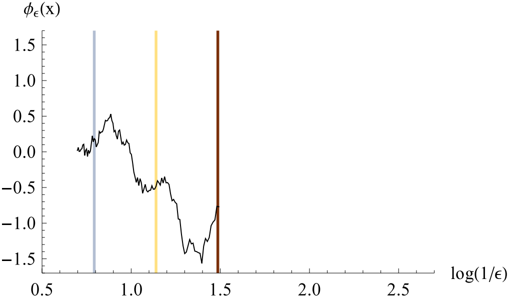

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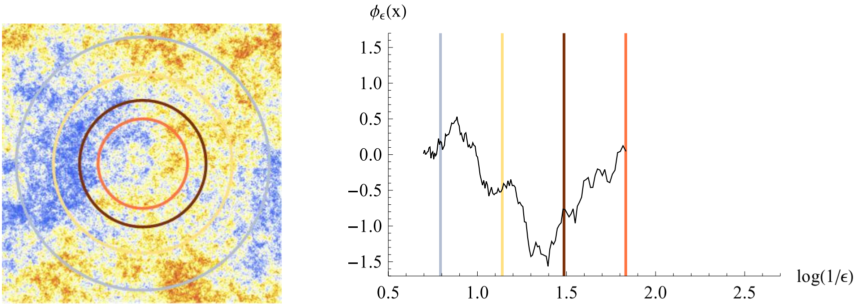

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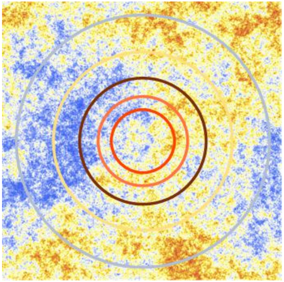

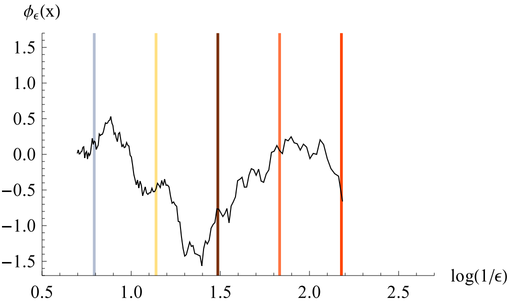

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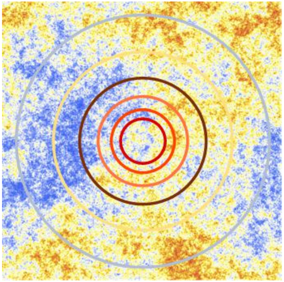

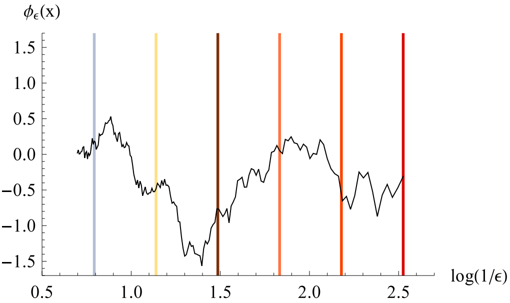

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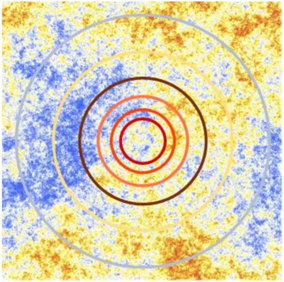

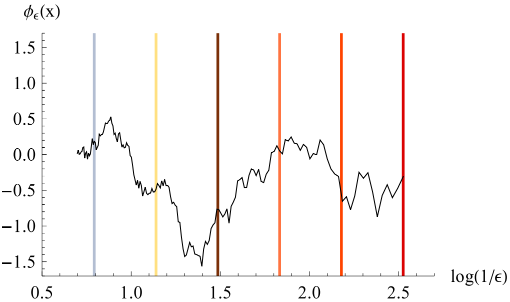

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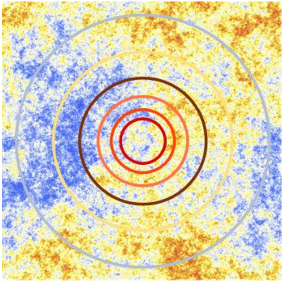

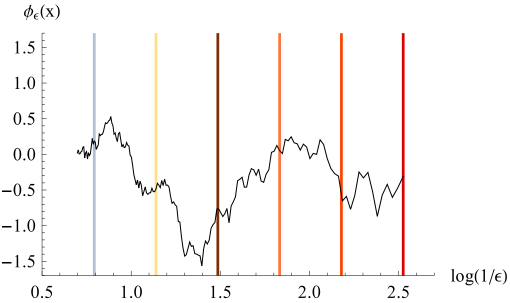

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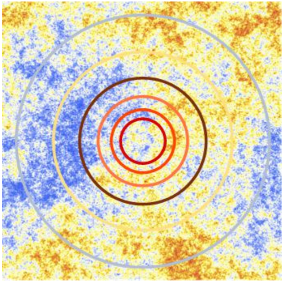
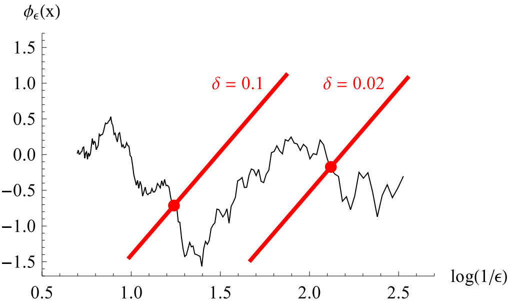
- Hence we need to solve

$$\delta = \pi\epsilon^2\epsilon^{\gamma^2/2}e^{\gamma\phi_\epsilon(x)} = \pi\epsilon^{\gamma Q}e^{\gamma\phi_\epsilon(x)}$$





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$$\left\langle \epsilon(\delta)^{2x-2} \right\rangle \propto \int dT\, e^{-(2x-2)T} P_\delta(T) \;\propto\; \delta^{\frac{1}{\gamma}(\sqrt{Q^2+4x-4}-Q)} = \delta^{\Delta_x - 1}$$

where Δ_x satisfies the famous KPZ relation [Knizhnik, Polyakov, Zamolodchikov, '88][Duplantier, Sheffield, '10]

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- In particular,

$$\left\langle \int d^2x\, e^{-\gamma\phi^\delta(x)} \right\rangle = \left\langle \frac{\pi\epsilon(\delta)^2}{\delta} \right\rangle \propto \delta^{\Delta_2 - 2}.$$

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- Hence, d_{TD} is given by Watabiki's formula,

$$d_{TD} = \frac{2}{\Delta_2 - 1} = 1 + \frac{\gamma^2}{4} + \sqrt{1 + \frac{3}{2}\gamma^2 + \frac{1}{16}\gamma^4}$$

How about the Hausdorff dimension?

- Watabiki's derivation of d_h relies on a similar derivation. [Watabiki, '93]
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- Maybe interpret differently? Also, Liouville Brownian motion under active investigation. [Garban, Rhodes, Vargas, ..., '13]

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  - Is it possible that shortest cycles and generic geodesic distances scale differently? Then $d_h = 4$ for $0 < c < 1$ is not yet ruled out, but the continuum random surface would be pinched.

Thanks! Questions? Slides available at `http://www.nbi.dk/~budd/`