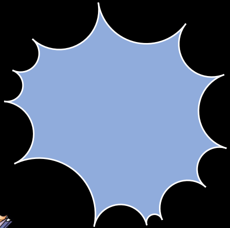
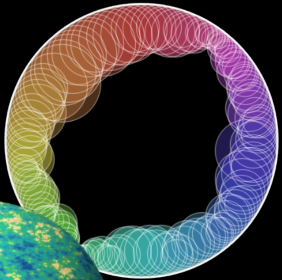
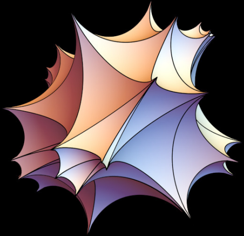
Random Geometry in Math & Physics, SCGP, Stony Brook University 16-04-2026

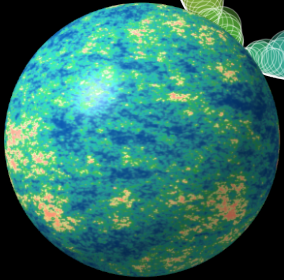
Random hyperbolic geometry and Schwarzian field theory

Timothy Budd









based on
w.i.p. with L. Chekhov
&
2508.18792 with N. Curien

Radboud University

Quantum Geometry gallery
https://qgeometry.com

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/

Euclidean Quantum Field Theories

$$Z = \int_{\{\phi:\mathcal{M}\to\Sigma\}} e^{-S(\phi)} \mathrm{D}\phi$$

Non-Gaussian QFTs constructed as (probability) measures on a function space:

1D

2D

3D

4D

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$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_Z = \mathbb{E}[\phi(x_1)\cdots\phi(x_n)]$$

and showing that they obey the QFT axioms

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- sine-Gordon$_1$

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- ϕ^4_2
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$1,000,000

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[Bauerschmidt, Losev, Wildemann, '24]

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Liouville CFT$_2$

[Kupiainen, Rhodes, Vargas,
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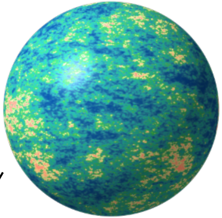
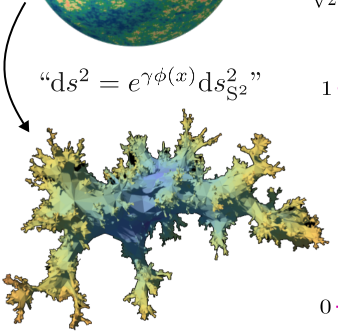
[Kupiainen, Rhodes, Vargas, David, Guillarmou, ..., '16-'20]

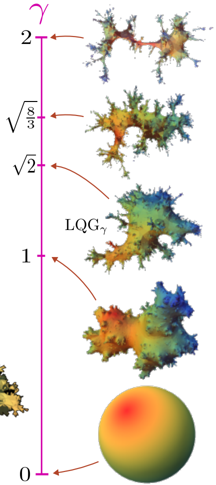
Liouville CFT$_2$: 2-dimensional quantum gravity

$$S_g(\phi) = \frac{1}{4\pi}\int_{S^2} (|\nabla \phi|_g^2 + QR_g\phi + 4\pi\mu e^{\gamma\phi})\mathrm{dVol}_g(x)$$



$$\text{“}\mathrm{d}s^2 = e^{\gamma\phi(x)}\mathrm{d}s^2_{\mathbb{S}^2}\text{”}$$





[Polyakov, Zamolodchikov, David, ..., '80s][Duplantier, Sheffield, Miller, Vargas, Rhodes, Kupiainen, Guillarmou, Gwynne, Holden, Sun, ..., '16-'26]

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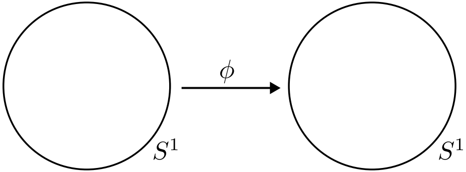
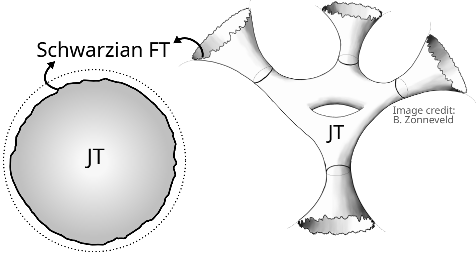
Schwarzian field theory

$$S_{\sigma^2}(\phi) = -\frac{1}{\sigma^2}\int_{S^1}\left(\frac{\phi'''(t)}{\phi'(t)} - \frac{3}{2}\frac{\phi''(t)^2}{\phi'(t)^2} + 2\pi^2\phi'(t)\right)\mathrm{d}t$$



Boundary dynamics of 2D Jackiw-Teitelboim gravity



[Sachdev, Kitaev, Maldacena, Stanford, Witten,
Suh, Verlinde, Mertens, Saad, Shenker, ..., '16 - '24]

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Approaches:

- Stochastic quantization / stochastic PDEs
- Rigorous renormalization group
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- Bijective methods (~field reparametrization)

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Today: via elementary random geometry

Hyperbolic geometry 101

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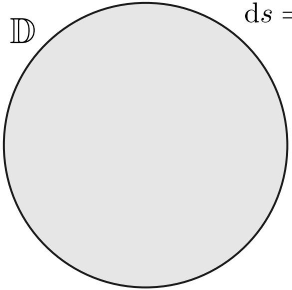
$\mathbb{D}$



$$\mathrm{d}s = 2\frac{|\mathrm{d}z|}{1-|z|^2}$$

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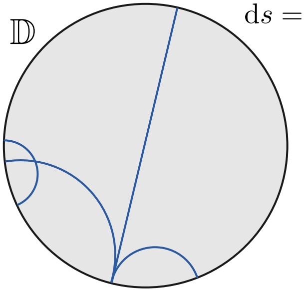

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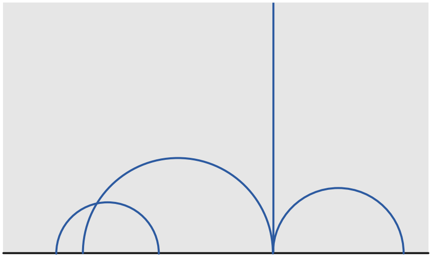

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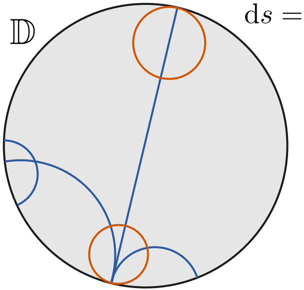

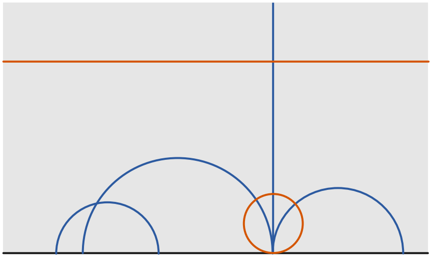

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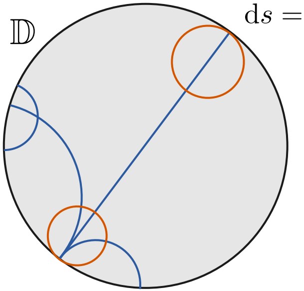

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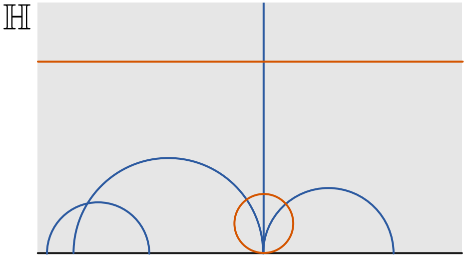

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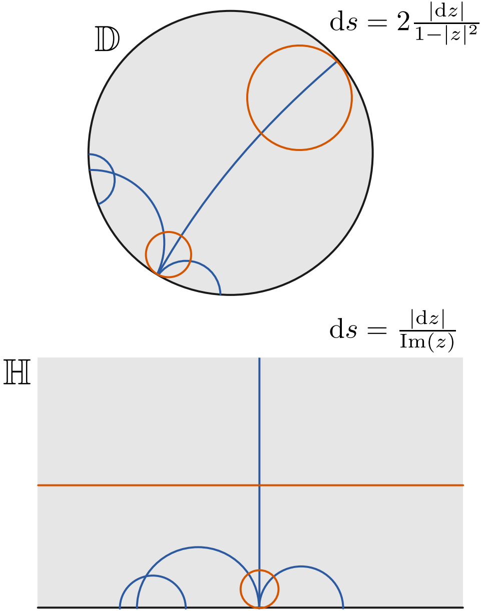

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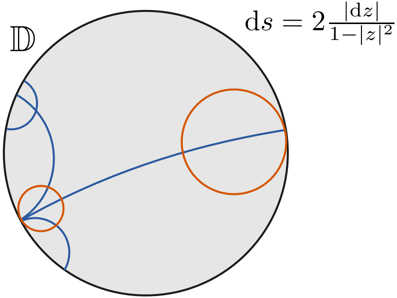

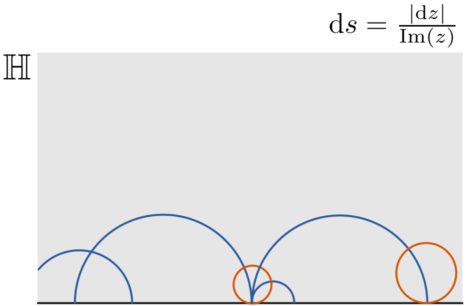

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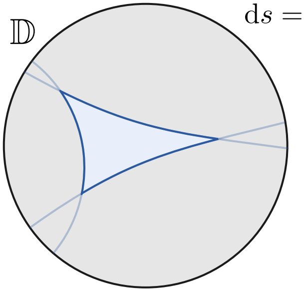

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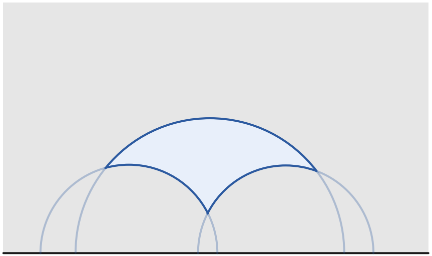

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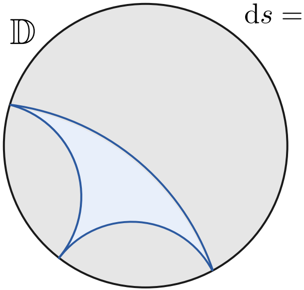

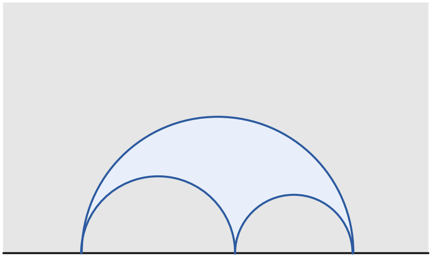

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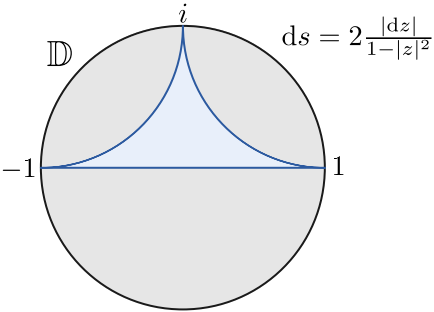

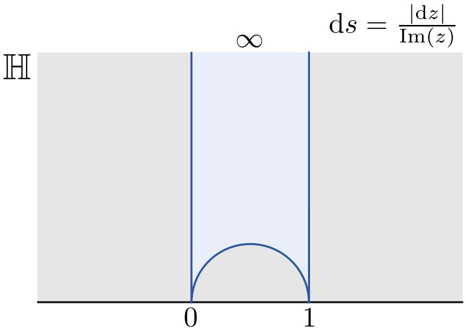

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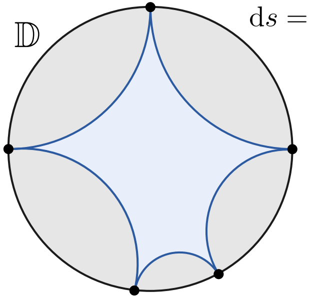

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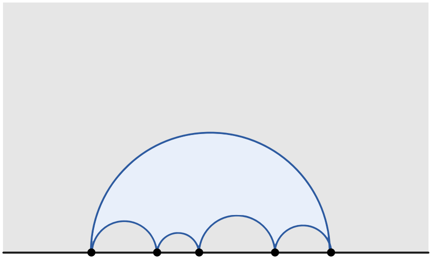

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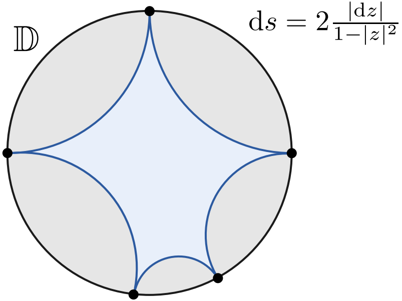

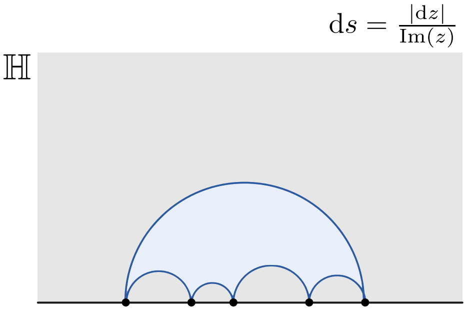

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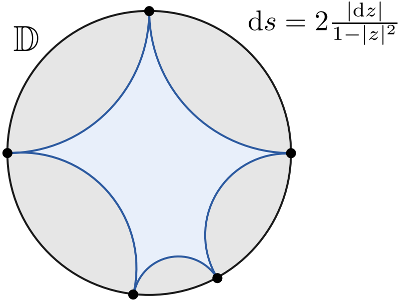
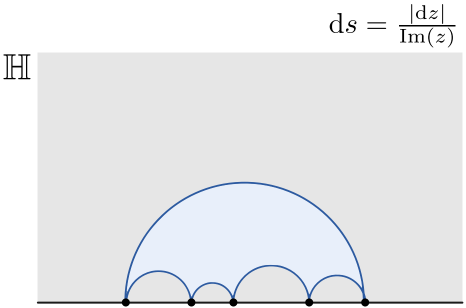
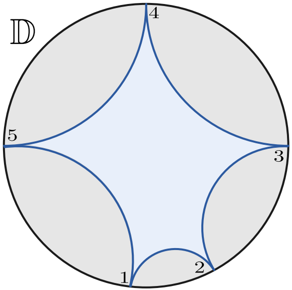
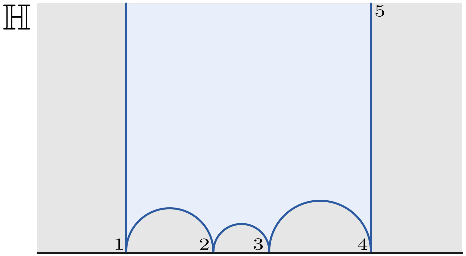
Any natural models of random ideal polygons?





A good measure on $\mathbb{M}_n$ [Chekhov, '24]





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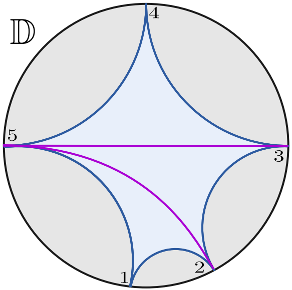

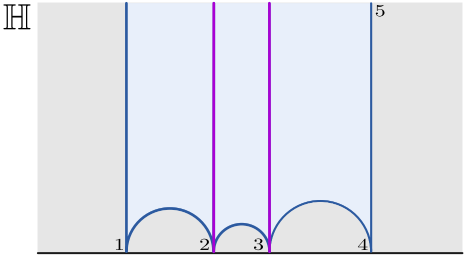

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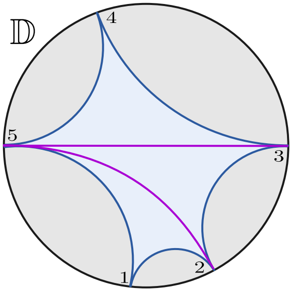

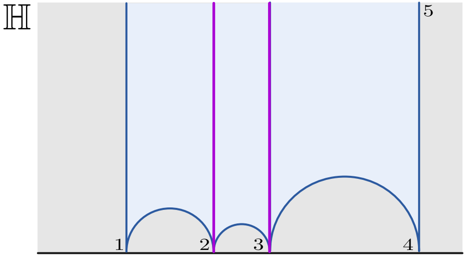

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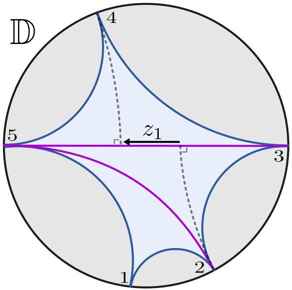

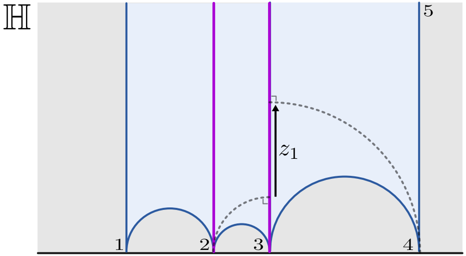

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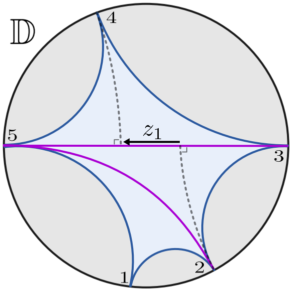

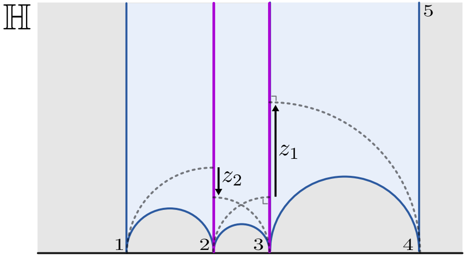

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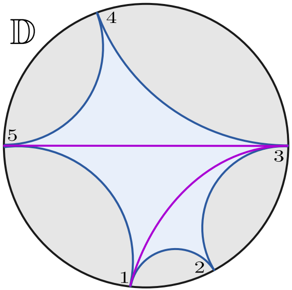

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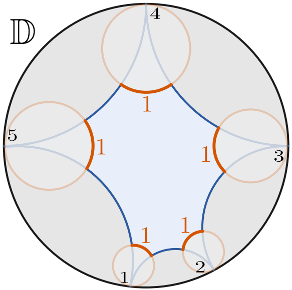

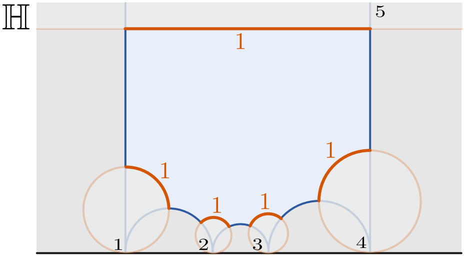

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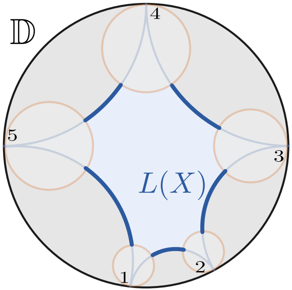

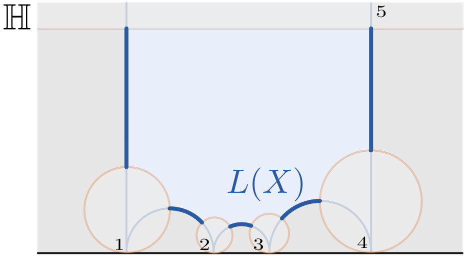

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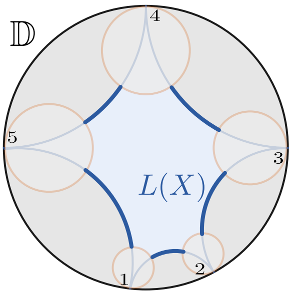

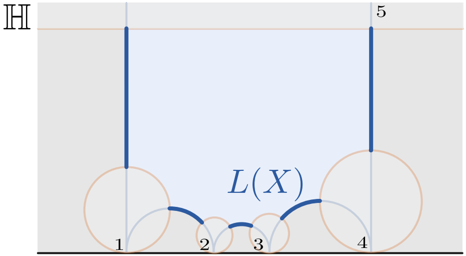

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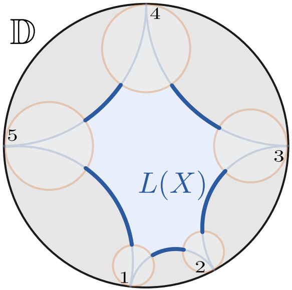

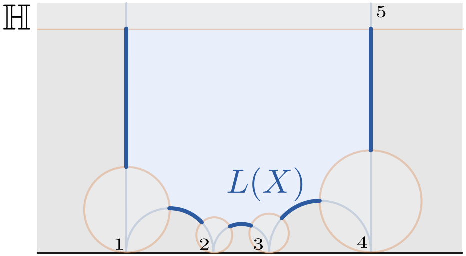

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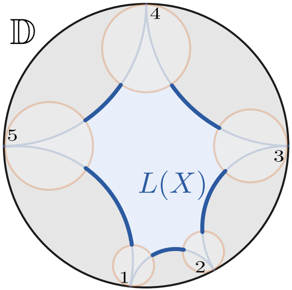

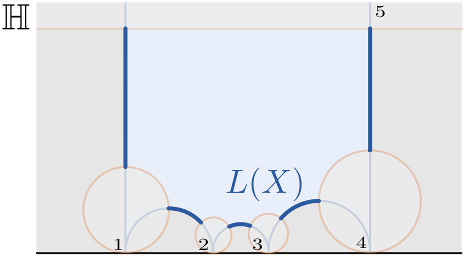

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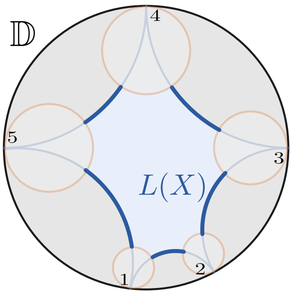

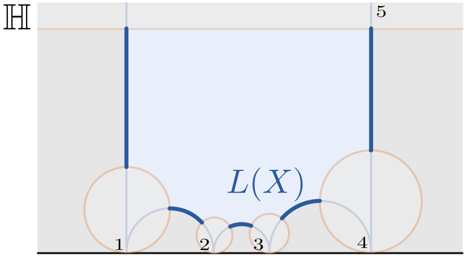

Simulations

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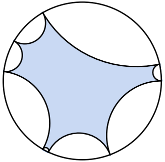
$n = 8$



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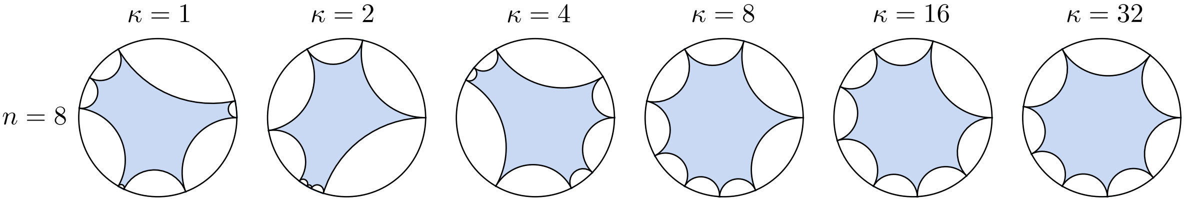

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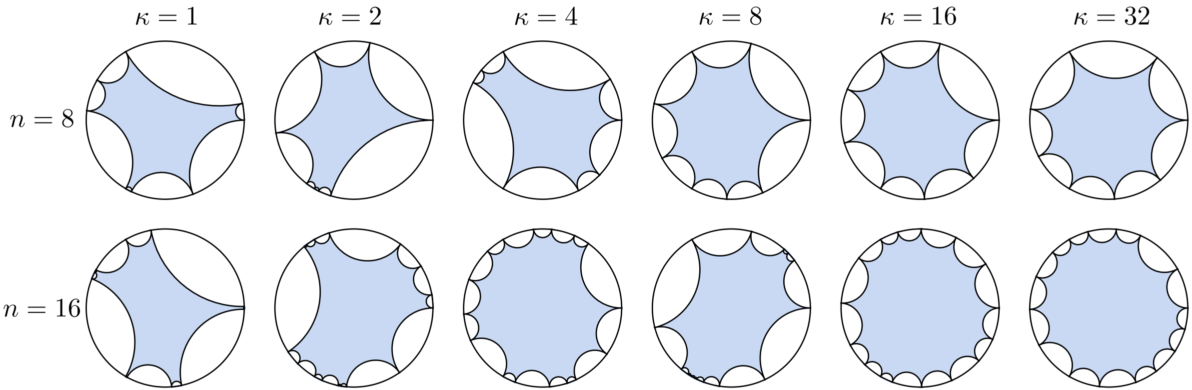

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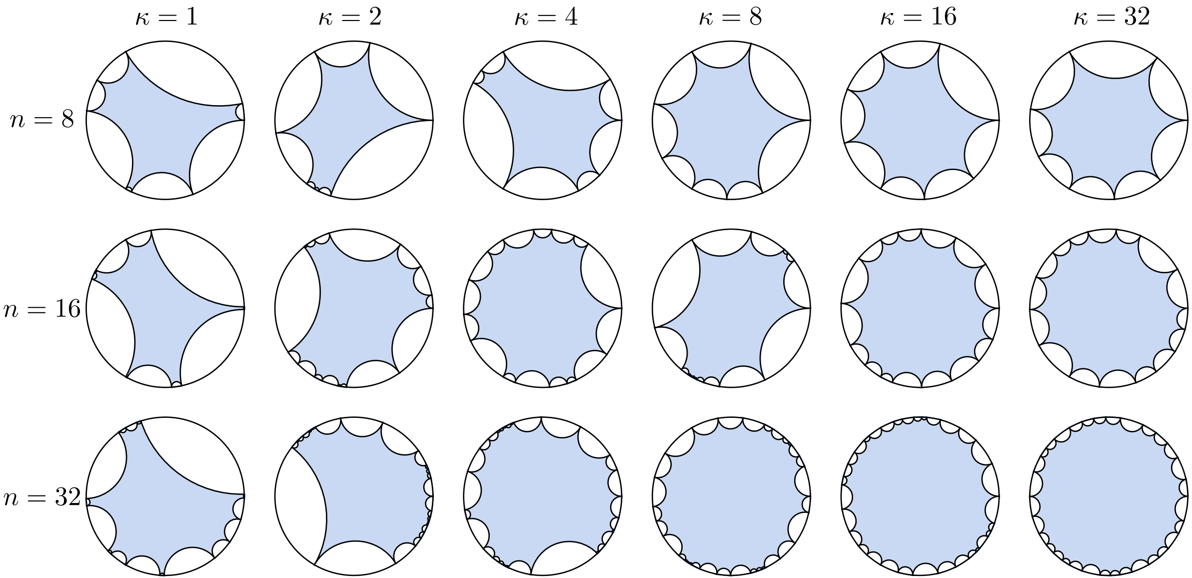
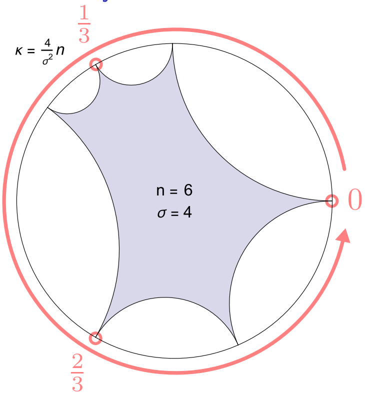
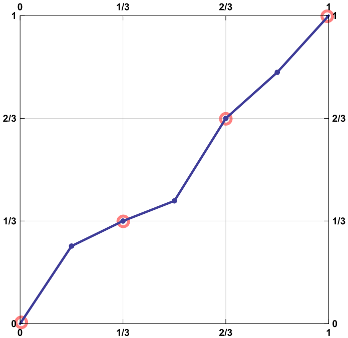


The Schwarzian field theory limit





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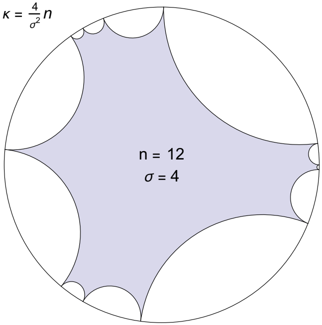

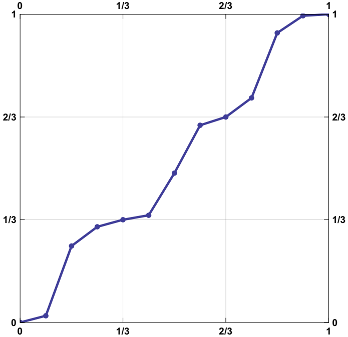

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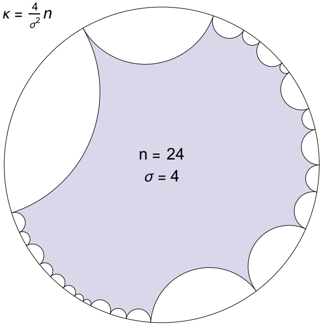

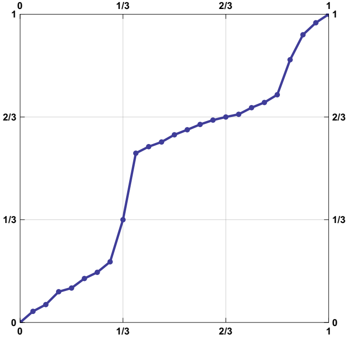

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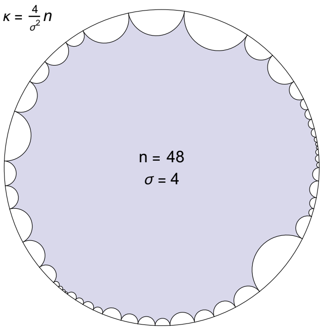

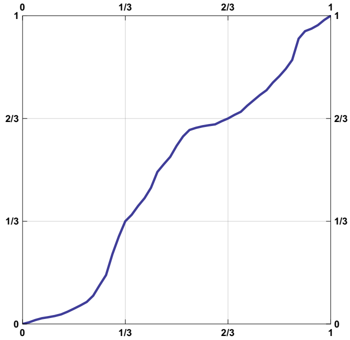

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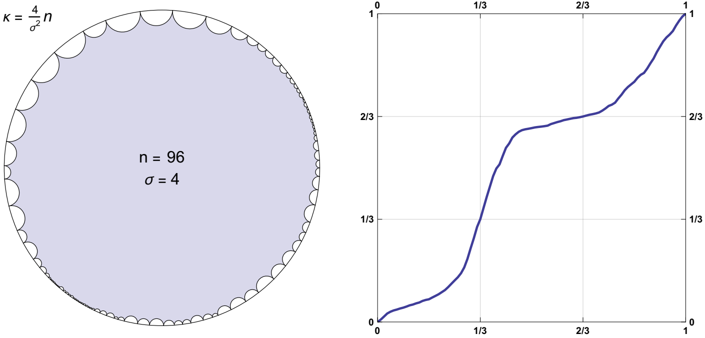

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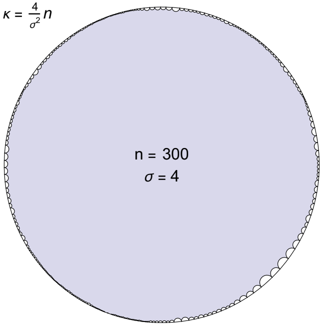

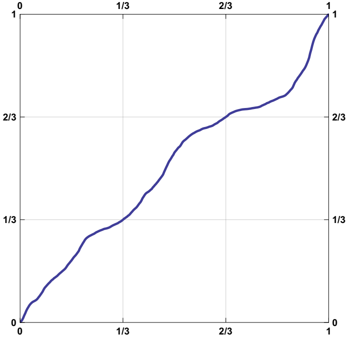

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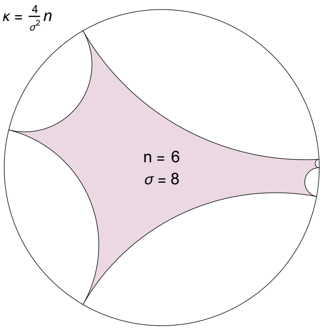

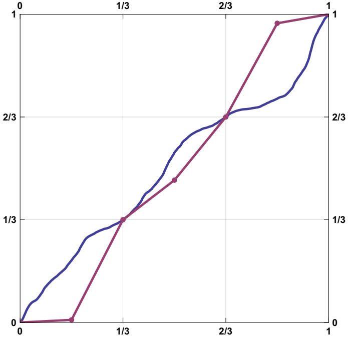

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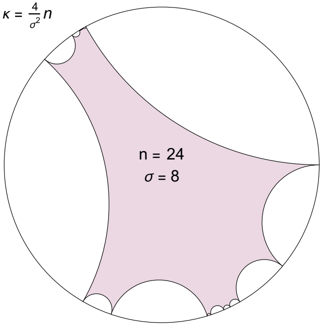

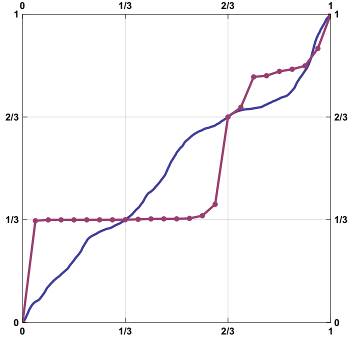

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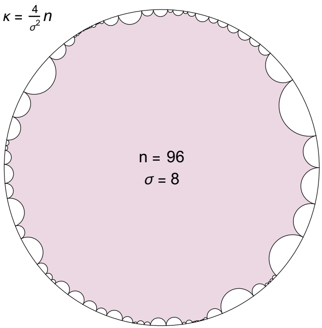

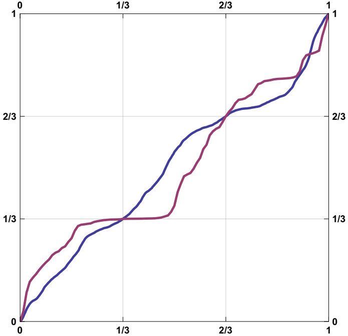

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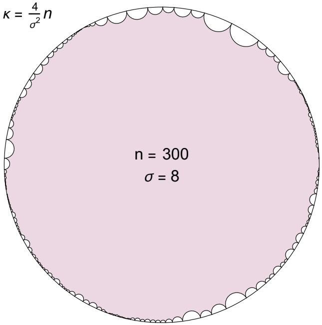

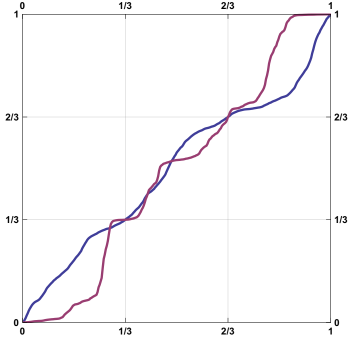

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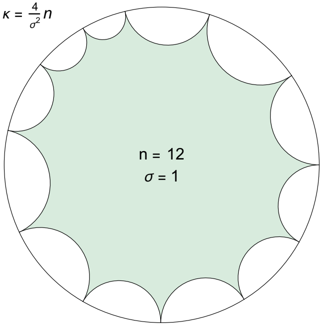

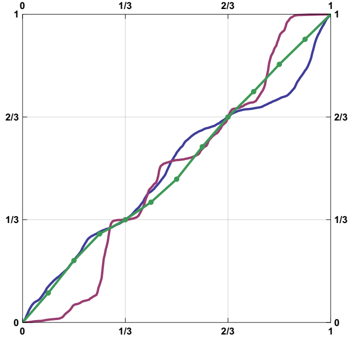

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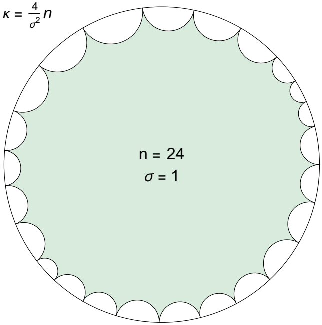

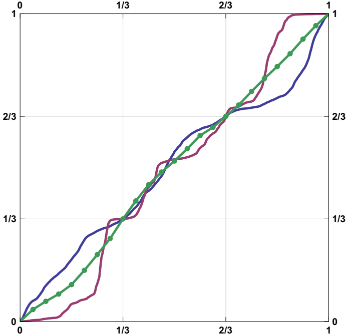

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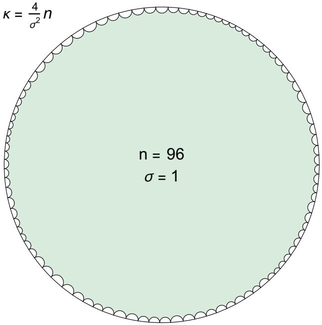

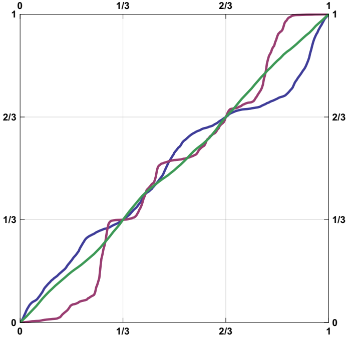

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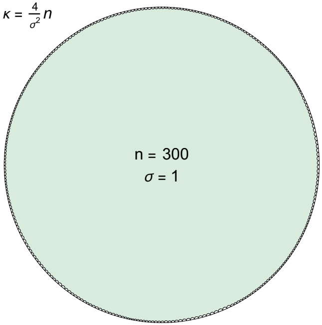

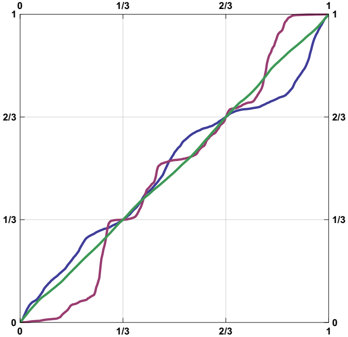

Theorem (Chekhov, TB, '26+)

If $\kappa, n \to \infty$ *and* $\frac{\kappa}{n} \to \frac{4}{\sigma^2} > 0$, *this random circle parametrization* $\Phi : \mathbb{M}_n \to \mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$ *converges to the Schwarzian field theory measure:* $\frac{\pi^2}{B(0)^n} \Phi_* \mathrm{d}\Omega_{\kappa,n} \xrightarrow[n\to\infty]{\text{weak}} \mathrm{d}\mathcal{M}_{\sigma^2}$.

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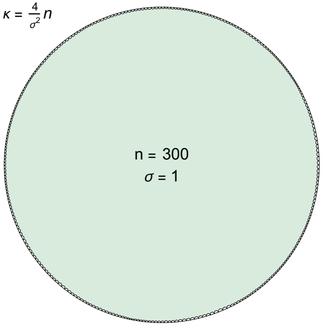

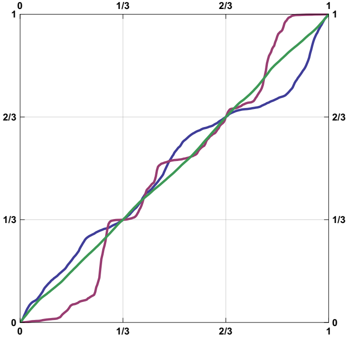

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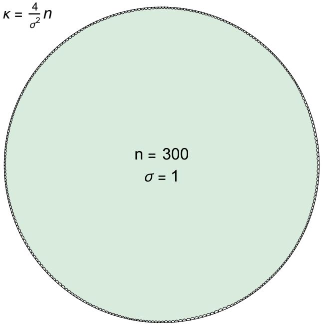
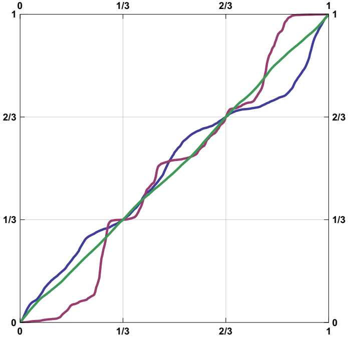
$$\frac{\pi^2}{B(0)^n}\int_{\mathbb{M}_n} \mathrm{d}\Omega_\kappa(X) = \int_0^\infty \tau \sinh(2\pi\tau) \left(\frac{B(\tau)}{B(0)}\right)^n \mathrm{d}\tau$$

The Schwarzian field theory limit



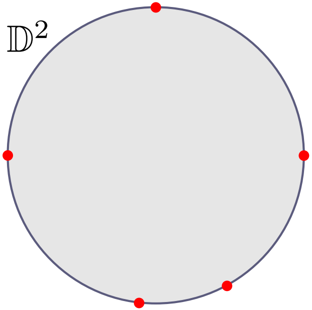


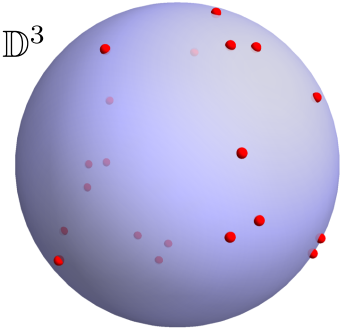
Theorem (Chekhov, TB, '26+)

If $\kappa, n \to \infty$ and $\frac{\kappa}{n} \to \frac{4}{\sigma^2} > 0$, this random circle parametrization $\Phi : \mathbb{M}_n \to \mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$ converges to the Schwarzian field theory measure: $\frac{\pi^2}{B(0)^n} \Phi_* \mathrm{d}\Omega_{\kappa,n} \xrightarrow[n\to\infty]{\text{weak}} \mathrm{d}\mathcal{M}_{\sigma^2}$.

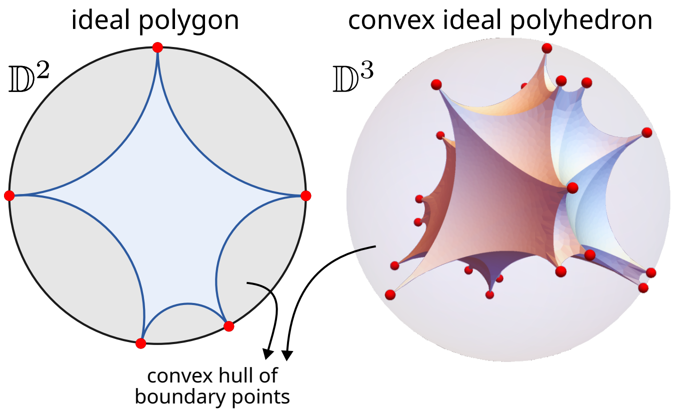
$$\frac{\pi^2}{B(0)^n} \int_{\mathbb{M}_n} \mathrm{d}\Omega_\kappa(X) = \int_0^\infty \tau \sinh(2\pi\tau) \left| \frac{\Gamma(\frac{\kappa}{2} + i\tau)}{\Gamma(\frac{\kappa}{2})} \right|^{2n} \mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^\infty \tau \sinh(2\pi\tau) e^{-\frac{1}{2}\tau^2\sigma^2} \mathrm{d}\tau = \left(\frac{2\pi}{\sigma^2}\right)^{3/2} e^{\frac{2\pi^2}{\sigma^2}}$$

Analogy with convex ideal polyhedra

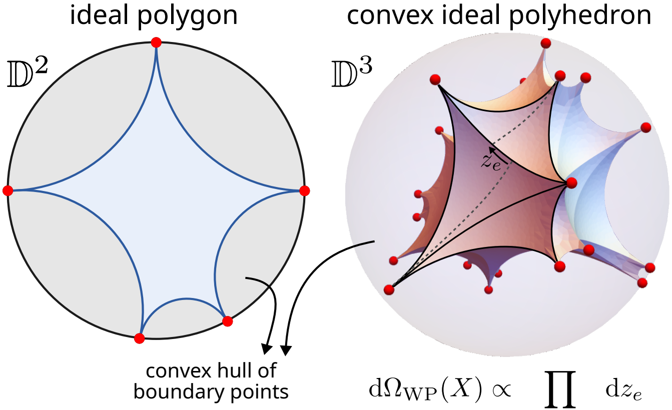




Analogy with convex ideal polyhedra

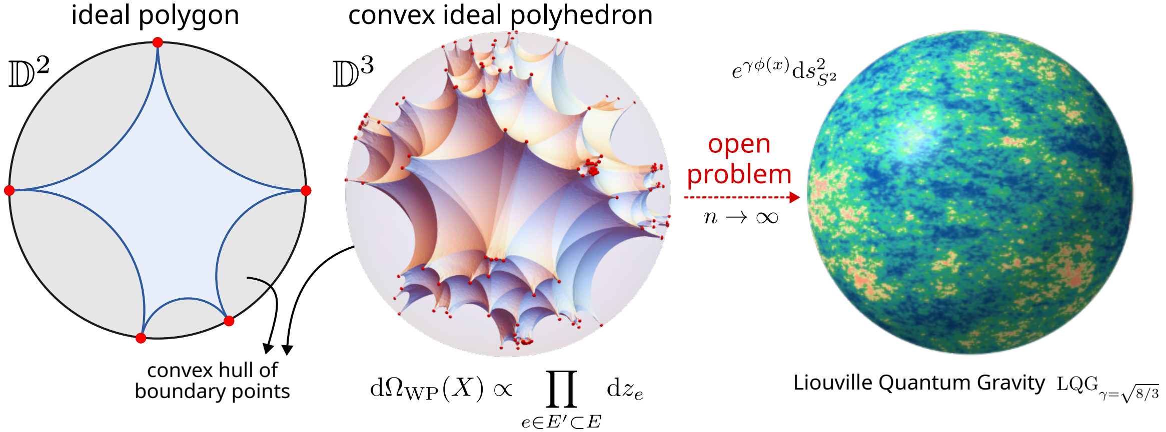


Analogy with convex ideal polyhedra



$$\mathrm{d}\Omega_{\mathrm{WP}}(X) \propto \prod_{e \in E' \subset E} \mathrm{d}z_e$$

Analogy with convex ideal polyhedra

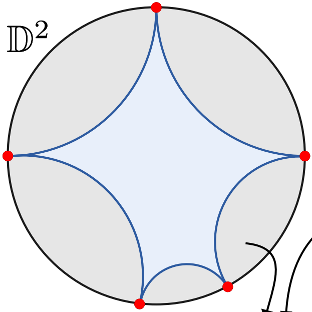


Analogy with convex ideal polyhedra

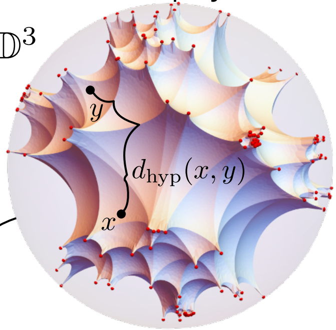
[Sheffield, Gwynne, Miller, Ding, Dubedat, Dunlap, Falconet, ..., '19]

$$d_\phi^\varepsilon(x,y) = \inf_{\Gamma: x\to y} \int \left(e^{\gamma\phi_\varepsilon(\Gamma(t))}\right)^{\frac{1}{d_\gamma}} |\Gamma'(t)| \mathrm{d}t$$

ideal polygon



convex ideal polyhedron

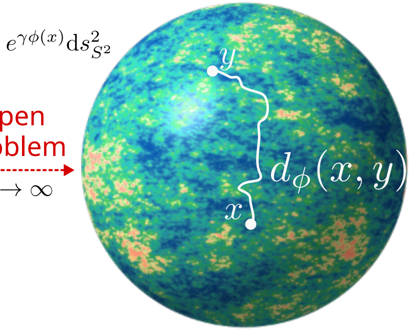


convex hull of boundary points

$$\mathrm{d}\Omega_{\mathrm{WP}}(X) \propto \prod_{e\in E'\subset E} \mathrm{d}z_e$$

open problem

$n \to \infty$



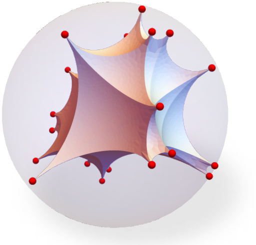
Liouville Quantum Gravity $\mathrm{LQG}_{\gamma=\sqrt{8/3}}$

Theorem (TB, Curien, '25)

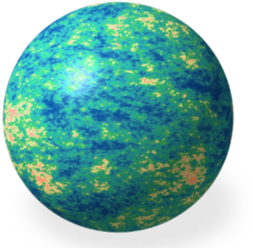
The intrinsic metric of the polyhedron boundary ∂X is the "Weil-Petersson random surface" and

$$\left(\partial X, \frac{1}{c_{\mathrm{WP}}} n^{-\frac{1}{4}} d_{\mathrm{hyp}}(\cdot,\cdot)\right) \xrightarrow[n\to\infty]{\text{law}} \text{Brownian sphere} \overset{\text{law}}{=} (S^2, d_\phi(\cdot,\cdot)) \text{ of } \mathrm{LQG}_{\gamma=\sqrt{\frac{8}{3}}}$$

convex ideal polyhedra



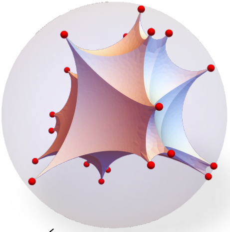
Liouville $_{\gamma=\sqrt{8/3}}$



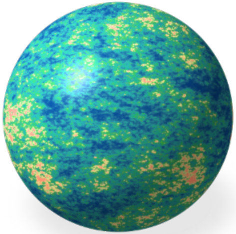
"bijective approach"

n-cusp hyperbolic spheres

convex ideal polyhedra



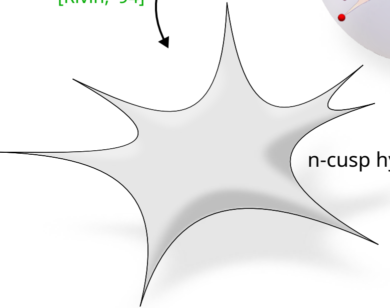
Liouville$_{\gamma=\sqrt{8/3}}$



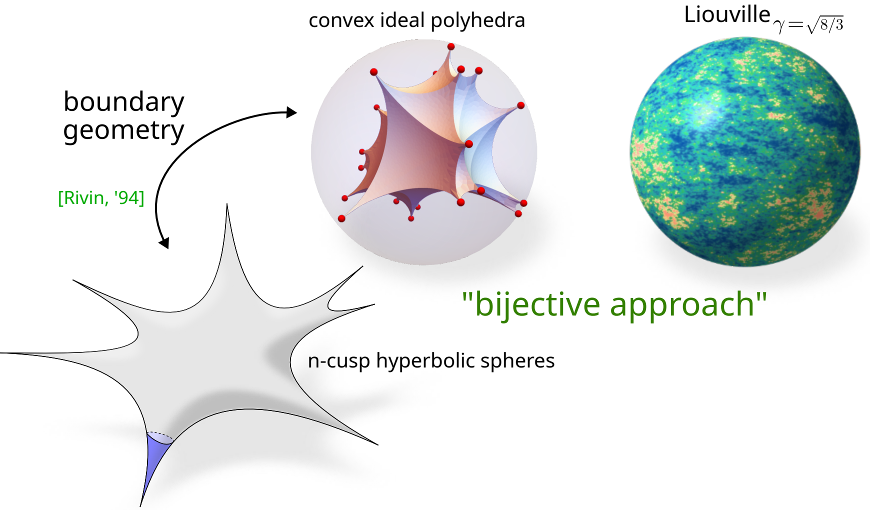
boundary geometry

[Rivin, '94]

"bijective approach"



n-cusp hyperbolic spheres


convex ideal polyhedra
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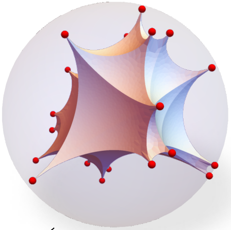

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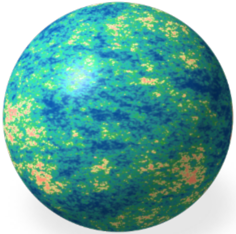

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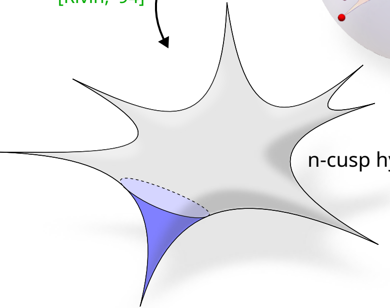

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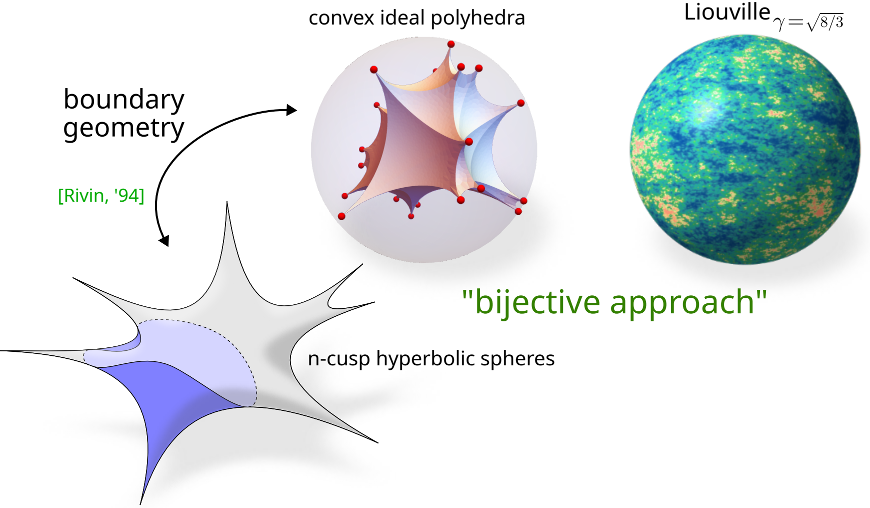

convex ideal polyhedra
Liouville$_{\gamma=\sqrt{8/3}}$
boundary geometry
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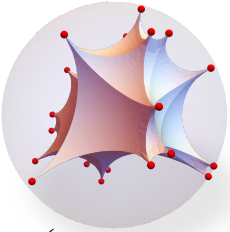

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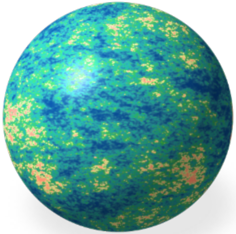
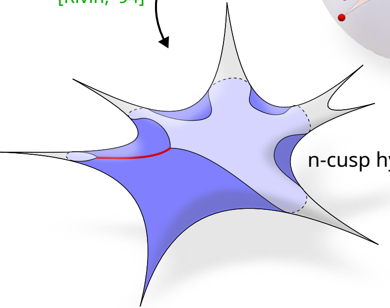


boundary geometry

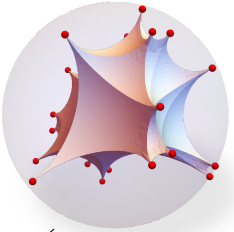
[Rivin, '94]



n-cusp hyperbolic spheres

"bijective approach"

convex ideal polyhedra



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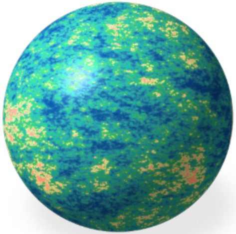

boundary geometry

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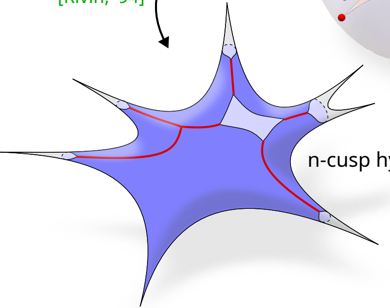
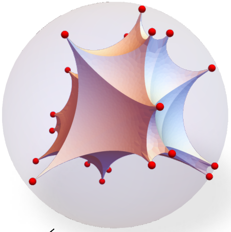


"bijective approach"

n-cusp hyperbolic spheres

convex ideal polyhedra



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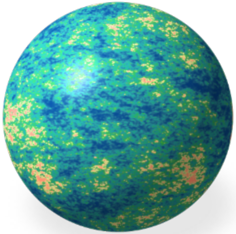
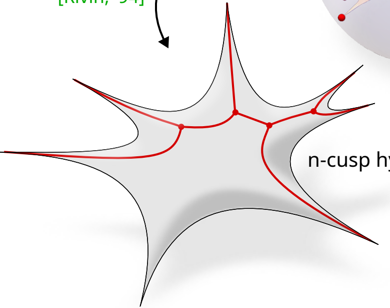


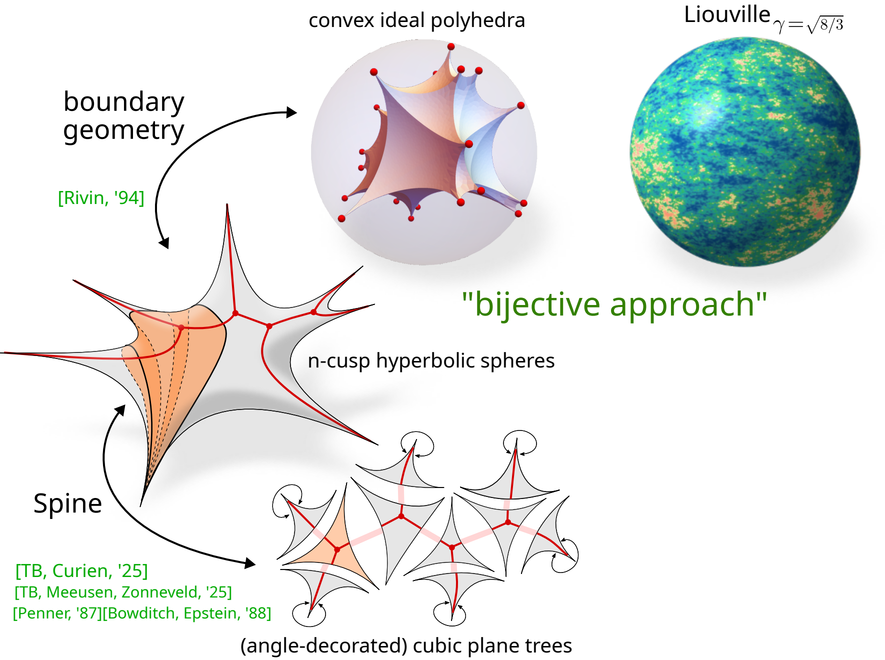
boundary geometry

[Rivin, '94]



"bijective approach"

n-cusp hyperbolic spheres


convex ideal polyhedra
Liouville $_{\gamma=\sqrt{8/3}}$
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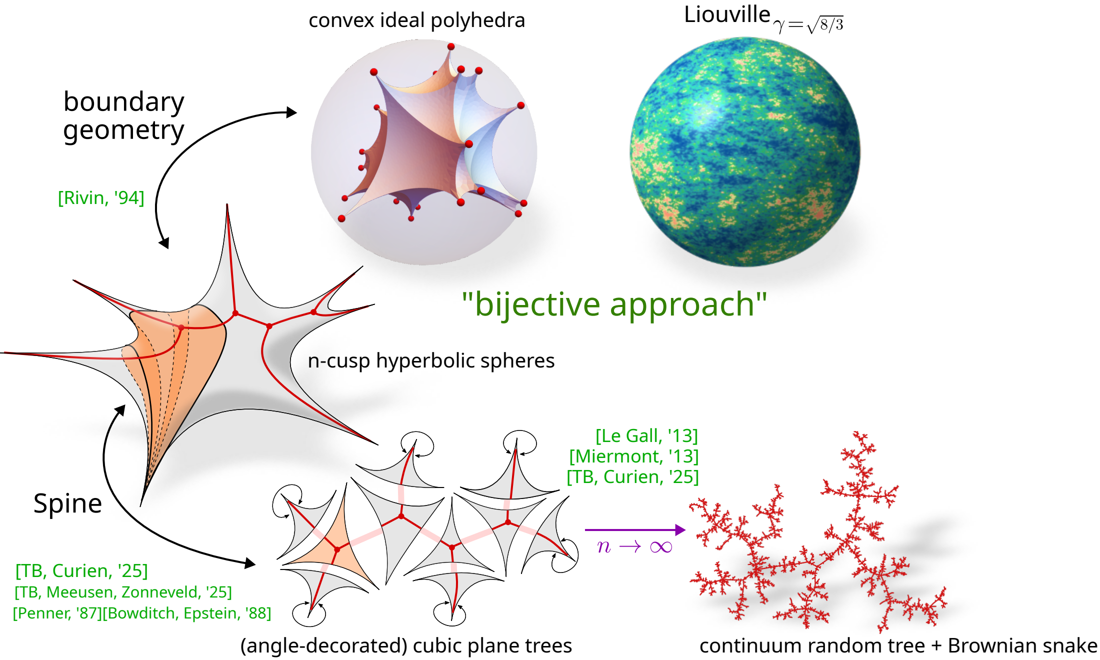
[Rivin, '94]
"bijective approach"
n-cusp hyperbolic spheres
Spine
[TB, Curien, '25]
[TB, Meeusen, Zonneveld, '25]
[Penner, '87][Bowditch, Epstein, '88]
(angle-decorated) cubic plane trees


convex ideal polyhedra
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[Le Gall, '13]
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$n \to \infty$
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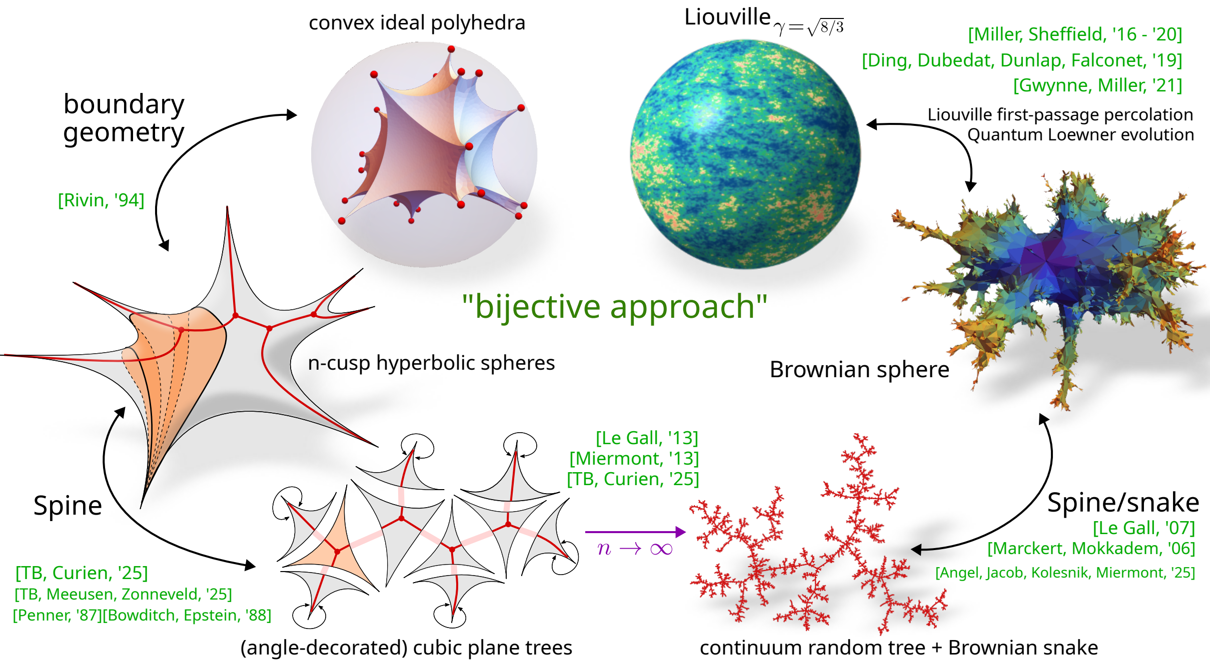
[Penner, '87][Bowditch, Epstein, '88]
(angle-decorated) cubic plane trees
continuum random tree + Brownian snake


convex ideal polyhedra
Liouville$_{\gamma=\sqrt{8/3}}$
[Miller, Sheffield, '16 - '20]
[Ding, Dubedat, Dunlap, Falconet, '19]
[Gwynne, Miller, '21]
Liouville first-passage percolation
Quantum Loewner evolution
boundary geometry
[Rivin, '94]
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Brownian sphere
[Le Gall, '13]
[Miermont, '13]
[TB, Curien, '25]
Spine
Spine/snake
[Le Gall, '07]
[Marckert, Mokkadem, '06]
[Angel, Jacob, Kolesnik, Miermont, '25]
$n \to \infty$
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continuum random tree + Brownian snake

What is Schwarzian field theory?

- First encountered in the low-energy limit of the Sachdev-Ye-Kitaev model
  [Sachdev, Ye, '93] [Sachdev, '10] [Maldacena, Stanford, '16] [Kitaev, Suh, '18]

$$H_{\mathrm{SYK}} = -\sum_{1 \le i_1 \le i_2 \le i_3 \le i_4 \le N} J_{i_1 i_2 i_3 i_4}\, \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4}\,.$$

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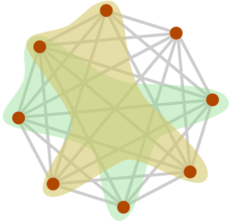

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Majorana fermions

i.i.d. random interactions

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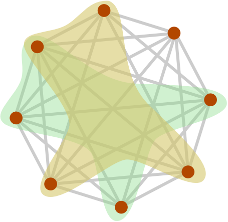

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Majorana fermions (ψ); i.i.d. random interactions (J)

- Formally, field theory of $\phi \in \mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$ for the action

$$S_{\sigma^2}(\phi) = -\frac{1}{\sigma^2} \int_{S^1} \left(\{\phi, t\} + 2\pi^2 \phi'(t)^2 \right) \mathrm{d}t, \quad \{\phi, t\} = \left(\frac{\phi''(t)}{\phi'(t)} \right)' - \frac{1}{2} \left(\frac{\phi''(t)}{\phi'(t)} \right)^2$$

with path integral

$$Z_{\sigma^2} = \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} \mathrm{d}\mathcal{M}_{\sigma^2}(\phi) \text{ “}=\text{” } \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} e^{-S_{\sigma^2}(\phi)} \frac{\prod_t \frac{\mathrm{d}\phi(t)}{\phi'(t)}}{\mathrm{PSL}(2,\mathbb{R})}$$

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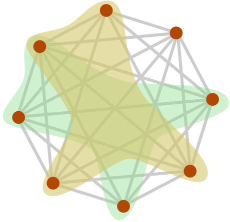

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Majorana fermions ($\psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4}$)

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Schwarzian derivative ($\{\phi, t\}$)

with path integral

$$Z_{\sigma^2} = \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} \mathrm{d}\mathcal{M}_{\sigma^2}(\phi) \text{ “}=\text{” } \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} e^{-S_{\sigma^2}(\phi)} \frac{\prod_t \frac{\mathrm{d}\phi(t)}{\phi'(t)}}{\mathrm{PSL}(2,\mathbb{R})}$$

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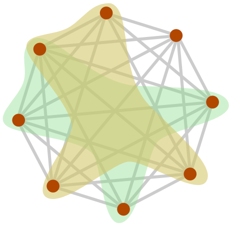

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ill-defined for $\phi \in \mathrm{Diff}^1(S^1)$ ($S_{\sigma^2}(\phi)$)

“Haar measure on $\mathrm{Diff}^1(S^1)$” ($\prod_t \frac{\mathrm{d}\phi(t)}{\phi'(t)}$)

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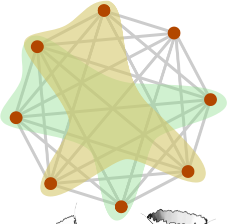

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$$Z_{\sigma^2} = \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} \mathrm{d}\mathcal{M}_{\sigma^2}(\phi) \text{ “}={\text{”}} \int_{\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})} e^{-S_{\sigma^2}(\phi)} \frac{\prod_t \frac{\mathrm{d}\phi(t)}{\phi'(t)}}{\mathrm{PSL}(2,\mathbb{R})}$$

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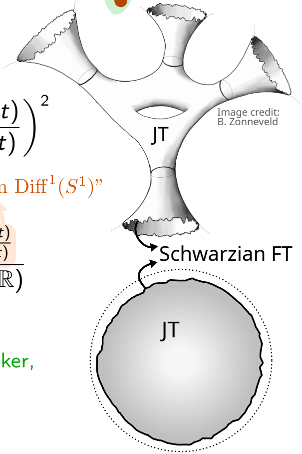

- Also captures the boundary dynamics of Jackiw-Teitelboim (JT) gravity.
  [Blommaert, Mertens, Verschelde, '18][Iliesu, Pufu, Verlinde, Wang, '19][Saad, Shenker, Stanford, '19]

A rigorous probabilistic construction [Bauerschmidt, Losev, Wildemann, '24] [Belokurov, Shavgulidze, '17]

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Theorem (Bauerschmidt, Losev, Wildemann, '24)

Up to normalization, there exists a unique $\mathrm{PSL}(2,\mathbb{R})$*-invariant measure* $\widetilde{\mathcal{M}}_{\sigma^2}$ *on* $\mathrm{Diff}^1(S^1)$ *such that*

$$\mathrm{d}\widetilde{\mathcal{M}}_{\sigma^2}(\psi\circ\phi) = \exp\left[S_{\sigma^2}(\phi) - S_{\sigma^2}(\psi\circ\phi)\right]\,\mathrm{d}\widetilde{\mathcal{M}}_{\sigma^2}(\phi) \quad \textit{for all } \psi \in \mathrm{Diff}^3(S^1).$$

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It is related to a Brownian bridge measure $\mathcal{B}^{0,1}_{\sigma^2}$ *via Malliavin–Shavgulidze map* $\phi(t) = \mathrm{P}\xi(t) + \phi(0)$,

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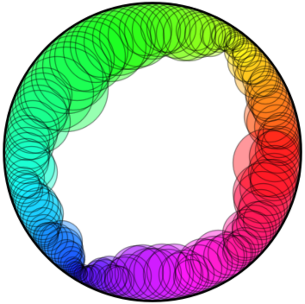
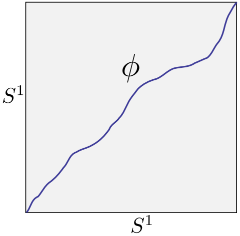
The Schwarzian field theory measure $\mathcal{M}_{\sigma^2} = \widetilde{\mathcal{M}}_{\sigma^2}/\mathrm{PSL}(2,\mathbb{R})$ obeys $Z_{\sigma^2} = \int \mathrm{d}\mathcal{M}_{\sigma^2} = \left(\frac{2\pi}{\sigma^2}\right)^{3/2} e^{\frac{2\pi^2}{\sigma^2}}$.

- Z_{σ^2} reproduces [Stanford, Witten, '17] which relied on formal Duistermaat-Heckman formula (viewing $\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$ as symplectic space and S_{σ^2} as $U(1)$-generator $\rightarrow$ Alekseev's talk).

Schwarzian field theory ⟶ Random geometry

[Epstein, '84] [Vargas Pallete, Wang, Wolfram, '25]





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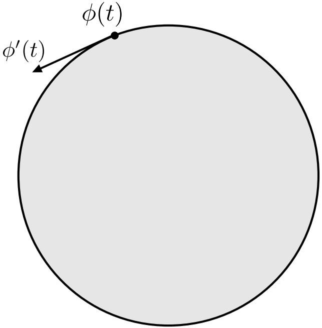
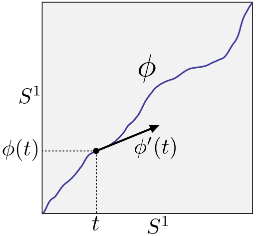
- Think of $\phi(t)$ as particle moving on $\partial\mathbb{D}$





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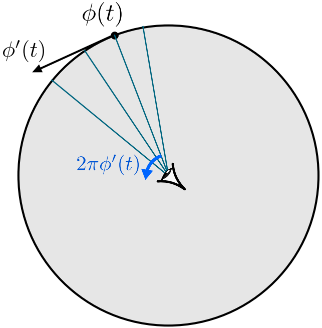

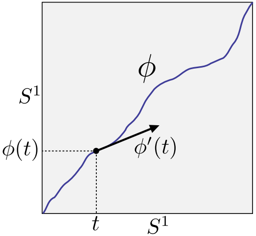

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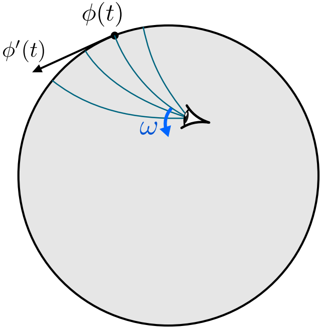

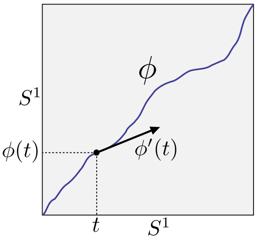

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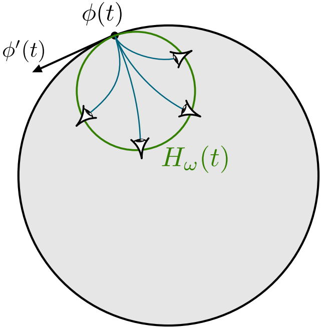

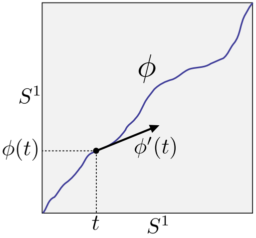

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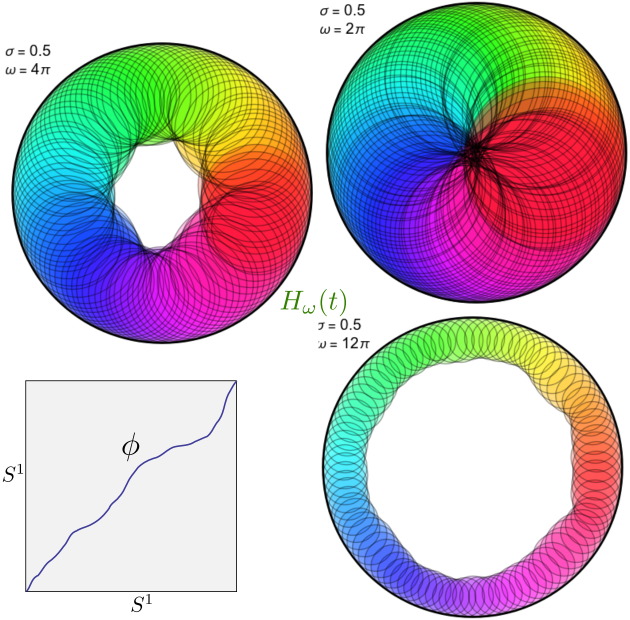

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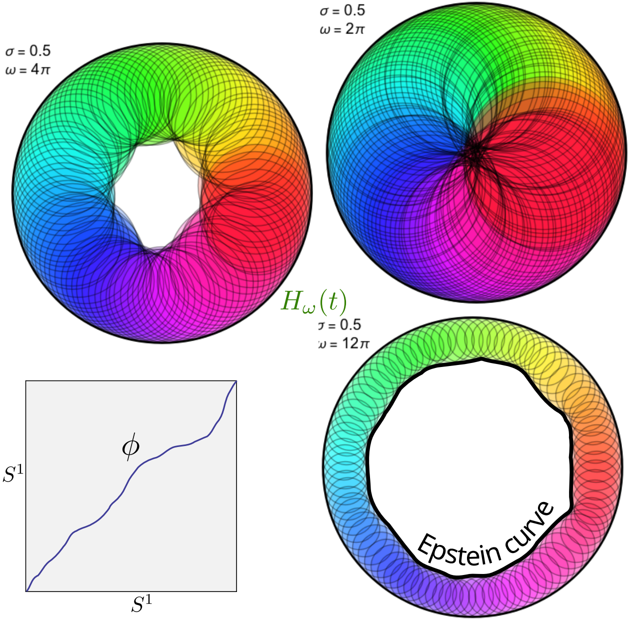

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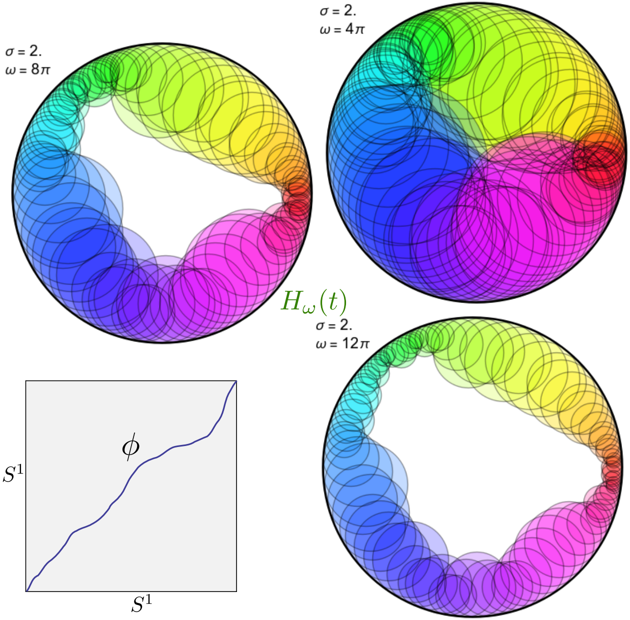

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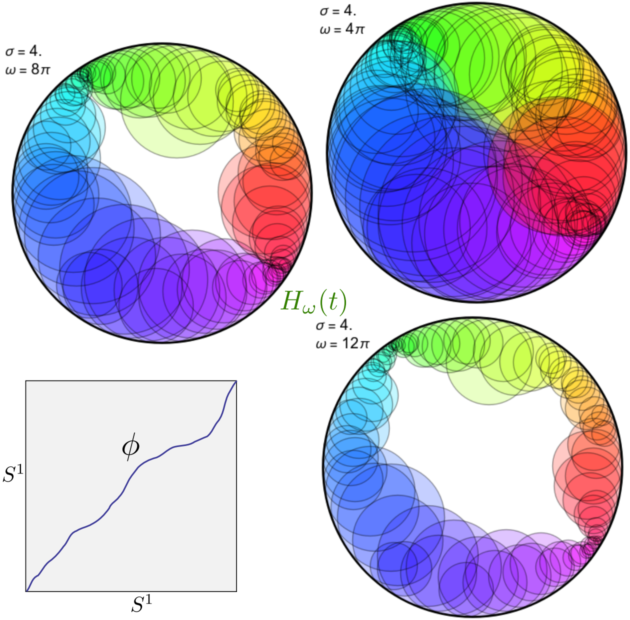

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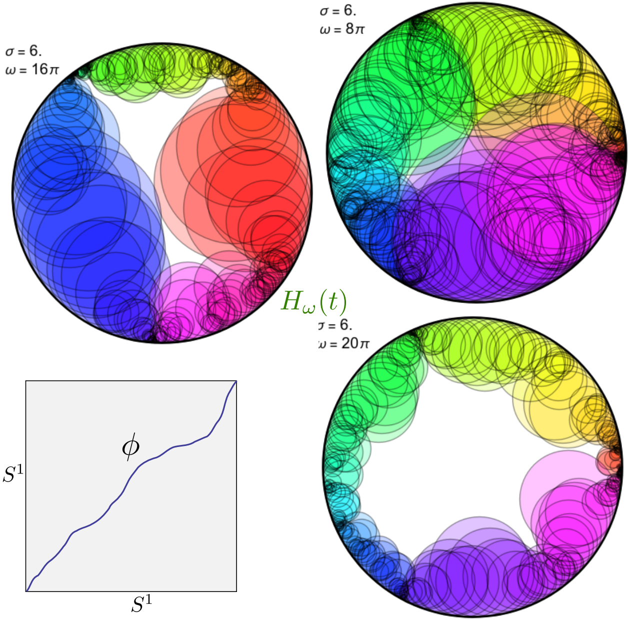

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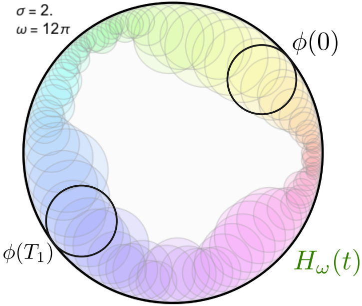

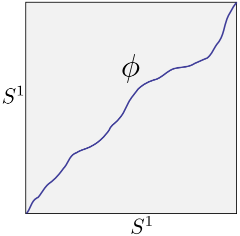

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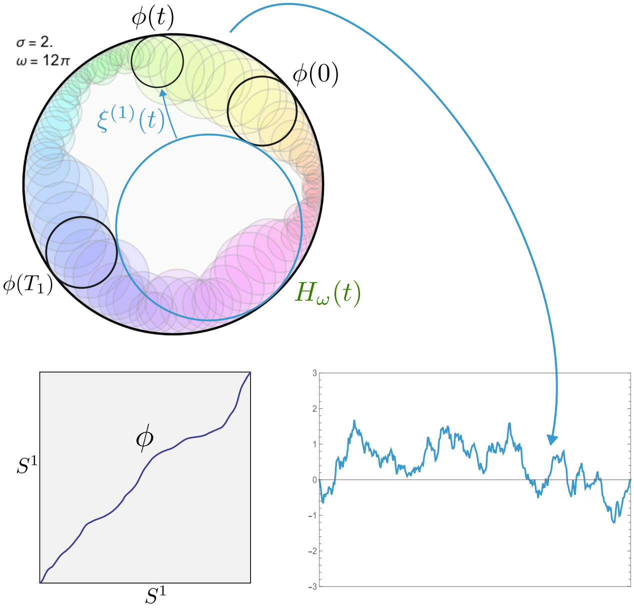

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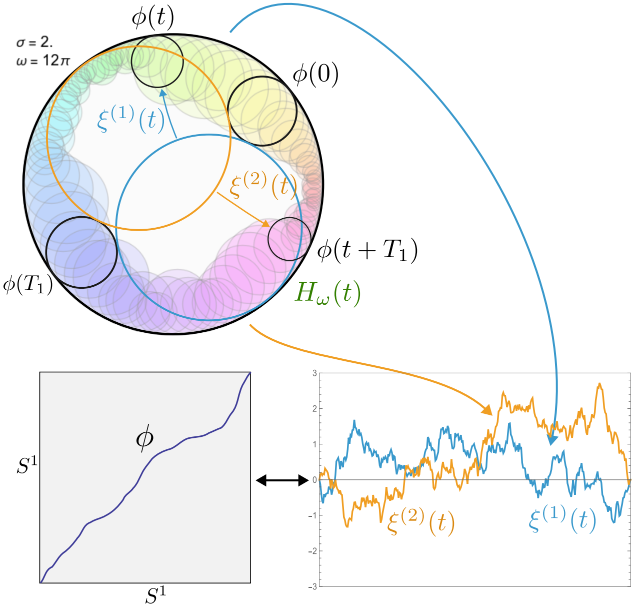

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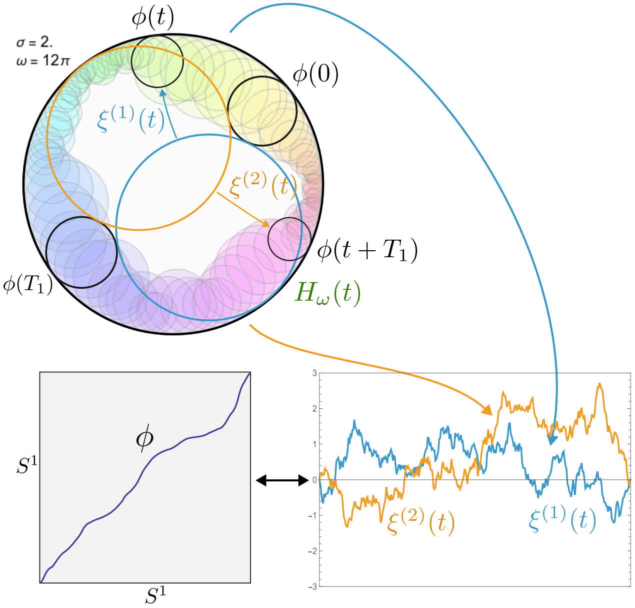
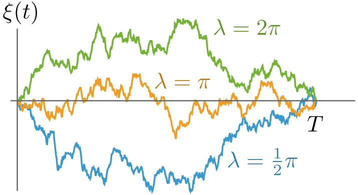
- This is a bijection

$$\mathrm{Diff}^1(S^1)/\mathrm{PSL}(2,\mathbb{R})$$
$$\updownarrow$$
$$\left\{ \begin{matrix} \xi^{(1)} \in C_0[0,T_1] \\ \xi^{(2)} \in C_0[0,T_2] \end{matrix} \,\middle|\, \int_0^{T_1} e^{\xi^{(1)}(t)}\mathrm{d}t = \int_0^{T_2} e^{\xi^{(2)}(t)}\mathrm{d}t \right\}$$



A practical characterization



- Consider *conditioned* Brownian bridge measure $\Theta^{\lambda,T}_{\sigma^2}$ on $\{\xi \in C_0[0,T] : \int_0^T e^{\xi(t)}\mathrm{d}t = \frac{\lambda}{\pi}\}$ normalized such that

$$\mathrm{d}\mathcal{B}^{0,T}_{\sigma^2}(\xi) = \int_{\mathbb{R}_+} \frac{\mathrm{d}\lambda}{\lambda}\, \tfrac{1}{2} e^{-\frac{4\pi}{\sigma^2\lambda}}\, \mathrm{d}\Theta^{\lambda,T}_{\sigma^2}(\xi)$$

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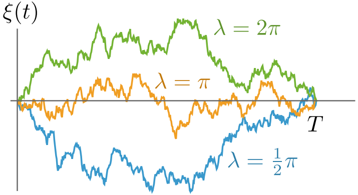

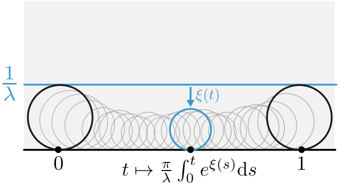

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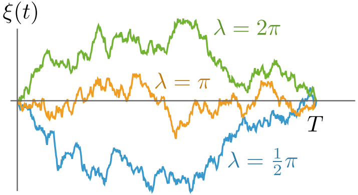

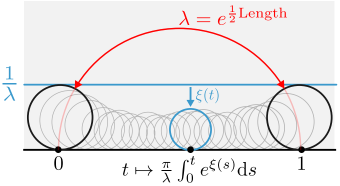

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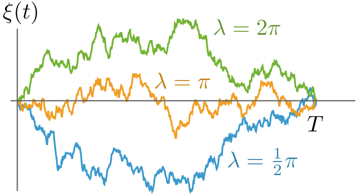
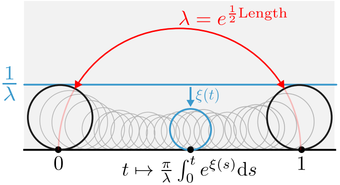
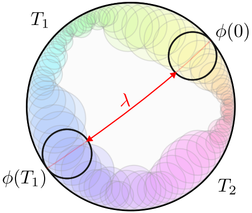
Theorem (TB, Chekhov, '26+)

For $T_1 + T_2 = 1$, the Schwarzian field theory measure is

$$\mathrm{d}\mathcal{M}_{\sigma^2}(\phi) = \pi^2 \int_{\mathbb{R}_+} \frac{\mathrm{d}\lambda}{\lambda}\, \mathrm{d}\Theta_{\sigma^2}^{\lambda,T_1}(\xi^{(1)})\, \mathrm{d}\Theta_{\sigma^2}^{\lambda,T_2}(\xi^{(2)}).$$







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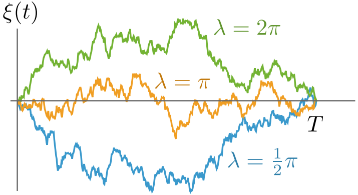
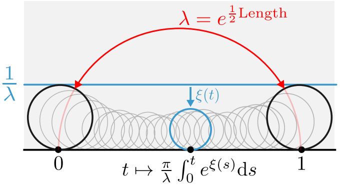
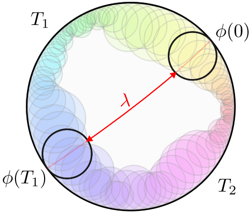
- $\int \mathrm{d}\Theta^{\lambda,T}_{\sigma^2} = \theta(\frac{4\pi}{\sigma^2\lambda}, \frac{\sigma^2}{4}T)$ is known as Yor's integral, and plays important role in exponential functionas of Brownian motion, $\mathcal{L}[\theta(r,\cdot)](x) = I_{\sqrt{2x}}(r)$ [Yor, '80]







A practical characterization

- Consider *conditioned* Brownian bridge measure $\Theta^{\lambda,T}_{\sigma^2}$ on $\{\xi \in C_0[0,T] : \int_0^T e^{\xi(t)}\mathrm{d}t = \frac{\lambda}{\pi}\}$ normalized such that

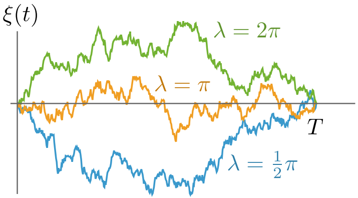
$$\mathrm{d}\mathcal{B}^{0,T}_{\sigma^2}(\xi) = \int_{\mathbb{R}_+} \frac{\mathrm{d}\lambda}{\lambda}\, \tfrac{1}{2} e^{-\frac{4\pi}{\sigma^2\lambda}}\, \mathrm{d}\Theta^{\lambda,T}_{\sigma^2}(\xi)$$

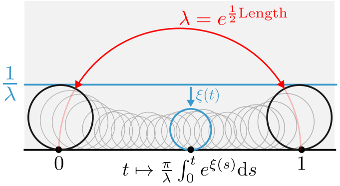
Theorem (TB, Chekhov, '26+)

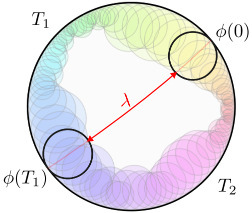
For $T_1 + T_2 = 1$, the Schwarzian field theory measure is

$$\mathrm{d}\mathcal{M}_{\sigma^2}(\phi) = \pi^2 \int_{\mathbb{R}_+} \frac{\mathrm{d}\lambda}{\lambda}\, \mathrm{d}\Theta^{\lambda,T_1}_{\sigma^2}(\xi^{(1)})\, \mathrm{d}\Theta^{\lambda,T_2}_{\sigma^2}(\xi^{(2)}).$$

- $\int \mathrm{d}\Theta^{\lambda,T}_{\sigma^2} = \theta(\frac{4\pi}{\sigma^2\lambda}, \frac{\sigma^2}{4}T)$ is known as Yor's integral, and plays important role in exponential functionas of Brownian motion, $\mathcal{L}[\theta(r,\cdot)](x) = I_{\sqrt{2x}}(r)$ [Yor, '80]
- Indeed $\pi^2 \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\, \theta(\frac{4\pi}{\sigma^2\lambda}, \frac{\sigma^2}{4}T_1)\, \theta(\frac{4\pi}{\sigma^2\lambda}, \frac{\sigma^2}{4}T_2) = Z_{\sigma^2}$



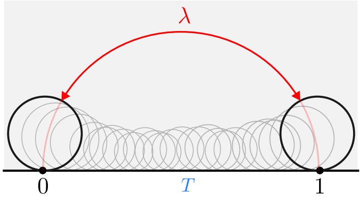


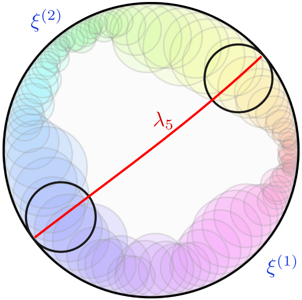


Correlation functions [Mertens, Turiaci, Verlinde, '17] [Losev, '24]

- The measure $\Theta^{\lambda,T}_{\sigma^2}$ enjoys the decomposition property $(T = T' + T'')$

$$\mathrm{d}\Theta^{\lambda,T}_{\sigma^2}(\xi) = \int_{\mathbb{R}^2_+} \frac{\mathrm{d}\lambda'}{\lambda'} \frac{\mathrm{d}\lambda''}{\lambda''} e^{-\frac{2\pi}{\sigma^2}\left(\frac{\lambda}{\lambda'\lambda''} + \frac{\lambda'}{\lambda''\lambda} + \frac{\lambda''}{\lambda\lambda'}\right)} \mathrm{d}\Theta^{\lambda',T'}_{\sigma^2}(\xi') \mathrm{d}\Theta^{\lambda'',T''}_{\sigma^2}(\xi'')$$





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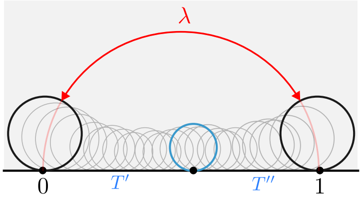

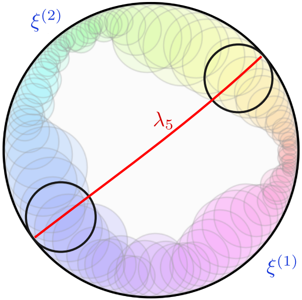

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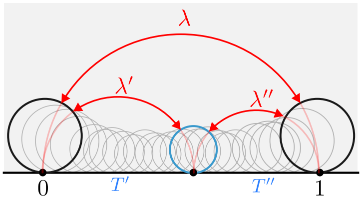

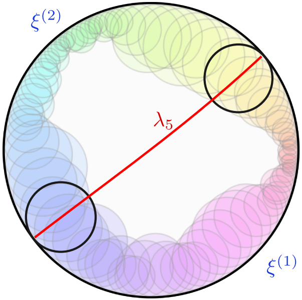

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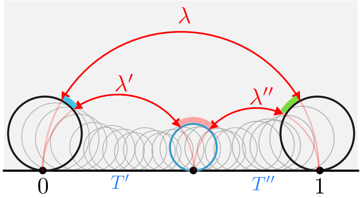
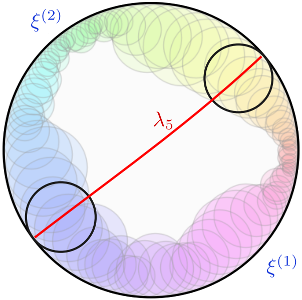
horocycle length





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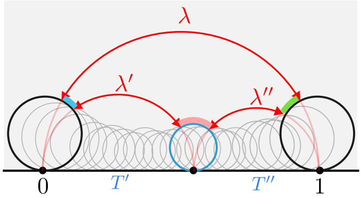

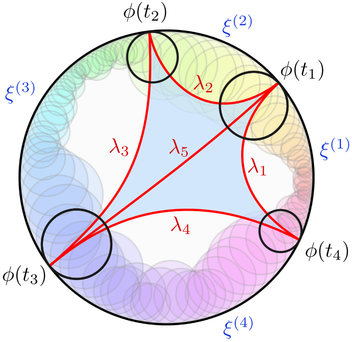

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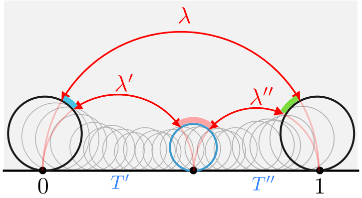
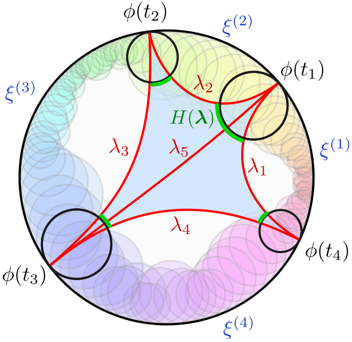
total horocycle length





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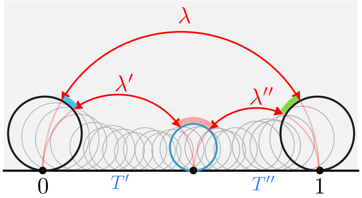
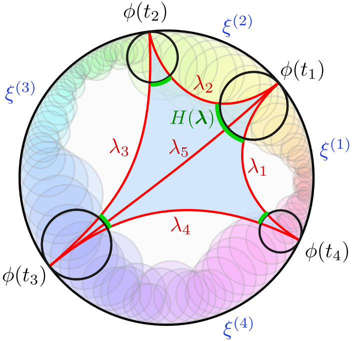
- Relation to bi-local observable → Wolfram's talk

$$\frac{\pi}{\lambda_i} = \mathcal{O}_i(\phi) \equiv \mathcal{O}(\phi; s,t) := \frac{\pi\sqrt{\phi'(s)\phi'(t)}}{\sin(\pi|\phi(s)-\phi(t)|)}$$





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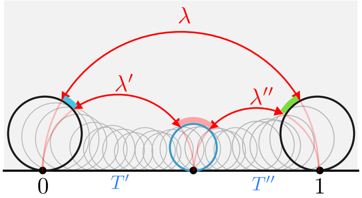
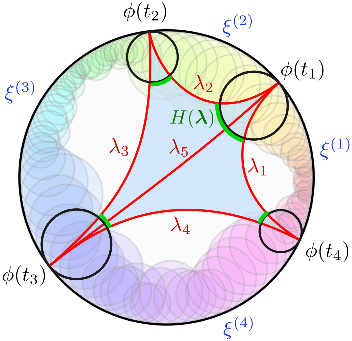
- n-pt correlation functions are $(2n-3)$-dim. integrals:

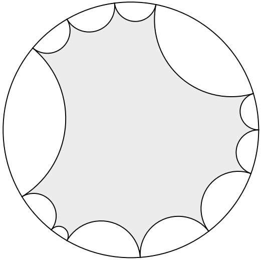
$$\left\langle \prod_{i=1}^{2n-3} \mathcal{O}_i(\phi)^{m_i} \right\rangle = \pi^2 \int_{\mathbb{R}^{2n-3}_+} \mathrm{d}\boldsymbol{\lambda}\, e^{-\frac{2\pi}{\sigma^2}H(\boldsymbol{\lambda})} \prod_{i=1}^{n} \theta\left(\tfrac{4\pi}{\sigma^2\lambda_i}, \tfrac{\sigma^2}{4} T_i\right) \prod_{i=1}^{2n-3} \left(\tfrac{\pi}{\lambda_i}\right)^{m_i}$$

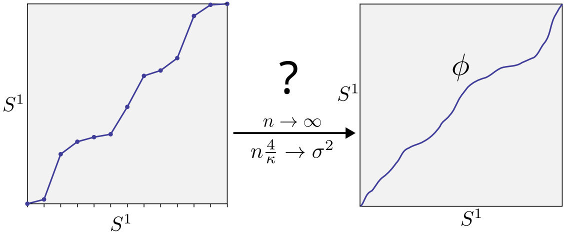
→ agreement with [Mertens, Turiaci, Verlinde, '17] [Losev, '24]

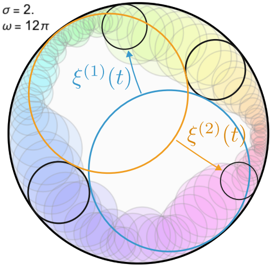




Idea of scaling limit proof [Chekhov, TB, '26+]







"Bijective approach"

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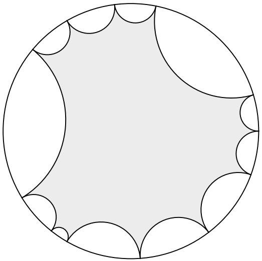
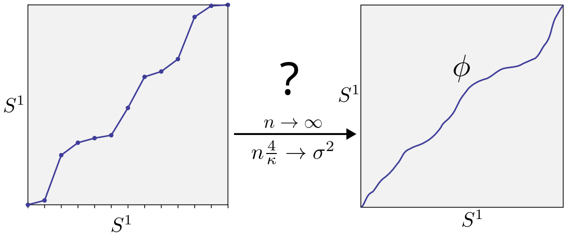
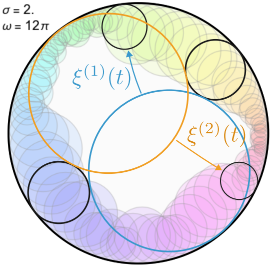
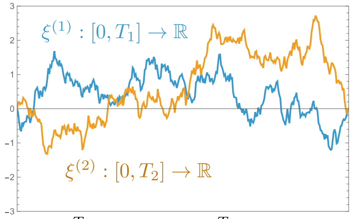






"Bijective approach"

$$1 = T_1 + T_2$$



$$\int_0^{T_1} e^{\xi^{(1)}(t)} \mathrm{d}t = \int_0^{T_2} e^{\xi^{(2)}(t)} \mathrm{d}t$$

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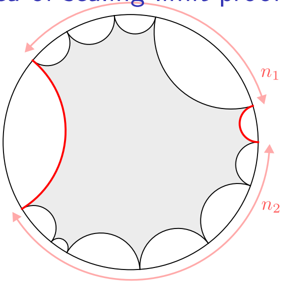
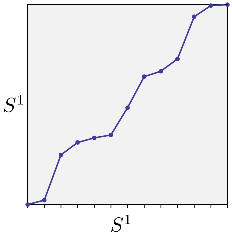


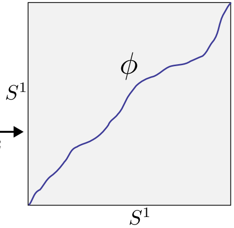
$n = n_1 + n_2$

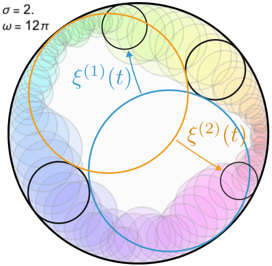


?

$n \to \infty$

$n^{\frac{4}{\kappa}} \to \sigma^2$





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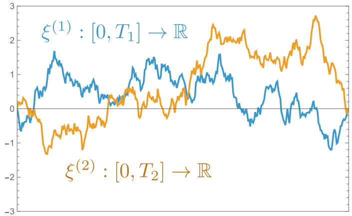

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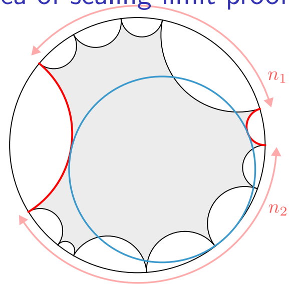
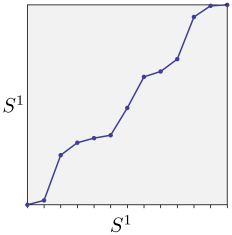
Idea of scaling limit proof [Chekhov, TB, '26+]





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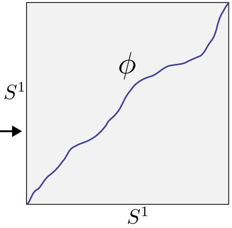

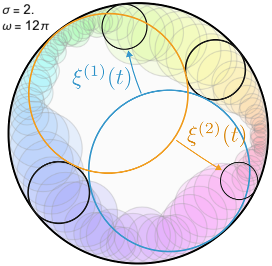

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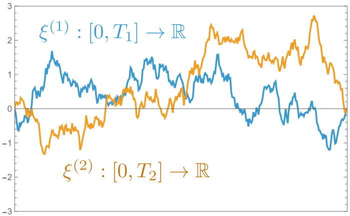

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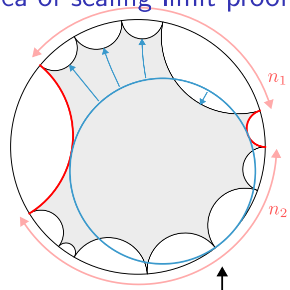

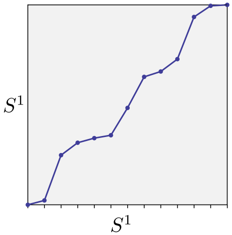

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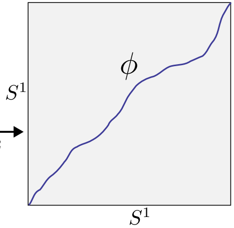

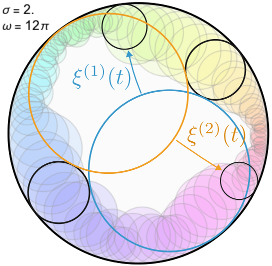

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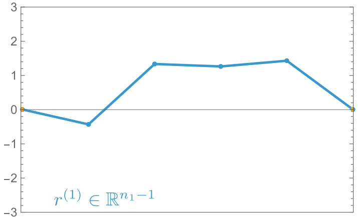

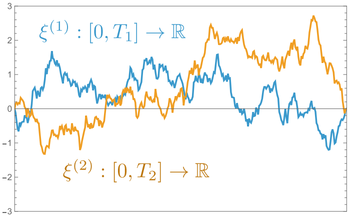

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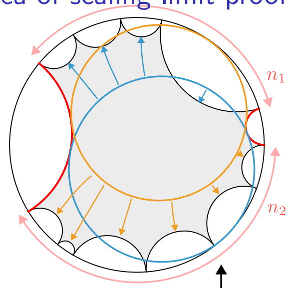
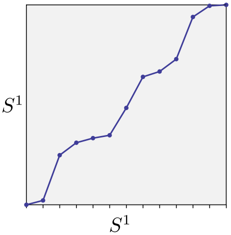
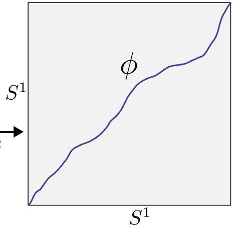


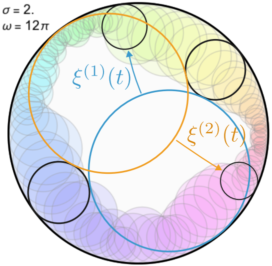


?

$n \to \infty$

$n^{\frac{4}{\kappa}} \to \sigma^2$

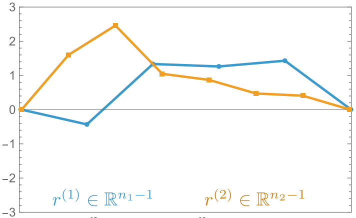




$n = n_1 + n_2$

"Bijective approach"

$1 = T_1 + T_2$



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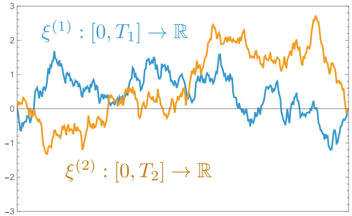

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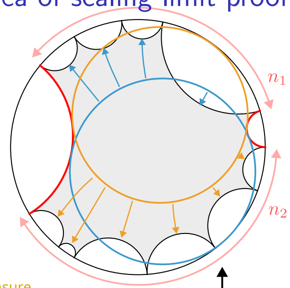

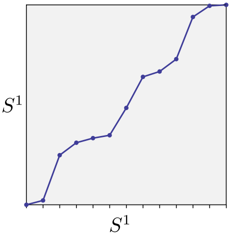

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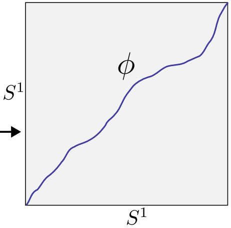

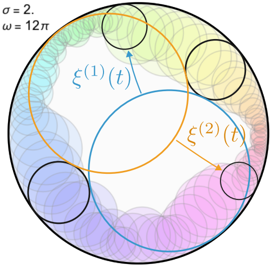

measure

$e^{-\frac{\kappa}{2}L(X)}\mathrm{d}\Omega(X)$

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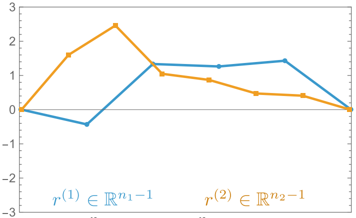
measure

Conditioned random walks with increments distributed as $\propto (2\cosh\frac{x}{2})^{-\kappa}\mathrm{d}x$



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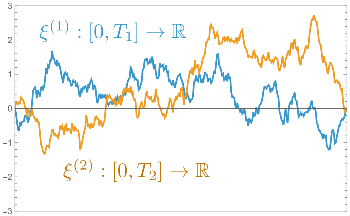

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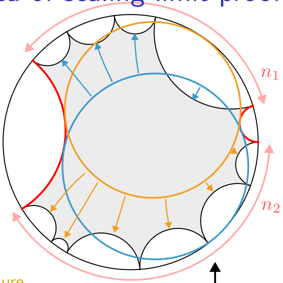

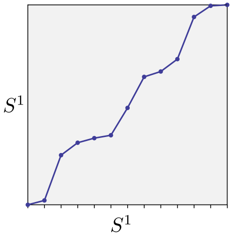

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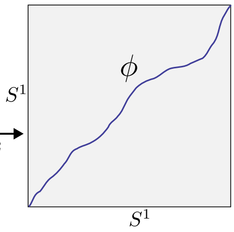

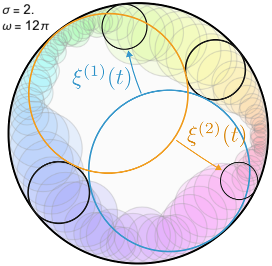

measure

Schwarzian field theory $\mathcal{M}_{\sigma^2}$

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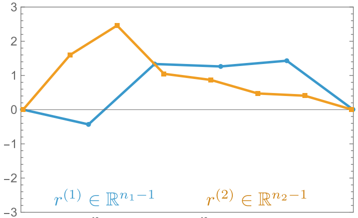
Conditioned Brownian bridges with variance σ^2



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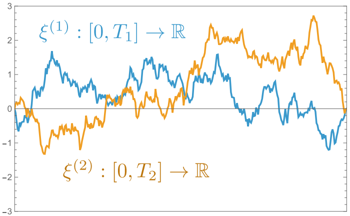

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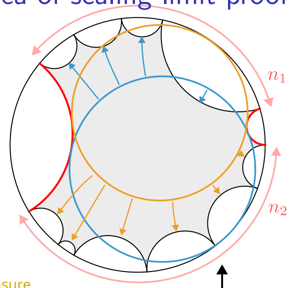

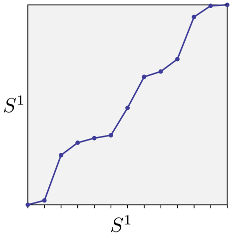

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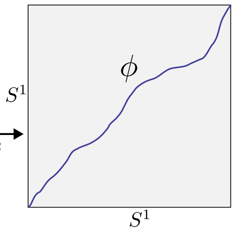

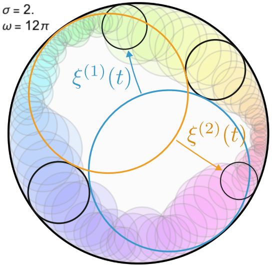

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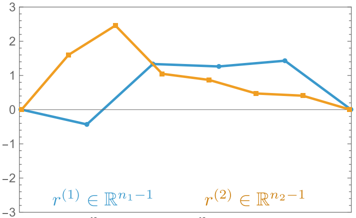
$\mathrm{Var} \overset{\kappa\to\infty}{\sim} \frac{4}{\kappa}$



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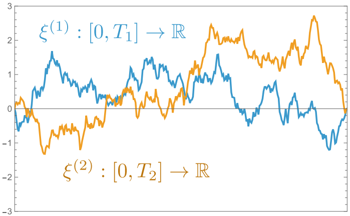
functional central limit theorem



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Integral transforms: metric vs spectral [Harlow, Jafferis, '20] [Yang, '20][Iliesiu, Levine, Lin, Maxfield, Mezei, '24]

- Hyperbolic ($y = \frac{4\pi}{\sigma^2\lambda}$) and spectral ($E$) side related by *Kontorovich-Lebedev transform*

$$\mathcal{K} : L^2(\mathbb{R}_+, \tfrac{\mathrm{d}y}{y}) \to L^2(\mathbb{R}_+, \tfrac{1}{\pi^2}\sinh(2\pi\sqrt{E})\mathrm{d}E)$$

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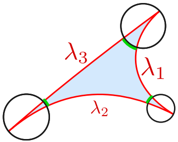

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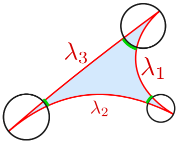

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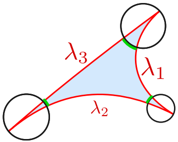

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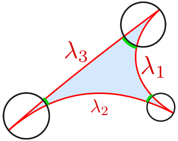

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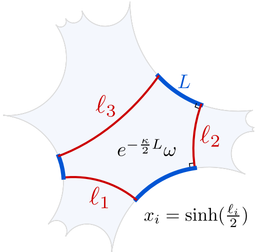

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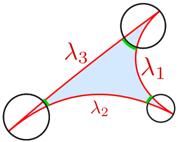

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