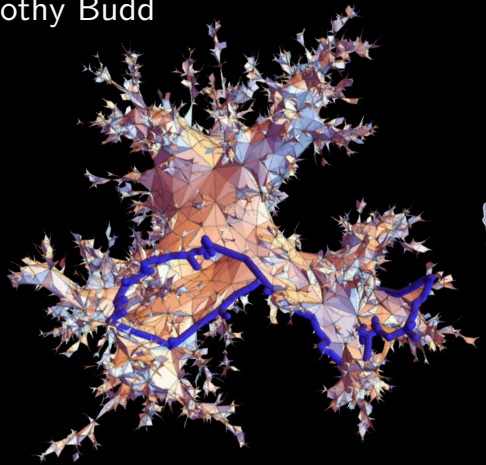
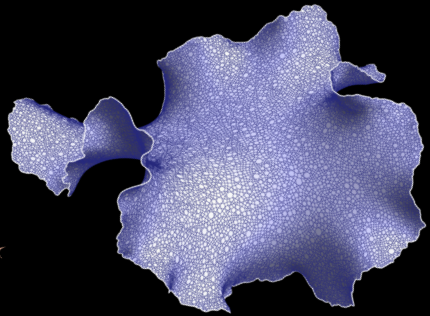
ESI Workshop "Combinatorics & Asymptotics of discrete planar structures", 19-10-2026

Planar maps and their boundary metric

Timothy Budd

based on
arXiv:2607.01995
and w.i.p.





Radboud University

Quantum Geometry gallery
https://qgeometry.com

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/

Holographic principle in physics

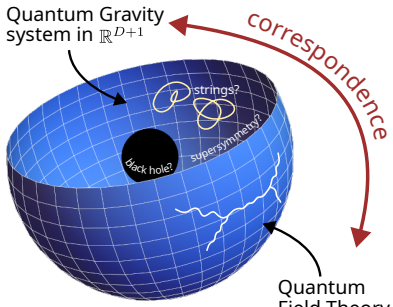
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[Holographic self-portrait
Venceslav Saynov]

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$\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$

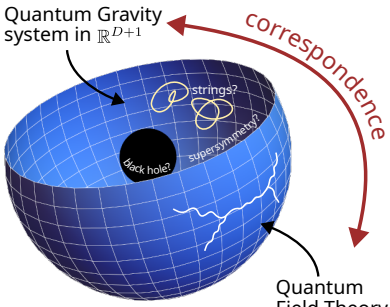
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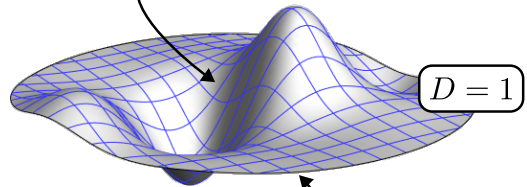
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[Sachdev, Ye, Kitaev, Maldacena, Stanford, Suh, Yang, Saad, Shenker, Lin, Witten, Verlinde, ..., '10 - present]

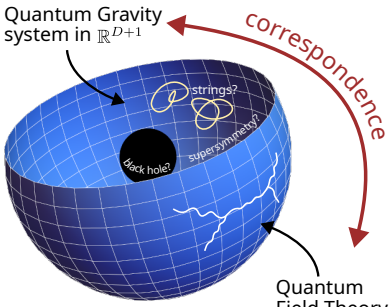
two-dimensional quantum gravity:
Liouville gravity, JT gravity, random geometry ....



Quantum mechanics
Sachdev-Ye-Kitaev, statistical physics on S^1...

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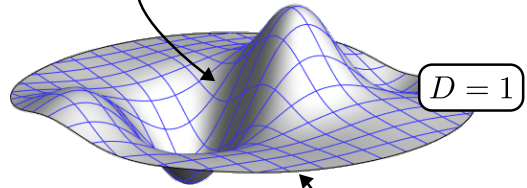
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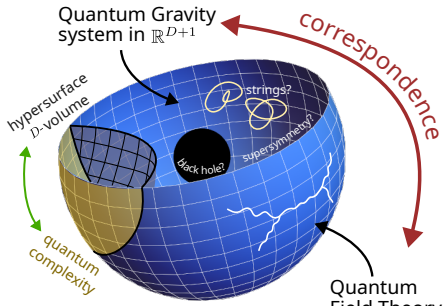
Stripped from exotic ingredients and reduced to 1d↔2d toy setting, does the holographic correspondence teach a non-trivial (combinatorial) lesson?

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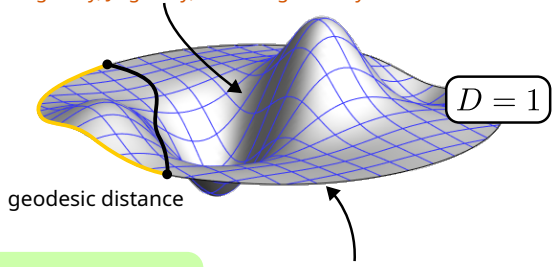


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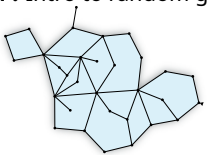


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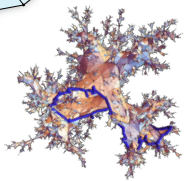
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Outline

1. Intro to random geometry from planar maps

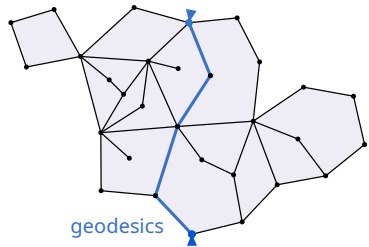


combinatorics

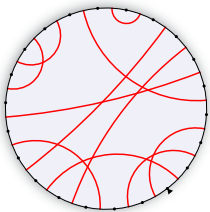


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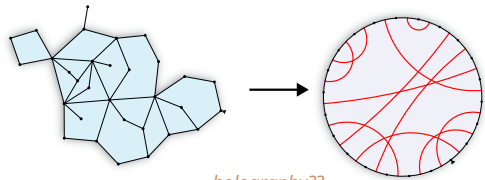
2. Boundary metric and orthogonal polynomials



3. Chord diagrams and DSSYK



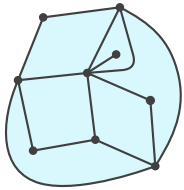
4. Relating planar maps and chord diagrams



holography??

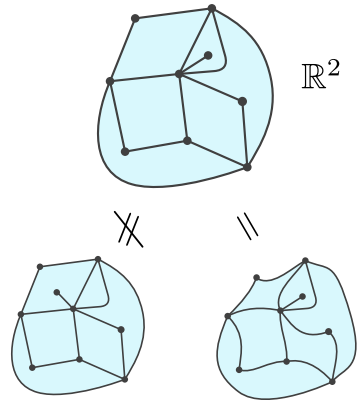
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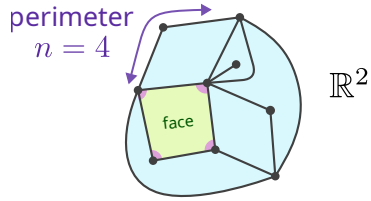
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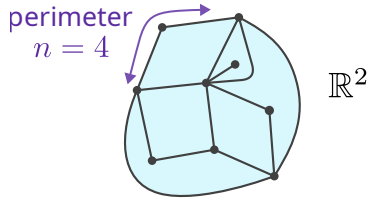
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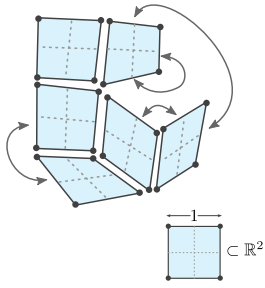
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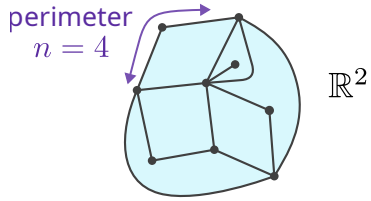
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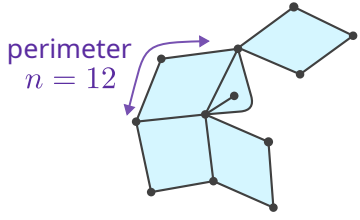




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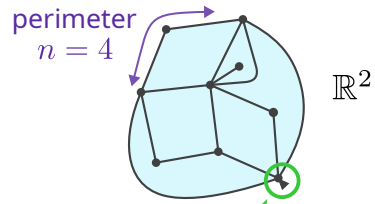
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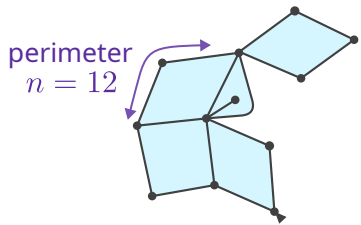




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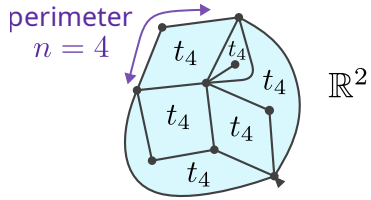


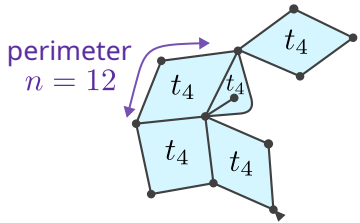


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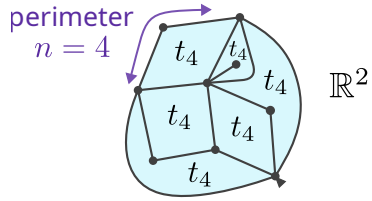


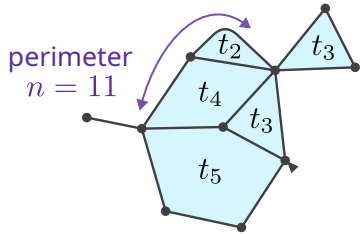


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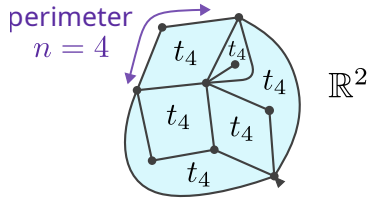


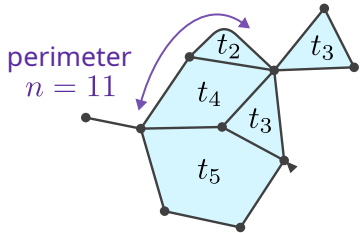


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Many map enumeration methods:

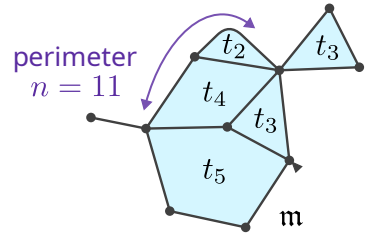
- Recursive methods and generating functions [Tutte, '60s]
  [Brown, Bender, Canfield, Goulden, Jackson, Ambjørn, Bousquet-Mélou, ...]
- Matrix integrals/models ['t Hooft, Brézin, Itzykson, Parisi, Zuber, Kazakov, ...]
- Probabilistic methods [Le Gall, Miermont, Curien, Sheffield, Miller, ...]
- Bijective method [Schaeffer, Bouttier, Di Francesco, Guitter, Fusy, Chapuy, Albenque, ...]

Planar maps: probability

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Planar maps: probability

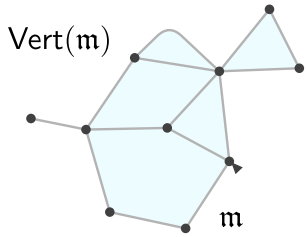
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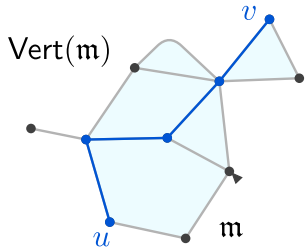
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$d_{\mathfrak{m}}^{\mathrm{graph}}(u, v) = 4$

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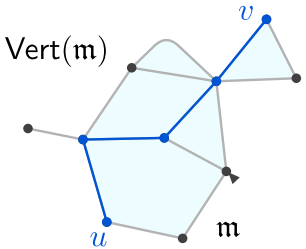
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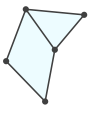
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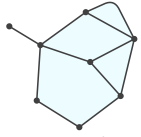
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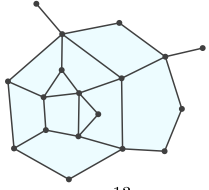
- Can we understand (Gromov-Hausdorff) limits in Met as $n \to \infty$?

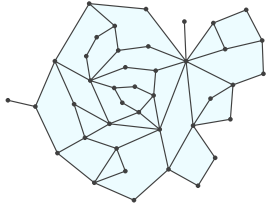


$d_{\mathfrak{m}}^{\mathrm{graph}}(u, v) = 4$

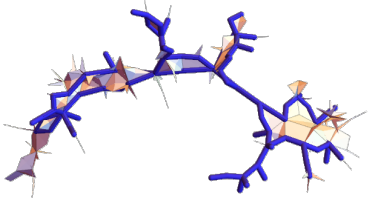

$n = 5$

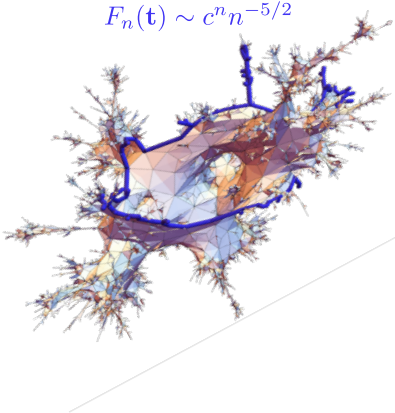

$n = 8$

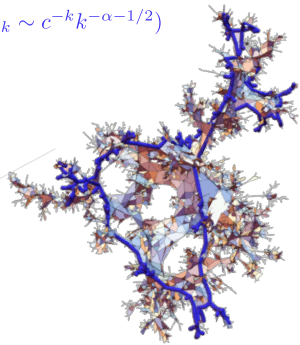

$n = 13$


$n = 22$

subcritical $\mathbf{t}$	generic critical $\mathbf{t}$	non-generic critical $\mathbf{t}$
$F_n(\mathbf{t}) \sim c^n n^{-3/2}$	$F_n(\mathbf{t}) \sim c^n n^{-5/2}$	$F_n(\mathbf{t}) \sim c^n n^{-\alpha-1/2}, 1 < \alpha < 2$
		$(t_k \sim c^{-k} k^{-\alpha-1/2})$

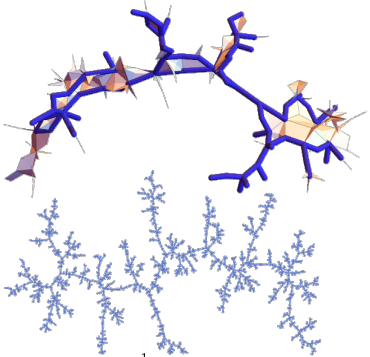






subcritical $\mathbf{t}$

$F_n(\mathbf{t}) \sim c^n n^{-3/2}$



$(\mathrm{Vert}(\mathfrak{m}), n^{-\frac{1}{2}} d_{\mathfrak{m}}) \xrightarrow[n\to\infty]{} \mathrm{CRT}$

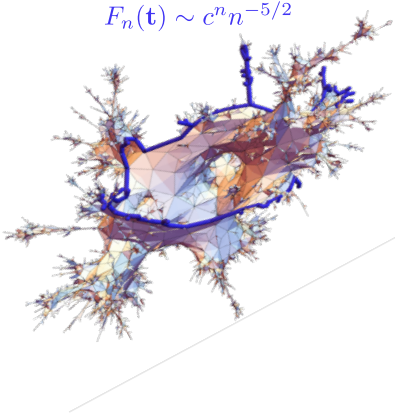
topology of self-similar real tree

Hausdorff dimension 2

[Aldous, '91] [Bettinelli, '11]

generic critical $\mathbf{t}$

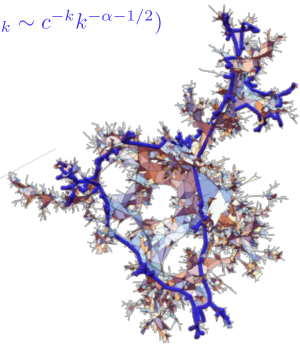
$F_n(\mathbf{t}) \sim c^n n^{-5/2}$



non-generic critical $\mathbf{t}$

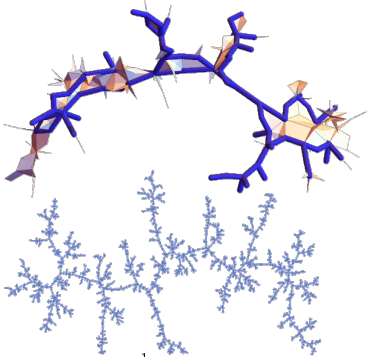
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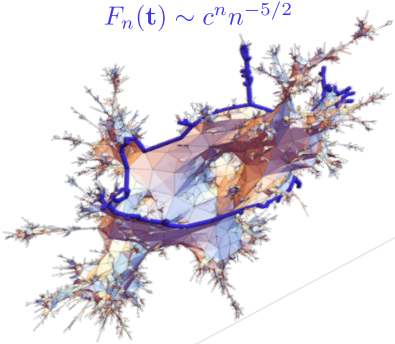
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$(\mathrm{Vert}(\mathfrak{m}), n^{-\frac{1}{2}} d_{\mathfrak{m}}) \xrightarrow[n\to\infty]{} \text{Brownian disk}$

topology of disk

Hausdorff dimension 4

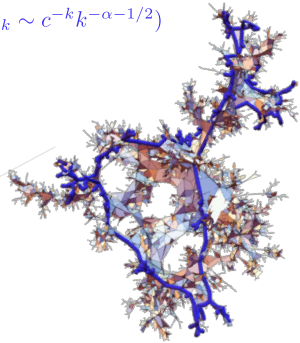
→ talks by Le Gall & Stufler

[Le Gall, '10] [Miermont, '10] [Bettinelli, '11]
[Bettinelli, Miermont, '15] [Gwynne, Miller, '17]

non-generic critical $\mathbf{t}$

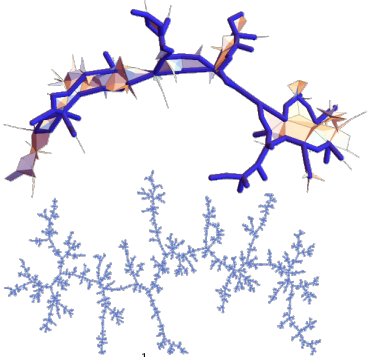
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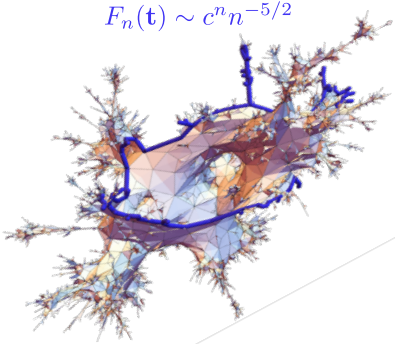
topology of self-similar real tree

Hausdorff dimension 2

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topology of disk

Hausdorff dimension 4

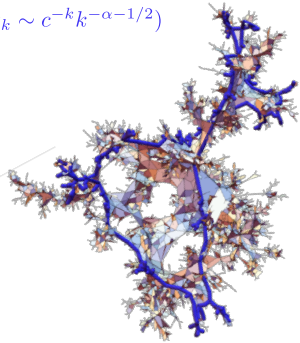
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α-stable disk

topology of Sierpinski carpet $(\alpha \geq \frac{3}{2})$

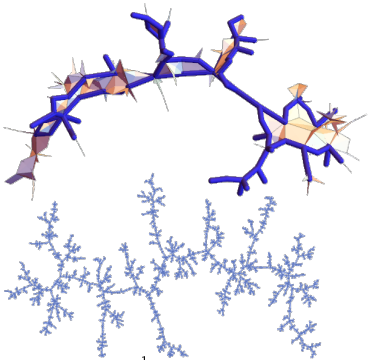
Hausdorff dimension 2α

→ Miermont's talk tomorrow

[Le Gall, Miermont, '11]
[Curien, Miermont, Riera, '25]

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$F_n(\mathbf{t}) \sim c^n n^{-3/2}$



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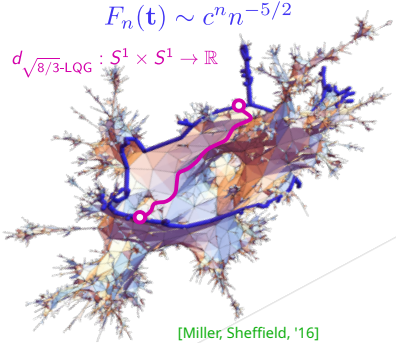
topology of self-similar real tree

Hausdorff dimension 2

[Aldous, '91] [Bettinelli, '11]

generic critical $\mathbf{t}$

$F_n(\mathbf{t}) \sim c^n n^{-5/2}$



[Miller, Sheffield, '16]

Liouville Quantum Gravity $_{\gamma=\sqrt{8/3}}$

$\wr$

$(\mathrm{Vert}(\mathfrak{m}), n^{-\frac{1}{2}} d_{\mathfrak{m}}) \xrightarrow[n\to\infty]{} \text{Brownian disk}$

topology of disk

Hausdorff dimension 4

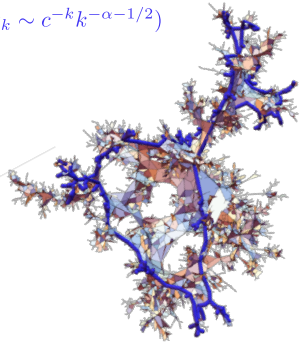
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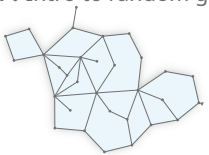
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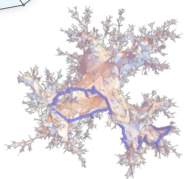
The $\sqrt{8/3}$-LQG boundary metric space on S^1 is accessible via $d_{\mathfrak{m}}(\cdot,\cdot)|_{\text{boundary vertices}}$.

Outline

1. Intro to random geometry from planar maps

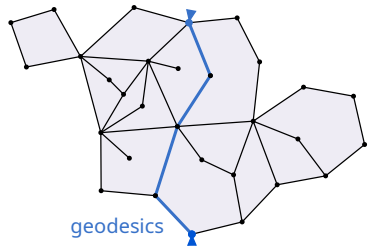


combinatorics

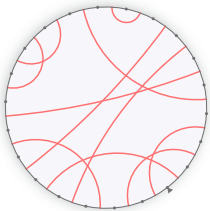


probability

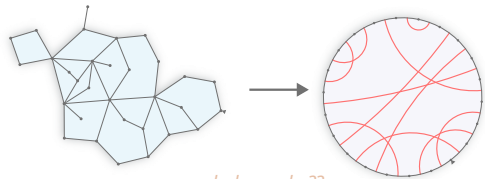
2. Boundary metric and orthogonal polynomials



3. Chord diagrams and DSSYK



4. Relating planar maps and chord diagrams



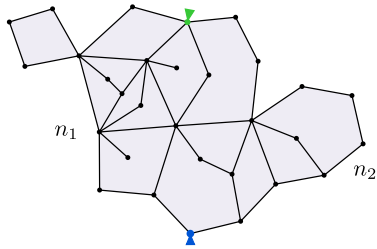
holography??

Planar maps: boundary perspective reformulating [Bouttier, Guitter, '10]

$$F_n \equiv F_n(\mathbf{t}) = \sum_{\mathfrak{m} \in \mathcal{M}_n} w(\mathfrak{m}) < \infty, \quad w(\mathfrak{m}) = \prod_{\text{faces } f \text{ of } \mathfrak{m}} t_{\deg(f)}$$

- Consider maps of perimeter $n = n_1 + n_2$ and $\ell \geq 0$,

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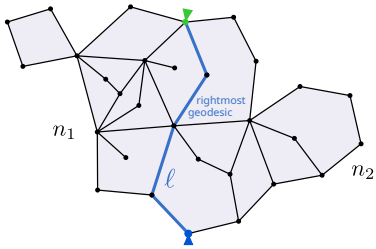


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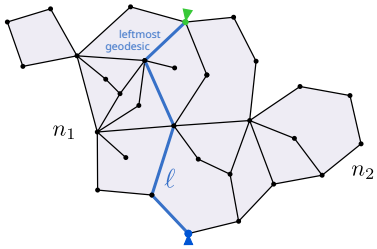


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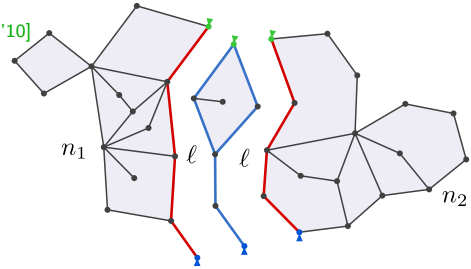
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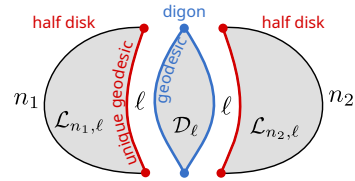
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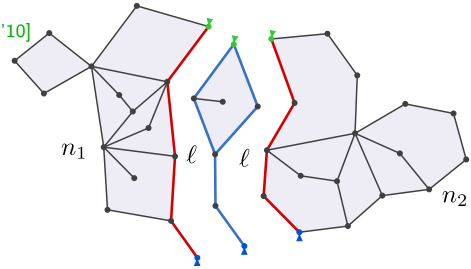
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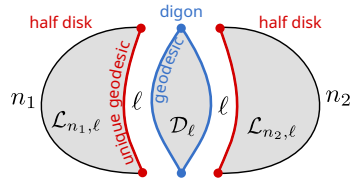
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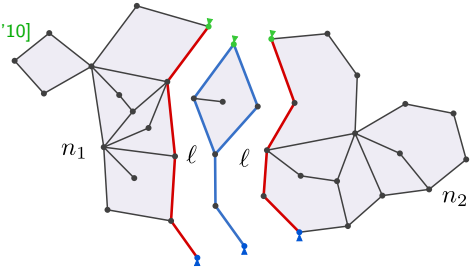
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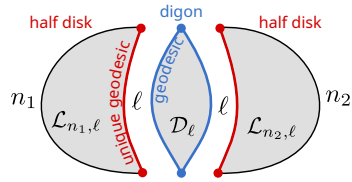
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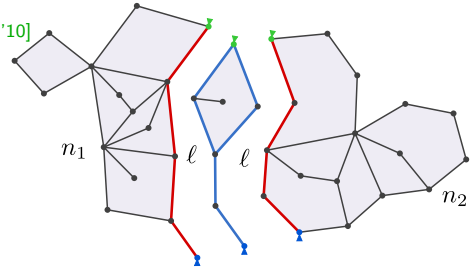
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- Monic polynomials $P_\ell(x) := \sum_{n=0}^{\ell} (L^{-1})_{\ell n} x^n = x^\ell + \cdots$ obey

$$\langle P_\ell(x) P_k(x)\rangle = D_\ell\, \delta_{\ell,k}, \quad \langle\, \cdot\, \rangle : \mathbb{R}[x] \to \mathbb{R}, \quad \langle x^n\rangle := F_n.$$



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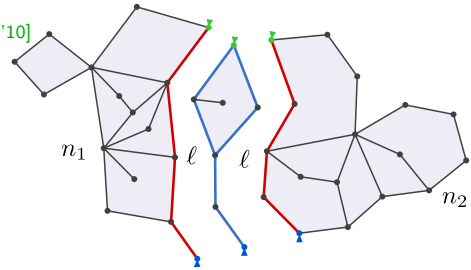
$$\{\mathfrak{m} \in \mathcal{M}_{n_1+n_2} : d_{\mathfrak{m}}(\text{♟}, \text{♟}) = \ell\} \overset{\text{1-to-1}}{\longleftrightarrow} \mathcal{L}_{n_1,\ell} \times \mathcal{D}_\ell \times \mathcal{L}_{n_2,\ell}$$

- Bijective slicing into geodesic *digons* $\mathcal{D}_\ell$ and *half disks* $\mathcal{L}_{n,\ell}$.

$$F_{n_1+n_2} = \sum_{\ell\geq 0} L_{n_1,\ell}\, D_\ell\, L_{n_2,\ell}, \quad L_{n,\ell} = \sum_{\mathfrak{m}\in\mathcal{L}_{n,\ell}} w(\mathfrak{m}), \quad D_\ell = \sum_{\mathfrak{m}\in\mathcal{D}_\ell} w(\mathfrak{m})$$

- Unique LDL^T-decomp. of Hankel matrix $(F_{n_1+n_2})_{n_1,n_2\geq 0}$!
  $L_{\ell,\ell} = 1, \quad L_{n,\ell} = 0$ for $n < \ell$.
- Monic polynomials $P_\ell(x) := \sum_{n=0}^{\ell} (L^{-1})_{\ell n} x^n = x^\ell + \cdots$ obey

$$\langle P_\ell(x) P_k(x)\rangle = D_\ell\, \delta_{\ell,k}, \quad \langle\,\cdot\,\rangle : \mathbb{R}[x] \to \mathbb{R}, \quad \langle x^n\rangle := F_n.$$



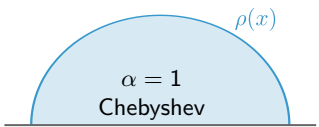
Historical note on Matrix Models: [Bouttier, Guitter, '10]

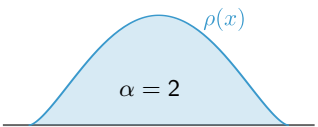
$F_n = \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\operatorname{Tr}(H^n)]$ for "random" $N \times N$ Hermitian matrix sampled $\propto e^{-N\operatorname{Tr}\left(\frac{1}{2}H^2 - \sum_{k\geq 1}\frac{1}{k}t_k H^k\right)} \mathrm{d}H$.

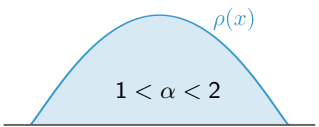
Then $\langle \bullet \rangle = \int_a^b \bullet\, \rho(x)\mathrm{d}x$ with $\rho(x)$ the large-N spectral density of H.

Example: Gegenbauer / ultraspherical polynomials

$$\langle \bullet \rangle = \frac{1}{c_\alpha} \int_{-2\sqrt{\alpha}}^{2\sqrt{\alpha}} \bullet \, (4\alpha - x^2)^{\alpha - 1/2} \mathrm{d}x, \qquad P_\ell(x) \overset{\text{monic}}{\propto} (4\alpha - x^2)^{\frac{1}{2}-\alpha} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} (4\alpha - x^2)^{\ell+\alpha-\frac{1}{2}}$$





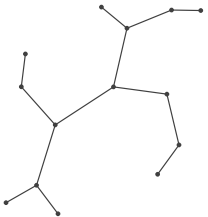


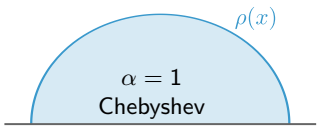
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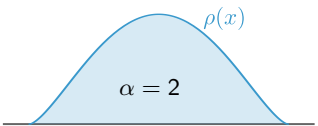
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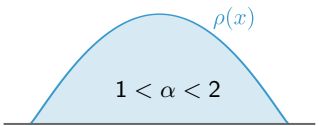
Plane trees

$t_k = 0$







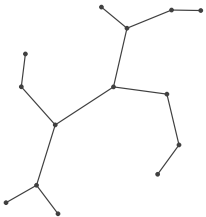


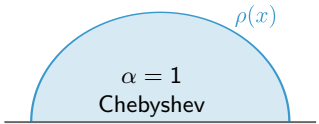
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Plane trees

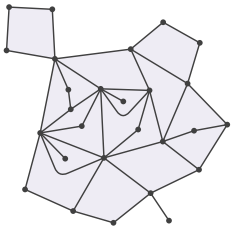
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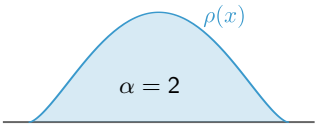


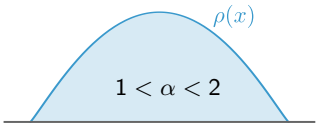


Critical quadrangulations

$t_k = \frac{1}{12}\delta_{k,4}$





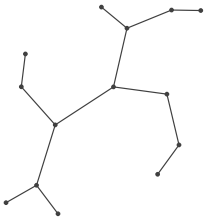


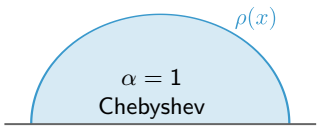
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Plane trees

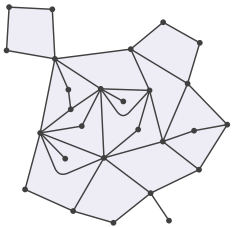
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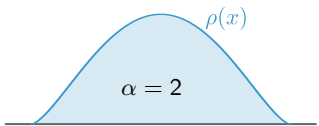




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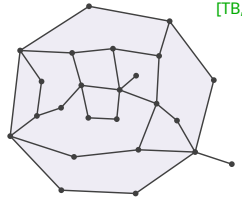


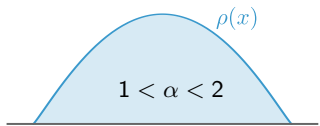


Non-generic critical bipartite maps

$t_{2k} \sim c(4\alpha)^{-k} k^{-\alpha - \frac{1}{2}}$

[Kazakov, '89]
[Ambjorn, TB, Makeenko, '16]
[TB, Curien, '17]



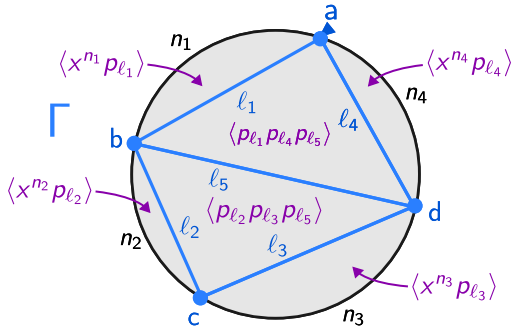


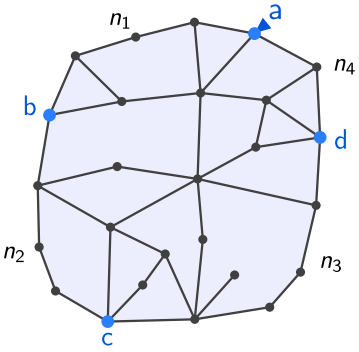
Complete handle on non-crossing geodesic distances

Theorem

For $\mathbf{t}$ *admissible and non-crossing diagonals* Γ *of* k*-gon, the random perim-*$(n_1+\ldots+n_k)$ *map* $\mathfrak{m}$ *obeys*

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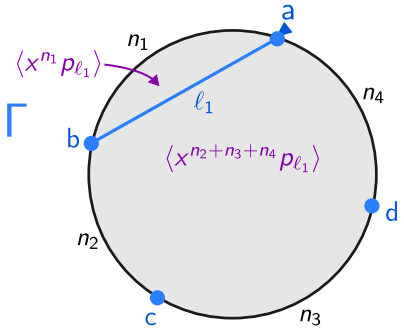
$n_1 = 4$
$n_2 = 4$
$n_3 = 5$
$n_4 = 2$

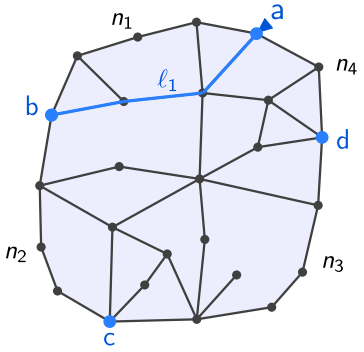
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$n_1 = 4$
$n_2 = 4$
$n_3 = 5$
$n_4 = 2$

$\ell_1 = 3$

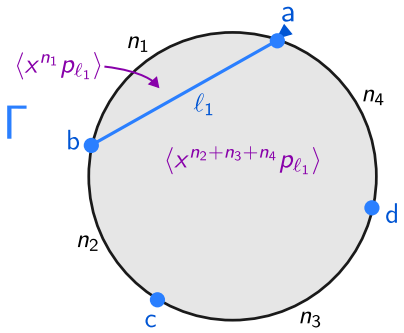
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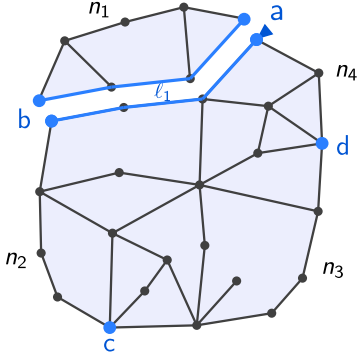
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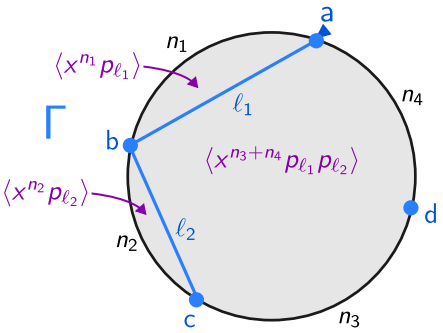
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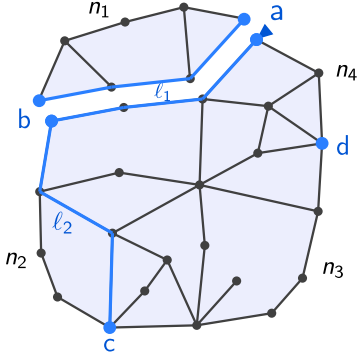
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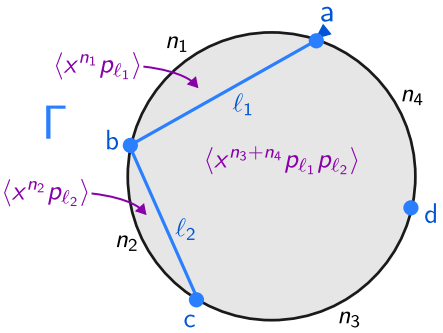
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$\ell_2 = 3$

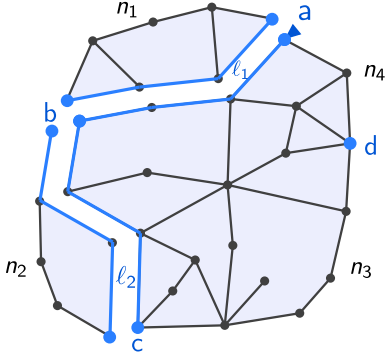
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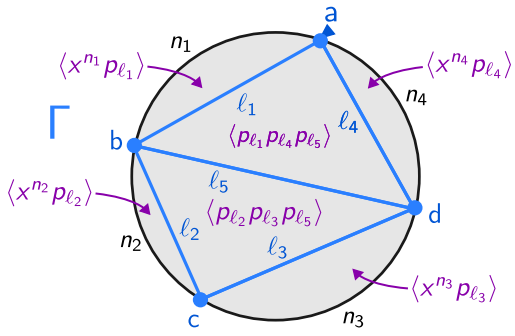
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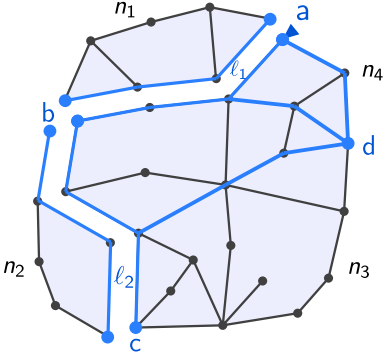
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$\ell_5 = 4$

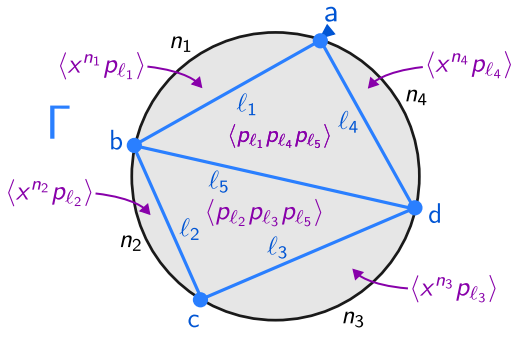
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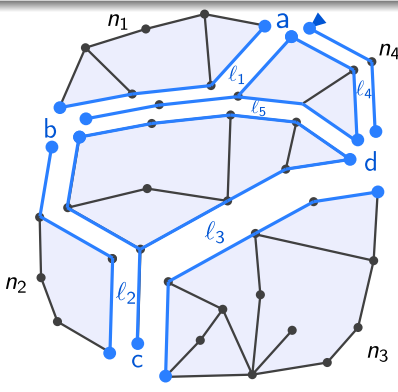
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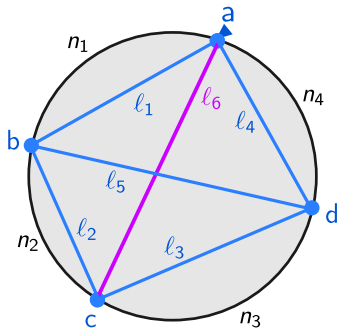
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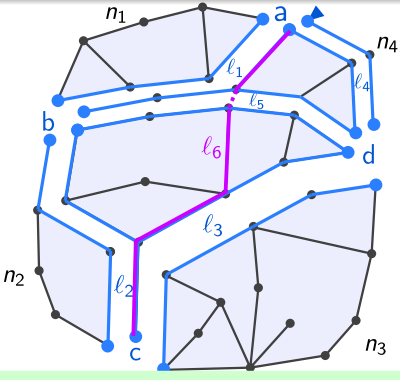
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Γ





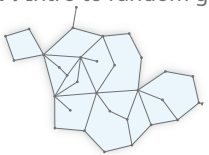
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$\ell_4 = 2$
$\ell_5 = 4$

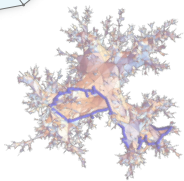
How about crossing geodesic distances? No exact formulas known (but scaling limits exist $\leftarrow$ GH)

Outline

1. Intro to random geometry from planar maps

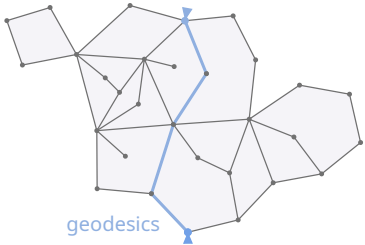


combinatorics

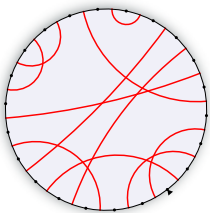


probability

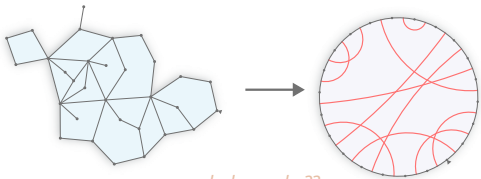
2. Boundary metric and orthogonal polynomials



3. Chord diagrams and DSSYK



4. Relating planar maps and chord diagrams



DSSYK & Chord diagrams

- Quantum system in context of holography: Sachdev-Ye-Kitaev (SYK)

  [Sachdev, Ye, '93] [Sachdev, '10] [Kitaev, '15] [Maldacena, Stanford, '16] [Kitaev, Suh, '18]

$$H_{\mathrm{SYK}} = i^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} \mathcal{J}_{i_1,\ldots,i_p}\, \psi_{i_1} \cdots \psi_{i_p}.$$

DSSYK & Chord diagrams

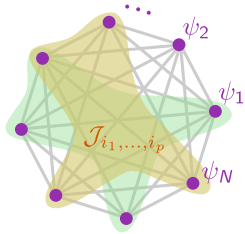
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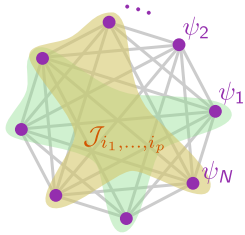
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i.i.d. normal random variables



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- Relation to chord diagrams: Double-Scaled SYK (*DSSYK*) for $q > 0$

  [Cotler, . . . , '17][Berkooz, Narayan, Simon, '18][Berkooz, Isachenkov, Narayan, Narovlansky, '22]

$$\mathbb{E}\left[\mathrm{Tr}\, H_{\mathrm{SYK}}^{2n}\right] \xrightarrow[2p^2/N=\log(1/q)]{N\to\infty} m_{2n}(q)$$

DSSYK & Chord diagrams

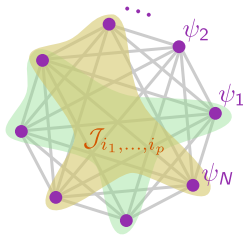
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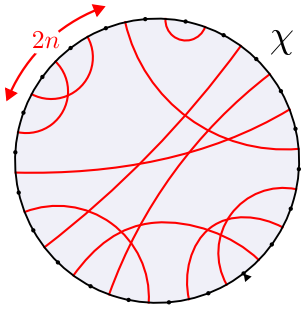
  $\mathcal{J}_{i_1,\ldots,i_p}$: i.i.d. normal random variables



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$$\mathbb{E}\left[\mathrm{Tr}\, H_{\mathrm{SYK}}^{2n}\right] \xrightarrow[2p^2/N=\log(1/q)]{N\to\infty} m_{2n}(q) = \sum_{\substack{\text{chord diagrams} \\ \text{of } 2n\text{-gon}}} \chi\; q^{\mathrm{cr}(\chi)}.$$



DSSYK & Chord diagrams

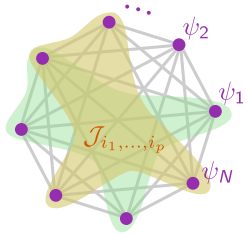
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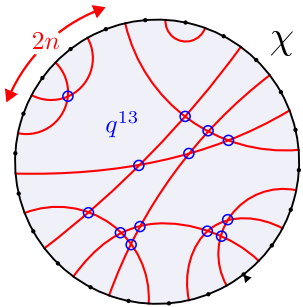


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  $\mathrm{cr}(\chi)$: crossing number



DSSYK & Chord diagrams

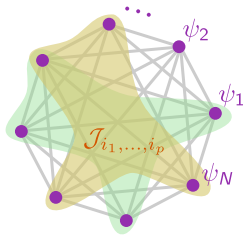
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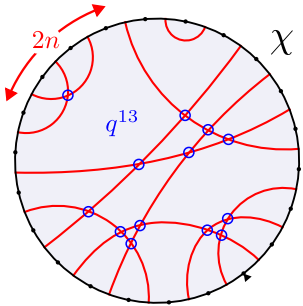


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- Solution known already in combinatorics: Touchard-Riordan formula

  [Touchard, '52][Riordan, '75]

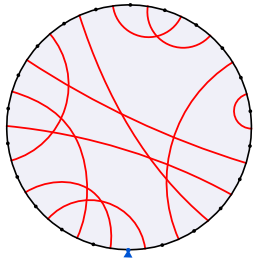
$$m_{2n}(q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] q^{\binom{k+1}{2}}.$$

LDL^T-decomposition of chord diagrams and orthogonal polynomials

- Chord diagram enumeration linked to q-Hermite polynomials

  [Askey, Wilson, '85] [Ismail, Stanton, Viennot, '87]

$$m_n(q) = \langle x^n \rangle = c_q \int_{-\frac{2}{\sqrt{1-q}}}^{\frac{2}{\sqrt{1-q}}} x^n \frac{\prod_{k\geq 0}[1+(2-x^2(1-q))q^k+q^{2k}]}{\sqrt{\frac{4}{1-q}-x^2}} \mathrm{d}x.$$



LDL^T-decomposition of chord diagrams and orthogonal polynomials

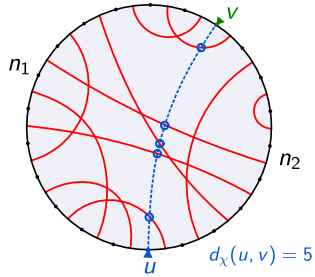
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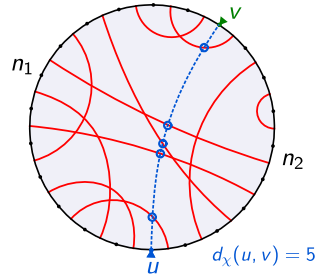
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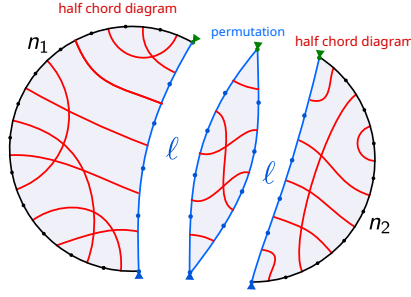
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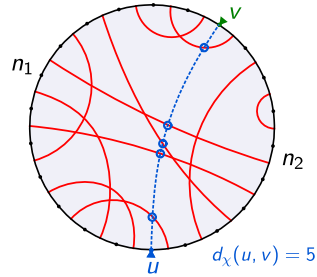


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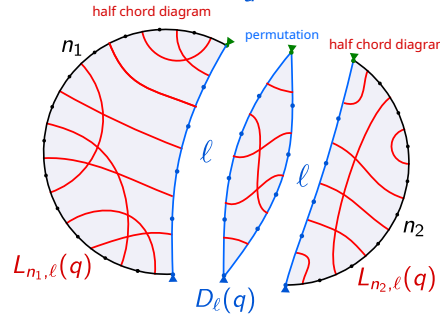
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$$m_{n_1+n_2}(q) = \sum_{\ell\geq 0} L_{n_1,\ell}(q) D_\ell(q) L_{n_2,\ell}(q)$$

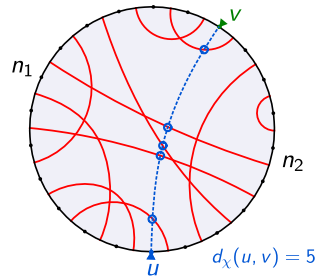


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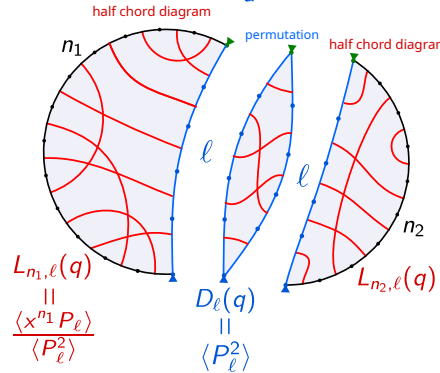
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- The underlying orthogonal polynomials are the *continuous q-Hermite polynomials*

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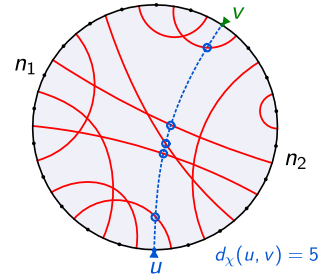
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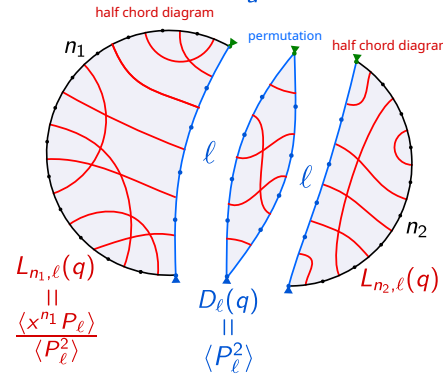
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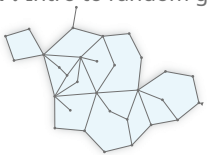
Is this analogy with planar map enumeration more than a coincidence?



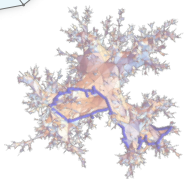


Outline

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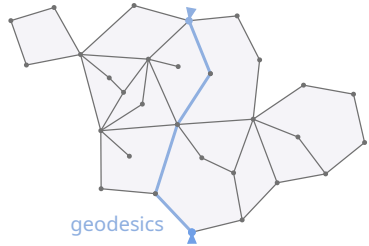


combinatorics

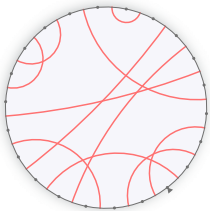


probability

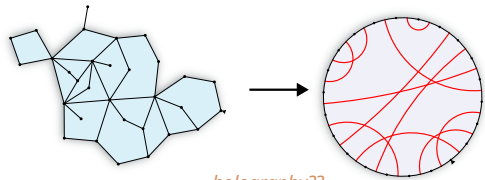
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4. Relating planar maps and chord diagrams



holography??

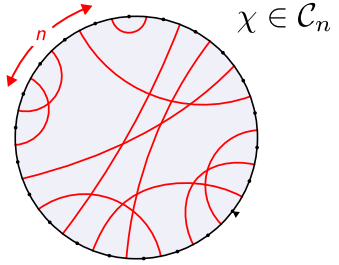
Maps and chord diagrams

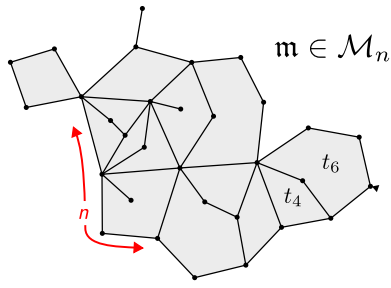
- There exists face weights $\mathbf{t}(q)$ such that $F_n(\mathbf{t}(q)) = m_n(q)$,
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$$t_{2k}(q) = \delta_{k,1} + (q-1)^k \sum_{\ell \geq k} q^{\binom{\ell}{2}} \frac{2\ell - 1}{2k-1} \binom{\ell + k - 2}{\ell - k}$$

  For $0 < q \leq 1$ signature of $\mathbf{t}(q)$ is $(0, -, 0, +, 0, -, \ldots)$.





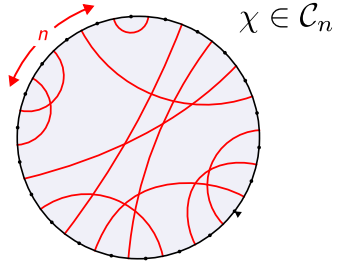
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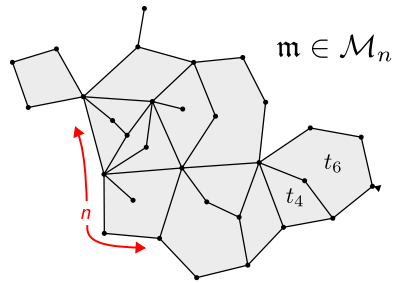
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  For $0 < q \leq 1$ signature of $\mathbf{t}(q)$ is $(0, -, 0, +, 0, -, \ldots)$.
- Non-probabilistic, but for $q < 0.04\ldots$, this is random $|\mathbf{t}(q)|$-admissible bipartite map weighted by $(-1)^{\mathrm{Vert}-n-1}$.
- By LDL^T-uniqueness, orthogonal polynomials coincide...





Maps and chord diagrams

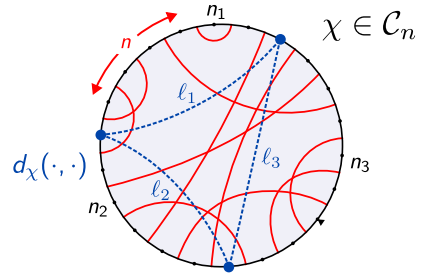
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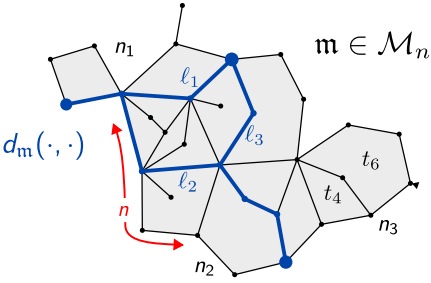
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- Non-probabilistic, but for $q < 0.04\ldots$, this is random $|\mathbf{t}(q)|$-admissible bipartite map weighted by $(-1)^{\mathrm{Vert}-n-1}$.
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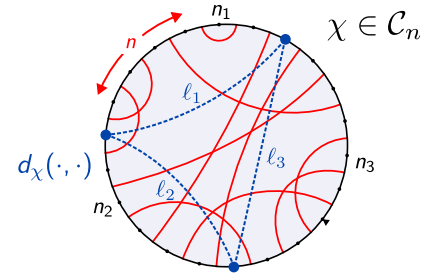
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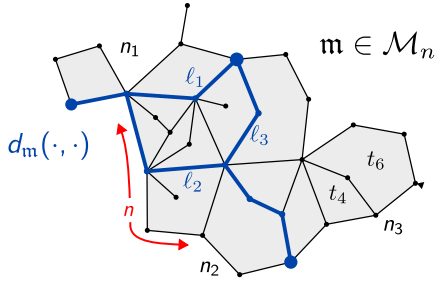
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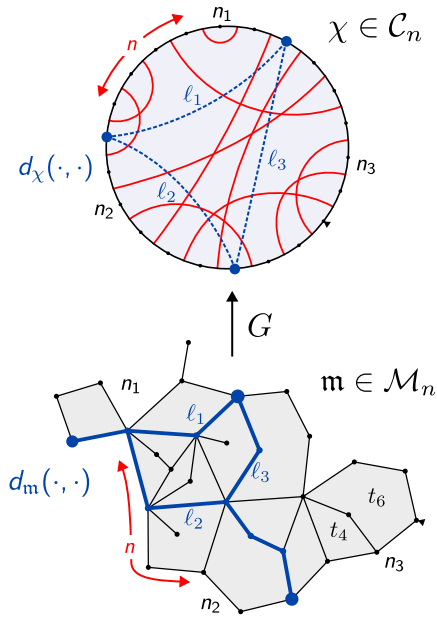
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- But $d_{\chi}(\cdot,\cdot)$ determines χ, so only possible if there exists a mapping $\mathsf{G} : \mathcal{M}_n \to \mathcal{C}_n$ such that $\sum_{\substack{\mathfrak{m} \in \mathcal{M}_n \\ \mathsf{G}(\mathfrak{m}) = \chi}} w(\mathfrak{m}) = q^{\mathrm{cr}(\chi)}$.

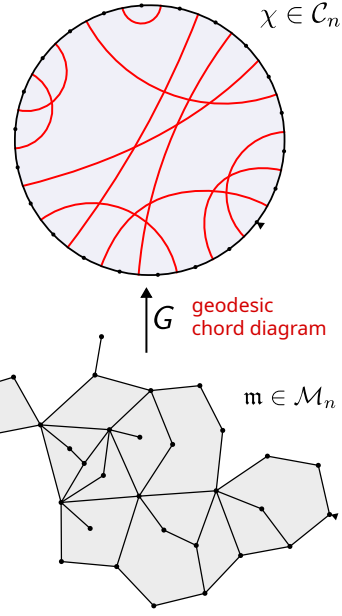


Main combinatorial theorem

Theorem (TB, arXiv:2607.01995)

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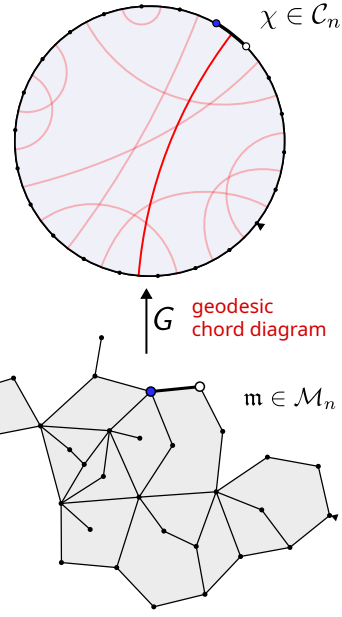
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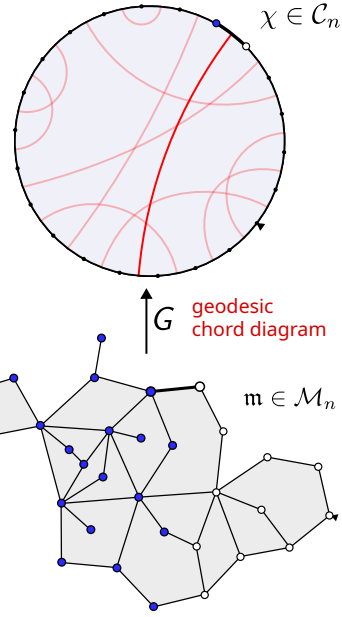
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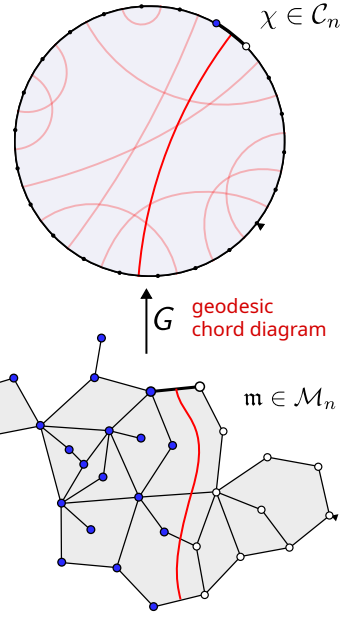
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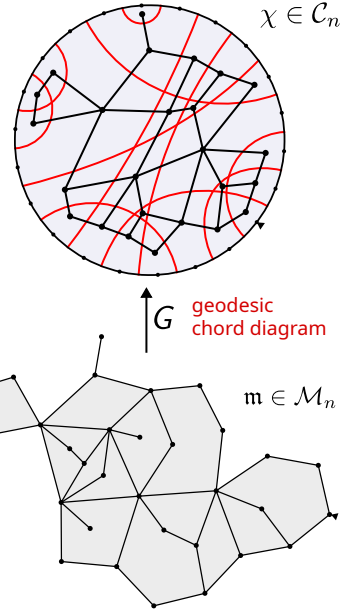
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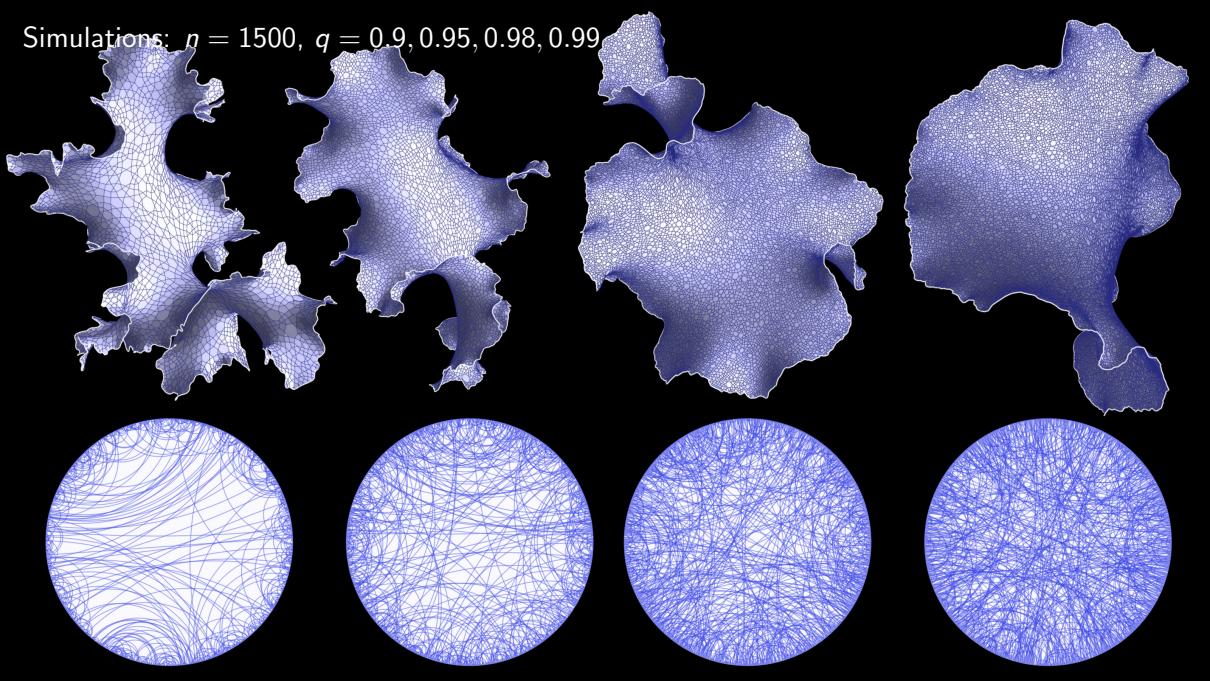
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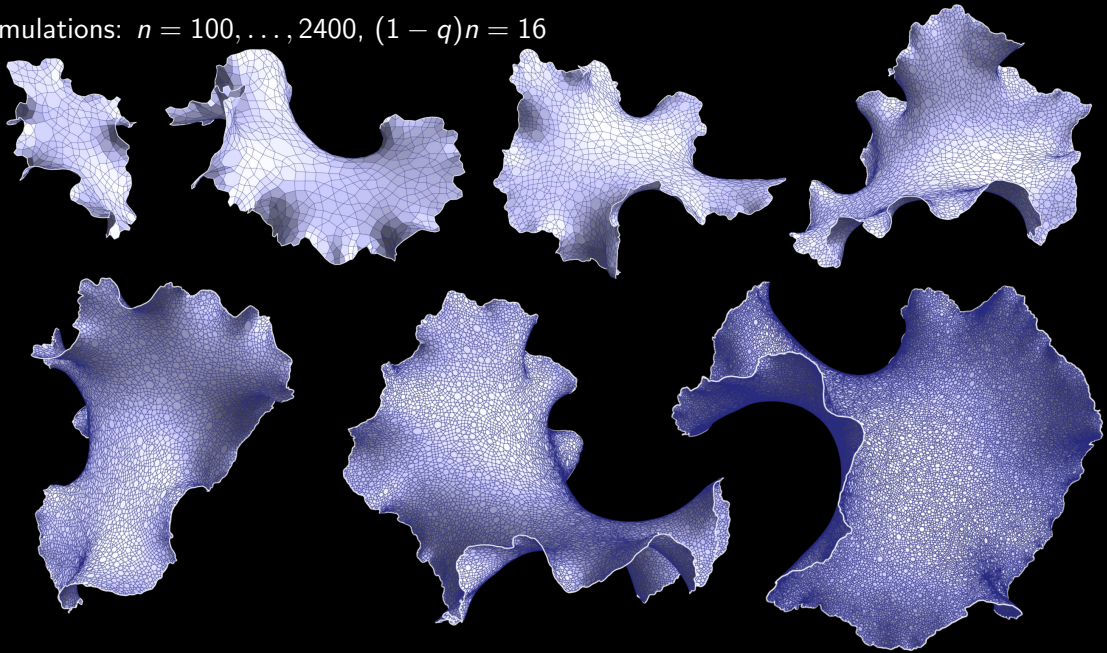
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Simulations: $n = 1500,\ q = 0.9, 0.95, 0.98, 0.99$



Simulations: $n = 100, \ldots, 2400$, $(1-q)n = 16$



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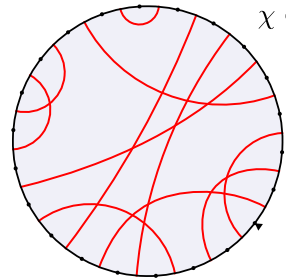
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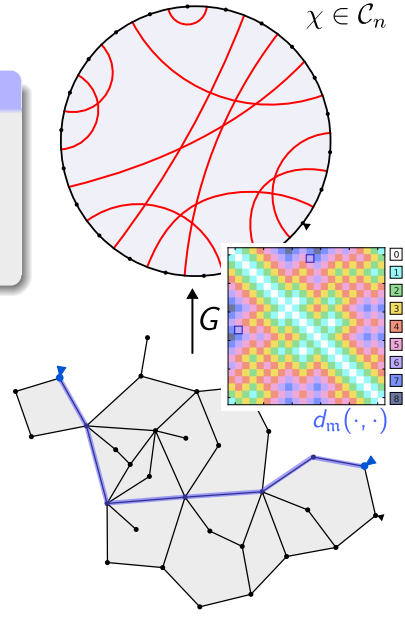
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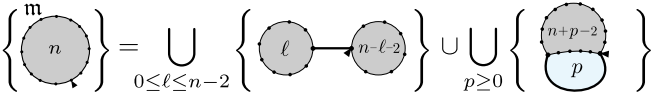
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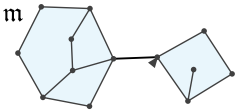
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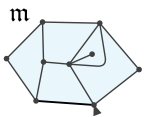


Idea of the proof: refining Tutte equations

- The classic method of Tutte uses the bijective decomposition

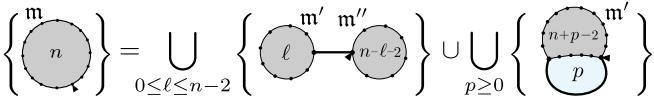






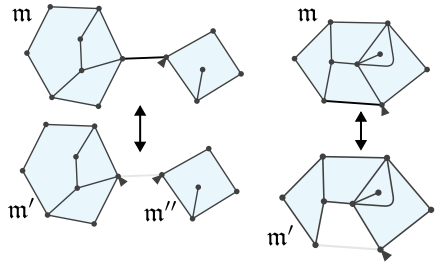
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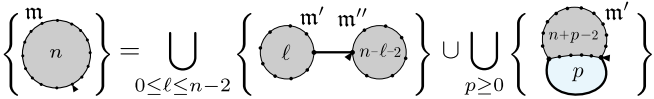
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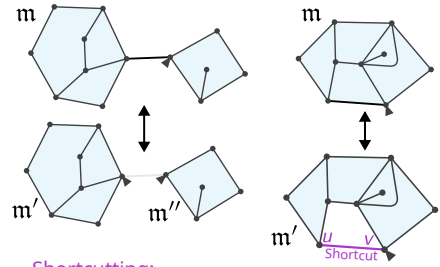


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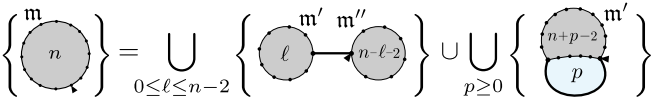


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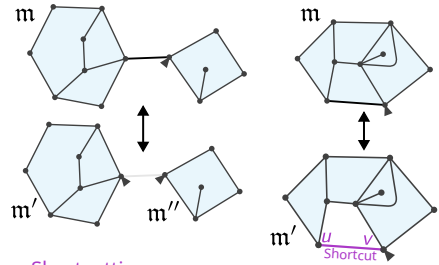


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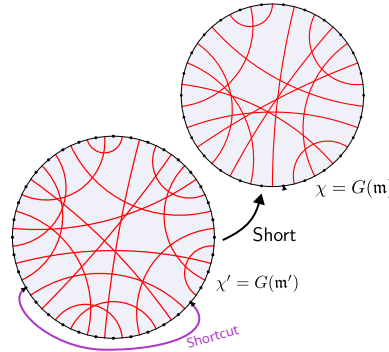
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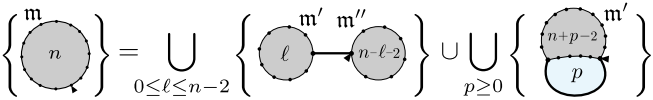
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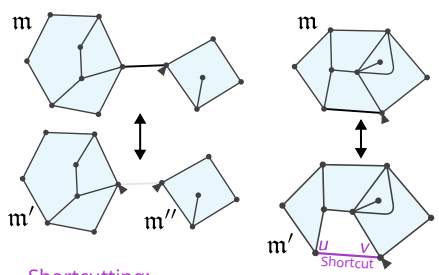
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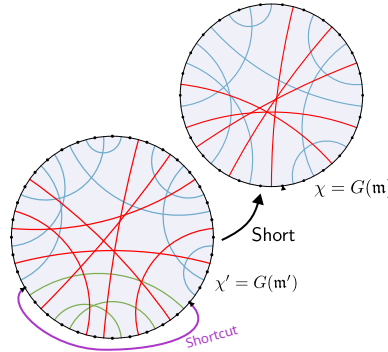
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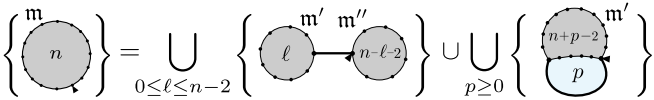
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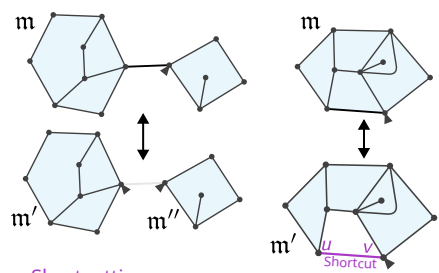
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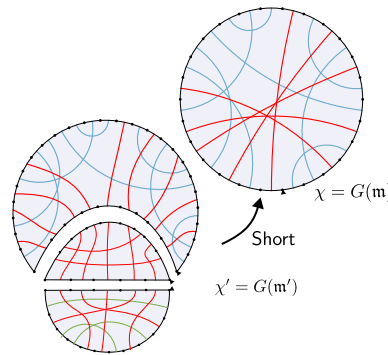
$$F_\chi := \sum_{\substack{\mathfrak{m}\in\mathcal{M} \\ G(\mathfrak{m})=\chi}} w(\mathfrak{m}) = \sum_{p\ge 1} t_{2p} \sum_{\substack{\mathfrak{m}'\in\mathcal{M}_{2n+2p-2} \\ G(\text{shortcutted } \mathfrak{m}')=\chi}} w(\mathfrak{m}') = \sum_{p\ge 1} t_{2p} \sum_{\substack{\chi'\in\mathcal{C}_{2n+2p-2} \\ \text{Short}(\chi')=\chi}} F_{\chi'}$$

- Then show that $F_\chi = q^{\mathrm{cr}(\chi)}$ is unique solution.



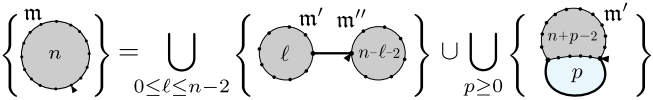
Shortcutting:

$d_{\mathfrak{m}}(\cdot,\cdot) = d_{\mathfrak{m}'}(\cdot,\cdot) \wedge (d_{\mathfrak{m}'}(\cdot,u) + 1 + d_{\mathfrak{m}'}(v,\cdot))$



Idea of the proof: refining Tutte equations

- The classic method of Tutte uses the bijective decomposition



- It translates into the *Tutte equations* [Tutte, '60s]

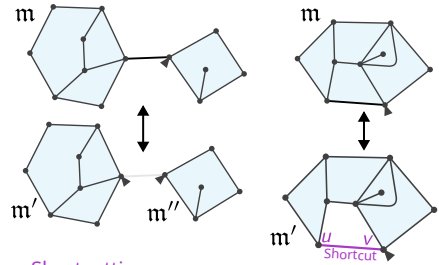
$$F_n(\mathbf{t}) = \sum_{0\le\ell\le n-2} F_\ell(\mathbf{t}) F_{n-\ell-2}(\mathbf{t}) + \sum_{p\ge 1} t_p F_{n+p-2}(\mathbf{t})$$

- Can be refined to prescribed boundary metric / chord diagram χ,

$$F_\chi := \sum_{\substack{\mathfrak{m}\in\mathcal{M}\\ G(\mathfrak{m})=\chi}} w(\mathfrak{m}) = \sum_{p\ge 1} t_{2p} \sum_{\substack{\mathfrak{m}'\in\mathcal{M}_{2n+2p-2}\\ G(\text{shortcutted } \mathfrak{m}')=\chi}} w(\mathfrak{m}') = \sum_{p\ge 1} t_{2p} \sum_{\substack{\chi'\in\mathcal{C}_{2n+2p-2}\\ \text{Short}(\chi')=\chi}} F_{\chi'}$$

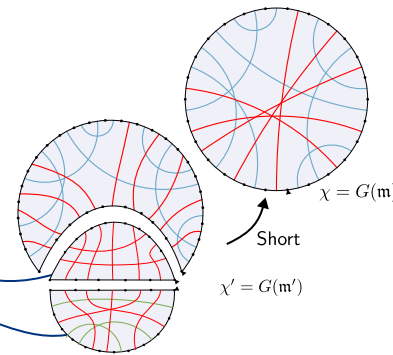
- Then show that $F_\chi = q^{\mathrm{cr}(\chi)}$ is unique solution.

orthogonality of
- q-Stirling number $S_q(k,\ell)$ of the 2nd kind
- q-Stirling number $s_q(\ell,1)$ of the 1st kind

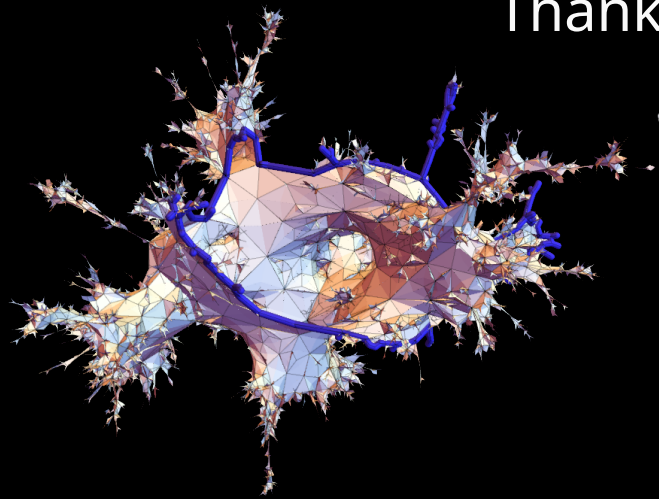


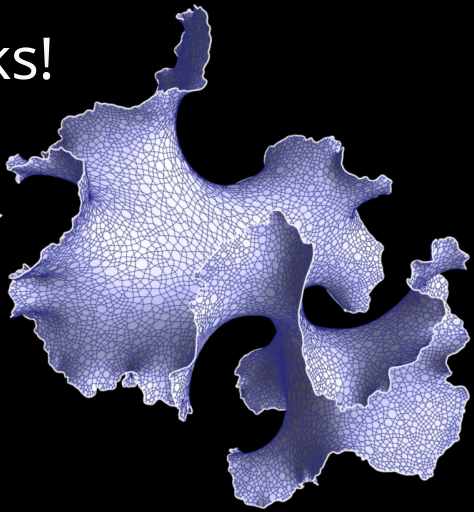
Shortcutting:

$$d_{\mathfrak{m}}(\cdot,\cdot) = d_{\mathfrak{m}'}(\cdot,\cdot) \wedge \left(d_{\mathfrak{m}'}(\cdot,u) + 1 + d_{\mathfrak{m}'}(v,\cdot)\right)$$



Thanks!





Quantum Geometry gallery

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