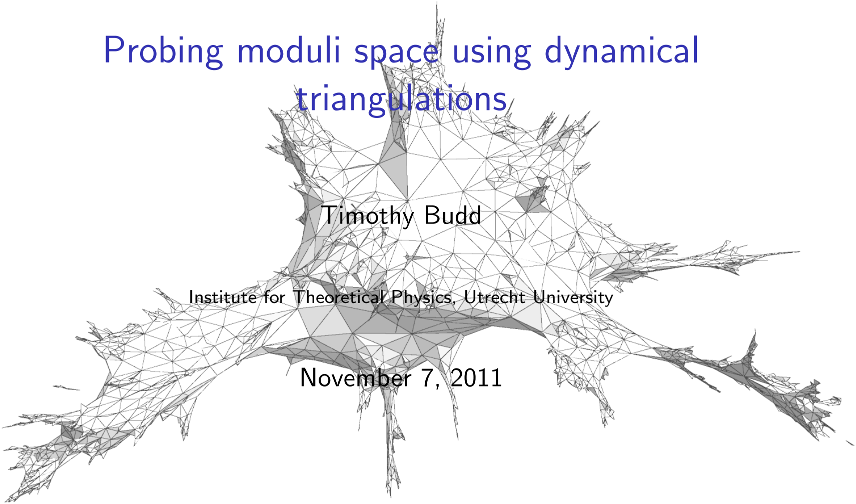

Probing moduli space using dynamical triangulations

Timothy Budd

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November 7, 2011

in collaboration with J. Ambjørn, J. Barkley

Based on: arXiv:1110.4649

Outline

- Introduction to 2d gravity and Liouville theory
- Derive the moduli integrand of the Liouville partition function on the torus
- 2D Dynamical triangulations
- Moduli parameters for torus triangulations
- Results for pure gravity
- Coupling to $c = -2$ matter
- Conclusions and outlook

2D gravity and Liouville theory

- Classical pure gravity in 2d is not very interesting

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- Bosonic string in d-dimensional Euclidean space (a conformal field theory with central charge $c = d$)

$$S_m[X,g] = \int d^2x\sqrt{g}g^{ab}\partial_a X^i\partial_b X^j\delta_{ij} \quad (4)$$

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- Requiring the partition function to be independent of $\hat{g}$ leads to the Liouville partition function

$$Z = \int d\tau \, \mathcal{D}_{\hat{g}}\phi \, \mathcal{D}_{\hat{g}}X \; J[\hat{g}] \exp\left(-S_L[\hat{g},\phi] - S_m[X,\hat{g}]\right) \quad (6)$$

$$S_L[\hat{g},\phi] = \frac{1}{4\pi}\int d^2x \sqrt{\hat{g}} \left(\phi\Delta\phi + Q\hat{R}\phi + \mu e^{2\beta\phi}\right) \quad (7)$$

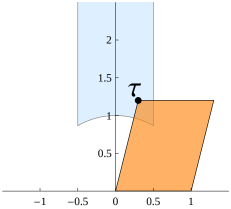
$$Q = \sqrt{\frac{25-d}{6}} = \frac{1}{\beta} + \beta \quad (8)$$

Liouville partition function for the torus

- Take flat metric on the torus

$$\hat{g}_{ab}(\tau, x) = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{pmatrix} \qquad (9)$$

and fix volume $\delta\left(V - \int d^2x \sqrt{\hat{g}} e^{2\beta\phi}\right)$



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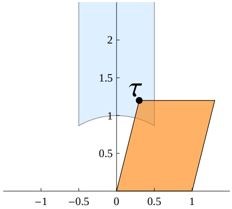

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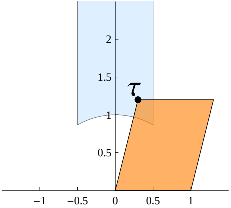

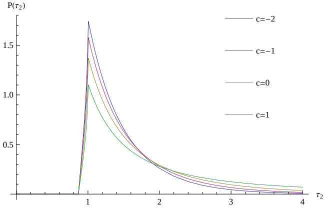

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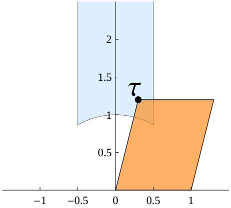

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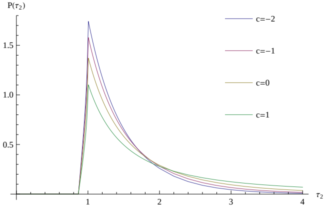

- Main goal of this project: Test whether this distribution of moduli also appears in discretized gravity, i.e. dynamical triangulations.

2D dynamical triangulation

- Discretize the pure gravity partition function ($d = c = 0$)

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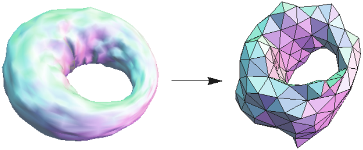
$$\rightarrow \quad Z(\mu) = \sum_{T \in \mathcal{T}} \frac{1}{C_T} e^{-\mu N}$$



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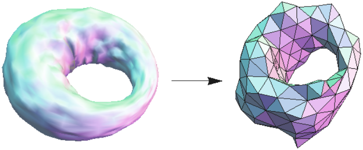

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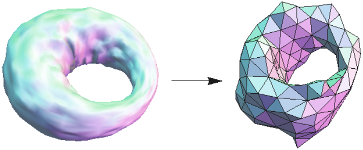

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- For genus g:

$$Z^{(g)}(N) \sim N^{(5g-7)/2} e^{\mu_0 N} \quad \text{as } N \to \infty. \tag{11}$$

2D dynamical triangulation - Monte Carlo simulations

- Fixed volume partition function

$$Z(N) \propto \sum_{T \in \mathcal{T}(N)} \frac{1}{C_T} = \frac{1}{N!} \sum_{\text{labelled } T \in \mathcal{T}(N)} 1. \tag{12}$$

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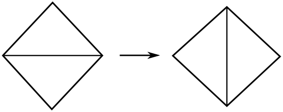

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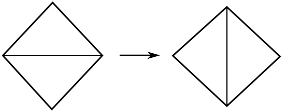

- Use this to measure moduli distribution in $Z^{g=1}(N)$. More precisely we want to measure the fraction $P(\tilde{\tau}_2)\Delta\tilde{\tau}_2$ of randomly generated triangulations with τ_2 between $\tilde{\tau}_2$ and $\tilde{\tau}_2 + \Delta\tilde{\tau}_2$.

Measurement of modular parameter τ - continuum

- How do we determine τ for a metric g_{ab} on the torus? Find periodic coordinates $x^1, x^2 \in [0,1)$ such that $ds^2 = \Omega^2(x)\hat{g}_{ab}dx^a dx^b$ for $\hat{g}_{ab}$ constant.

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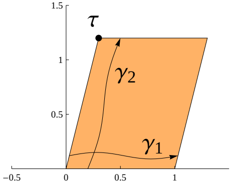

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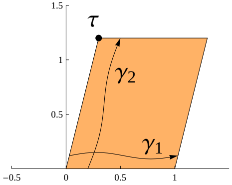

Measurement of τ on triangulation

- Recipe:
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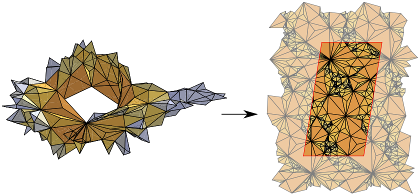
- Once we have those ingredients we can do stuff like this:



Discrete differential forms

- In 2d triangulations we have
  - Vertices: 0-simplices denoted by i,
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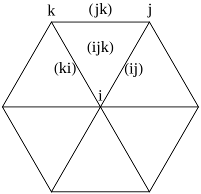

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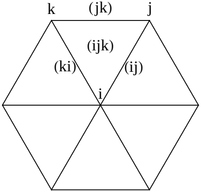

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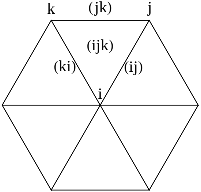

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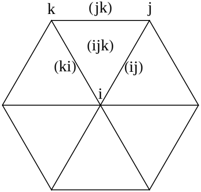

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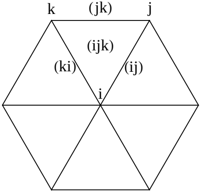

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- $\Delta = d\delta + \delta d$ becomes a matrix of which we can determine the nullspace $\Delta\alpha = 0$ ($\Leftrightarrow d\alpha = 0$ and $\delta\alpha = 0$)

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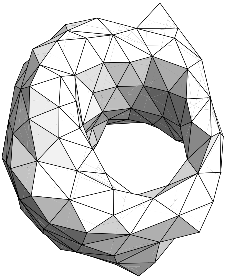
- To find curves γ_j that generate the fundamental group we grow a spanning tree:



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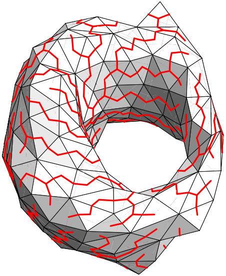

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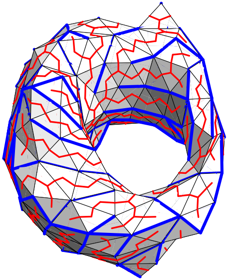

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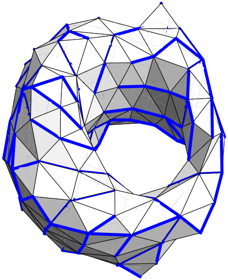

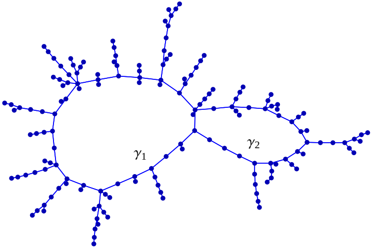

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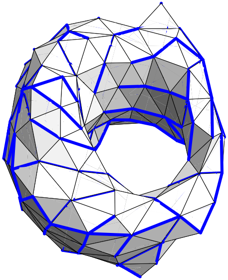

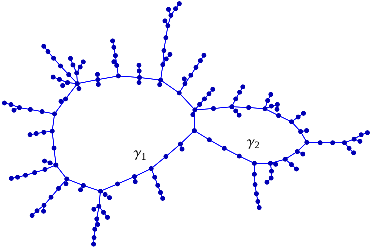

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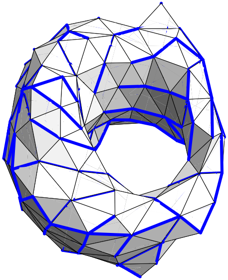

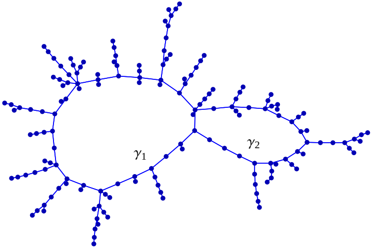

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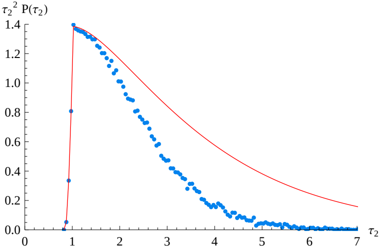

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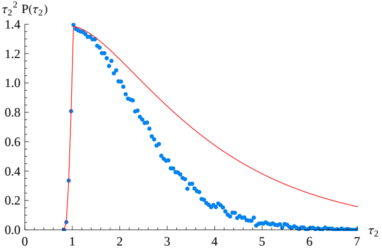

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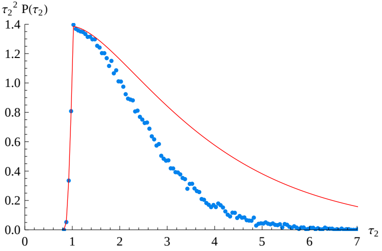

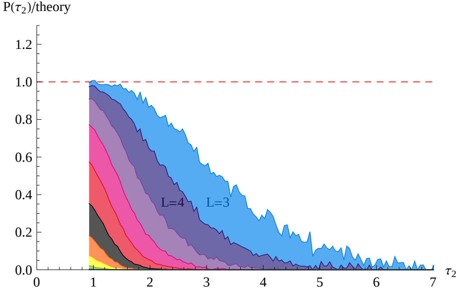

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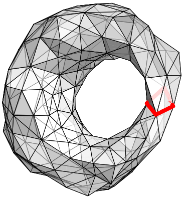

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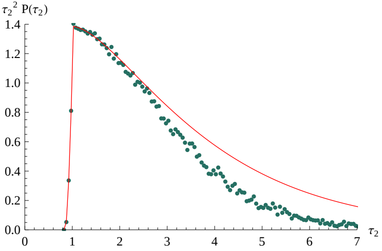

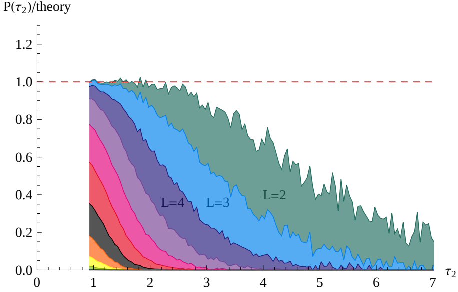

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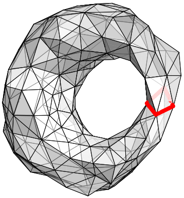

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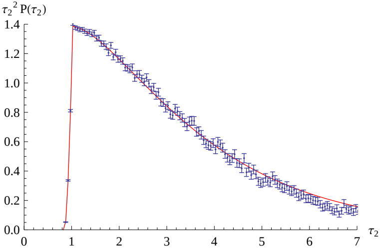

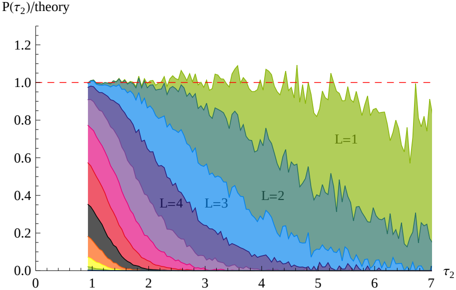

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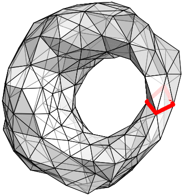
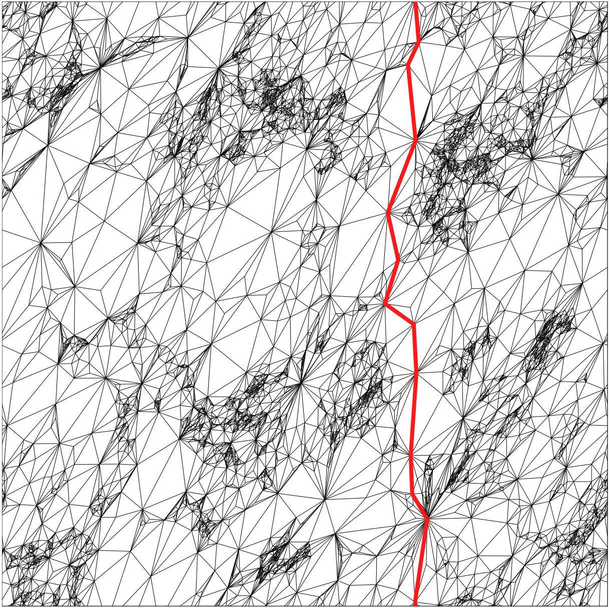


Example of a harmonic embedding



Dynamical triangulations coupled to $c = -2$ matter

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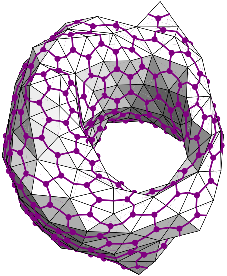
$$\Delta_{st} = \begin{cases} 3 & s = t \\ -1 & s, t \text{ adjacent} \\ 0 & \text{else} \end{cases}$$



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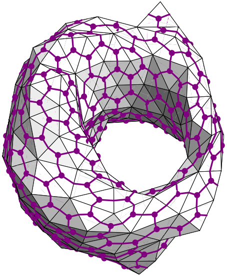

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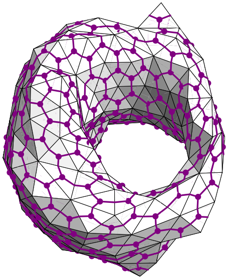

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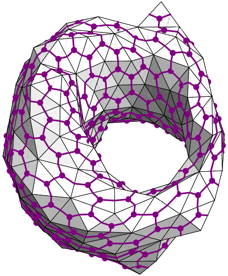
- Take formally $c = -2$:

$$Z_{c=-2}(N) = \sum_{T\in\mathcal{T}} \frac{1}{C_T} \sum_{\text{spanningtrees}} 1$$



Generating random $c = -2$ triangulations

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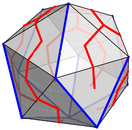

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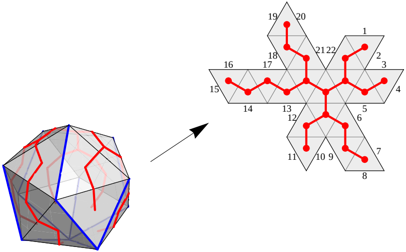

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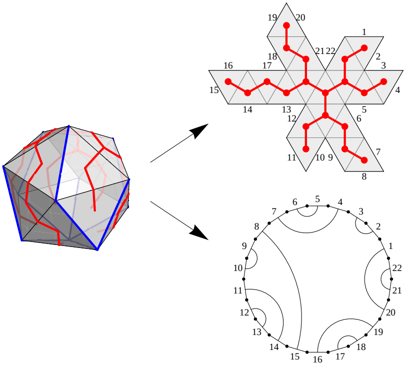

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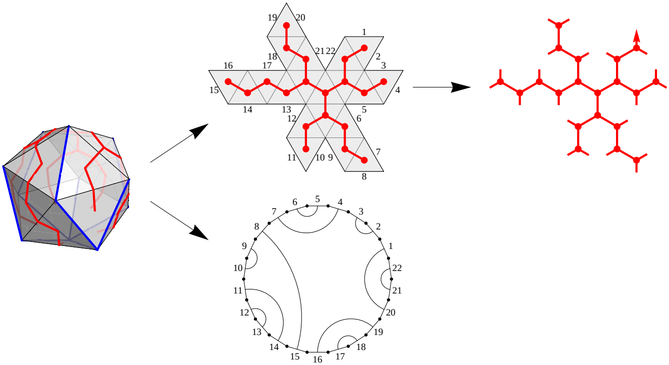

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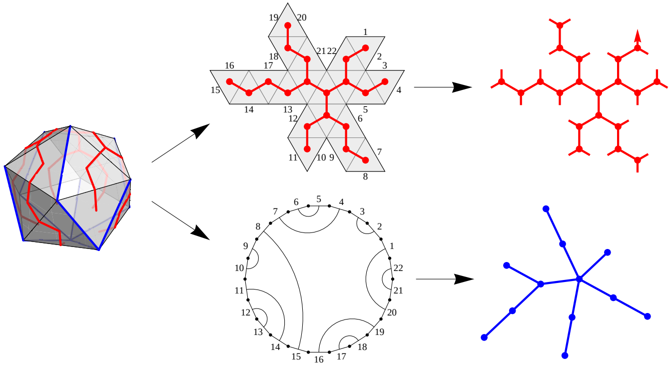
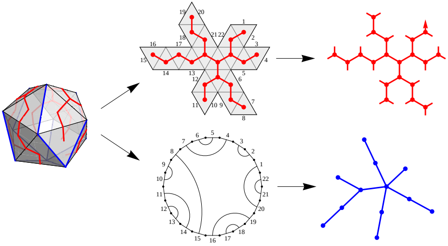


- Both trees can be easily generated randomly.

Generating random $c = -2$ triangulations - Torus



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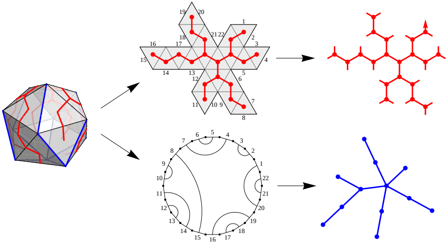

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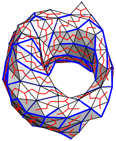

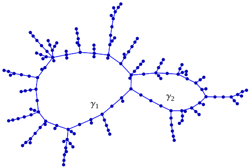

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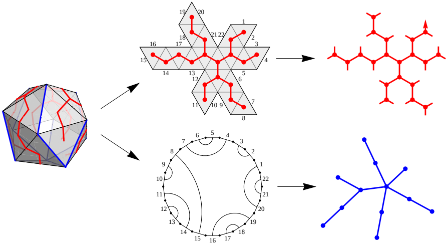

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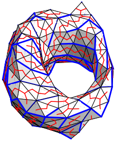
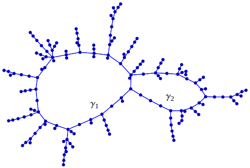
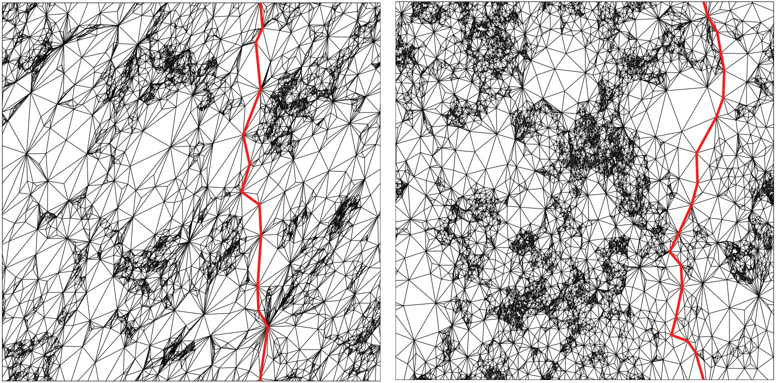




- We can construct these from planar trees by identifying three vertices [Chapuy, 2011].

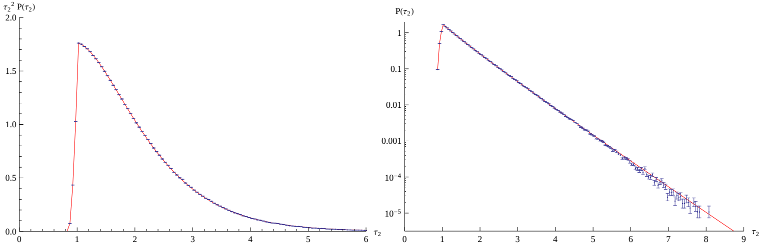
Examples of harmonic embeddings



$c = 0$ $\qquad\qquad$ $c = -2$

Results for $c = -2$ dynamical triangulations

- Smaller c means more regular. Data fits theory "perfectly" already at $N = 8000$:



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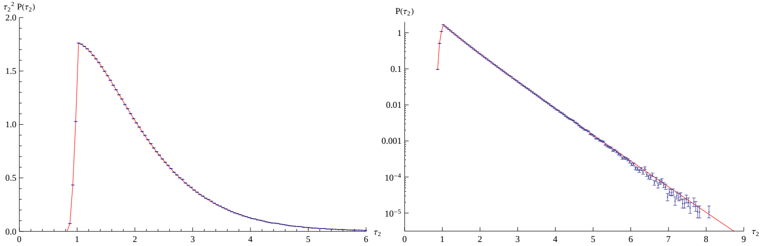

- For example we measure $P(\tau_2 < 1.5) = 0.64067 \pm 0.00014$, while Liouville theory predicts

$$P(\tau_2 < 1.5) = \frac{\int_{\tau_2 < 1.5} \frac{d^2\tau}{\tau_2^2} F(\tau)^{c-1}}{\int \frac{d^2\tau}{\tau_2^2} F(\tau)^{c-1}} = 0.640648\ldots \qquad (15)$$

Conclusions and outlook

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  - The same moduli measurements are also used in CDT in 2+1 dimensions[1]. We have more confidence now that these measurements make sense.

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