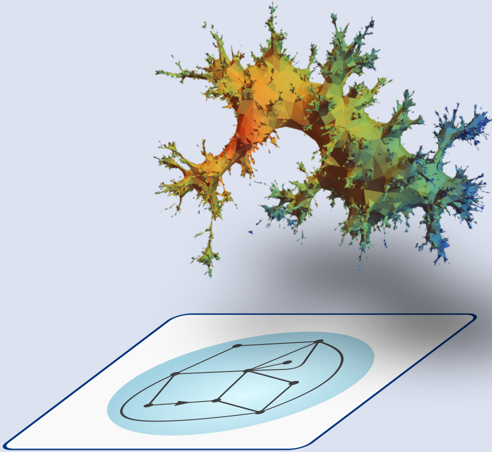
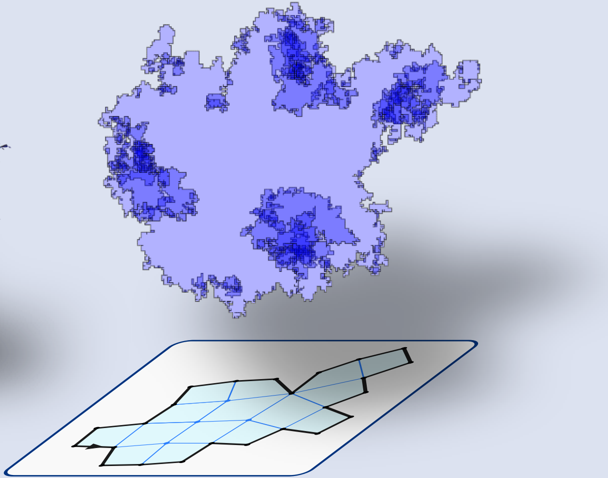
YOUNGST@RS - Combinatorics in Fundamental Physics, 26-11-2024

Random geometry and the enumerative combinatorics of maps

Timothy Budd



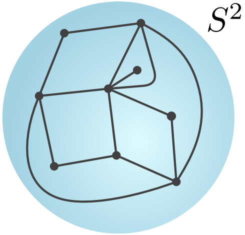


Radboud University

T.Budd@science.ru.nl
http:// hef.ru.nl/ ~tbudd/

Discrete geometry by planar maps

- Planar map is a connected graph embedded in S^2 viewed up to continuous deformation.



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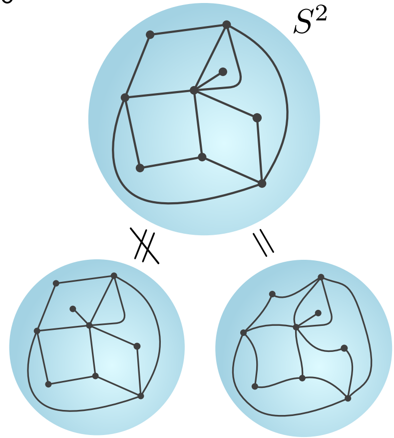

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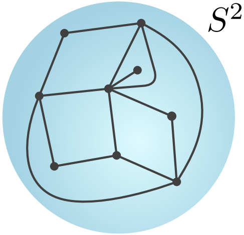

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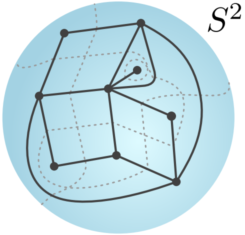

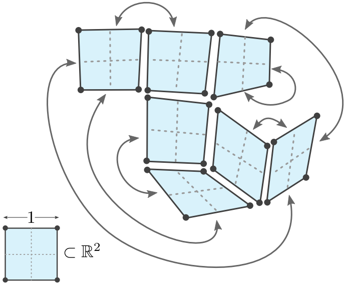

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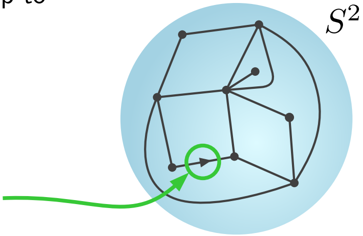

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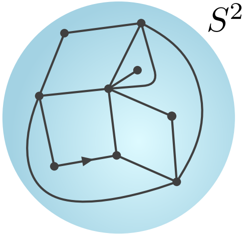

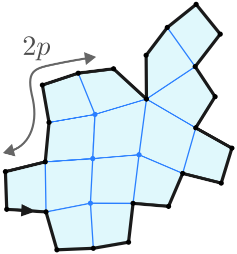

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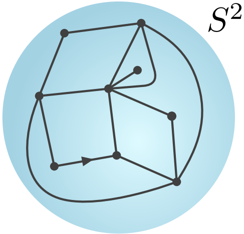

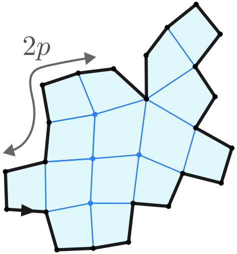

Many map enumeration methods:

- Recursive methods and generating functions [Tutte, '60s] [Brown, Bender, Canfield, Goulden, Jackson, Ambjørn, Bousquet-Mélou, . . .]
- Matrix models ['t Hooft, Brézin, Itzykson, Parisi, Zuber, Kazakov, Kostov, Ginsparg, Zinn-Justin, . . .]
- Probabilistic methods [Le Gall, Miermont, Curien, Bettinelli, Sheffield, Miller, Gwynne, Budzinski, . . .]
- Bijective method [Cori, Vauqelin, Schaeffer, Bouttier, Di Francesco, Guitter, Fusy, Chapuy, Bernardi, Miermont, . . .]

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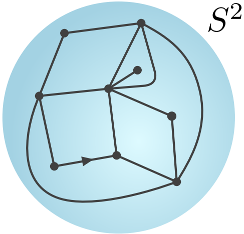

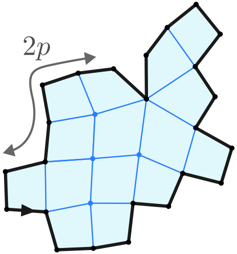

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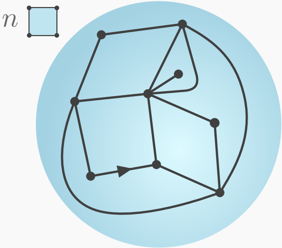
Quadrangulations (of the 2-sphere)



[Tutte, '63]

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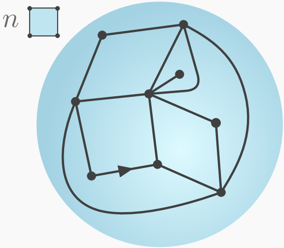

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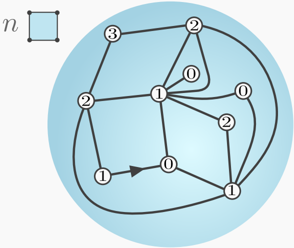
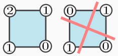
Height-labeled quadrangulations (without alternating faces)





[Kostov, Zinn-Justing, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

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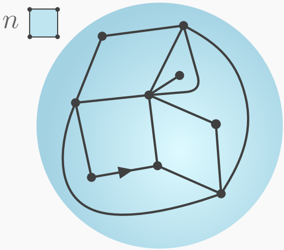

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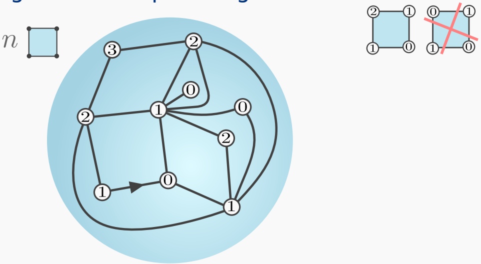

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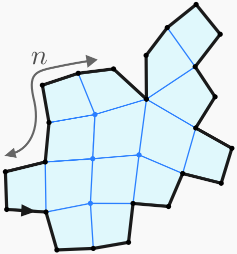
Flat quadrangulations (of the disk)



[TB, '24+]

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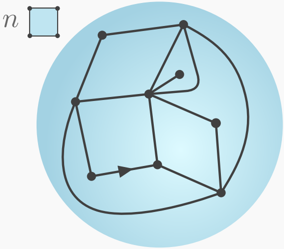

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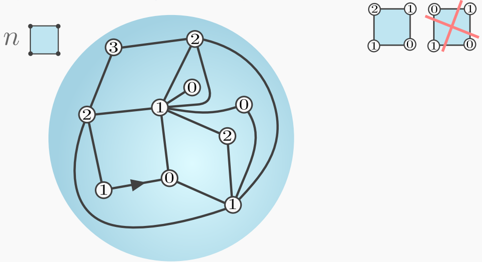

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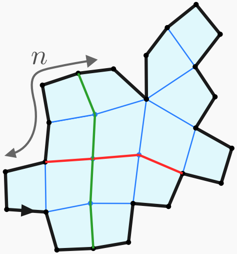

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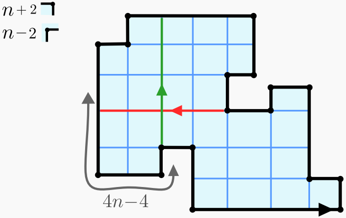
Rigid (flat) quadrangulations (of the disk)



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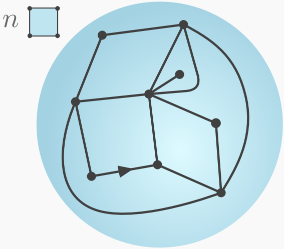

$$\underset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}} \frac{\mu_{\mathrm{Q}}^{n}}{n^{5/2}}$$

$$\mu_{\mathrm{Q}} = 12$$

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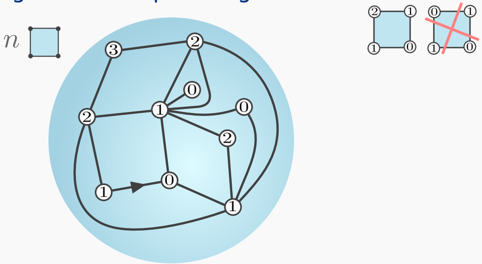

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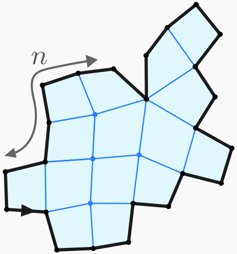

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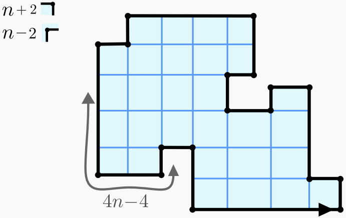

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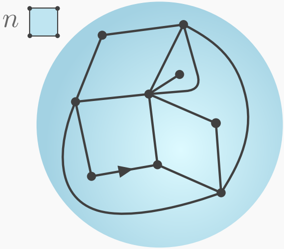

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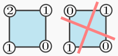

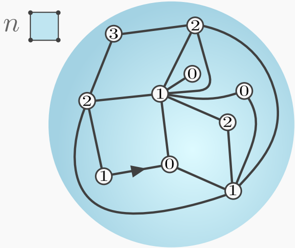

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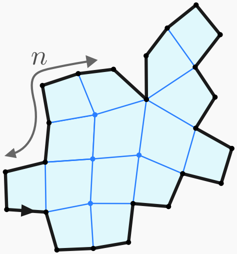

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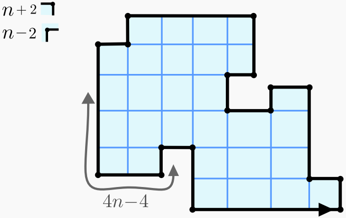

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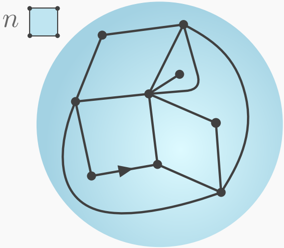

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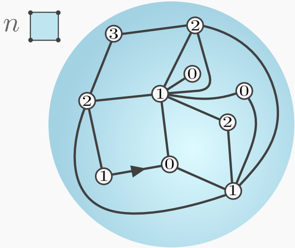

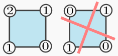

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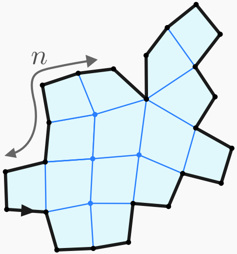

$$\overset{n\to\infty}{\sim} \frac{\pi}{4} \frac{\mu_{\mathrm{F}}^n}{n^2 \log^2 n}$$

$$\mu_{\mathrm{F}} = e^{\frac{4G}{\pi}}$$

[TB, '24+]

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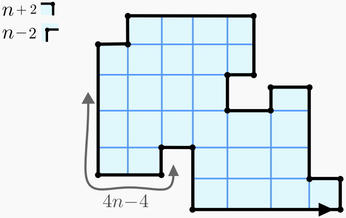

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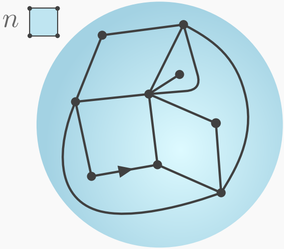

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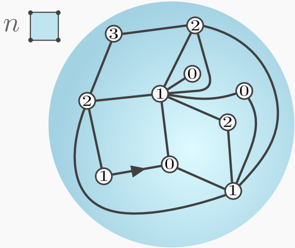

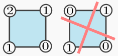

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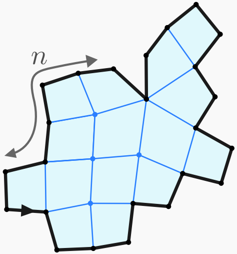

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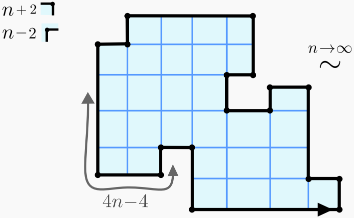

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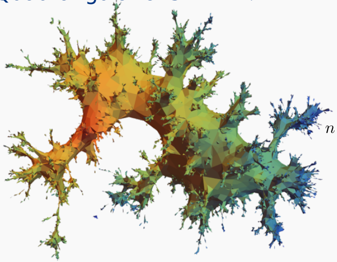
$$\mu_{\mathrm{R}} = 4\pi$$

[TB, Zonneveld, '24+]

Quadrangulations (of the 2-sphere)



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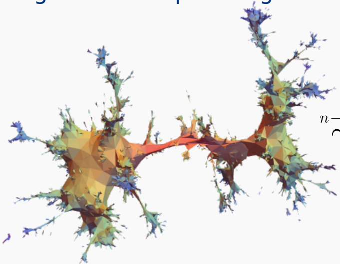

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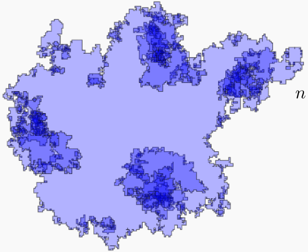

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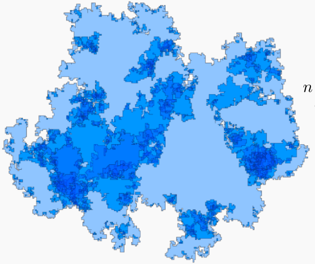

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2D (Euclidean) quantum gravity

$$\text{“}\int \mathcal{D}g_{ab}(x)\, e^{-\Lambda \mathrm{Area}[g]}\text{”}$$

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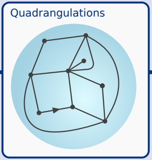
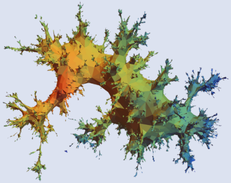
$g_{ab}(x)$

discretize



Quadrangulations

scaling limit



Brownian sphere

[Le Gall, '10] [Miermont, '10]

Liouville Quantum Gravity $\gamma = \sqrt{8/3}$

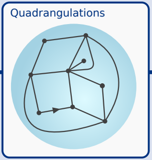
[Polyakov, '81] [Miller, Sheffield, '15]
[Duplantier, Miller, Sheffield, Gwynne, Bernardi, Holden, Sun, Rhodes, Vargas, Kuppiainen, ..., '10s-'20s]

2D (Euclidean) quantum gravity

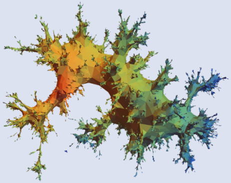
$$“\int \mathcal{D}g_{ab}(x)\, e^{-\Lambda \mathrm{Area}[g]}”$$

$g_{ab}(x)$

discretize



scaling limit



Brownian sphere

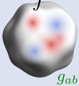
[Le Gall, '10] [Miermont, '10]

Liouville Quantum Gravity $\gamma = \sqrt{8/3}$

[Polyakov, '81] [Miller, Sheffield, '15]
[Duplantier, Miller, Sheffield, Gwynne, Bernardi, Holden, Sun, Rhodes, Vargas, Kuppiainen, ..., '10s-'20s]

2D quantum gravity coupled to scalar field

$$“\int \mathcal{D}g_{ab}(x)\mathcal{D}\varphi(x)\, e^{-\Lambda \mathrm{Area}[g] - \int \mathrm{d}^2x \sqrt{g}\, \partial^a\varphi \partial_a\varphi}”$$



$\varphi(x)$

$g_{ab}(x)$

2D (Euclidean) quantum gravity

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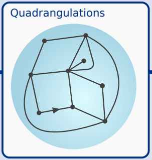
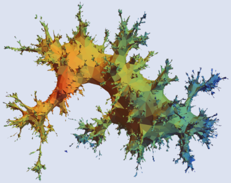


scaling limit



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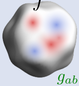

$\varphi(x)$

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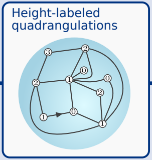
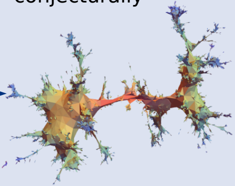
discretize



scaling limit

conjecturally



Critical
Liouville Quantum Gravity $\gamma = 2$

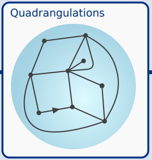
+ Gaussian Free Field

[Gwynne, Ding, Aru, Holden Powell, Sun, ..., '20s]

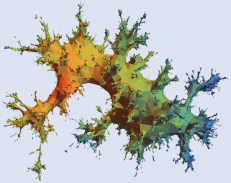
2D (Euclidean) quantum gravity

“$\int \mathcal{D}g_{ab}(x)\, e^{-\Lambda\mathrm{Area}[g]}$”

discretize



scaling limit



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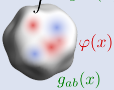
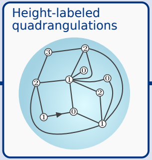
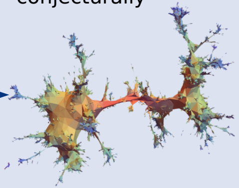


discretize



scaling limit

conjecturally



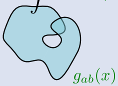
Critical Liouville Quantum Gravity $\gamma = 2$

\+ Gaussian Free Field

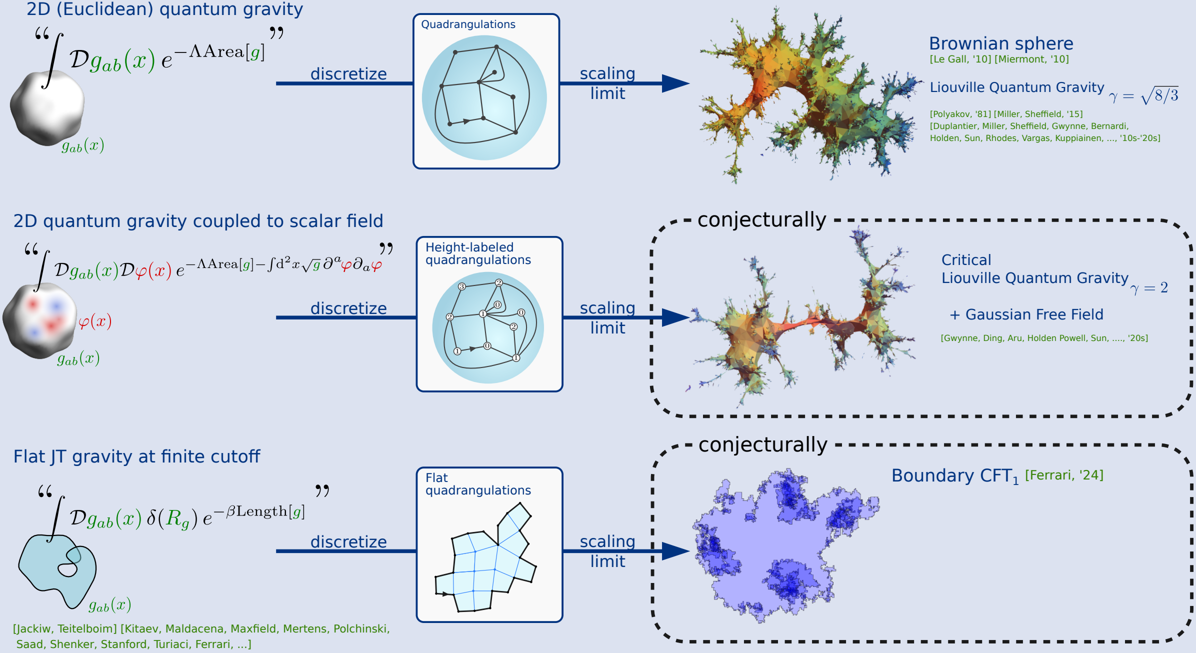
[Gwynne, Ding, Aru, Holden Powell, Sun, ..., '20s]

Flat JT gravity at finite cutoff

“$\int \mathcal{D}g_{ab}(x)\,\delta(R_g)\, e^{-\beta\mathrm{Length}[g]}$”



[Jackiw, Teitelboim] [Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Ferrari, ...]



2D (Euclidean) quantum gravity

$$“\int \mathcal{D}g_{ab}(x)\, e^{-\Lambda \mathrm{Area}[g]}”$$

$g_{ab}(x)$

discretize

Quadrangulations

scaling limit

Brownian sphere [Le Gall, '10] [Miermont, '10]

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$g_{ab}(x)$

discretize

Height-labeled quadrangulations

scaling limit

conjecturally

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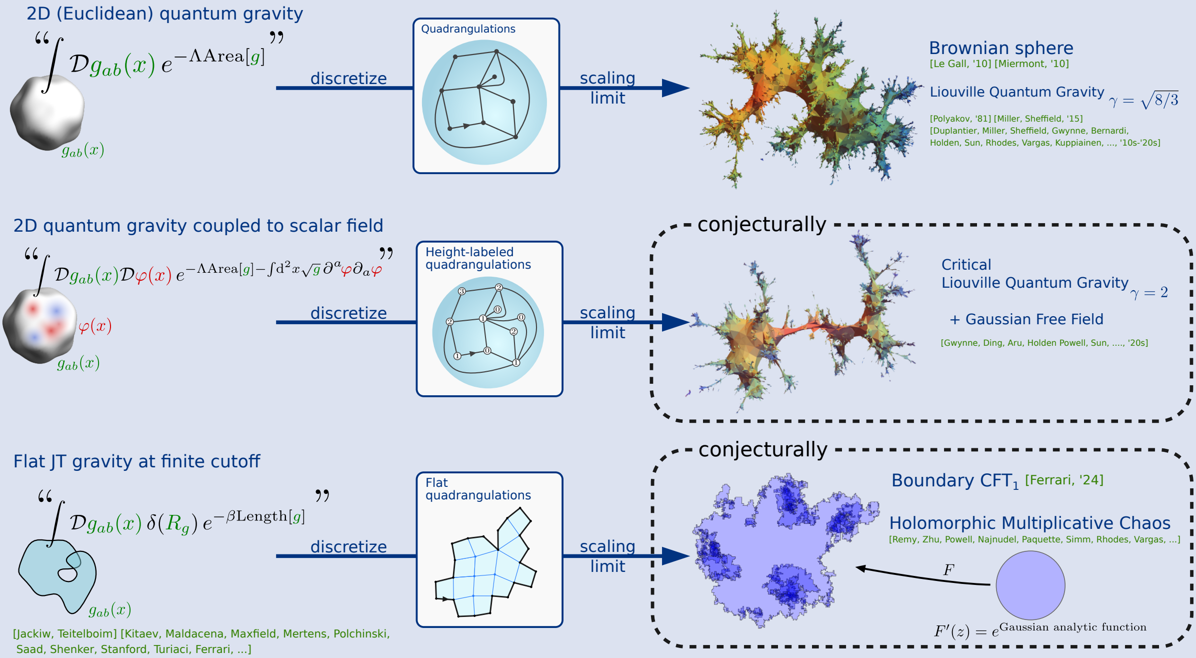
discretize

Flat quadrangulations

scaling limit

conjecturally

Boundary CFT_1 [Ferrari, '24]


2D (Euclidean) quantum gravity
"$\int \mathcal{D}g_{ab}(x)\, e^{-\Lambda \mathrm{Area}[g]}$"
$g_{ab}(x)$
discretize
Quadrangulations
scaling limit
Brownian sphere
[Le Gall, '10] [Miermont, '10]
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$\varphi(x)$
$g_{ab}(x)$
discretize
Height-labeled quadrangulations
3
2
2
1
0
0
2
1
0
1
scaling limit
conjecturally
Critical Liouville Quantum Gravity $\gamma = 2$
+ Gaussian Free Field
[Gwynne, Ding, Aru, Holden Powell, Sun, ..., '20s]
Flat JT gravity at finite cutoff
"$\int \mathcal{D}g_{ab}(x)\, \delta(R_g)\, e^{-\beta \mathrm{Length}[g]}$"
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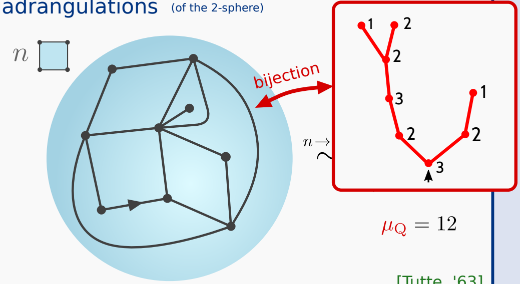
[Jackiw, Teitelboim] [Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Ferrari, ...]
discretize
Flat quadrangulations
scaling limit
conjecturally
Boundary CFT$_1$ [Ferrari, '24]
Holomorphic Multiplicative Chaos
[Remy, Zhu, Powell, Najnudel, Paquette, Simm, Rhodes, Vargas, ...]
F
$F'(z) = e^{\text{Gaussian analytic function}}$

Quadrangulations (of the 2-sphere)

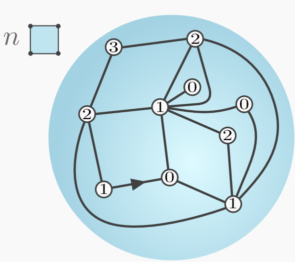


$\overset{n\to}{\sim}$

$\mu_{\mathrm{Q}} = 12$

[Tutte, '63]

Height-labeled quadrangulations (without alternating faces)

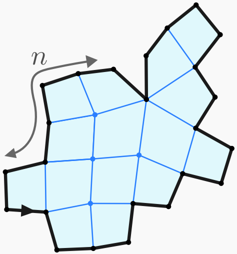


$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{H}}^n}{n^2 \log^2 n}$$

$\mu_{\mathrm{H}} = 4\pi$

[Kostov, Zinn-Justing, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

Flat quadrangulations (of the disk)

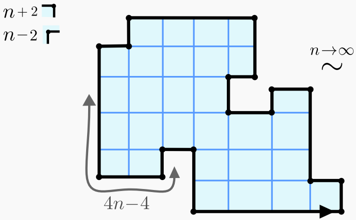


$$\overset{n\to\infty}{\sim} \frac{\pi}{4} \frac{\mu_{\mathrm{F}}^n}{n^2 \log^2 n}$$

$\mu_{\mathrm{F}} = e^{\frac{4G}{\pi}}$

[TB, '24+]

Rigid (flat) quadrangulations (of the disk)

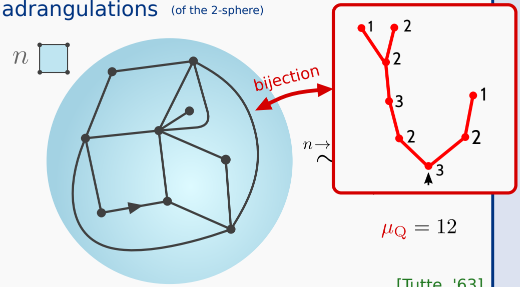


$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{R}}^n}{n^2 \log^2 n}$$

$\mu_{\mathrm{R}} = 4\pi$

[TB, Zonneveld, '24+]

Quadrangulations (of the 2-sphere)



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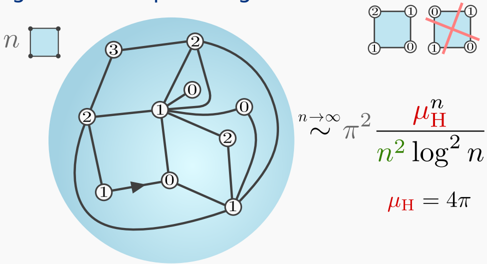
[Tutte, '63]

Height-labeled quadrangulations (without alternating faces)



$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{H}}^n}{n^2 \log^2 n}$$

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[Kostov, Zinn-Justing, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

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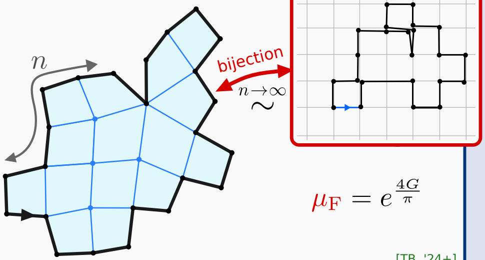

$$\overset{n\to\infty}{\sim}$$

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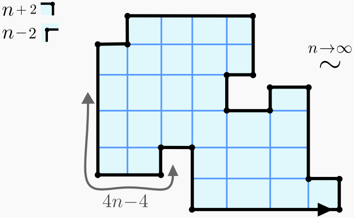
Rigid (flat) quadrangulations (of the disk)

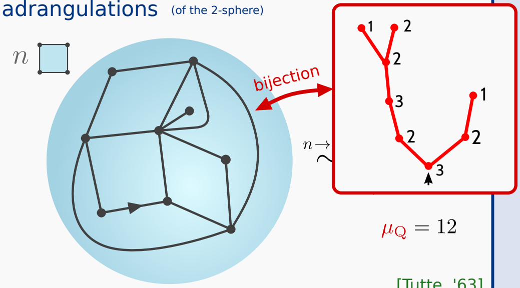


$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{R}}^n}{n^2 \log^2 n}$$

$\mu_{\mathrm{R}} = 4\pi$

[TB, Zonneveld, '24+]

Quadrangulations (of the 2-sphere)

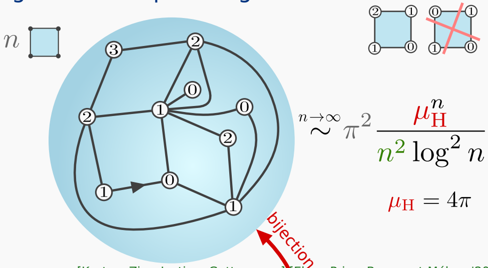


$\overset{n\to}{\sim}$

$\mu_{\mathrm{Q}} = 12$

[Tutte, '63]

Height-labeled quadrangulations (without alternating faces)



$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{H}}^n}{n^2 \log^2 n}$$

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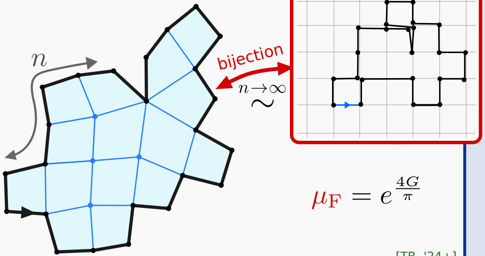
[Kostov, Zinn-Justing, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

Flat quadrangulations (of the disk)

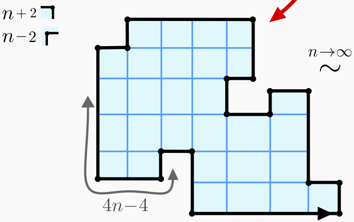


$\overset{n\to\infty}{\sim}$

$\mu_{\mathrm{F}} = e^{\frac{4G}{\pi}}$

[TB, '24+]

Rigid (flat) quadrangulations (of the disk)



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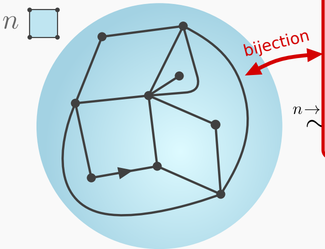
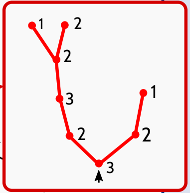
$\mu_{\mathrm{R}} = 4\pi$

[TB, Zonneveld, '24+]

Quadrangulations (of the 2-sphere)

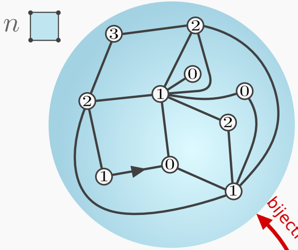


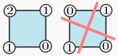


$\mu_{\mathrm{Q}} = 12$

[Tutte, '63]

Height-labeled quadrangulations (without alternating faces)





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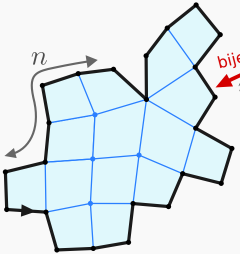
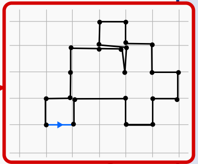
[Kostov, Zinn-Jusling, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

Flat quadrangulations (of the disk)

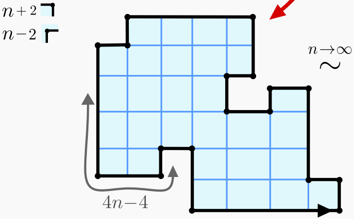




$\mu_{\mathrm{F}} = e^{\frac{4G}{\pi}}$

[TB, '24+]

Rigid (flat) quadrangulations (of the disk)



$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{R}}^n}{n^2 \log^2 n}$$

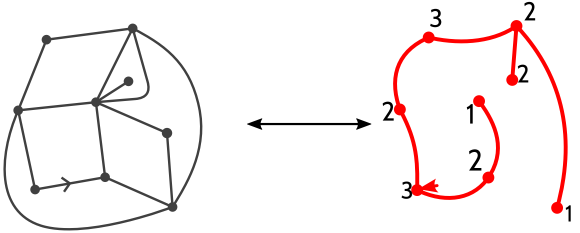
$\mu_{\mathrm{R}} = 4\pi$

[TB, Zonneveld, '24+]

Bijection between quadrangulations and labeled trees

- There exists a relation [Cori, Vauquelin] [Schaeffer, '99]

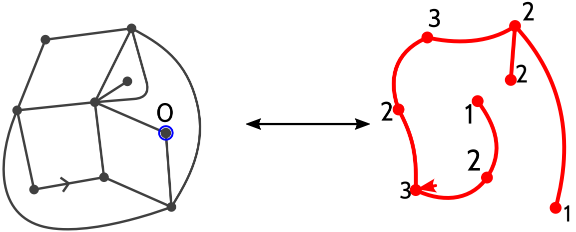
$$\left\{\begin{array}{c}\text{rooted quadrangulations}\\ \text{with } n \text{ faces}\end{array}\right\} \xleftrightarrow{\text{2-to-}(n+2)} \left\{\begin{array}{c}\text{rooted plane trees with labels in } \mathbb{N}\\ \text{that vary by at most 1 along its } n \text{ edges}\end{array}\right\}$$



Bijection between quadrangulations and labeled trees

- There exists a bijection [Cori, Vauquelin] [Schaeffer, '99]

$$\left\{\begin{array}{c}\text{rooted quadrangulations}\\ \text{with } n \text{ faces and marked vertex}\end{array}\right\} \xleftrightarrow{\text{1-to-1}} \left\{\begin{array}{c}\text{rooted plane trees with labels in } \mathbb{N}\\ \text{that vary by at most 1 along its } n \text{ edges}\end{array}\right\}\times\{\pm1\}$$



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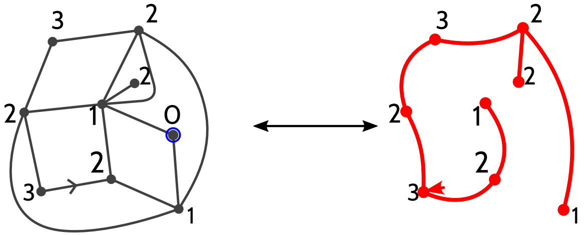

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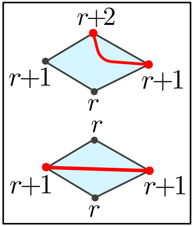
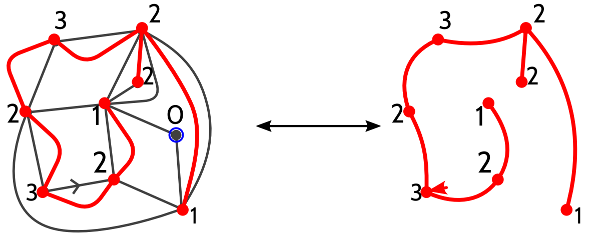


Recipe



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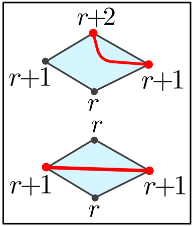
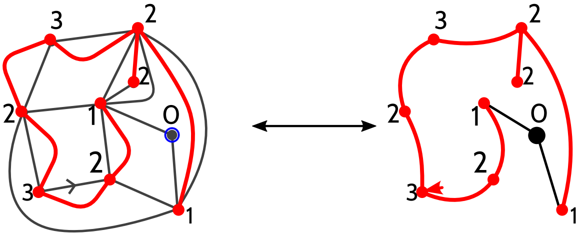


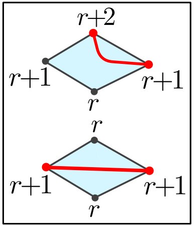
Recipe



Bijection between quadrangulations and labeled trees

- There exists a bijection [Cori, Vauquelin] [Schaeffer, '99]

$$\left\{\begin{array}{c}\text{rooted quadrangulations}\\ \text{with } n \text{ faces and marked vertex}\end{array}\right\} \overset{\text{1-to-1}}{\longleftrightarrow} \left\{\begin{array}{c}\text{rooted plane trees with labels in } \mathbb{N}\\ \text{that vary by at most 1 along its } n \text{ edges}\end{array}\right\}\times\{\pm1\}$$



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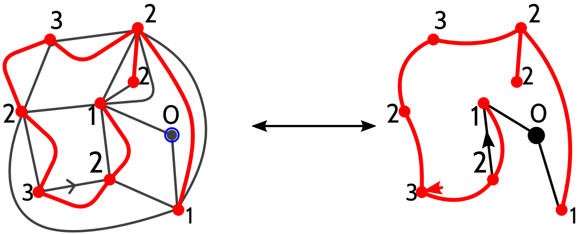

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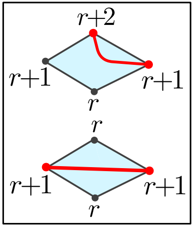

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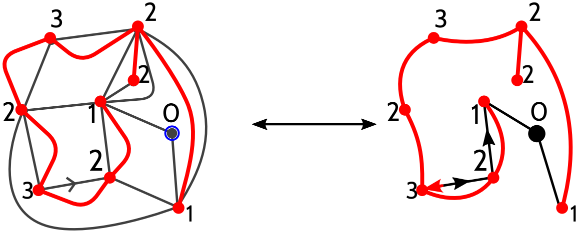

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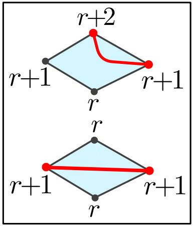

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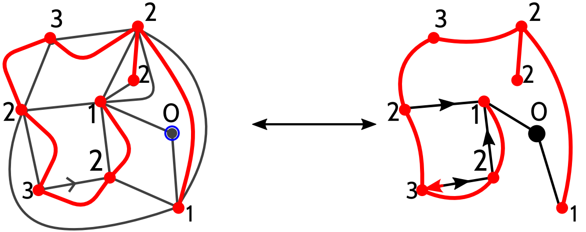

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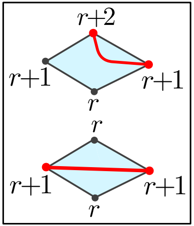

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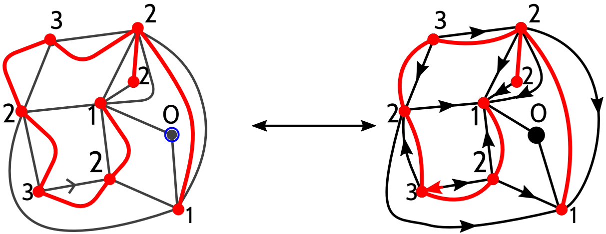

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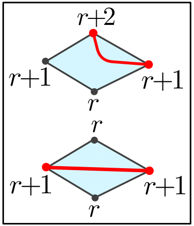
- Hence, $\#\text{quadrangulation} = \frac{2}{n+2}\, 3^n \,\frac{1}{n+1}\binom{2n}{n}$



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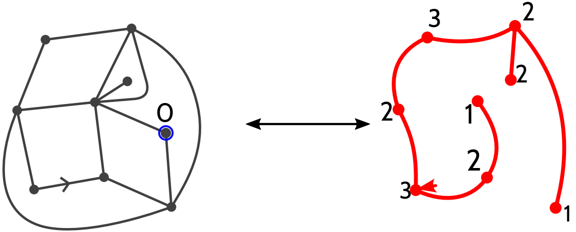

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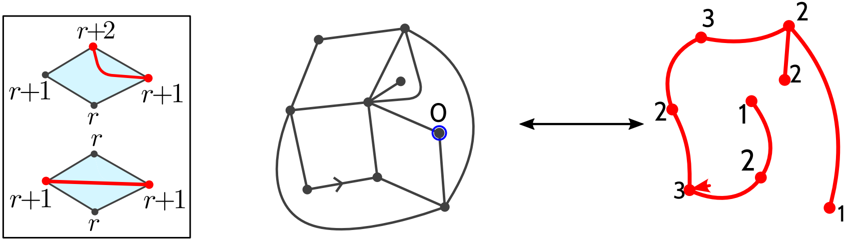

Recipe

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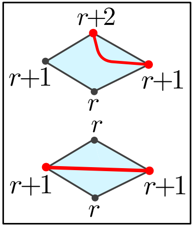

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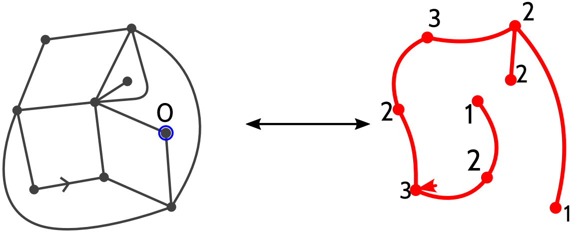

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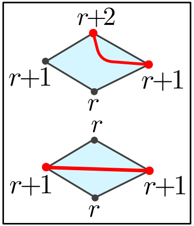
- Uniform quadrangulation $\iff$ uniform plane tree + uniform labeling.



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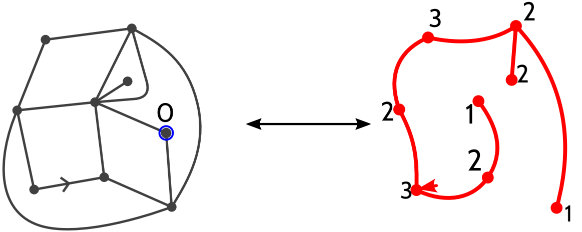
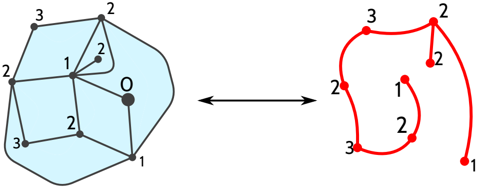


Beyond enumeration: metric information preserved under bijection



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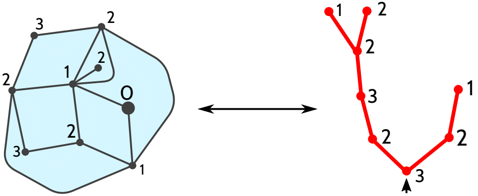

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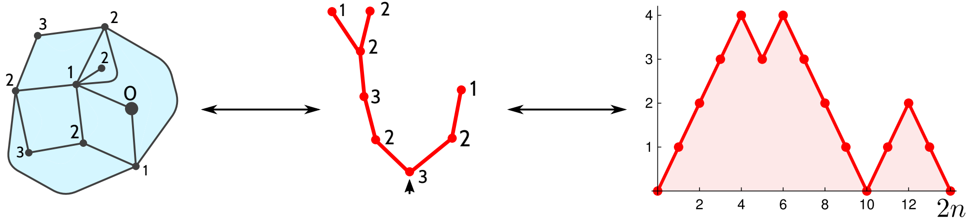

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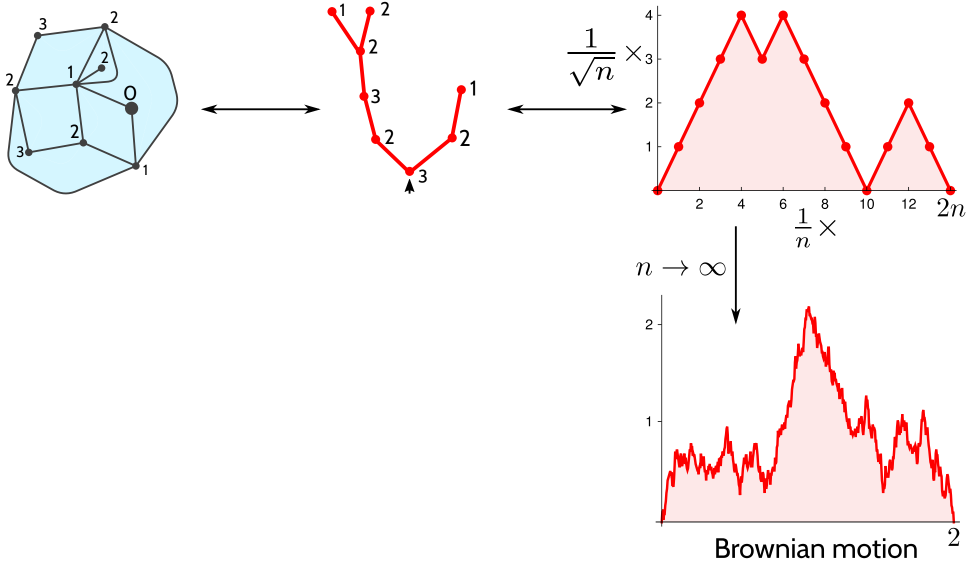

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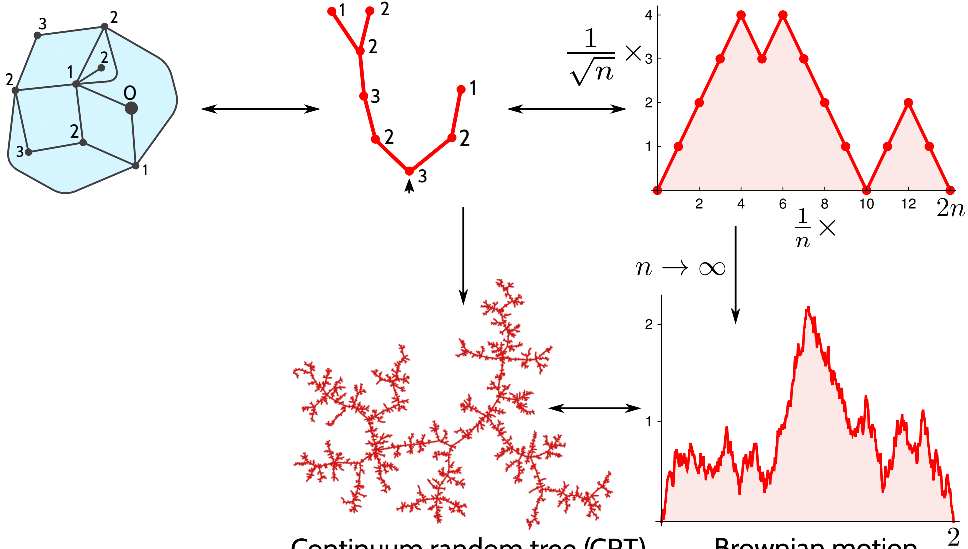

Continuum random tree (CRT)

[Aldous, '92]

Brownian motion

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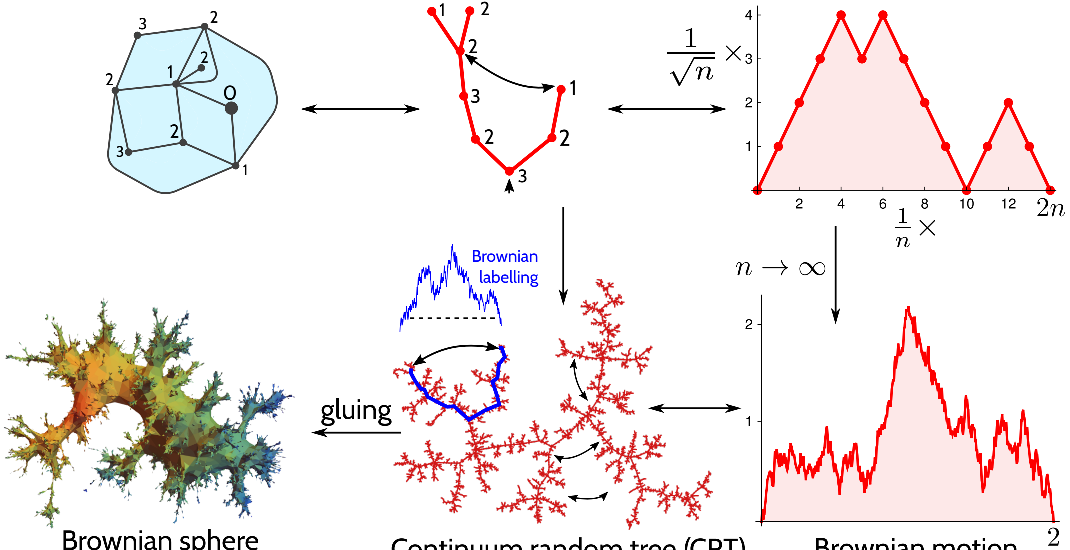

Brownian sphere

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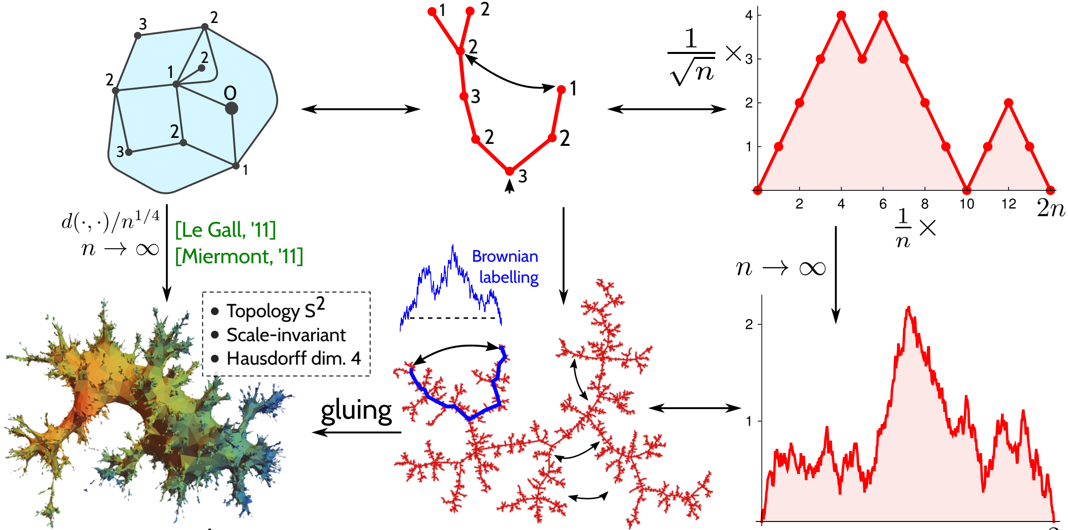

[Le Gall, '11]
[Miermont, '11]

- Topology S^2
- Scale-invariant
- Hausdorff dim. 4

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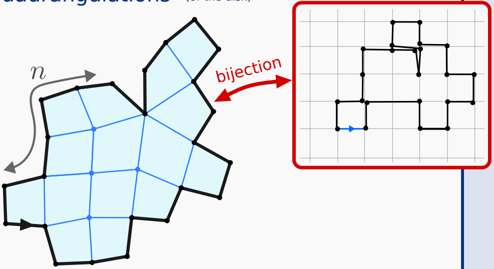
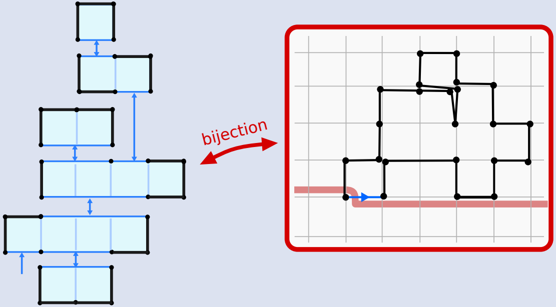
Flat quadrangulations (of the disk)





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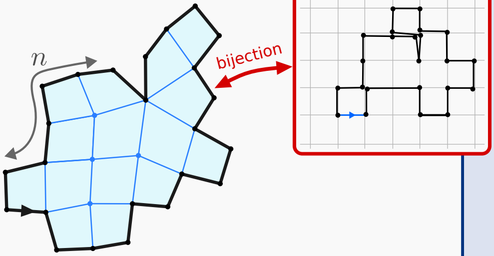
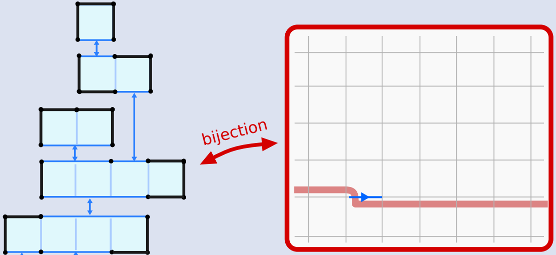
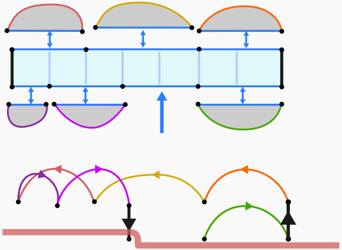




recipe



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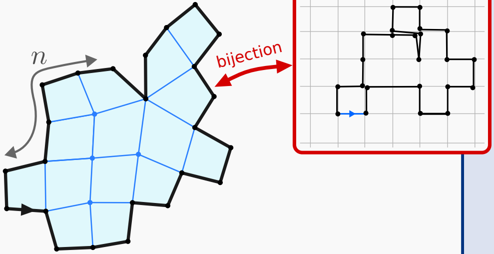

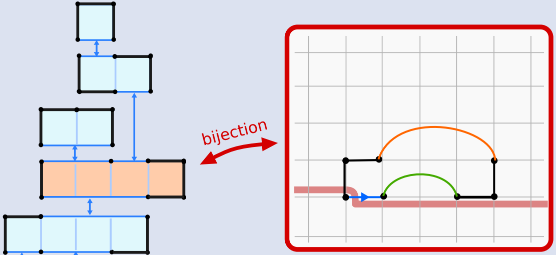

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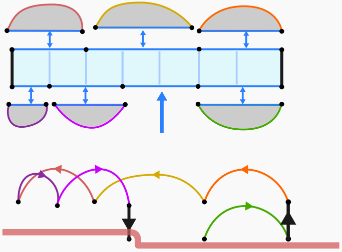

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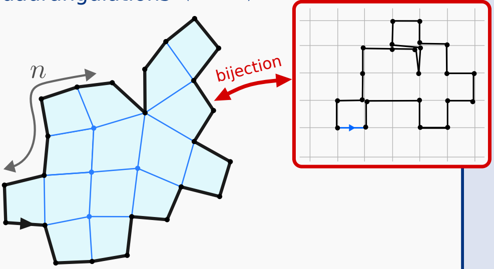

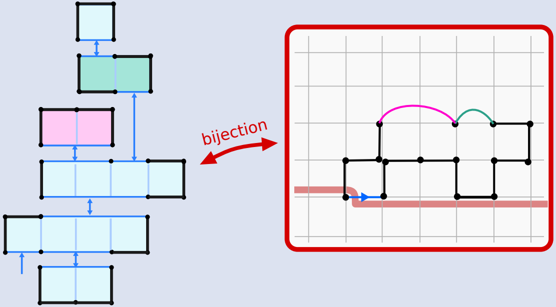

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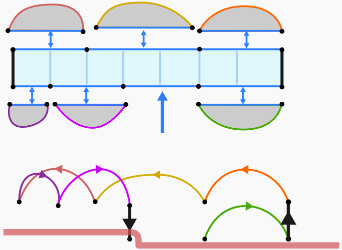

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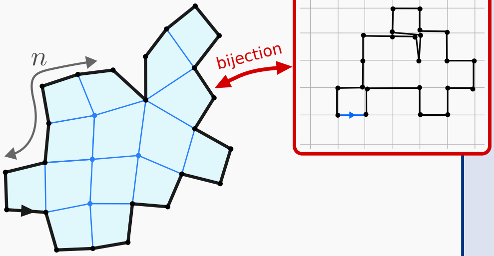

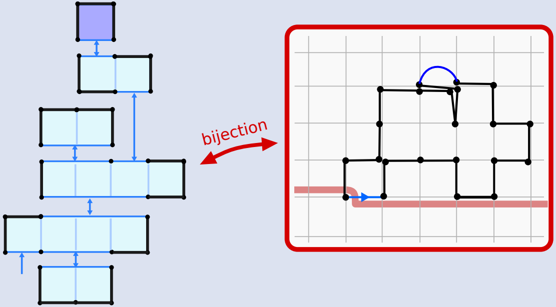

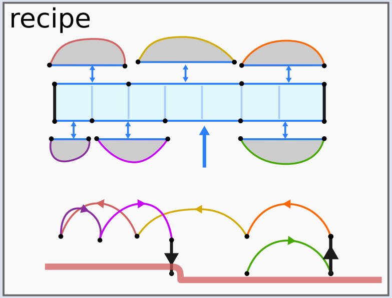

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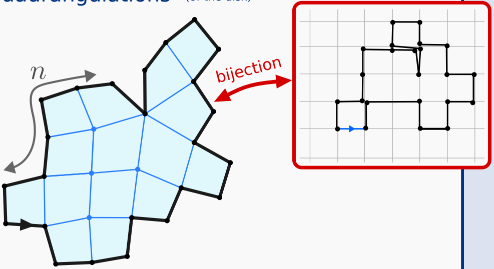

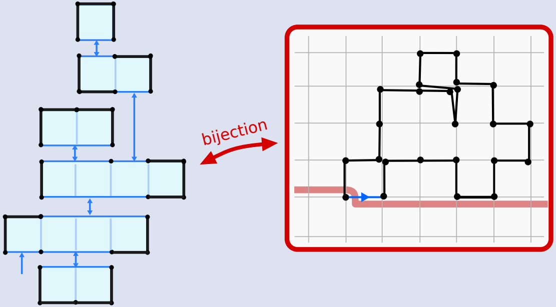

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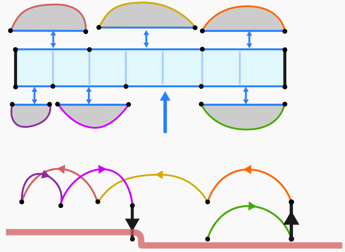

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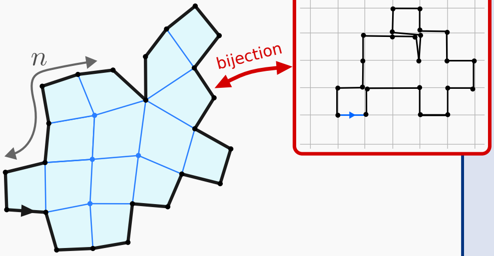
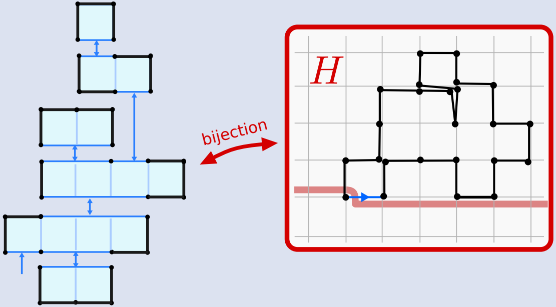
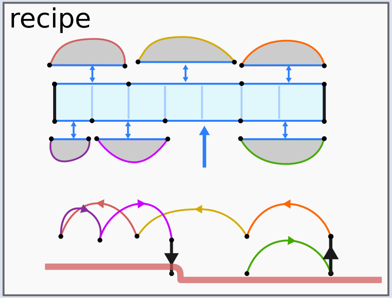
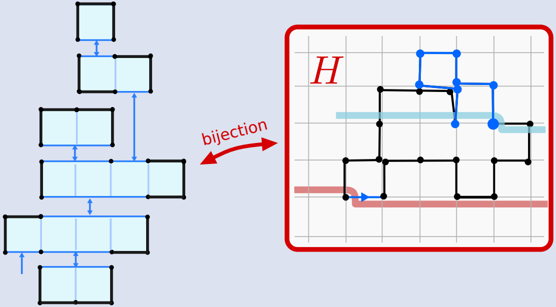


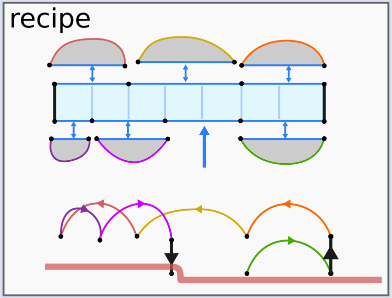




Theorem (TB, '24+)

Flat quadrangulations of perimeter n are in bijection with $(n-1)$-step walks $(1,0) \to (0,0)$ in half plane H with no point subexcursions.





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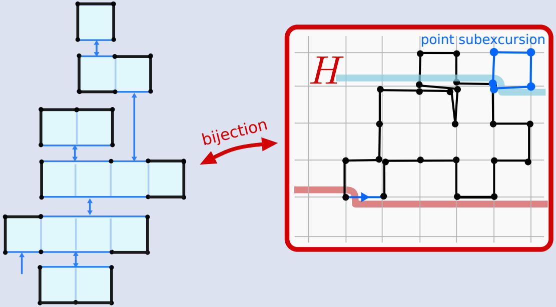

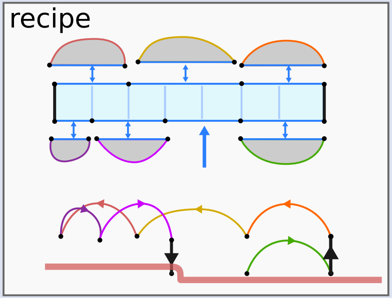

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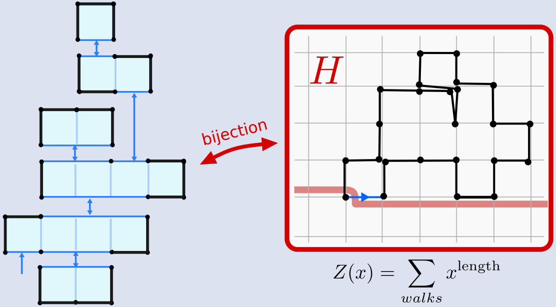
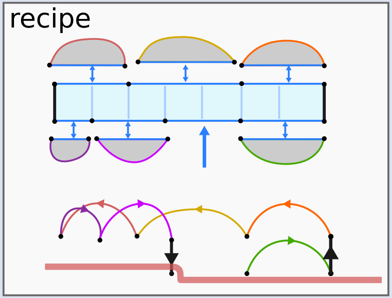


$$Z(x) = \sum_{walks} x^{\text{length}}$$



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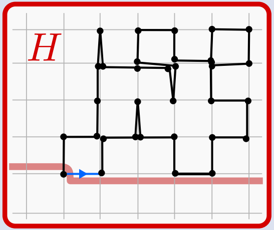

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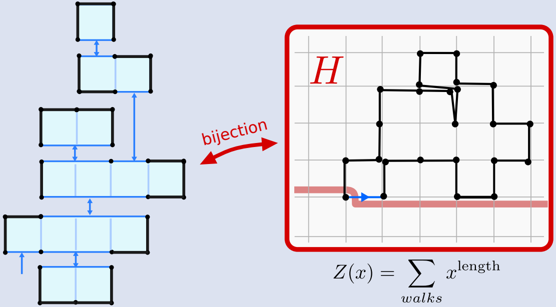

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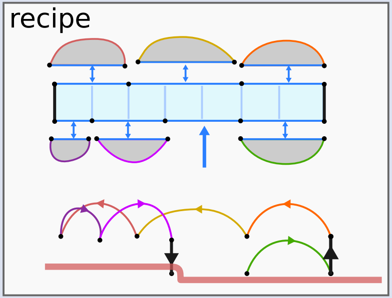

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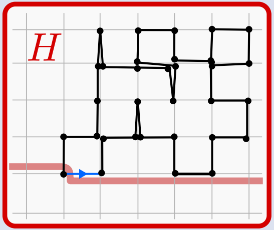

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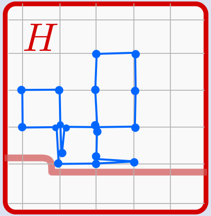
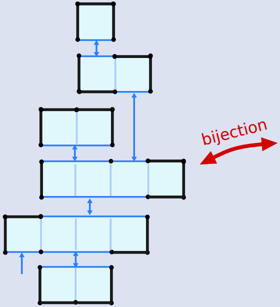
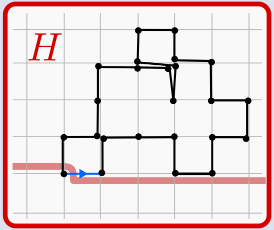


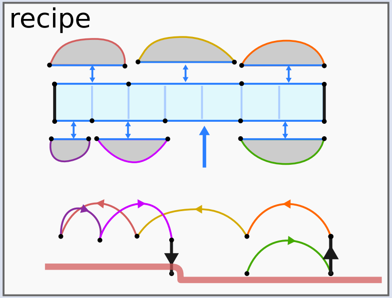
$$E(x) = \sum_{walks} x^{\text{length}+1}$$

$$F(x) = Z(E(x))$$



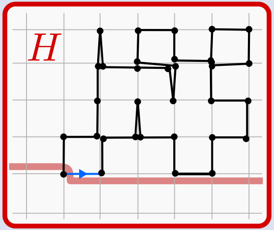


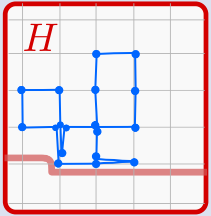
$$Z(x) = \sum_{walks} x^{\text{length}}$$



Theorem (TB, '24+)

Flat quadrangulations of perimeter n are in bijection with $(n-1)$-step walks $(1,0) \to (0,0)$ in half plane H with no point subexcursions. Its generating function is $Z(x) = F(E^{-1}(x))$ with radius of convergence $\frac{1}{\mu_{\mathrm{F}}} = E(\frac{1}{4}) = e^{-4G/\pi}$.

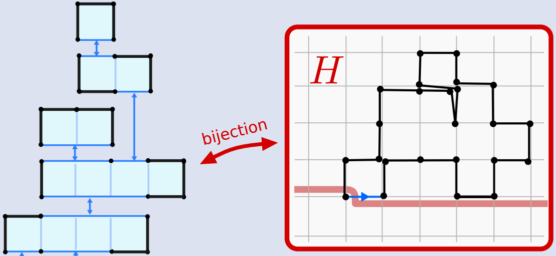




$$F(x) = \sum_{walks} x^{\text{length}} = \sum_{m=1}^{\infty} \frac{x^{2m-1}}{2m-1}\binom{2m-1}{m}^2$$

$$E(x) = \sum_{walks} x^{\text{length}+1}$$

$$F(x) = Z(E(x))$$



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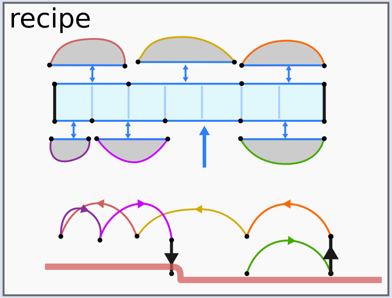
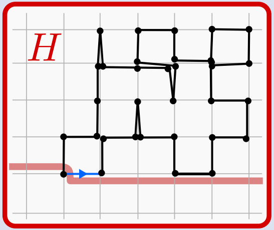
Flat quadrangulations of perimeter n are in bijection with $(n-1)$-step walks $(1,0) \to (0,0)$ in half plane H with no point subexcursions. Its generating function is $Z(x) = F(E^{-1}(x))$ with radius of convergence $\frac{1}{\mu_{\mathrm{F}}} = E(\frac{1}{4}) = e^{-4G/\pi}$.

Note: horizontal side ↔ horizontal step with same relative x-coordinate.





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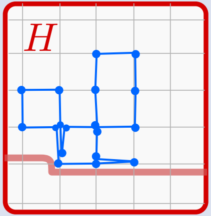

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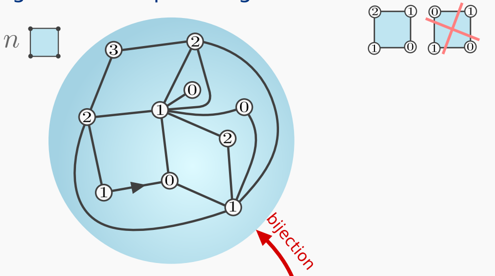
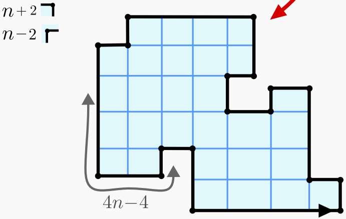
$$F(x) = Z(E(x))$$

Height-labeled quadrangulations (without alternating faces)

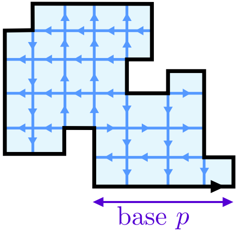


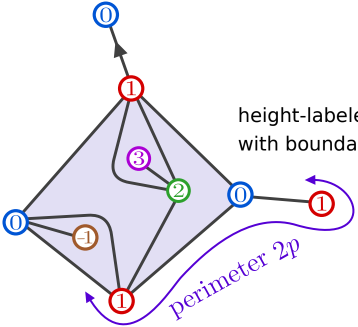
Rigid (flat) quadrangulations (of the disk)

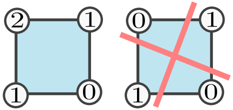


An extended bijection

rigid quadrangulation



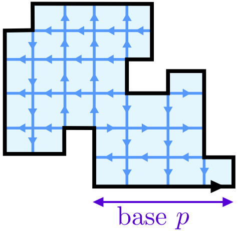


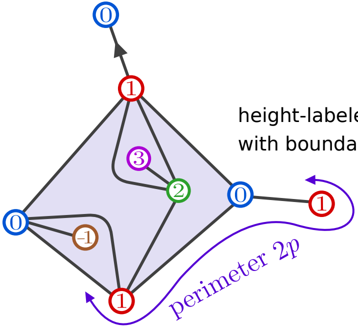


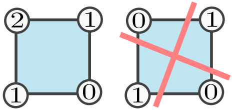
height-labeled quadrangulation
with boundary labeled 0101...

An extended bijection

rigid quadrangulation







height-labeled quadrangulation
with boundary labeled 0101...

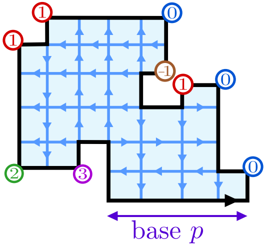
Theorem (TB, '24+)

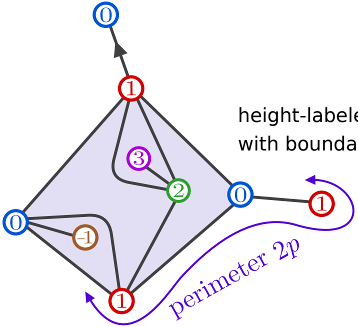
For $n \geq 2$ and $p \geq 1$ there exists a bijection

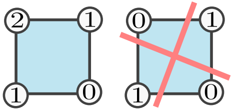
$$\left\{ \begin{array}{c} \textit{rigid quadrangulations with} \\ \textit{2n corners and base p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{height-labeled quadrangulations} \\ \textit{with n vertices and perimeter 2p} \end{array} \right\}$$

An extended bijection

rigid quadrangulation







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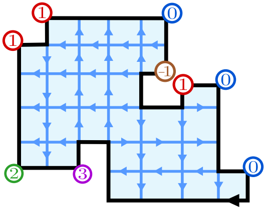
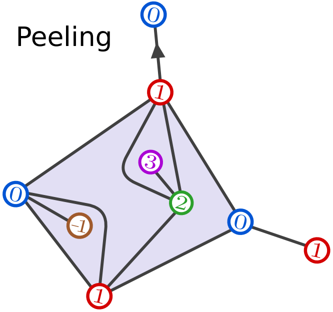
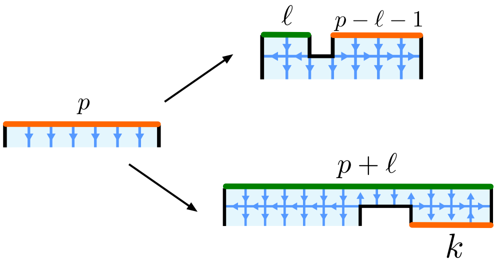
$$\tfrac{\pi}{2}\textit{-corner with } \underbrace{\textit{turning number } \ell}_{\#\textit{left}-\#\textit{right}} \longleftrightarrow \textit{vertex with label } \ell$$

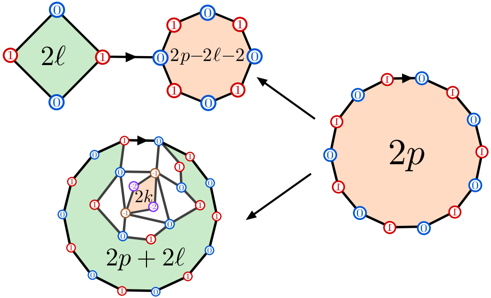
Bijection via exploration

Scanning vs Peeling



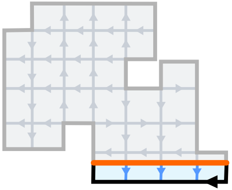


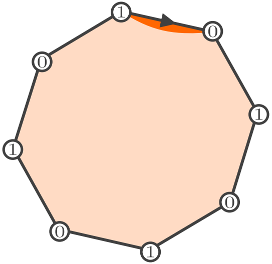


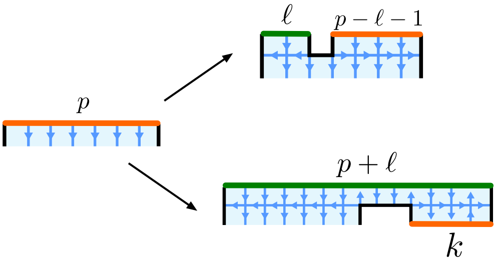


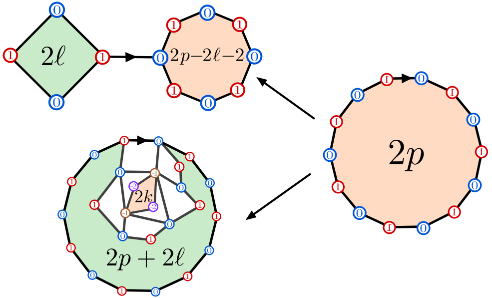
Bijection via exploration

Scanning vs Peeling



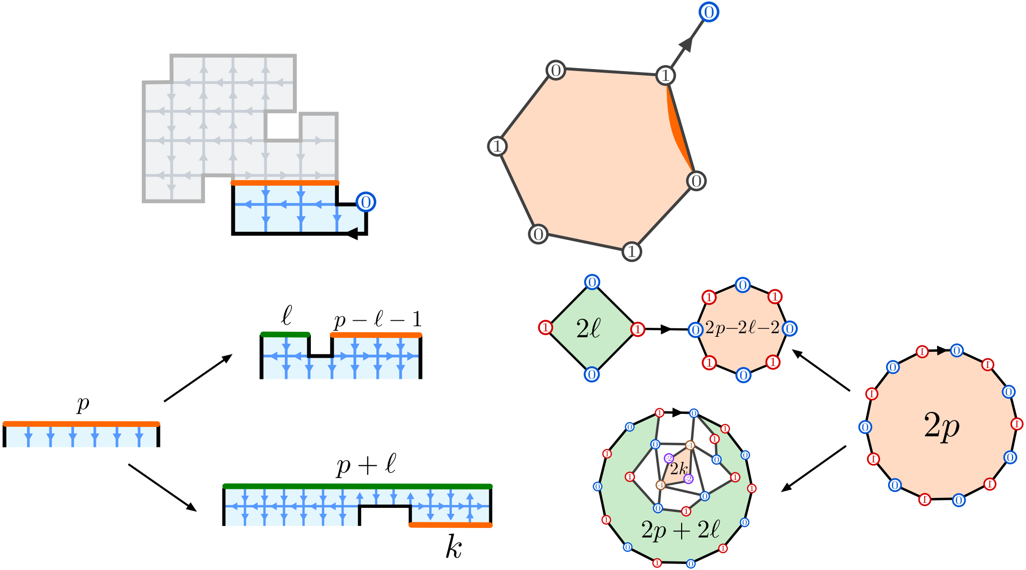






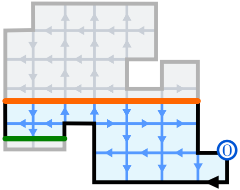
Bijection via exploration

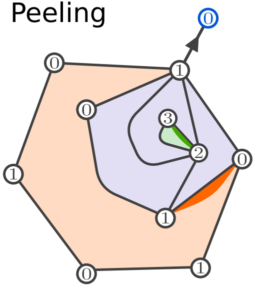
Scanning vs Peeling

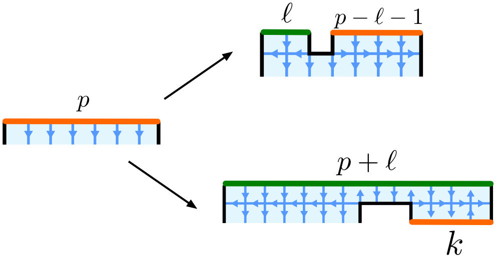


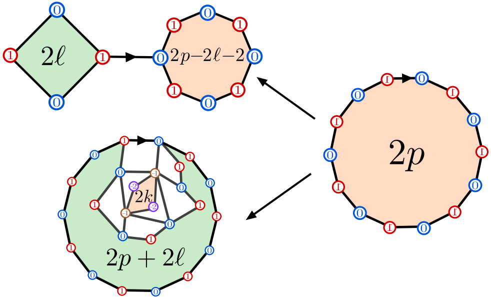
Bijection via exploration

Scanning vs Peeling



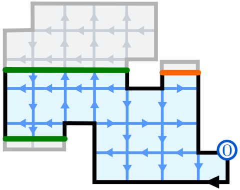


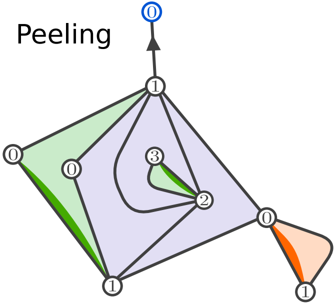


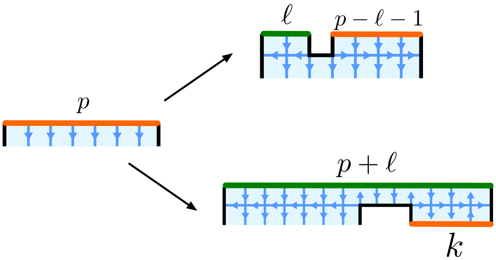


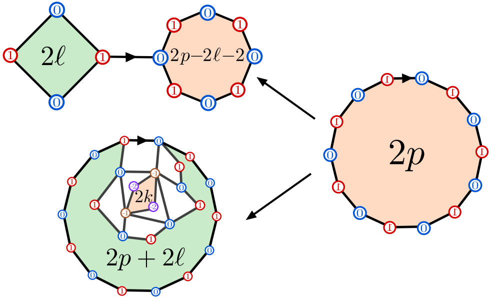
Bijection via exploration

Scanning vs Peeling



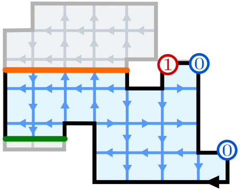


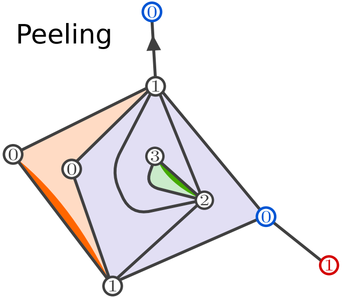


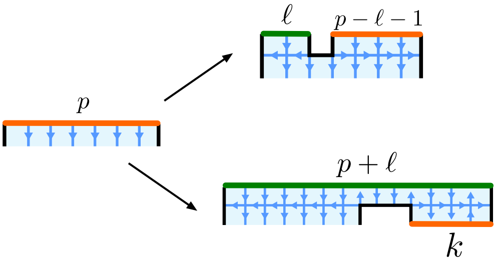


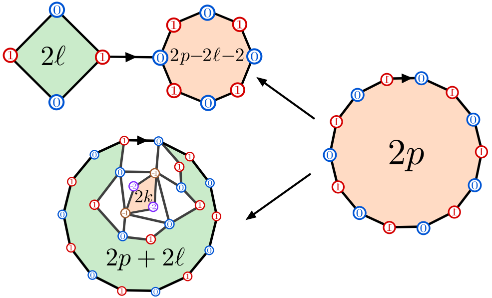
Bijection via exploration

Scanning vs Peeling









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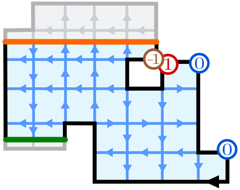

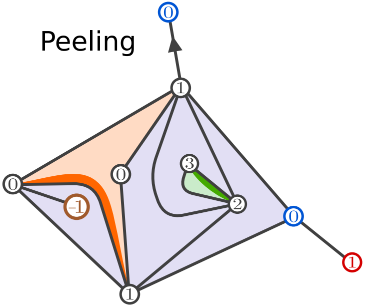

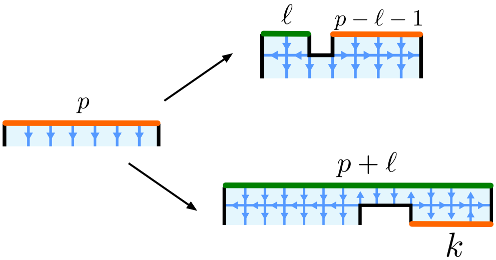

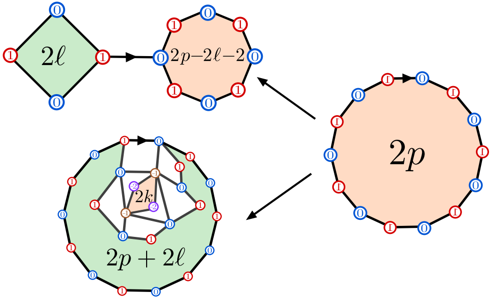

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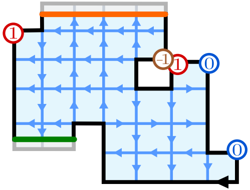

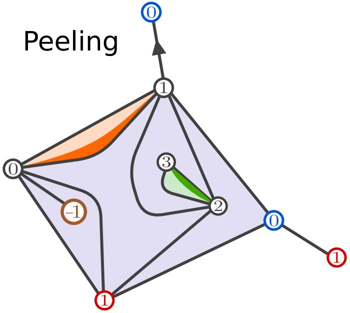

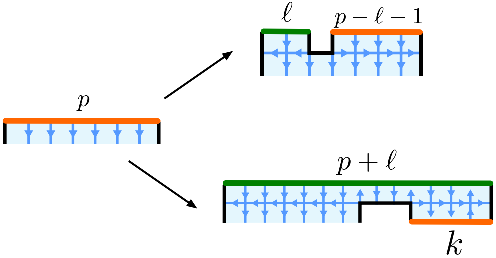

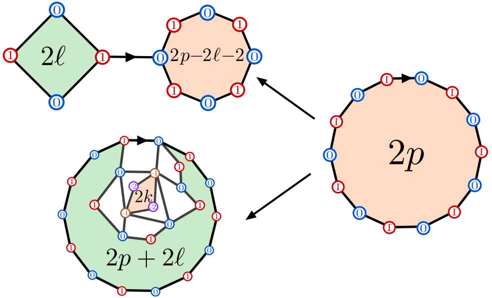

Bijection via exploration

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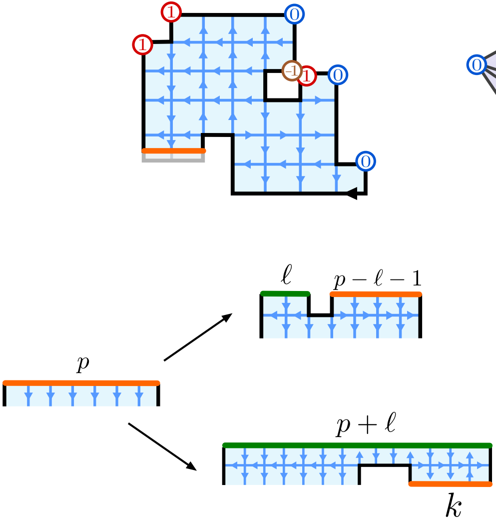

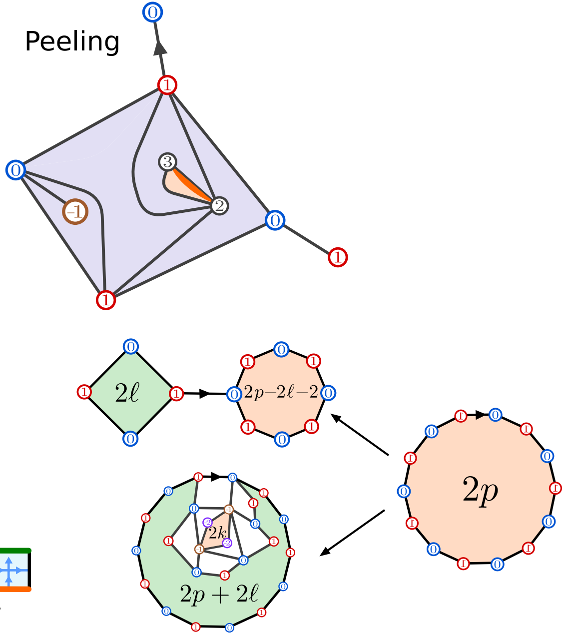

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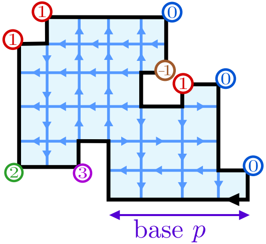
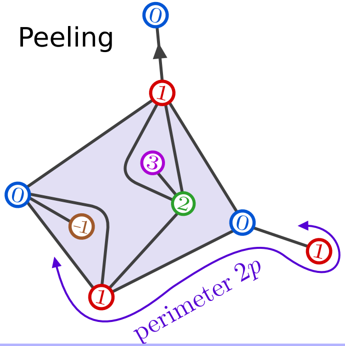




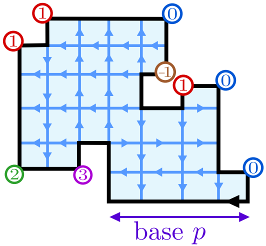
Theorem (Bousquet-Mélou, Elvey Price, ’20)

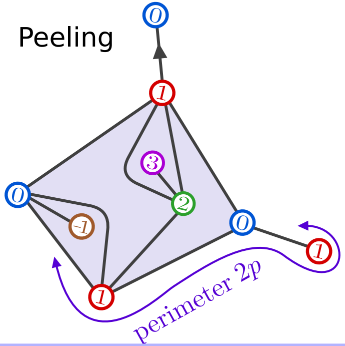
The generating function of perimeter-2p height-labeled quadrangulations is

$$Q^{(p)}(x) = \sum_{k\geq p} \frac{1}{k+1}\binom{2k}{k}\binom{2k-p}{k} R(x)^{k+1}, \qquad \textit{when} \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}^2 R(x)^{k+1} = x.$$

Bijection via exploration

Scanning vs Peeling





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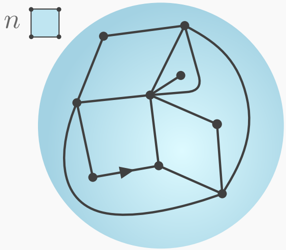
The generating function of rigid quadrangulations is $Q^{(1)}(x)$ with radius of convergence $\frac{1}{\mu_{\mathrm{R}}} = \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k}^2 16^{-k-1} = \frac{1}{4\pi}$.

Quadrangulations (of the 2-sphere)

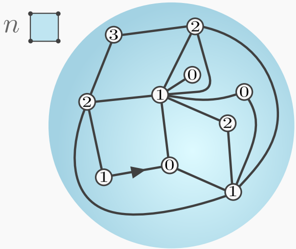


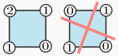
$$\overset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}} \frac{\mu_{\mathrm{Q}}^n}{n^{5/2}}$$

$$\mu_{\mathrm{Q}} = 12$$

[Tutte, '63]

Height-labeled quadrangulations (without alternating faces)



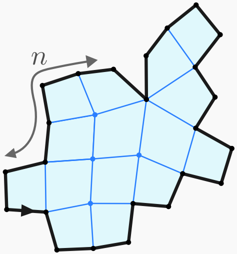


$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{H}}^n}{n^2 \log^2 n}$$

$$\mu_{\mathrm{H}} = 4\pi$$

[Kostov, Zinn-Justing, Guttman, ...] [Elvey Price, Bousquet-Mélou, '20]

Flat quadrangulations (of the disk)

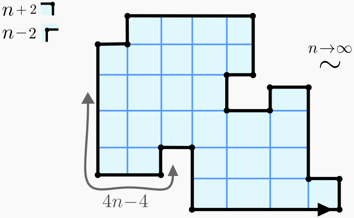


$$\overset{n\to\infty}{\sim} \frac{\pi}{4} \frac{\mu_{\mathrm{F}}^n}{n^2 \log^2 n}$$

$$\mu_{\mathrm{F}} = e^{\frac{4G}{\pi}}$$

[TB, '24+]

Rigid (flat) quadrangulations (of the disk)



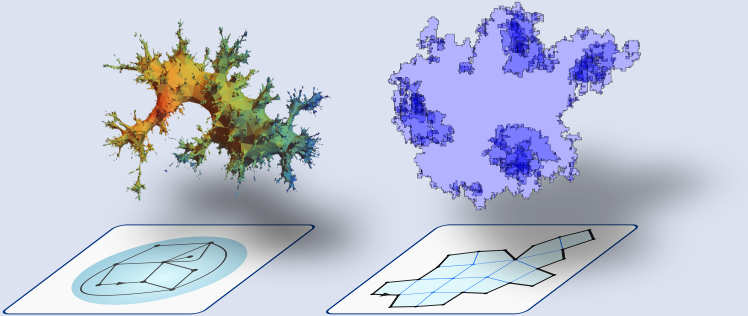
$$\overset{n\to\infty}{\sim} \pi^2 \frac{\mu_{\mathrm{R}}^n}{n^2 \log^2 n}$$

$$\mu_{\mathrm{R}} = 4\pi$$

[TB, Zonneveld, '24+]

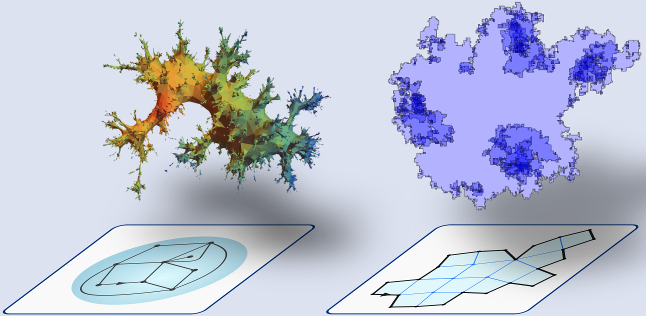
Take home messages

- Mathematical construction of path integrals in QG via the lattice approach requires fine handle on combinatorics.
- The bijective method often brings more than enumeration:
  - a dictionary to read of statistics;
  - if the other side is suitably simple: a construction of the continuum object.
- Being in bijection does not imply being in the same universality class.



Take home messages

- Mathematical construction of path integrals in QG via the lattice approach requires fine handle on combinatorics.
- The bijective method often brings more than enumeration:
  - a dictionary to read of statistics;
  - if the other side is suitably simple: a construction of the continuum object.
- Being in bijection does not imply being in the same universality class.



Thanks!