Geometric Observables in 2+1 Dimensional Quantum Gravity

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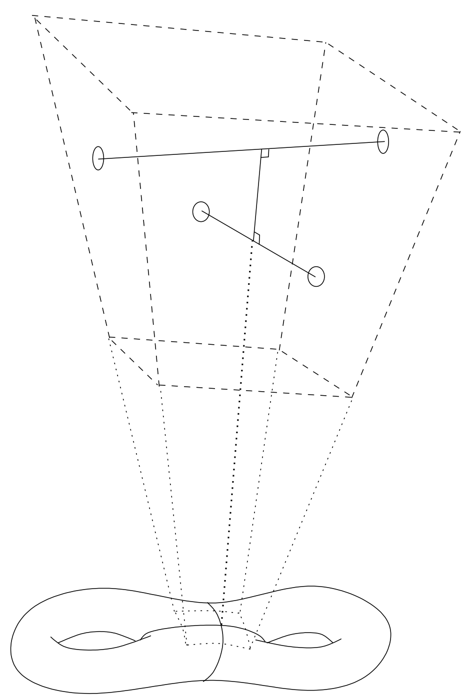

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Abstract

We discuss general relativity in $2+1$ dimensions with vanishing cosmological constant and in absence of matter. The phase space is identified with the cotangent bundle of Teichmüller space. We show that certain gauge invariant length observables in space-time arise as derivatives of geometric functions on Teichmüller space. The Poisson structure of these observables is established. After this we discuss the geometric quantisation of the reduced phase space. The length observables are recognised as derivative operators on wave functions on Teichmüller space. The spectrum of two of these observables is calculated. The first observable, which measures a space-like distance, turns out to have the whole real line as its spectrum. The second observable, which measures a time-like distance, is quantized, but the eigenvalue separation varies between zero and the Planck length depending on the sector of phase space. Finally we relate our results to claims made in the literature.

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Contents

Chapter 1

Introduction

In this thesis we will focus on the combination of four notions which are already mentioned in the title: *gravity*, *quantisation*, *2+1 dimensions* and *geometric observables*.

Since Einstein wrote down his general theory of relativity, we know that the force of gravity is different from the other forces we know. The three fundamental forces, the electromagnetic, the weak and the strong forces, we can understand as fields living in space and changing in time. The gravitational field contrarily must itself describe space and time. This produces difficulties in interpreting general relativity, perhaps not so much classically, but definitely the last century has shown that quantisation is far from straightforward. Indeed, up to now no quantum theory of gravity is known or at least we have not yet identified a theory as such.

Since the full theory of gravity is really hard to quantize, we will examine an easier toy-model, namely gravity in 2+1 dimensions (with vanishing cosmological constant). By 2+1 dimensions we actually mean two space dimensions and one time dimension, as opposed to the 3+1 dimensions we live in. This terminology is chosen such, because term “3 dimensional gravity” is usually reserved for general relativity in 3 space dimensions.

As we will show in chapter 2 gravity in $2 + 1$ dimensions is much simpler than in $3 + 1$ dimensions because it has no local degrees of freedom, in particular no gravitational waves exist in $2+1$ dimensions. It might seem that the theory becomes trivial in absence of matter, but this is not entirely true, since there will be global degrees of freedom when we take the spatial topology to be non-trivial. In chapter 2 we will identify the whole phase space of space-time solutions. It will turn out that the phase space of space-time solutions of a particular spatial topology is closely related to the Teichmüller space of complex structures on a surface of the same topology. After a review of Teichmüller spaces and hyperbolic geometry in chapter 3 we will establish the relation in chapter 4.

In chapter 5 we will provide a quantisation of gravity in 2+1 dimensions in absence of matter. There are several inequivalent ways of quantising but we will choose for a geometric quantisation of the reduced phase space we have constructed. It is certainly not guaranteed that different quantisations will have the same features. To make statements about physics of $2+1$ dimensional quantum gravity, either one should argue that one of the quantisations is most desired in some way or one should look for common features. One of these features will play a central role in this thesis, namely the quantum geometrical structure of space-time. More precisely, we will investigate the spectrum of geometric length observables.

An intriguing question in quantum gravity, in whatever dimension, is whether we should consider space-time as a continuous or a discrete substance. Does it make sense to talk about measuring arbitrary small distances or does there exist a minimal length scale? Should we consider space as being built from elementary building blocks, atoms of space-time? These questions tend to become a little philosophical, but there are ways of probing the microscopic structure. Indeed we would expect that discretization of space-time should have its impact on the possible values we can measure for distances, for instance between particles or around non-contractible loops.

Several other approaches to quantum gravity in $2 + 1$ dimensions have raised evidence for some sort of discretization. Loop quantum gravity was one of the first and is one of the most studied approaches to quantum gravity in $3+1$ dimensions. A Hilbert space is obtained which has a basis given by spin-networks, labeled graphs embedded in space. These basis elements are precisely the eigenstates of the volume and area operators in $3+1$ dimensions. See [26] for more details. The spin-networks in loop quantum gravity in $2+1$ dimensions are eigenstates of area and length operators. Freidel, Livine and Rovelli [13] have calculated the spectra of these operators and they were lead to conclude that space-like distances have continuous spectra, while time-like distances have discrete spectra (although not evenly spaced). Somewhat longer ago 't Hooft [17] found indications that the time variable in his polygon approach should take discrete values.

In chapter 6 we will compare these findings with our calculations of spectra of chapter 5. A main difference between our approach and the ones mentioned above is the fact that we are considering fully gauge-invariant, physical observables[1]. Indeed we actually have not much choice since we start our quantisation with a phase space with all gauge symmetry divided out. Although it seems reasonable to expect that gauge invariant length observables and those which are not gauge invariant have consistent spectra, it is not clear whether is always the case. A recent paper [10] by Dittrich and Thiemann demonstrates situations in which the spectra do not coincide. We will return to this issue in chapter 6.

1.1 Conventions

Throughout this thesis we will use units in which the speed of light $c = 1$ and the Newton constant $G = 1$. We will not put $\hbar$ to 1, such that it remains clear where we are dealing with

[1]There is however a subtlety concerning a remaining symmetry of the phase space. We will discuss this in paragraph 5.1.4 and in chapter 6.

quantum behaviour. In terms of these units the Planck mass is[2]

$$M_{Pl} = 1 \tag{1.1}$$

and the Planck length is

$$l_{Pl} = \hbar. \tag{1.2}$$

We write the Minkowski metric

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

which we use to raise and lower latin indices (from the beginning of the alphabet). Space-time indices are denoted by Greek indices and are raised and lowered by the space-time metric $g_{\mu\nu}$.

The isometry group of Minkowski space we denote by $ISO(2,1)$. We choose a basis of $\mathfrak{so}(2,1)$,

$$J_0^{\mathfrak{so}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad J_1^{\mathfrak{so}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_2^{\mathfrak{so}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

We choose a basis of $\mathfrak{sl}(2,\mathbb{R})$,

$$J_0^{\mathfrak{sl}} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J_1^{\mathfrak{sl}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_2^{\mathfrak{sl}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

Both the $J_a^{\mathfrak{so}}$ and the $J_a^{\mathfrak{sl}}$ satisfy the commutation relation

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \tag{1.6}$$

where $\epsilon_{ab}{}^c = \epsilon_{abd}\eta^{dc}$. They form an orthonormal basis with respect to the trace forms $B^{\mathfrak{sl}}(X,Y) = 2\mathrm{Tr}(XY)$ and $B^{\mathfrak{so}}(X,Y) = \frac{1}{2}\mathrm{Tr}(XY)$ respectivily,

$$B(J_a, J_b) = \eta_{ab}. \tag{1.7}$$

[2]Notice that the Planck mass does not depend on $\hbar$ and hence is a classical notion. In paragraph 4.4 we will see that a classical space-time solution with a particle of mass M has a cone-singularity with deficit angle proportional to M. It follows immediately that we can make the mass dimensionless without using $\hbar$.

Chapter 2

Gravity in 2+1 dimensions

Gravity or rather general relativity is a theory describing the *dynamics* of space-time metric on a space-time manifold M. A solution of the theory consists of a metric g on M of Lorentzian signature $(-,+,+)$, which allows us to measure distances and time intervals. In particular the metric determines a light-cone at each point of M and therefore the causality of space-time.

A large part of this chapter is based on Carlip's book [6] and we refer to this book for more details.

2.1 Topology

Before we start analysing the restrictions put on our space-time metric by Einstein's equations of general relativity we first consider the properties of Lorentzian metrics on manifolds and the significance of the topology of space-time. Although three-dimensional manifolds of almost all topologies admit Lorentzian metrics, only a small subset will be suitable to act as topology for physical space-time. It will turn out that certain restrictions on the Lorentzian space-time metric will highly restrict the allowed topologies. The physical restrictions amount to two related aspects of causality: absence of closed time-like curves and global hyperbolicity.

We consider a space-time manifold M with boundary $\partial M = \Sigma^+ \cup \Sigma^-$, where we view Σ^+ as lying in the future and Σ^- in the past. It has been proven that if Σ^+ and Σ^- have different topology (i.e. are not homeomorphic), that M necessarily contains closed time-like curves. Since we can cut a space-time in all kinds of ways and apply this theorem repeatedly, we conclude that if M satisfies the above causality conditions, it must have topology equal to $\Sigma^+ \times \mathbb{R} = \Sigma^- \times \mathbb{R}$.

In principle Σ can have all sorts of topologies. But we will restrict ourselves to the case where Σ is compact and orientable. The topology is then fixed by the genus of Σ. We will see that for genus 0 (topology of a two-sphere) the theory becomes trivial. The genus 1 case (topology of a torus) provides the simplest non-trivial theory and is therefore most studied in literature. In chapter 3, however, we will see that the geometry of the torus is actually in some sense degenerate. Therefore and because less is known about it, we will be concerned with Σ of genus 2 or larger.

2.2 Einstein equations in 2+1 dimensions

Just as in 3+1 dimensions we start off with the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda) \tag{2.1}$$

and our total action S can be written as a sum of S_{EH} and some action S_{matter} describing matter in space-time. Completely analogous to the 3+1 dimensional case we can vary S with respect to the metric $g_{\mu\nu}$ to obtain the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{2.2}$$

where the stress-energy tensor is given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{2.3}$$

The peculiar feature of gravity in 2+1 dimensions is that the Riemann tensor is completely fixed by its Ricci tensor. We can express the Riemann tensor $R_{\mu\nu\rho\sigma}$ in terms of the metric $g_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ by

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{2.4}$$

It follows from the Einstein equations (2.2) that $R_{\mu\nu} = -8\pi G(T_{\mu\nu} - Tg_{\mu\nu}) + 2\Lambda g_{\mu\nu}$. Hence, in absence of matter $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ and

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}). \tag{2.5}$$

The consequence of this is that the local geometry is completely fixed by the cosmological constant. If $\Lambda = 0$ the cosmological constant must vanish and space-time will locally look just like Minkowski space. If $\Lambda > 0$ or $\Lambda < 0$ the space-time metric will locally be that of de Sitter and Anti-de Sitter space respectively. In this thesis we will be mainly concerned with the flat $\Lambda = 0$ case. In the appendix we will rederive some properties for non-vanishing cosmological constant.

What we have actually derived above is that the gravitational field in 2+1 dimensions has no local degrees of freedom. Put differently, no gravitational waves exist contrarily to 3+1 dimensions. We could also have established this fact by counting local degrees of freedom or by explicitly gauging away wave solutions of the linearized Einstein equations. We refer to [6] for more details.

It may seem that classical general relativity has become a trivial theory, but this is not entirely true. No local degrees of freedom exist, but when we consider space-time manifolds with non-trivial topology, we will see that we obtain a finite number of global degrees of freedom. In paragraph 2.4.1 we will identify them and write down the phase space of general space-time solutions.

2.3 First-order formalism: Chern-Simons theory

In the previous paragraph we have described general relativity using the metric $g_{\mu\nu}$ and its derivatives as fundamental variables. Although in many cases this is a convenient description, there is an alternative set of variables which is useful especially for quantization and even inevitable when for instance we want to add fermions. Let's consider a one-form e^a with values in Minkowski space, i.e.

$$e^a = e^a_\mu dx^\mu \tag{2.6}$$

where we use the convention that Greek letters denote space-time indices and latin indices denote Lorentz indices. The e^a is sometimes called *dreibein*, *triad* or just gravitational field. The triad actually just maps the tangent space of the space-time manifold to Minkowski space, hence we can write the metric $g_{\mu\nu}$ in terms of e^a and η_{ab},

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{2.7}$$

The analogue of the Christoffel connection in the first-order formalism is the *spin connection* ω, which is a $\mathfrak{so}(2,1)$-valued one-form. We can write ω in indices by choosing the basis (1.4),

$$\omega = \omega^a_\mu J_a dx^\mu. \tag{2.8}$$

In the Palatini formalism the triad e^a and the spin-connection ω are viewed as independent variables which will be linked by the equations of motion following from the Einstein-Hilbert action. In terms of the new variables the Einstein-Hilbert action becomes

$$S_{EH} = -2\int e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c\right) + \frac{\Lambda}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c. \tag{2.9}$$

Variation with respect to ω ensures that spin-connection becomes metric-compatible and it allows us to express ω in terms of the triad e^a and its derivatives. We denote the curvature of ω by

$$R_a = d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \tag{2.10}$$

which is a $\mathfrak{so}(2,1)$-valued two-form. It is related to the Riemann tensor by $R^{\mu\nu}_{\rho\sigma}e^a_\mu e^b_\nu = \epsilon^{abc}R_{c\rho\sigma}$. Variation of (2.9) with respect to e^a now gives Einstein's equations

$$R_a = -\frac{\Lambda}{2}\epsilon_{abc}e^b \wedge e^c. \tag{2.11}$$

Classically the first order formalism in terms of e^a and ω is equivalent to the second order formalism in terms of $g_{\mu\nu}$. There is however a slight difference which may become important on quantizing the theory. Above we mentioned that the equations of motion allow us to solve ω in terms of e^a but this is only possible as long as the triad e^a_μ is invertible. Invertibility however is not a constraint following from the Einstein-Hilbert action (2.9).

To obtain a well-known form for the action with vanishing cosmological constant we combine the triad and the spin connection, which takes valued in $\mathfrak{so}(2,1)$, into a single connection one-form A with values in $\mathfrak{iso}(2,1)$ (see paragraph 2.4.2 for details),

$$A = e^a P_a^{\mathfrak{iso}} + \omega^a J_a^{\mathfrak{iso}}. \tag{2.12}$$

If we write (2.9) in terms of the connection A, we obtain

$$S_{CS} = -\frac{1}{16\pi G}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right), \tag{2.13}$$

where Tr is defined by the invariant inner-product on $\langle\cdot,\cdot\rangle$ given by[1]

$$\left\langle P_a^{\mathfrak{iso}}, J_b^{\mathfrak{iso}}\right\rangle = \eta_{ab},\ \left\langle P_a^{\mathfrak{iso}}, P_b^{\mathfrak{iso}}\right\rangle = 0,\ \left\langle J_a^{\mathfrak{iso}}, J_b^{\mathfrak{iso}}\right\rangle = 0. \tag{2.14}$$

The action (2.13) is precisely the *Chern-Simons* action for gauge group $ISO(2,1)$.

Chern-Simons theory is a prime example of a *topological field theory*, a field theory without local degrees of freedom. The theory has been and still is widely studied by physicists and mathematicians[2]. It is well-known that a complete set of observables for a (classical) Chern-Simons theory is given by the *Wilson loops*. Given a closed path γ in the manifold M, we define the Wilson loop

$$W_\gamma = \mathrm{Tr}\left(\mathcal{P}\exp\int_\gamma A\right), \tag{2.15}$$

where $\mathcal{P}\exp$ denotes the path-ordered exponential[3]. It turns out that W_γ is invariant under (infinitesimal) deformations of γ. Therefore we conclude that W_γ only depends on the homotopy class $[\gamma]$ in M. In paragraph 2.4.1 we will see that the W_γ is nothing else than the trace of the holonomy along γ.

2.4 Phase space of gravity in 2+1 dimensions

The main goal of this thesis is to quantize gravity in 2+1 dimensions in absence of matter. We will do this by considering the phase space of solution of the Einstein equations on a manifold M with as many unphysical degrees of freedom gauged away as possible. Then we have to determine the symplectic or Poisson structure which will be our starting point for quantization in chapter 5.

[1] If we identify $ISO(2,1)$ with $SO(2,1)\ltimes\mathfrak{so}(2,1)^*$ (see paragraph 2.4.2), $\langle\cdot,\cdot\rangle$ coincides with the orthogonal structure H in (4.11).

[2] For a report on recent progress and relation with knot theory see [20]. For Chern-Simons theory and topological field theory see [4].

[3] The path-ordered exponential is defined by its Taylor expansion, $\mathcal{P}\exp\left(\int_0^s A_\mu \frac{dx^\mu}{ds}\right) = \sum_{n=0}^\infty \int_0^s ds_1 A_{\mu_1}\frac{dx^{\mu_1}}{ds_1}\int_0^{s_1} ds_2 A_{\mu_2}\frac{dx^{\mu_2}}{ds_2}\cdots\int_0^{s_{n-1}} ds_n A_{\mu_n}\frac{dx^{\mu_n}}{ds_n}$.

2.4.1 Geometric structures

We have seen that the Einstein equations in vacuum imply that locally space-time is flat. Equivalently, any simply-connected open subset of the space-time manifold must be isometric to an open subset of Minkowski space. We could therefore reconstruct our manifold by appropriately patching together pieces of Minkowski space. We say our space-time is *locally modeled* on Minkowski space.

A (topological) manifold is actually a space locally modeled on $\mathbb{R}^n$ with its standard topology, which means that it is constructed or can be reconstructed by patching together open subsets of $\mathbb{R}^n$ by transition functions which conserve the structure. In this case the transition functions must therefore be homeomorphisms. Likewise a differentiable manifold is modeled on $\mathbb{R}^n$ with its standard differentiable structure and the transition functions should be diffeomorphisms. Analogously our space-time manifold M is modeled on $\mathbb{R}^3$ with a metric structure and therefore the transition functions are isometries of Minkowski space.

These are all examples of *geometric structures*. Given a space X and a group G with an action on X, we say a manifold M has a geometric structure (G, X) if M is locally modeled on X and the transition functions are in G. Hence our space-time manifold has a geometric structure $(ISO(2,1), \mathbb{R}^{2,1})$. Next chapter we will extensively study two-dimensional manifolds with a geometric structure $(PSL(2,\mathbb{R}), \mathbb{H})$, which are just Riemann surfaces.

A geometric structure on a manifold M allows us to define the import notion of the *holonomy* along a closed path γ in M. Let $\{U_1, U_2, \ldots, U_n, U_{n+1} = U_1\}$ be a collection of (simply-connected) open subsets of M covering γ, such that $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n$, together with charts $\phi_i : U_i \to X$. The transition functions $\phi_{i+1} \circ \phi_i^{-1} : \phi_i(U_i \cap U_{i+1}) \to \phi_{i+1}(U_i \cap U_{i+1})$ correspond to group elements $g_i \in G$. We then define the holonomy $H(\gamma) \in G$ as

$$H(\gamma) = g_1 \cdot \ldots \cdot g_n. \tag{2.16}$$

Clearly small deformations of γ will leave its holonomy invariant. It can indeed be shown that the holonomy only depends on the homotopy class of γ. The composition $\gamma_1 \circ \gamma_2$ of two closed paths γ_1 and γ_2 based at the same point will of course have holonomy

$$H(\gamma_1 \circ \gamma_2) = H(\gamma_1) \circ H(\gamma_2). \tag{2.17}$$

We conclude that H is actually a homomorphisms from the fundamental group $\pi_1(M, p)$ of homotopy classes of closed paths based at $p \in M$ to G

$$H : \pi_1(M, p) \to G. \tag{2.18}$$

Now H still depends on the base point and choice of the charts, but this freedom only amounts to overall conjugation of H.

Mess [22] has shown that the Lorentzian structure on a space-time manifold M of topology $\Sigma \times \mathbb{R}$ is uniquely determined by its holonomies. Moreover, given a homomorphism $\phi : \pi_1(\Sigma) \to ISO(2,1)$, Mess has shown that there exists a space-time with topology $\Sigma \times \mathbb{R}$ with holonomy ϕ if and only if the projection $\pi \circ \phi : \pi_1(\Sigma) \to SO(2,1)$ has a Fuchsian subgroup of $SO(2,1)$ as image. We will return to the notion of Fuchsian groups in paragraph 3.1.7, but essentially a Fuchsian group must be a subgroup whose elements form a discrete subset of $SO(2,1)$. We have now found a full correspondence between space-time solutions and holonomies. Hence, we can write down our reduced phase space as

$$\mathcal{P} = \mathrm{Hom}_0(\pi_1(\Sigma), ISO(2,1))/ISO(2,1), \tag{2.19}$$

where $ISO(2,1)$ acts on $\mathrm{Hom}_0(\pi_1(\Sigma), ISO(2,1))$ by overall conjugation and the subscript 0 means we restrict to homomorphisms with a Fuchsian projection.

Our characterization of the phase space in (2.19) is such that almost all gauge symmetry is removed, which means that distinct points of $\mathcal{P}$ correspond to physically distinct space-time solutions. Indeed we have gauged away the local diffeomorphism invariance by modelling our space-times on Minkowski space. But it turns out that the local diffeomorphisms alone only generate a subgroup of the whole diffeomorphism group. The quotient of the diffeomorphism group by this subgroup consists of so-called *large diffeomorphisms* (in contrast to local diffeomorphisms which are also called *small diffeomorphisms*). When we consider the action of the large diffeomorphisms on the surface Σ we see that they precisely correspond to the *Mapping class transformations* which we will discuss in paragraph 3.4.

Although two space-time solutions which are connected by a large diffeomorphism are classically indistinguishable, it is still an open question whether we should view large diffeomorphisms as a true symmetry of the quantum theory as well. We will return to this question in chapter 5.

2.4.2 The gauge group

As is clear from considerations in the previous paragraph we are dealing with a theory with gauge group $ISO(2,1)$, the group of Poincaré transformations of 2+1 dimensional Minkowski space. Before we continue we should examine some properties of this group.

Actually $ISO(2,1)$ is constructed as a semi-direct product of two groups. Given two groups G and H and a homomorphism $\rho : G \to \mathrm{Aut}(H)$, i.e. an action of G on H, we define the *semidirect product* $G \ltimes H$ to be the Cartesian product $G \times H$ with the group action

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot \rho(g_1)(h_2)). \tag{2.20}$$

Now we can write $ISO(2,1)$ as $SO(2,1) \ltimes \mathbb{R}^3$ where the group action on $\mathbb{R}^3$ is just addition and $SO(2,1)$ acts on $\mathbb{R}^3$ in the fundamental representation. Hence the group action on $SO(2,1) \ltimes \mathbb{R}^3$ can be written as

$$(g_1, X_1) \cdot (g_2, X_2) = (g_1 g_2, X_1 + g_1 X_2), \tag{2.21}$$

which is indeed the group multiplication of the Poincaré group.

Actually we will use a slightly different characterisation of $ISO(2,1)$. Notice that the Lie algebra $\mathfrak{so}(2,1)$ as a vector space is isomorphic to $\mathbb{R}^3$ and of course the same holds for the dual $\mathfrak{so}(2,1)^*$. Now if we choose the right isomorphism, the fundamental representation of $SO(2,1)$ on $\mathbb{R}^3$ becomes identified with its adjoint representation on $\mathfrak{so}(2,1)$. A suitable isomorphism is given by mapping the standard basis elements of $\mathbb{R}^3$ to $J_i^{\mathfrak{so}}$ defined in (1.4). We conclude that $ISO(2,1)$ is isomorphic to $SO(2,1) \ltimes \mathfrak{so}(2,1)$ with group action

$$(g_1, X_1) \cdot (g_2, X_2) = (g_1 \cdot g_2, X_1 + \mathrm{Ad}(g_1)X_2), \tag{2.22}$$

which is again isomorphic to $SO(2,1) \ltimes \mathfrak{so}(2,1)^*$ with group action

$$(g_1, X_1) \cdot (g_2, X_2) = (g_1 \cdot g_2, X_1 + \mathrm{Ad}^*(g_1^{-1})X_2). \tag{2.23}$$

If G is a Lie group and $\mathfrak{g}$ its Lie algebra, we will sometimes refer to $G \ltimes \mathfrak{g}$ and $G \ltimes \mathfrak{g}^*$ as the *tangent group* and *cotangent group* respectively. The reason is that the tangent bundle TG of G has a natural multiplication map given by the tangent map of the multiplication map on G. It is not hard to see that TG and $G \ltimes \mathfrak{g}$ are actually isomorphic[4]. The same holds for the cotangent bundle T^*G and $G \ltimes \mathfrak{g}^*$.

Now let us focus on the Lie group $SO(2,1)$ which maps the light cone in $2+1$ dimensions to itself. Clearly $SO(2,1)$ is not connected because some elements preserve the time direction and other elements swap the future and the past. Since we are concerned with holonomies of space-times with a definite time-direction, we are interested only in the subgroup $SO^0(2,1)$ of future preserving transformations. It turns out that $SO^0(2,1)$ is also precisely the image of the exponential map $\exp : \mathfrak{so}(2,1) \to SO(2,1)$.

In the following chapters we will be interested in the relation of gravity to hyperbolic geometry in which the relevant transformation groups are $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ and $PSU(1,1) = SU(1,1)/\{\pm I\}$. It is easily seen that the Lie algebras $\mathfrak{so}(2,1)$, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$ are isomorphic, i.e. they have the same structure constants. As we will see in paragraph 3.1.6 $SL(2,\mathbb{R})$ and $SU(1,1)$ are indeed isomorphic to each other. To see that $SO^0(2,1)$ is also isomorphic to $SL(2,\mathbb{R})$, we consider the adjoint representation of $SL(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$, which turns out to be faithful. All elements of the adjoint representation leave the bilinear form (1.7) invariant, but in the basis (1.5) B is just the Minkowski metric. It can be shown that indeed the whole $SO^0(2,1)$ is obtained in this way. We have thus established the equivalence

$$SO^0(2,1) \cong PSL(2,\mathbb{R}) \cong PSU(1,1). \tag{2.24}$$

2.4.3 Poisson algebra

Above we have found a nice description of the phase space manifold $\mathcal{P}$. But to do physics we need an additional structure on $\mathcal{P}$ which allows us to identify configuration and momentum variables. Either we define a symplectic structure on $\mathcal{P}$, i.e. a non-degenerate closed two-form on $\mathcal{P}$, or a Poisson algebra structure on the space of smooth functions on $\mathcal{P}$.

[4] Since $\mathfrak{g}$ is just the tangent space at the origin an isomorphism $G \ltimes \mathfrak{g} \to TG$ is given by translating it to tangent space at other points, i.e. $(g, X) \to (g, T_e r_g(X))$ where r_g is the right multiplication by g.

To identify momentum variables we normally look for time derivatives in the action, so let us write (2.9) in coordinates t and x^i where the x^i parametrize Σ. Then it is not hard to calculate the functional derivative with respect to the time-derivatives of e_i^a and ω_i^a,

$$\begin{aligned}\frac{\delta S}{\delta \dot{e}_i^a(x)} &= 0, \\ \frac{\delta S}{\delta \dot{\omega}_i^a(x)} &= 2\epsilon^{ij}\eta_{ab}e_i^a(x).\end{aligned} \tag{2.25}$$

We see that the triad e_i^a can be viewed as momentum conjugate to the spin-connection ω_{ja}. Hence, the equal time Poisson brackets are

$$\left\{e_i^a(x), \omega_i^b(x')\right\} = -\frac{1}{2}\epsilon_{ij}\eta^{ab}\delta^{(2)}(x-x'), \tag{2.26}$$

where indices i and j now label the two spatial directions.

To find the Poisson structure on $\mathcal{P}$ we evaluate the Poisson brackets of the holonomies expressed in terms of the path-ordered exponential of the connection A,

$$H(\gamma) = \mathcal{P}\exp\int_\gamma A. \tag{2.27}$$

In terms of the connection variable A, defined by (2.12), (2.26) becomes

$$\left\{A_i^a(x), A_i^b(x')\right\} = -\frac{1}{2}\epsilon_{ij}\left\langle T^a, T^b\right\rangle\delta^{(2)}(x-x'), \tag{2.28}$$

where the T^a denote the generators of $ISO(2,1)$.

Let γ_1 and γ_2 be two closed paths in M. Since the holonomy $H(\gamma_i)$ only depends on the homotopy class of γ_i and we want to consider equal time Poisson brackets, we view γ_1 and γ_2 as closed paths in Σ. Then it is clear that the Poisson bracket between $H(\gamma_1)$ and $H(\gamma_2)$ only has contributions from intersection points of γ_1 and γ_2. Let us for the moment therefore assume that γ_1 and γ_2 have precisely one intersection point $p \in \Sigma$. We can cut γ_i into three pieces σ_i, η_i and τ_i such that η_i contains the intersection point and has infinitesimal length. Since $H(\gamma_i) = H(\sigma_i)\cdot H(\eta_i)\cdot H(\tau_i)$ we can write

$$\{H(\gamma_1), H(\gamma_2)\} = (H(\sigma_1)\otimes H(\sigma_2))\cdot\{H(\eta_1), H(\eta_2)\}\cdot(H(\tau_1)\otimes H(\tau_2))\,. \tag{2.29}$$

Since the η_i have infinitesimal length, we only consider the linear term in the holonomy

$$H(\eta_i) = 1 + \int_{\eta_i} A. \tag{2.30}$$

It is not hard to show that we obtain

$$\{H(\eta_1), H(\eta_2)\} = -\frac{1}{2}\eta_{ab}(J^a\otimes P^b + P^a\otimes J^b)\epsilon(p;\gamma_1,\gamma_2), \tag{2.31}$$

where $\epsilon(p;\gamma_1,\gamma_2)$ equals $+1$ or -1 depending on the orientation of γ_1 and γ_2 at p.

Let's denote the $SL(2,\mathbb{R})$ part of $H(\gamma) \in SL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$ by $\rho_0(\gamma)$. Then clearly ρ_0 is just given by the holonomy of the spin-connection (as a $\mathfrak{sl}(2,\mathbb{R})$-valued one-form),

$$\rho_0(\gamma) = \mathcal{P}\exp\int_\gamma \omega. \tag{2.32}$$

The Ashtekar-Rovelli-Smolin loop variables are then defined as

$$\begin{aligned} T^0(\gamma) &= \frac{1}{2}\mathrm{Tr}\rho_0(\gamma) && (2.33)\\ T^1(\gamma) &= \int_\gamma \mathrm{Tr}\left(\rho_0(\gamma_x)e^a(x)J_a\right), && (2.34)\end{aligned}$$

where x runs along the path and γ_x is the path γ with starting point x.

We claim that we can write $T^1(\gamma)$ simpler in terms of the holonomy $H(\gamma)$, namely if we write $H(\gamma) = (\rho_0(\gamma), X_0(\gamma)) \in SL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$ then

$$T^1(\gamma) = \frac{1}{2}\mathrm{Tr}(X_0(\gamma)\rho_0(\gamma)). \tag{2.35}$$

We will not prove this here, but we give a hand-waving argument. In the previous paragraph we found that the gauge group $ISO(2,1)$ is isomorphic to the tangent group $TPSL(2,\mathbb{R}) \cong PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$. As a matter of fact, this allows us to write the (reduced) phase space (2.19) as the tangent bundle of the space of $PSL(2,\mathbb{R})$ holonomies,

$$\mathcal{P} = T\left(\mathrm{Hom}_0(\pi_1(\Sigma), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})\right). \tag{2.36}$$

Analogously, we consider the not-yet-reduced (infinite dimensional) phase space of spin connections and triads as the tangent bundle to the space $\mathcal{C}$ of all spin connections. Clearly $T^0(\gamma)$ is a function on $\mathcal{C}$ and $T^1(\gamma)$ from (2.34) is then just the derivative $dT^0(\gamma) : T\mathcal{C} \to \mathbb{R}$,

$$\begin{aligned} dT^0(\gamma) &= \frac{1}{2}d\left(\mathrm{Tr}\mathcal{P}\exp\int_\gamma \omega^a J_a\right)\\ &= \frac{1}{2}\int_\gamma dx^i\mathrm{Tr}\left(\rho_0(\gamma_x)d\omega_i^a(x^i)J_a\right). && (2.37)\end{aligned}$$

Then the same must hold on $\mathcal{P}$ in the holonomy formulation. But now $dT^0(\gamma) = \frac{1}{2}\mathrm{Tr}(X_0(\gamma)\rho_0(\gamma))$ since $X_0(\gamma)$ is the $\mathfrak{so}(2,1)$ part corresponding to $\rho_0(\gamma)$ in the $TPSL(2,\mathbb{R}) = PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$ decomposition. The significance of the phase space being a cotangent bundle will become clear in chapter 4.

The Poisson brackets between the T^0 and T^1 functions can be calculated from (2.31), but we will show that they arise naturally in the mathematical framework of chapter 4. Now we just refer to literature [6],

$$\begin{aligned} \left\{T^0(\gamma_1), T^0(\gamma_2)\right\} &= 0 && (2.38)\\ \left\{T^1(\gamma_1), T^0(\gamma_2)\right\} &= -\frac{1}{2}\sum_i \epsilon(p_i;\gamma_1,\gamma_2)\left(T^0(\gamma_1\circ_i\gamma_2) - T^0(\gamma_1\circ_i\gamma_2^{-1})\right) && (2.39)\\ \left\{T^1(\gamma_1), T^1(\gamma_2)\right\} &= -\frac{1}{2}\sum_i \epsilon(p_i;\gamma_1,\gamma_2)\left(T^1(\gamma_1\circ_i\gamma_2) - T^1(\gamma_1\circ_i\gamma_2^{-1})\right). && (2.40)\end{aligned}$$

Here $\gamma_1 \circ_i \gamma_2$ denotes the path obtained by cutting γ_1 and γ_2 at p_i and composing them.

Obviously, if we take the T^0 and the T^1 of all simple closed paths in Σ we obtain an over-complete set of observables. Hence, we should be able to find relations between them. Indeed they follow from identities for traces of 2×2 matrices[5],

$$T^0(\gamma_1)T^0(\gamma_2) = \frac{1}{2}\left(T^0(\gamma_1 \circ \gamma_2) + T^0(\gamma_1 \circ \gamma_2^{-1})\right) \tag{2.41}$$

$$T^0(\gamma_1)T^1(\gamma_2) + T^1(\gamma_1)T^0(\gamma_2) = \frac{1}{2}\left(T^1(\gamma_1 \circ \gamma_2) + T^1(\gamma_1 \circ \gamma_2^{-1})\right). \tag{2.42}$$

Notice that the second identity is just the derivative of the first one (in the sense discussed above).

[5] Let $g \in SL(2,\mathbb{R})$, then its characteristic polynomial equals $\lambda^2 - \mathrm{Tr}(g)\lambda + 1$. It is well-known that g itself must be a root of its characteristic polynomial, hence $g^2 - \mathrm{Tr}(g)g + 1 = 0$ and therefore $g + g^{-1} = \mathrm{Tr}(g)1$. If $h \in SL(2,\mathbb{R})$ as well, then $hg + hg^{-1} = \mathrm{Tr}(g)h$, hence $\mathrm{Tr}(hg) + \mathrm{Tr}(hg^{-1}) = \mathrm{Tr}(h)\mathrm{Tr}(g)$.

Chapter 3

Hyperbolic geometry and Teichmüller theory

We will start this chapter by recalling some theory on Riemann surfaces. There is a lot of literature on this subject, therefore I will omit most of the proofs and discussions and refer to the literature where necessary. The first half of this chapter we will mainly follow the definitions and derivations from Imayoshi and Taniguchi's book [18].

3.1 Riemann Surfaces and Teichmüller space

3.1.1 Definition of Riemann surface

Definition 3.1.1. *A* Riemann surface *R is a one-dimensional connected complex manifold. Specifically this means that R is a connected Hausdorff space together with a collection of maps $\{z_i\}_{i\in I}$ from open subsets U_i to $\mathbb{C}$ with the following properties:*

(i) $R = \bigcup_{i\in I} U_i$

(ii) $z_i : U_i \to \mathbb{C}$ *is a homeomorphism onto its image* $z(U_i)$.

(iii) If $i, j \in I$ *and* $U_j \cap U_i \neq \emptyset$ *the transition map* $z_j \circ z_i^{-1} : z(U_j \cap U_i) \to \mathbb{C}$ *is a biholomorphism.*

Clearly every Riemann surface can be viewed as a two-dimensional oriented real-analytic manifold by forgetting the complex structure. Actually we are only interested in the compact case, so in the remainder we will assume our Riemann surfaces to be *closed*. It is well known from topology that oriented compact topological surfaces are totally classified up to homeomorphisms by their *genus*, i.e. the number of holes or handles of the surface. As a consequence any closed Riemann surface is homeomorphic to a sphere with g handles attached and we will call g the *genus* of the Riemann surface.

3.1.2 Fundamental group

Let's fix a basepoint p_0 in our Riemann surface R. The *fundamental group* $\pi(R, p_0)$ of R has as its elements homotopy classes of closed curves based at p_0 and group multiplication is given by path composition (of homotopty representatives). Suppose R has genus g, then we can cut R along $2g$ simple closed paths $\{A_i, B_i\}_{i=1,\dots,g}$ to obtain a $2g$ sided polygon as in figure 3.1. We can reconstruct our space topologically by gluing the corresponding edges.

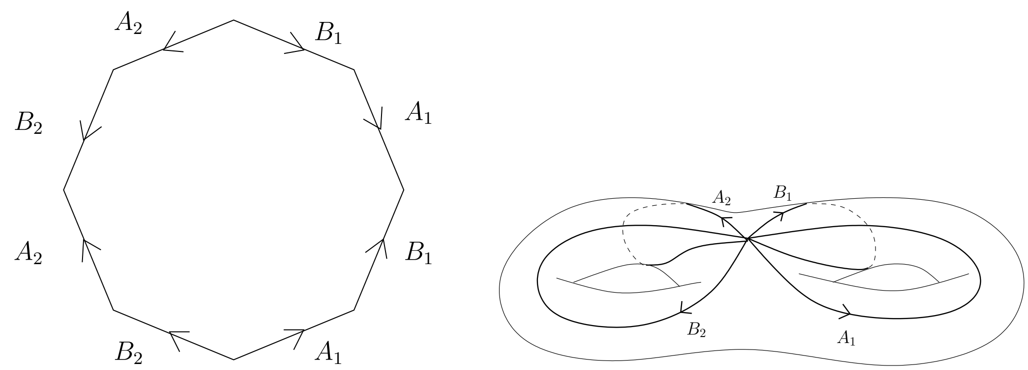


Figure 3.1: *Polygon corresponding to a surface of genus 2.*

Now it is clear that the fundamental group $\pi(R, p_0)$ is generated by the homotopy classes $[A_i], [B_i]$ corresponding to the paths A_i and B_i and that they satisfy the relation

$$\prod_{i=1}^{g} [A_i][B_i][A_i]^{-1}[B_i]^{-1} = 1. \tag{3.1}$$

Notice that the fundamental group $\pi(R, p_0)$ with base-point p_0 is isomorphic to the fundamental group $\pi(R, p_1)$ with base-point p_1. Indeed, given a path C from p_1 to p_0 we can map $[A] \in \pi(R, p_0)$ to $[C^{-1}AC] \in \pi(R, p_1)$. But this isomorphism is certainly not unique, because non-homotopic paths from p_1 to p_0 yield different isomorphisms. Nevertheless we will talk about *the* fundamental group $\pi(R)$ when we are just interested in the group structure or when the choice of generators is unimportant.

3.1.3 Teichmüller space

One would like to consider what variety of inequivalent closed Riemann surfaces exist. We have already seen that we can distinguish Riemann surfaces by their genus. What does the space of closed Riemann surfaces of genus g look like? At first sight it would seem natural to consider the following space.

Definition 3.1.2. *The* moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus g *is the space of all equivalence classes of biholomorphic Riemann surfaces of genus g.*

It turns out that there exists a slightly larger space, known as Teichmüller space, which has some nicer properties and from which we can construct the moduli space without too much effort. The idea is that we restrict the equivalence to particular biholomorphisms, namely those that are in some sense homotopic to the identity map. There exist several equivalent definitions of Teichmüller space, but all of them involve adding a little extra structure to the Riemann surfaces.

Definition 3.1.3. *Let S be a two-dimensional oriented differentiable manifold of genus g. A* marked Riemann surface of genus g *is a pair (R, f) of a Riemann surface R of genus g and an orientation-preserving diffeomorphism $f : S \to R$. Two marked Riemann surfaces (R, f) and (R', f') are said to be equivalent if there exists a biholomorphism $\phi : R \to R'$ such that the diffeomorphism $f'^{-1} \circ \phi \circ f : S \to S$ is homotopic to the identity map on S. The set of all equivalence classes $[(R, f)]$ of marked Riemann surfaces is called* Teichmüller space $\mathcal{T}_g$.

Equivalently we could have defined a marked Riemann surface as a Riemann surface R together with a base point p and a set of generators $\Gamma_p = \{[A_i], [B_i]\}_{i=1,\ldots,g}$ of the fundamental group $\pi(R, p)$. Given two sets of generators $\Gamma_{p_0} = \{[A_i], [B_i]\}$ and $\Gamma'_{p_1} = \{[A'_i], [B'_i]\}$, we have seen above that a path connecting p_0 and p_1 defines a isomorphism between $\pi(R, p_0)$ and $\pi(R, p_1)$ and we call Γ_{p_0} and Γ'_{p_1} equivalent if such an isomorphism exists that maps $[A_i]$ and $[B_i]$ to $[A'_i]$ and $[B'_i]$ respectively. A biholomorphism of Riemann surfaces extends naturally to a map between marked Riemann surfaces and therefore we can define two marked Riemann surfaces (R, Γ_p) and $(R', \Gamma'_{p'})$ to be equivalent if there exists a biholomorphism ϕ which maps Γ_p to a set of generators $\phi_*(\Gamma_p)$ equivalent to $\Gamma'_{p'}$.

In the next sections we will study some properties of Teichmüller space. In particular we will construct global coordinates on Teichmüller space in two different ways.

3.1.4 Relation between Riemann surfaces and Riemannian surfaces

Let us consider a closed two-dimensional smooth manifold M with Riemannian metric ds^2. It can be shown that in each coordinate neighbourhood $U \subset M$ there exist *isothermal coordinates* (u, v), which means

$$ds^2 = \rho(du^2 + dv^2) \tag{3.2}$$

for some smooth function $\rho : U \to \mathbb{R}^+$. The complex coordinate $w = u + iv$ now defines a unique complex structure on M, called the *conformal structure* induced by the metric ds^2. An orientation-preserving map $f : M \to N$ between Riemannian manifolds (M, ds_1^2) and (N, ds_2^2) is called a *conformal map* if the pull-back of ds_2^2 along f is equal to ρds_1^2 for some smooth function ρ. Now M and N are *conformally equivalent*, i.e. there exists a conformal map between them, if and only if the M and N with their conformal structures are biholomorphic.

We conclude that the moduli space of closed Riemann surfaces of genus g can be identified with the space of conformal equivalence classes of Riemannian metrics on a two-dimensional manifold of genus g. Define two Riemannian metrics to be *strongly equivalent* if there exists a conformal map that is homotopic to the identity. Then it can be shown that Teichmüller space can be identified with the space of equivalence classes of strongly equivalent Riemannian metrics.

3.1.5 Universal coverings

A powerful method of examining Riemann surfaces is by considering their universal coverings.

Definition 3.1.4. *Let R and $\tilde{R}$ be Riemann surfaces and $\pi : \tilde{R} \to R$ a surjective holomorphic map. If for every $p \in R$ there exists an open neighbourhood $U \subset R$ such that π restricts to a biholomorphism $V \to U$ for any connected component V of $\pi^{-1}(U)$, then we call π a* covering map *and $(\tilde{R}, \pi)$ a* covering *of R. If $\tilde{R}$ is simply connected, $(\tilde{R}, \pi)$ is called a* universal covering*. A biholomorphic mapping $\gamma : \tilde{R} \to \tilde{R}$ with $\pi \circ \gamma = \pi$ is called a* covering transformation *and the set Γ of covering transformations form a group under composition, the* covering transformation group.

By the uniformization theorem (see [18] paragraph 2.1) every simply connected Riemann surface is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the complex upper half-plane $\mathbb{H}$. Every closed Riemann surface has a universal covering, hence every closed Riemann surface has $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$ as universal covering. To be more precise, a genus 0 Riemann surface is simply-connected itself and compact, hence already biholomorphic to $\hat{\mathbb{C}}$. A genus 1 surface has as universal covering $\mathbb{C}$ and any surface with genus larger than 1 has $\mathbb{H}$ as universal covering. In the latter case the covering transformation group Γ is also called a *Fuchsian model* of the Riemann surface.

Let R be a closed Riemann surface. It can be shown that the fundamental group $\pi(R)$ of R is isomorphic to the covering transformation group Γ of the universal covering $\tilde{R}$. Furthermore R itself is biholomorphically equivalent to the quotient surface $\tilde{R}/\Gamma$. As a consequence we can consider Teichmüller space of a surface R to be the set of inequivalent quotients of the universal covering $\tilde{R}$. We will make this construction more precise in the next section.

For later reference we notice that there is a biholomorphically equivalent representation of the upper half-plane $\mathbb{H}$ given by the *Poincaré disc* Δ. This representation is particularly useful for drawing pictures. A specific biholomorphism is given by the map

$$\mathbb{H} \to \Delta : z \to \frac{z-i}{z+i}. \tag{3.3}$$

3.1.6 Möbius transformations

The covering transformation group Γ of a Riemann surface R is clearly a subgroup of the group $Aut(\tilde{R})$ of biholomorphic automorphisms of $\tilde{R}$. Therefore we would like to know exactly what the automorphisms of $\hat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{H}$ are.

Lemma 3.1.5. *The automorphism group of $\tilde{R}$ is given by fractional linear transformations, or* Möbius transformations*,*

$$z \to \frac{az+b}{cz+d}, \tag{3.4}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{cases} PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\} & \text{if } \tilde{R} = \hat{\mathbb{C}} \\ \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{C}, a \neq 0 \right\} & \text{if } \tilde{R} = \mathbb{C} \\ PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\} & \text{if } \tilde{R} = \mathbb{H} \\ PSU(1,1) = SU(1,1)/\{\pm I\} & \text{if } \tilde{R} = \Delta \end{cases} . \tag{3.5}$$

Using this lemma we see that the biholomorphism (3.3) defines an isomorphism between $PSL(2,\mathbb{R})$ and $PSU(1,1)$, given by

$$PSL(2,\mathbb{R}) \to PSU(1,1) : A \to \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot A \cdot \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} . \tag{3.6}$$

From here on we will be mainly concerned with the transformation groups of $\mathbb{H}$ and Δ. We can classify Möbius transformations according to the number of fixed points:

Definition 3.1.6. *Let $\gamma \in \Gamma$ be a Möbius transformation of $\mathbb{H}$ or Δ. Denote by $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and S the boundaries of $\mathbb{H}$ and Δ respectively.*

(i) γ is called parabolic *if γ has a single fixed point on the boundary.*

(ii) γ is called hyperbolic *if γ has two distinct fixed points on the boundary.*

(iii) γ is called elliptic *if γ has a single fixed point outside the boundary.*

Every element of $Aut(\mathbb{H})$ or $Aut(\Delta)$ which is not the identity is either parabolic or hyperbolic or elliptic. There exists an easy way of deciding which type a particular element is by considering the trace of the matrix.

Lemma 3.1.7. *If γ is an element of $Aut(\mathbb{H})$ or $Aut(\Delta)$ which is not the identity, then $Tr(\gamma) \in \mathbb{R}$ and*

(i) γ is hyperbolic if and only if $|Tr(\gamma)| > 2$,

(ii) γ is parabolic if and only if $|Tr(\gamma)| = 2$,

(iii) γ is elliptic if and only if $|Tr(\gamma)| < 2$.

3.1.7 Fuchsian groups

In paragraph 3.1.5 we mentioned that any Riemann surface R of genus $g \geq 2$ has a Fuchsian model Γ and that R is biholomorphic to $\mathbb{H}/\Gamma$. But conversely not all subgroups Γ of $Aut(\mathbb{H})$ can act on $\mathbb{H}$ to obtain a Riemann surface.

Definition 3.1.8. *A subgroup Γ of $Aut(\mathbb{H})$ or $Aut(\Delta)$ is called a* Fuchsian group *if it is a discrete subgroup, with respect to the standard topology of $PSL(2,\mathbb{R})$ or $PSU(1,1)$ as Lie groups.*

Equivalently, Γ is a Fuchsian group if it acts *properly discontinuously* on $\mathbb{H}$ or Δ. Hence, if Γ is a Fuchsian group, the quotient $\mathbb{H}/\Gamma$ is a Riemann surface. In particular Γ cannot have elliptic elements, which leave a point of $\mathbb{H}$ fixed. Furthermore it can be shown that $\mathbb{H}/\Gamma$ is compact if and only if Γ has no parabolic elements. As a consequence a Fuchsian model of R is a Fuchsian group consisting only of hyperbolic elements and the identity element.

Note that the Fuchsian model is not uniquely defined by the Riemann surface. Indeed, given a Fuchsian model Γ, we conjugate it with any element $\alpha \in Aut(\mathbb{H})$ to obtain another Fuchsian model $\Gamma' = \alpha\Gamma\alpha^{-1}$. Taking this into account we can identify Teichmüller space of surface R with genus g with conjugacy classes of Fuchsian groups which are isomorphic to the fundamental group $\pi(R)$,

$$\mathcal{T}_g \equiv \mathrm{Hom}_0(\pi(R), Aut(\mathbb{H}))/Aut(\mathbb{H}). \tag{3.7}$$

Here the subscript 0 means that we restrict to injective homomorphisms which have as image a Fuchsian group without parabolic elements and $Aut(\mathbb{H})$ acts on $\mathrm{Hom}_0(\pi(R), Aut(\mathbb{H}))$ by overall conjugation.

3.2 Hyperbolic geometry

In this section we will see how geometry on Riemann surfaces can be understood by considering hyperbolic geometry.

3.2.1 Poincaré metric

If we want to consider distances in the upper half-plane $\mathbb{H}$ or in the Poincaré disc Δ, we have to define a natural metric on them. We have seen in paragraph 3.1.4 that a metric on a two-dimensional Riemannian manifold defines a complex structure on it, the conformal structure. Now it turns out that up to a constant there exists a unique metric on Δ whose induced conformal structure coincides with the standard complex structure and which is invariant under $Aut(\Delta)$. If we fix the constant factor such that the curvature of the metric is everywhere -1, we obtain the *Poincaré metric* on Δ

$$ds^2_\Delta = \frac{4|dz|^2}{(1-|z|^2)^2}. \tag{3.8}$$

The Poincaré metric on the upper half-plane is obtained by the pull-back of ds^2_Δ under (3.3),

$$ds^2_{\mathbb{H}} = \frac{|dz|^2}{\mathrm{Im}^2 z}. \tag{3.9}$$

Since the Poincaré metric is invariant under the automorphism group, it is in particular invariant under the action of a Fuchsian model Γ of a closed Riemann surface R. Hence, we can define a Riemannian structure on R by copying it from $\mathbb{H}/\Gamma$. This is the unique Riemannian metric on R with constant curvature -1 and with the correct conformal structure, which we will call the *hyperbolic metric*.

3.2.2 Geodesics

Given a piecewise smooth curve α in the Poincaré disc Δ, we define its length with respect to the metric (3.8),

$$l(\alpha) = \int_\alpha ds_\Delta. \tag{3.10}$$

It is not hard to check that geodesics in Δ (and in $\mathbb{H}$) correspond to arcs of circles or line segments which are orthogonal to the boundary, see figure 3.2.

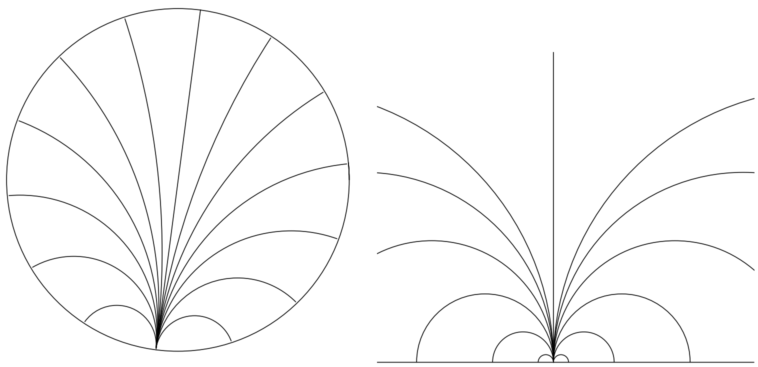


Figure 3.2: *Some geodesics in Δ (left) and in $\mathbb{H}$ (right).*

Clearly there exists a unique geodesic connecting any two points of Δ. Because the metric is invariant under the automorphism group $Aut(\Delta)$, geodesics are mapped to geodesics by $Aut(\Delta)$. In particular, if $\gamma \in Aut(\Delta)$ is a hyperbolic element, it leaves two points on the boundary fixed and therefore the geodesic connecting the two points is mapped to itself by γ. We call this unique invariant geodesic the *axis* A_γ of γ.

Given a Riemann surface R and a Fuchsian model $\Gamma \subset Aut(\Delta)$, we find that geodesics in R are given by the projections of geodesics in Δ onto Δ/Γ. The projection of a geodesic is closed if and only if it is invariant under a non-trivial element of $Aut(\Delta)$. We conclude that for any $\gamma \in Aut(\Delta)$ there exists a unique closed geodesic in R which originates from projecting the axis A_γ. It can be shown that the length l_γ of this closed geodesic is easily expressed in terms of the trace of the matrix corresponding to γ (see (3.5)),

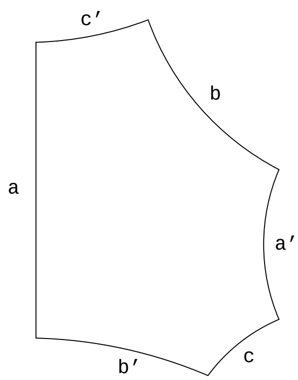
$$|\mathrm{Tr}(\gamma)| = 2\cosh\left(\frac{l_\gamma}{2}\right). \tag{3.11}$$

3.2.3 Hyperbolic trigonometry

Just as in Euclidean geometry there are powerful trigonometric relations between sides and angles of hyperbolic polygons. We will state some analogues of the usual trigonometric identities which we will need later on, see [12] for proofs.

Lemma 3.2.1. *We have the following relations between lengths and angles of geodesic edges of polygons (with respect to the Poincaré metric (3.8) in the disc or (3.9) in the upper half-plane).*

(i)



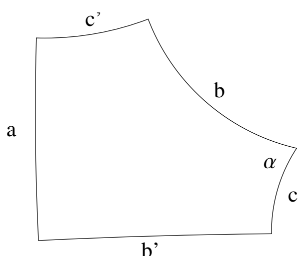
Given any three numbers $a, b, c \in \mathbb{R}_{>0}$ *there exists a unique convex right-angled hexagon with alternating sides of length* a*,* b *and* c*. The lengths of the sides (with the notation as in figure to the left) satisfy the relations*

$$\frac{\sinh a}{\sinh a'} = \frac{\sinh b}{\sinh b'} = \frac{\sinh c}{\sinh c'}, \tag{3.12}$$

$$\cosh a' = \frac{\cosh b \cosh c + \cosh a}{\sinh b \sinh c} \tag{3.13}$$

and analogues for b' *and* c'*.*

(ii)

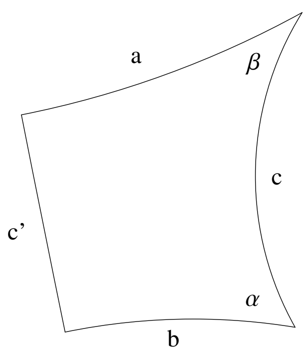


A pentagon with four right angles and remaining angle α *satisfies*

$$\frac{\sinh a}{\sin \alpha} = \frac{\cosh b}{\sinh b'} = \frac{\cosh c}{\sinh c'}, \tag{3.14}$$

$$\cosh a = \sinh b \sinh c - \cosh b \cosh c \cos \alpha. \tag{3.15}$$

(iii)

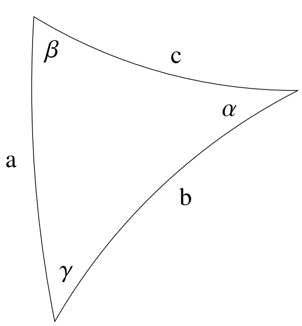


A quadrangle with two adjacent right angles and remaining angles α *and* β *satisfies*

$$\frac{\cosh a}{\sin \alpha} = \frac{\cosh b}{\sin \beta} = \frac{\sinh c}{\sinh c'}, \tag{3.16}$$

$$\sinh a = \sinh b \cosh c - \cosh b \sinh c \cos \alpha. \tag{3.17}$$

(iv)



Triangle with arbitrary angles satisfies

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}, \tag{3.18}$$

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha. \tag{3.19}$$

Finally we mention that there exists a nice relation between the area and the angles of a

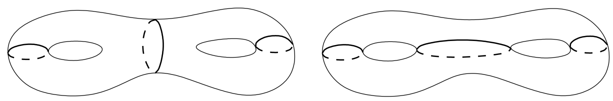


Figure 3.3: *Two inequivalent pants decompositions of a genus 2 surface.*

polygon. Because the Poincaré metric has constant curvature -1, the difference between π and the sum of the angles of a triangle is equal to its area. So for an n-sided (non-intersecting) polygon with inner angles θ_i the area A is given by

$$A = (2n-2)\pi - \sum_{i=1}^{n} \theta_i. \tag{3.20}$$

In particular, there exists an upper bound $(2n-2)\pi$ for the area of a polygon, which is reached when all vertices lie on the boundary of Δ or $\mathbb{H}$.

3.3 Coordinates on Teichmüller space

Because Teichmüller space will play the role of configuration space in 2 + 1-dimensional gravity, we would like to identify useful global coordinates. First we will construct the Fenchel-Nielsen coordinates and we will generalize them to Riemann surfaces with punctures and holes. Another set of coordinates, the Penner coordinates, will be introduced for Riemann surfaces with at least one puncture or hole.

3.3.1 Pants decomposition

Given a Riemann surface R of genus g we would like to decompose it into a set of elementary building blocks. It turns out that for genus $g \geq 2$ a Riemann surface can be cut along closed geodesic to obtain $2g-2$ copies of a *pair of pants*. So from now on we will assume R to be of genus $g \geq 2$.

Definition 3.3.1. *A* pair of pants P *of a Riemann surface* R *is a simply connected subsurface of* R *whose boundary* ∂P *in* R *consists of three simple closed geodesics.*

It can be shown that there exists a collection $\mathcal{L} = \{\alpha_i\}_{i=1}^{3g-3}$ of simple closed geodesics on R which decomposes R into pairs of pants. Clearly this collection $\mathcal{L}$ is not unique, see figure 3.3.

Now we claim that the complex structure on a pair of pants P is uniquely determined by the lengths l_1, l_2, l_3 of the three geodesic boundary components. To show this we decompose P into two right angled hexagons by cutting along three shortest geodesics with lengths d_{12}, d_{13}, d_{23} connecting the boundary components, as in figure 3.4.

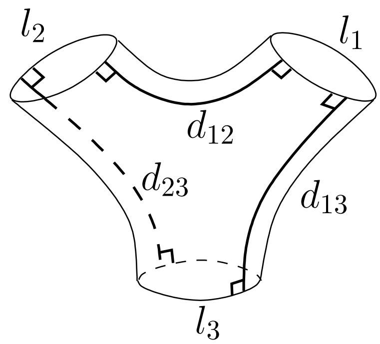


Figure 3.4: *Dividing a pair of pants into two right-angled hexagons.*

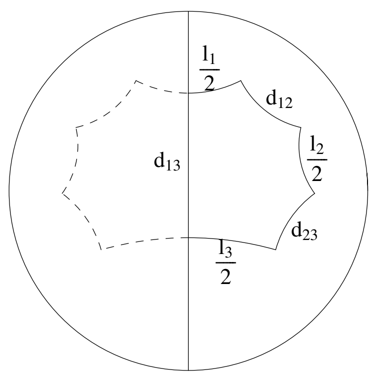


Figure 3.5: *A pair of pants given by a region of the Poincaré disc.*

Because the hexagons have the sides d_{12}, d_{13}, d_{23} in common, by lemma 3.2.1 (i) they must actually be identical (or mirrored to be more precise). As a consequence the remaining sides of the hexagon must have lengths $\frac{1}{2}l_1, \frac{1}{2}l_2, \frac{1}{2}l_3$. Applying lemma 3.2.1 (i) once more, we find that our hexagon is uniquely determined by l_1, l_2, l_3. Consequently, our pair of pants P is uniquely determined by l_1, l_2, l_3 too. See also figure 3.5.

3.3.2 Fenchel-Nielsen coordinates

Since we have already fixed the complex structure, to reconstruct our Riemann surface we only need to specify how to glue together the pairs of pants. To this end we define the *twisting parameters* τ_i corresponding to the closed geodesics l_i in $\mathcal{L}$. Notice that, after choosing an ordering of the boundary components of P, the connecting geodesics d_{12}, d_{23}, d_{13} define distinguished points on l_1, l_2, l_3 respectively. Now we can define τ_i modulo l_i to be the distance along l_i between the two distinguished points corresponding to the two pairs of pants glued along l_i.

Due to a result known as Teichmüller Theorem we know that Teichmüller space is simply

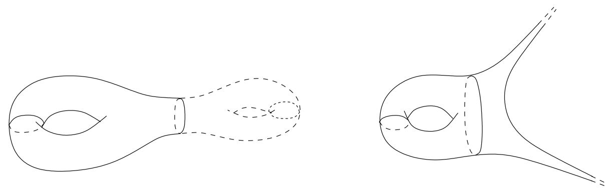


Figure 3.6: *A one-holed torus and a two-punctured torus with their pants decomposition.*

connected, therefore we conclude that we must allow τ_i to run over $\mathbb{R}$. Hence we have the following:

Lemma 3.3.2. *Given a collection $\mathcal{L}$ of decomposing simple closed geodesics on R and fixing the zeroes of the twisting parameters, we obtain a diffeomorphism*

$$\Psi : \mathcal{T}_g \to (\mathbb{R}_{>0})^{3g-3} \times \mathbb{R}^{3g-3} : R \to (l_i, \tau_i) \tag{3.21}$$

and hence a global set of coordinates on Teichmüller space, the Fenchel-Nielsen coordinates*.*

3.3.3 Punctures and holes

Using the pants decompositions we can easily generalize Teichmüller space to Riemann surfaces with holes and punctures. Notice that when we decompose a Riemann surface R of genus g into pairs of pants such that one of them forms a handle and we throw away this handle we obtain a Riemann surface of genus $g-1$ with a hole with geodesic boundary length equal to the corresponding Fenchel-Nielsen coordinate (figure 3.6).

Define $\mathcal{T}_{g,s}^{l_1,\ldots,l_s}$ to be the Teichmüller space of Riemann surfaces of genus g with s holes of geodesic boundary lengths $l_1, \ldots, l_s$ respectively. From considering the pants decomposition we find that $\mathcal{T}_{g,s}^{l_1,\ldots,l_s}$ is a space of dimension $6g-6+2s$ and the Fenchel-Nielsen coordinates define again global coordinates.

Furthermore we would like to admit zero boundary lengths. We then obtain Riemann surfaces with *punctures*. When we compare figure 3.5 to lemma 3.2.1 (i) we find

$$\frac{\sinh d_{23}}{\sinh \frac{1}{2} l_1} = \frac{\sinh d_{13}}{\sinh \frac{1}{2} l_2} = \frac{\sinh d_{12}}{\sinh \frac{1}{2} l_3}. \tag{3.22}$$

Hence, if we fix l_2 and l_3 and let l_1 approach zero, d_{12} and d_{13} will go to infinity. Punctures therefore correspond to infinitally long 'spikes' on a Riemann surface, as in figure 3.6.

Let γ be the Möbius transformation corresponding to a path around a puncture. Then from (3.11) we find $|\mathrm{Tr}(\gamma)| = 2$ and therefore due to lemma 3.1.7 γ must be parabolic, i.e. it has one fixed point on the boundary of $\mathbb{H}$ which is of course the puncture itself. In case of punctures we must revise our statement in paragraph 3.1.7 that a Fuchsian model consists only of the identity and hyperbolic elements: a Fuchsian model of R with s punctures consists

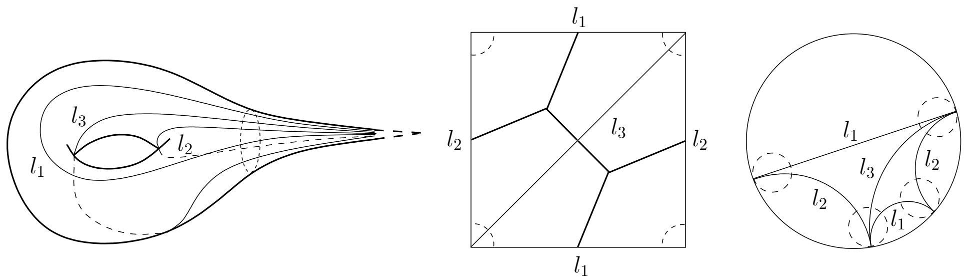


Figure 3.7: *Geodesic triangulation of a once-punctured torus. The second picture shows schematically the two constituating triangles together with the dual fat graph. The third picture shows the same triangles but now embedded in the Poincaré disc. The dashed circles correspond to the horocycle.*

only of the identity, exactly s distinct parabolic conjugacy classes and hyperbolic conjugacy classes.

3.3.4 Penner and Fock coordinates

In presence of punctures another useful set of coordinates exists on Teichmüller space. Here we will shortly review how to construct them (see, for instance, [27] for more details).

Let R be a Riemann surface of genus g with $s > 0$ punctures. It can be shown that there exist $6g - 6 + 3s$ disjoint geodesics running between punctures of R which decompose R into $4g - 4 + 2s$ triangles. Dual to this triangulation is a trivalent graph, which is called a *fat graph* on R. See figure 3.7 for an example of a triangulation of a once-punctured torus.

We would like to view the lengths of the geodesic edges of the triangulation as coordinates, but obviously these lengths are all infinite. There is a way out of this, namely by choosing a *horocycle* for all punctures. A horocycle for a puncture is a path around the puncture which is perpendicular to all geodesics originating at the puncture. So in some sense a horocycle gives a set of points which have the same distance to the puncture at infinity. In the Poincaré disc horocycles are given by circles tangent to the boundary.

Now define the length l_e of an edge e of the triangulation to be the distance along e between the horocycles of the punctures which e connects. Shifting a horocycle for some puncture just corresponds to adding a constant to the lengths of all edges emanating from that puncture. It turns out that modulo this symmetry the set of lengths l_e constitute global coordinates on $\mathcal{T}_{g,s}$, which are called *Penner coordinates*.

To get rid of the symmetry in the Penner coordinates it is sometimes more convenient to assign coordinates to the edges of the dual fat graph. Define the *Fock coordinate* corresponding to the edge e of the fat graph as

$$z_e = l_a + l_c - l_b - l_d \tag{3.23}$$

with the labelling as in figure 3.8. Notice that not all Fock coordinates are independent: for

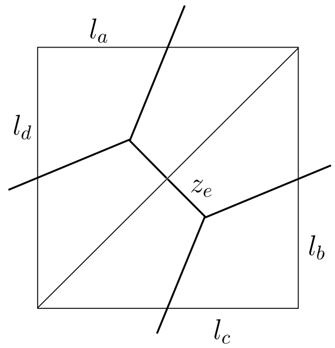


Figure 3.8: *Fat graph of two adjacent triangles.*

a set $\{e_i\}$ of edges forming a closed path in the fat graph we have

$$\sum_i z_{e_i} = 0. \tag{3.24}$$

If we use this type of relations to eliminate s of the z_e's corresponding to the $6g - 6 + 3s$ edges we obtain a set of $6g - 6 + 2s$ global coordinates.

One advantage of using Fock coordinates instead of Fenchel-Nielsen coordinates is that we can easily construct a Fuchsian model in terms of the Fock coordinates. To achieve this we assign to each edge e of the fat graph a matrix

$$X_e = E(z_e) = \begin{pmatrix} 0 & e^{\frac{z_e}{2}} \\ -e^{-\frac{z_e}{2}} & 0 \end{pmatrix} \tag{3.25}$$

and we blow up the vertices by inserting a triangle (with counter-clockwise oriented edges) with associated matrices

$$X_e = R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{3.26}$$

Given a closed path c of edges $\{e_i\}_{i=1,\dots,r}$ in the blown-up fat graph, the element γ_c of the Fuchsian model corresponding to the homotopy class of c is given by

$$X_c = X_{e_r} \cdots X_{e_1}. \tag{3.27}$$

See figure 3.9 for an example.

It turns out not to be too difficult to generalize Fock coordinates to surfaces with holes instead of punctures. Notice that it is sufficient to impose the constraints (3.24) for paths which encircle a single puncture. If we calculate the element of the Fuchsian model corresponding to such a path c in clockwise direction we find that its trace is equal to

$$\mathrm{Tr}(\gamma_c) = \mathrm{Tr}\left(\prod_{e\in c} RX(z_e)\right) = 2\cosh\left(\frac{1}{2}\sum_{e\in c} z_e\right). \tag{3.28}$$

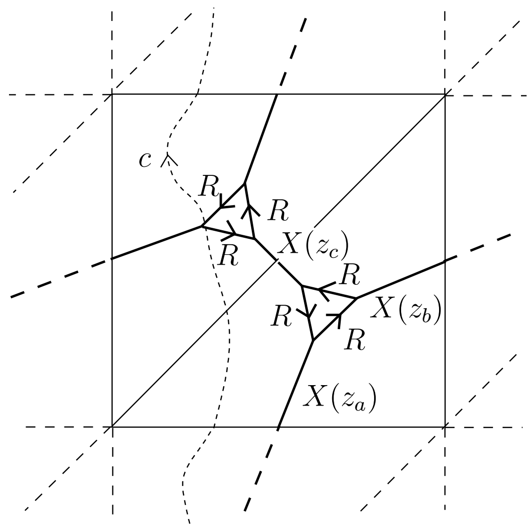


Figure 3.9: *Construction of Fuchsian model from Fock coordinates for a once-punctured torus. For example, in this case the element corresponding to the closed path c is* $\gamma_c = R^{-1}X(z_c)R^{-1}X(z_a)R^{-1}$.

We compare this to (3.11) and conclude that if we instead impose the constraint

$$\sum_{e \in c} z_e = 2l_c, \tag{3.29}$$

we obtain a Fuchsian model with geodesic boundary length l_c. So, for example, coordinates for a torus with one hole of boundary length l_c are given by z_a, z_b, z_c under the constraint $\sum_{e\in c} z_e = 2z_a + 2z_b + 2z_c = 2l_c$.

3.4 Mapping class group

In paragraph 3.1.3 we gave the definition of Moduli space and Teichmüller space. It is not entirely clear which is the more physical one. Should we consider any two Riemann surfaces which are biholomorphic as physically equivalent or only those which are connected by a biholomorphism homotopic to the identity? We will return to this question later. But meanwhile let's see how we can retrieve the Moduli space from Teichmüller space.

From the definition it is clear that Moduli space must be a quotient of Teichmüller space by some group acting on it. We will call this group the *Mapping Class Group* (MCG). In some literature this group is also called *Modular group*, while often this terminology is only used for the torus ($g = 1$) case.

It turns out that a set of generators can nicely be characterized by geometric operations known as *Dehn twists*. Given a simple closed geodesic α on a Riemann surface R, we can cut R along α, twist the ends by 2π and then reglue them to obtain a Riemann surface R'.

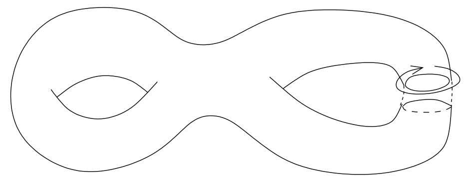


Figure 3.10: *Dehn twist along a simple closed geodesic.*

Since no local complex structure has changed, R and R' will be biholomorphic. But if we had considered a marked Riemann surface, R together with a set of generators of its fundamental group, we would have found that the marked surfaces R and R' are not equivalent. By Dehn twisting certain generators will have mixed up, i.e. a Dehn twist acts as a automorphism on the fundamental group. It can be shown that the Dehn twists actually generate all automorphisms $Aut(\pi(R))$ modulo overall conjugations. Hence, the mapping class group is isomorphic to

$$Aut(\pi(R))/Inn(\pi(R)) \tag{3.30}$$

where $Inn(\pi(R))$ denotes the conjugations, also known as the *inner automorphisms*.

3.4.1 MCG in Fenchel-Nielsen coordinates

Given a Riemann surface R together with a pants decomposition $\mathcal{L}$, we can easily write down the MCG element corresponding to a Dehn twist along some simple closed geodesic $l_i \in \mathcal{L}$. We just have to shift the twisting parameter τ_i by a distance l_i,

$$\tau_i \to \tau_i + l_i \tag{3.31}$$

and keep the remaining parameters fixed.

Unfortunately the Dehn twist along the $l_i \in \mathcal{L}$ do not generate the entire MCG. The best we can do is to select two different pants decompositions in such a way that the Dehn twists along their closed geodesics generate the MCG. See for instance [19] for minimal sets of generators. Because there is no simple relation between Fenchel-Nielsen coordinates corresponding to different pants decompositions, we cannot easily express the whole MCG in terms of one set of Fenchel-Nielsen coordinates.

3.4.2 MCG in Penner/Fock-coordinates

Recall from paragraph 3.3.4 that the Penner- and Fock-coordinates were defined with respect to a triangulation of a Riemann surface. Clearly such a triangulation is not uniquely defined, but it turns out that we can define a set of elementary moves on the dual fat graph which can transform any triangulation to any other. Apart from renaming edges and changing the orientation of vertices (see [27] for more details) the elementary moves boil down to *flipping* edges of the fat graph, see figure 3.11.

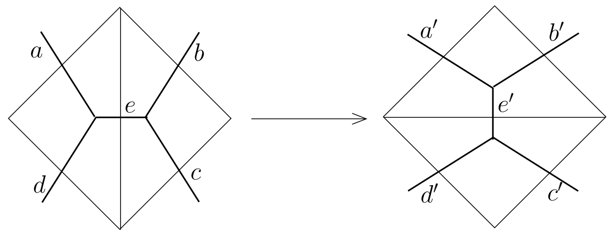


Figure 3.11: *Elementary move given by flipping two adjacent triangles in the triangulation.*

After flipping the Penner coordinates in the new triangulation can easily be expressed in terms of the old ones. The lengths of the edges which the triangulations have in common remain unchanged and

$$e^{l_{e'}} = e^{-l_e}\left(e^{l_a}e^{l_c} + e^{l_b}e^{l_d}\right). \qquad (3.32)$$

Hence, in terms of Fock coordinates we have

$$\begin{aligned} z_{e'} &= -z_e, \\ z_{a'} &= z_a + \log\left(1+e^{z_e}\right), \\ z_{b'} &= z_b - \log\left(1+e^{-z_e}\right), \\ z_{c'} &= z_c + \log\left(1+e^{z_e}\right), \\ z_{d'} &= z_d - \log\left(1+e^{-z_e}\right). \end{aligned} \qquad (3.33)$$

Now a mapping class transformation acting on a triangulated Riemann surface will send the triangulation to another one. As a consequence we can write it as a sequence of flippings and relabeling of the edges.

As an example we will construct two generators of the MCG for the once-punctured torus. Denote the lengths of the edges of triangulation l_1, l_2, l_3 as in figure 3.7 and the Fock coordinates z_1, z_2, z_3 accordingly. Then

$$\begin{aligned} z_1 &= 2l_2 - 2l_3 \\ z_2 &= 2l_3 - 2l_1 \\ z_3 &= 2l_1 - 2l_2. \end{aligned} \qquad (3.34)$$

Comparing figure 3.12 to figure 3.11 and using (3.32) we see that

$$\begin{aligned} l'_1 &= l_1, \\ l'_2 &= l_3, \\ l'_3 &= -l_2 + \log\left(e^{2l_1} + e^{2l_3}\right). \end{aligned} \qquad (3.35)$$

And the Fock-coordinates get transformed as

$$\begin{aligned} z'_1 &= z_1 - 2\log\left(e^{-z_2}+1\right), \\ z'_2 &= z_3 + 2\log\left(e^{z_2}+1\right), \\ z'_3 &= -z_2. \end{aligned} \qquad (3.36)$$

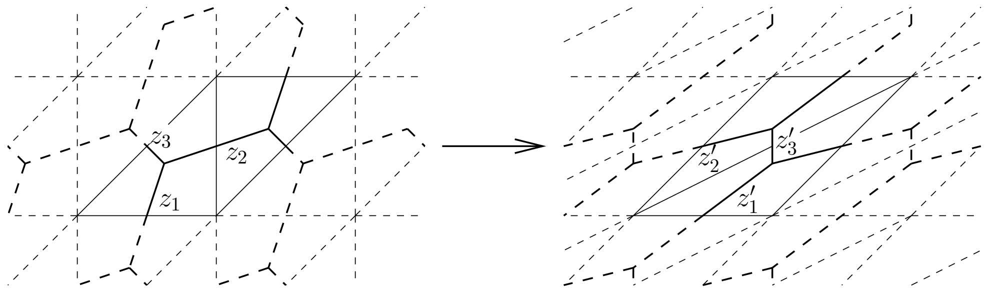


Figure 3.12: *A flip corresponding to a Dehn twist in case of a once-punctured torus.*

Notice that $z'_1+z'_2+z'_3 = z_1+z_2+z_3$ and therefore that the length of the boundary component, which is zero in case of a puncture, is preserved.

If we view this Dehn twist as being in the horizontal direction, then we have another one in the vertical direction. Analogously we find the transformation

$$\begin{aligned} l'_1 &= l_3, \\ l'_2 &= l_2, \\ l'_3 &= -l_1 + \log\left(e^{2l_2} + e^{2l_3}\right) \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} z'_1 &= z_3 - 2\log\left(e^{-z_1} + 1\right), \\ z'_2 &= z_2 + 2\log\left(e^{z_1} + 1\right), \\ z'_3 &= -z_1. \end{aligned} \tag{3.38}$$

It is not hard to see that these transformations together generate the whole MCG of the once-punctured (or one-holed) torus.

3.5 Weil-Petersson symplectic structure on Teichmüller space

We know that the Teichmüller space of marked Riemann surfaces forms a manifold of dimension $6g-6+2s$ and we have identified some sets of global coordinates on it. It turns out that Teichmüller space carries some natural additional structure, namely, a symplectic structure which is called the *Weil-Petersson* 2-form. Actually it is even the imaginary part of a natural Kählerian metric (see for instance [18]). We will not construct the Weil-Petersson symplectic form here, but in the next chapter we will demonstrate how to construct a symplectic form for a general automorphism group of which the Weil-Petersson symplectic form is a special case.

It can be shown that the Weil-Petersson symplectic form is invariant under the action of the mapping class group. Therefore it turns the moduli space into a symplectic manifold as well.

3.5.1 Weil-Petersson form in Fenchel-Nielsen coordinates

Due to a theorem of Wolpert, the Weil-Petersson 2-form takes on a particularly simple form in terms of the Fenchel-Nielsen coordinates,

$$\omega_{WP} = \sum_{i=1}^{3g-3+s} dl_i \wedge d\tau_i, \tag{3.39}$$

with respect to any pants decomposition. We immediately see that the Hamiltonian flow corresponding to the length l_i of a simple closed geodesic corresponds to shifting τ_i, which is often called an *infinitesimal Dehn twist*. Actually Wolpert only proved this result for compact closed Riemann surfaces, but it remains true for surfaces with holes or punctures, see for instance [23].

Of course the Weil-Petersson symplectic form also defines a Poisson algebra on the space of smooth functions on Teichmüller space. From (3.39) we conclude that lengths of non-intersecting simple closed geodesics Poisson-commute. The Poisson-bracket bracket of lengths of intersecting closed geodesics is given by ([28] or [14])

$$\{l_\alpha, l_\beta\}_{WP} = \sum_{p\in\alpha\sharp\beta} \cos\theta_p, \tag{3.40}$$

where the sum runs over the intersection points p and θ_p denotes the counterclockwise angle between α and β at p.

It is well known that a symplectic form defines a corresponding non-vanishing volume form by taking its appropriate exterior power. In case of Teichmüller space we therefore obtain a natural volume form, the *Weil-Petersson volume form*, which in Fenchel-Nielsen coordinates is given by

$$\omega_{WP}^{3g-3+s} = dl_1 \wedge d\tau_1 \wedge \cdots \wedge dl_{3g-3+s} \wedge d\tau_{3g-3+s}. \tag{3.41}$$

3.5.2 Weil-Petersson form in Penner and Fock coordinates

It can be shown [27] that in terms of the Penner coordinates the Weil-Petersson symplectic form becomes

$$\omega_{WP} = -\sum_{\text{triangles}\, t} \left(dl_{e_1(t)} \wedge dl_{e_2(t)} + dl_{e_2(t)} \wedge dl_{e_3(t)} + dl_{e_3(t)} \wedge dl_{e_1(t)}\right), \tag{3.42}$$

where $e_1(t)$, $e_2(t)$ and $e_3(t)$ denote the edges of t in counter-clockwise order. Notice that this 2-form only becomes non-degenerate once we divide out the symmetry in the Penner coordinates.

In Fock coordinates

$$\omega_{WP} = \sum_{\text{vertices}\, v} \left(dz_{e_1(v)} \wedge dz_{e_2(v)} + dz_{e_2(v)} \wedge dz_{e_3(v)} + dz_{e_3(v)} \wedge dl_{e_1(v)}\right), \tag{3.43}$$

where $e_1(v)$, $e_2(v)$ and $e_3(v)$ denote the edges of the fat graph incident at vertex v in counter-clockwise order. Again we must first impose the constraints (3.24) for (3.43) to become non-degenerate.

Chapter 4

Gravity and Teichmüller theory

As we have seen in paragraph 2.4.2, the relevant gauge group in gravity in $2+1$ dimensions, $ISO(2,1)$, is closely related to the lower dimensional groups $SO(2,1)$ and especially $SL(2,\mathbb{R})$. Therefore we will consider how the properties of a gauge theory change when we enlarge the gauge group, for instance by replacing it with its cotangent bundle. We will first do some group theory with a general Lie group, after which we will apply the results to gravity and Teichmüller space.

4.1 Some general group theory

4.1.1 Variations on Lie groups

In this paragraph we will follow the construction of *variations* as presented by Goldman in [15].

In the following we will consider a general Lie group G, not necessarily compact, and its Lie algebra $\mathfrak{g} = T_eG$. Let us denote $l_g, r_g : G \to G$ the *left* and *right multiplication* by $g \in G$ respectively. The *conjugation map* $C_g = l_g \circ r_{g^{-1}} = r_{g^{-1}} \circ l_g : G \to G$ is an isomorphism of G to itself which fixes the identity $e \in G$. A *class function* f on G is a smooth function $f : G \to \mathbb{R}$ which is invariant under conjugation by elements of G, i.e. $C_g^* f = f$ for all $g \in G$.

Notice that l_g and r_g provide us with two different canonical isomorphisms between $\mathfrak{g}$ and the tangent space T_gG at g which are given by their tangent maps at the origin. Given a smooth function $f : G \to \mathbb{R}$ we can consider the differential form $df : G \to T^*G$ which determines a map

$$\hat{\xi}_f : G \to \mathfrak{g}^* : g \to (T_e l_g)^* df(g). \tag{4.1}$$

From now on suppose f is a class function. Notice that we just as well could have replaced l_g by r_g in (4.1), because $l_g^* f(h) = f(gh) = f(g^{-1}ghg) = f(hg) = r_g^* f(h)$. We have $f(C_h g) = f(g)$, hence $df(hgh^{-1}) = T_{hgh^{-1}} C_{h^{-1}}^* df(g)$ and we find $\hat{\xi}_f(hgh^{-1}) = T_e l_{hgh^{-1}}^* (df(hgh^{-1})) =$

$T_e l^*_{hgh^{-1}} \circ T_{hgh^{-1}} C^*_{h^{-1}}(df(g)) = T_e C^*_{h^{-1}} \circ T_e l^*_g(df(g)) = T_e C^*_{h^{-1}} \hat{\xi}_f(g)$. We conclude that $\hat{\xi}_f$ is *equivariant*,

$$\hat{\xi}_f(C_h(g)) = \mathrm{Ad}^*(h^{-1})\hat{\xi}_f(g). \tag{4.2}$$

Notice that a path γ in G, where $\gamma(0) = g$, satisfies $df(g)(\dot{\gamma}(0)) = \frac{d}{dt} f(\gamma(t))_{t=0}$. Using this we can also write $\hat{\xi}_f$ as the following linear form on $\mathfrak{g}$,

$$\hat{\xi}_f(g) : X \to \left.\frac{d}{dt}\right|_{t=0} f(g \exp(tX)). \tag{4.3}$$

Suppose we are given an *orthogonal structure* on G, i.e. a non-degenerate symmetric bilinear form B on $\mathfrak{g}$ which is Ad-invariant, $B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X,Y)$ for all $X, Y \in \mathfrak{g}$ and $g \in G$. Let's write $\tilde{B} : \mathfrak{g} \to \mathfrak{g}^*$ for the associated isomorphism. Then we can construct the *variation* $\xi_f := \tilde{B}^{-1} \circ \hat{\xi}_f : G \to \mathfrak{g}$ of a smooth function $f : G \to \mathbb{R}$. The variation of f is again equivariant,

$$\xi_f(C_h(g)) = \mathrm{Ad}(h)\xi_f(g) \tag{4.4}$$

and (4.3) implies

$$B(\xi_f(g), X) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp(tX)). \tag{4.5}$$

For $g \in G$ we define the *centralizer of* g *in* $\mathfrak{g}$ as $Z_{\mathfrak{g}}(g) = \{X \in \mathfrak{g} | \mathrm{Ad}(g)(X) = X\}$. From (4.4) we see that if $hgh^{-1} = g$, i.e. $h \in Z_G(g)$, the centralizer of g in G, then $\xi_f(h) \in Z_{\mathfrak{g}}(g)$, so $\xi_f(Z_G(g)) \subset Z_{\mathfrak{g}}(g)$. In particular $\xi_f(g) \in Z_{\mathfrak{g}}(g)$, i.e. $\xi_f(g)$ is an eigenvector of $\mathrm{Ad}(g)$ with eigenvalue 1,

$$\mathrm{Ad}(g)\xi_l(g) = \xi_l(g). \tag{4.6}$$

Taking the variation of f can be viewed as taking the derivative of f while identifying the tangent space with the Lie algebra and its dual. As a consequence the variation satisfies a *Leibniz rule*. Let f_1 and f_2 be class functions, then the product $f_1 \cdot f_2$ is again a class function and we can compute its variation,

$$\begin{aligned} B(\xi_{f_1 \cdot f_2}(g), X) &= T_e((f_1 \cdot f_2) \circ l_g)(X) \\ &= T_e((f_1 \circ l_g) \cdot (f_2 \circ l_g))(X) \\ &= (f_1(g) \cdot T_e(f_1 \circ l_g) + f_2(g) \cdot T_e(f_2 \circ l_g))(X). \end{aligned} \tag{4.7}$$

We conclude that

$$\xi_{f_1 \cdot f_2} = f_1 \cdot \xi_{f_2} + f_2 \cdot \xi_{f_1}. \tag{4.8}$$

There exists another way of constructing from two class function f_1 and f_2 a new one by using the orthogonal structure on our Lie group, namely

$$G \to \mathbb{R} : g \to B(\xi_{f_1}(g), \xi_{f_2}(g)). \tag{4.9}$$

This is a class function due to the Ad-invariance of B and relation (4.4).

4.1.2 Cotangent group

As we saw in paragraph 2.4.2 we can naturally construct from G a group structure on its cotangent bundle T^*G. This group turned out to be canonically isomorphic by left translation to the semidirect product $G \ltimes \mathfrak{g}^*$, where the group structure is given by

$$G \times \mathfrak{g}^* \times G \times \mathfrak{g}^* \to G \times \mathfrak{g} : (g, X) \cdot (h, Y) \to (gh, X + \mathrm{Ad}^*(g^{-1})Y). \tag{4.10}$$

We denote the Lie algebra of $G \ltimes \mathfrak{g}^*$ by $\mathfrak{t} = T_e(G \ltimes \mathfrak{g}^*) \equiv \mathfrak{g} \times \mathfrak{g}^*$ and write $\pi : G \ltimes \mathfrak{g}^* \to G$ for the projection. There exists a canonical bilinear form $H : \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ given by

$$H((X_1, Y_1), (X_2, Y_2)) = Y_1(X_2) + Y_2(X_1). \tag{4.11}$$

We would like to show that this bilinear form defines an orthogonal structure on $G \ltimes \mathfrak{g}^*$. To verify that H is Ad-invariant we need to derive an expression for the adjoint map on $G \ltimes \mathfrak{g}^*$. We have $(g, X) \cdot (h, Y) \cdot (g, X)^{-1} = (g, X) \cdot (h, Y) \cdot (g^{-1}, -\mathrm{Ad}^*(g)X) = (ghg^{-1}, \mathrm{Ad}^*(g^{-1})Y + (1 - \mathrm{Ad}^*(g^{-1})\mathrm{Ad}^*(h^{-1})Ad^*(g))X)$ and taking the tangent map at the identity $(h, Y) = (e, 0)$ we get

$$\mathrm{Ad}(g, X) = \begin{pmatrix} \mathrm{Ad}(g) & 0 \\ \mathrm{Ad}^*(g^{-1})\mathrm{ad}^*\mathrm{Ad}^*(g)(X) & \mathrm{Ad}^*(g^{-1}) \end{pmatrix}. \tag{4.12}$$

Now

$$\begin{aligned}
& H(\mathrm{Ad}(g, Z)(X_1, Y_1), \mathrm{Ad}(g, Z)(X_2, Y_2)) \\
= \;& H((\mathrm{Ad}(g)(X_1), \mathrm{Ad}^*(g^{-1})\mathrm{ad}^*(X_1)\mathrm{Ad}^*(g)(Z) + \mathrm{Ad}^*(g^{-1})(Y_1)), \\
& (\mathrm{Ad}(g)(X_2), \mathrm{Ad}^*(g^{-1})\mathrm{ad}^*(X_2)\mathrm{Ad}^*(g)(Z) + \mathrm{Ad}^*(g^{-1})(Y_2)) \\
= \;& (\mathrm{Ad}^*(g^{-1})\mathrm{ad}^*(X_1)\mathrm{Ad}^*(g)(Z))(\mathrm{Ad}(g)(X_2)) + (\mathrm{Ad}^*(g^{-1})(Y_1))(\mathrm{Ad}(g)(X_2)) + \\
& (\mathrm{Ad}^*(g^{-1})\mathrm{ad}^*(X_2)\mathrm{Ad}^*(g)(Z))(\mathrm{Ad}(g)(X_1)) + (\mathrm{Ad}^*(g^{-1})(Y_2))(\mathrm{Ad}(g)(X_1)) \\
= \;& (\mathrm{Ad}^*(g)(Z))(\mathrm{ad}(X_1)X_2 + \mathrm{ad}(X_2)X_1) + Y_2(X_1) + Y_1(X_2) \\
= \;& Y_2(X_1) + Y_1(X_2),
\end{aligned} \tag{4.13}$$

where in the last identity we used the antisymmetry of $\mathrm{ad}(X_1)X_2 = [X_1, X_2]$. Non-degeneracy of H follows immediately from $\tilde{H} : \mathfrak{t} = \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{t}^* = \mathfrak{g}^* \times \mathfrak{g} : (X, Y) \to (Y, X)$ being an isomorphism. Hence, we conclude that H indeed defines an orthogonal structure on $G \ltimes \mathfrak{g}^*$.

We can now consider variations with respect to H of class functions on $G \ltimes \mathfrak{g}^*$. Let's first consider the class function f' which is the pull-back of the class function f on G along π. Explicitly $f' : G \ltimes \mathfrak{g}^* \to \mathbb{R} : (g, X) \to f(g)$. Now suppose we still have an orthogonal structure B defined on G. Then we can consider the variation $\xi_f : G \to \mathfrak{g}$ of f, which determines a function $F' : G \ltimes \mathfrak{g}^* \to \mathbb{R} : (g, X) \to X(\xi_f(g))$. It turns out that F' is a class function on $G \ltimes \mathfrak{g}^*$, since for $(g, X), (h, Y) \in G \ltimes \mathfrak{g}^*$ we have

$$\begin{aligned}
& F'((g, X) \cdot (h, Y) \cdot (g, X)^{-1}) \\
= \;& ((1 - \mathrm{Ad}^*(g^{-1})\mathrm{Ad}^*(h^{-1})\mathrm{Ad}^*(g))(X) + \mathrm{Ad}^*(g^{-1})(Y))(\xi_f(ghg^{-1})) \\
= \;& (\mathrm{Ad}^*(g)(X) - \mathrm{Ad}^*(h^{-1})\mathrm{Ad}^*(g)(X) + Y)(\xi_f(h)) = Y(\xi_f(h)) = F'(h, Y),
\end{aligned} \tag{4.14}$$

where we used (4.4) and $\mathrm{Ad}(h)\xi_f(h) = \xi_f(h)$.

We now have two class functions on $G \ltimes \mathfrak{g}^*$ and we would like to know what their variations look like. Clearly for the variation $\xi_{f'}$ of f' we have $\hat{\xi}_{f'} = (\hat{\xi}_f \circ \pi, 0)$ and therefore

$$\xi_{f'} = \tilde{H}^{-1} \circ \hat{\xi}_{f'} = (0, \hat{\xi}_f \circ \pi) : G \ltimes \mathfrak{g}^* \to \mathfrak{t}. \tag{4.15}$$

The variation of F' is a little more complicated. First, from the definition $F'(g, X) = X(\xi_f(g))$ we find

$$T_{(g,X)}F' : T_gG \times \mathfrak{g}^* \to \mathbb{R} : (Y, Z) \to X(T_g\xi_f(Y)) + Z(\xi_f(g)) \tag{4.16}$$

and the tangent map at the origin of the left translation $l_{(g,X)}(h, W) = (gh, X + \mathrm{Ad}^*(g^{-1})W)$ is given by

$$T_{(e,0)}l_{(g,X)} : \mathfrak{t} \to T_gG \times \mathfrak{g}^* : (Y, Z) \to (T_el_g(Y), \mathrm{Ad}^*(g^{-1})Z). \tag{4.17}$$

Now we can compute $\hat{\xi}_{F'}$,

$$\begin{aligned}
\hat{\xi}_{F'}(g, X) : (Y, Z) &\to dF'(g, X)(T_{(e,0)}l_{(g,X)}(Y, Z)) \\
&= T_{(g,X)}F'(T_el_g(Y), \mathrm{Ad}^*(g^{-1})Z) \\
&= X(T_g\xi_f \circ T_el_g(Y)) + (\mathrm{Ad}^*(g^{-1})Z)(\xi_f(g)) \\
&= X(T_e(\xi_f \circ l_g)(Y)) + Z(\xi_f(g)).
\end{aligned} \tag{4.18}$$

So we find for the variation $\xi_{F'}$ of F'

$$\xi_{F'} : G \ltimes \mathfrak{g}^* \to \mathfrak{t} : (g, X) \to (\xi_f(g), T_e(\xi_f \circ l_g)^*(X)). \tag{4.19}$$

Suppose now we have two class functions f_1 and f_2 on G, with corresponding class functions f_1', f_2', F_1' and F_2' on $G \ltimes \mathfrak{g}^*$. Let's see what we get when we apply the construction from (4.9) to them. We obtain from (4.15)

$$H(\xi_{f_1'}, \xi_{f_2'}) = 0 \tag{4.20}$$

and from (4.19)

$$H(\xi_{f_1'}, \xi_{F_2'})(g, X) = \hat{\xi}_{f_1}(g)(\xi_{f_2}(g)) = B(\xi_{f_1}, \xi_{f_2})(g). \tag{4.21}$$

The last one, $H(\xi_{F_1'}, \xi_{F_2'})$, requires some more work. First notice that (4.5) implies $B(\xi_f \circ l_g, X)(h) = \frac{d}{dt}\Big|_{t=0} f(gh\exp(tX))$, hence

$$\begin{aligned}
B(T_e(\xi_f \circ l_g)(Y), X) &= T_eB(\xi_f \circ l_g, X)(Y) \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} f(g\exp(sY)\exp(tX)) \\
&= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f(\exp(tX)g\exp(sY)) \\
&= T_eB(\xi_f \circ r_g, Y)(X) \\
&= T_eB(\xi_f \circ C_{g^{-1}} \circ l_g, Y)(X) \\
&= T_eB(\xi_f \circ l_g, \mathrm{Ad}(g)(Y))(X) \\
&= B(T_e(\xi_f \circ l_g)(X), \mathrm{Ad}(g)(Y)),
\end{aligned} \tag{4.22}$$

where in the sixth equality we used the equivariance (4.4) of ξ_f and the Ad-invariance of B. It follows that

$$\begin{aligned}
H(\xi_{F_1'},\xi_{F_2'})(g,X) &= T_e(\xi_{f_1}\circ l_g)^*(X)(\xi_{f_2}(g)) + T_e(\xi_{f_2}\circ l_g)^*(X)(\xi_{f_1}(g)) \\
&= B(T_e(\xi_{f_1}\circ l_g)(\xi_{f_2}(g)),\tilde{B}^{-1}(X)) + B(T_e(\xi_{f_2}\circ l_g)(\xi_{f_1}(g)),\tilde{B}^{-1}(X)) \\
&= B(T_e(\xi_{f_1}\circ l_g)(\tilde{B}^{-1}(X)),\mathrm{Ad}(g)\xi_{f_2}(g)) + B(T_e(\xi_{f_2}\circ l_g)(\tilde{B}^{-1}(X)),\mathrm{Ad}(g)\xi_{f_1}(g)) \\
&= B(T_g\xi_{f_1}(T_el_g(\tilde{B}^{-1}(X))),\xi_{f_2}(g)) + B(T_g\xi_{f_2}(T_el_g(\tilde{B}^{-1}(X))),\xi_{f_1}(g)) \\
&= T_gB(\xi_{f_1},\xi_{f_2})\circ T_el_g(\tilde{B}^{-1}(X)) \\
&= T_e(B(\xi_{f_1},\xi_{f_2})\circ l_g)(\tilde{B}^{-1}(X)) \\
&= X(\xi_{B(\xi_{f_1},\xi_{f_2})}(g)) \qquad (4.23)
\end{aligned}$$

In particular we observe that $H(\xi_{F_1'},\xi_{F_2'})$ is the class function on $G\ltimes\mathfrak{g}$ associated with the variation of the class function $H(\xi_{f_1'},\xi_{F_2'})$ restricted to G.

4.1.3 Generalized Weil-Petersson symplectic structure

Let Σ be a compact oriented surface of genus $g>1$. Let's denote the fundamental group of Σ by $\pi_1=\pi_1(\Sigma,p)$ where $p\in\Sigma$ is the base point. We are interested in homomorphisms from π_1 to a Lie group G. In particular we will consider the space $\mathrm{Hom}(\pi_1,G)/G$ where G acts on $\mathrm{Hom}(\pi_1,G)$ by overall conjugation. If $\phi:\pi_1\to G$ we write $[\phi]\in\mathrm{Hom}(\pi_1,G)/G$ for its equivalence class.

It is well known that $\mathrm{Hom}(\pi_1,G)$ is not everywhere a manifold, it contains singular points. Let's denote the non-singular part of $\mathrm{Hom}(\pi_1,G)$ by $\mathrm{Hom}(\pi_1,G)^-$, which is now a smooth manifold. $\mathrm{Hom}(\pi_1,G)/G$ is certainly not a manifold everywhere, but it is proven in [14] that G does act locally free precisely on $\mathrm{Hom}(\pi_1,G)^-$. This means that $\mathrm{Hom}(\pi_1,G)^-/G$ is again a manifold, but it is not necessarily Hausdorff.

However in [14] it was demonstrated that we can view $\mathrm{Hom}(\pi_1,G)/G$ as a singular algebraic variety and consider structures on its Zariski tangent space, which correspond to their common differential counterparts away from the singular points. It was also shown that an orthogonal structure on G provides us with a natural symplectic structure on $\mathrm{Hom}(\pi_1,G)/G$ which is a generalization of the Weil-Petersson form on Teichmüller space (see for instance [18]). We will briefly describe the construction.

First one notices that a Zariski tangent vector to $\mathrm{Hom}(\pi_1,G)$ at ϕ corresponds to a 1-cocycle $\pi_1\to\mathfrak{g}_{\mathrm{Ad}(\phi)}$ where $\mathfrak{g}_{\mathrm{Ad}(\phi)}$ is the π_1-module obtained by the action $\mathrm{Ad}(\phi):\pi_1\to GL(\mathfrak{g})$. The Zariski tangent space to $\mathrm{Hom}(\pi_1,G)/G$ at $[\phi]$ can then be identified with the cohomology group $H^1(\pi_1,\mathfrak{g}_{\mathrm{Ad}(\phi)})$. The cup-product in π_1 gives us a map $H^1(\pi_1,\mathfrak{g}_{\mathrm{Ad}(\phi)})\times H^1(\pi_1,\mathfrak{g}_{\mathrm{Ad}(\phi)})\to H^2(\pi_1,\mathbb{R})\cong\mathbb{R}$ using the orthogonal structure as pairing $B:\mathfrak{g}_{\mathrm{Ad}(\phi)}\times\mathfrak{g}_{\mathrm{Ad}(\phi)}\to\mathbb{R}$. This way we obtain a bilinear form ω_ϕ on the tangent space to $\mathrm{Hom}(\pi_1,G)/G$ at $[\phi]$. In [14] it is proved that ω is a closed non-degenerate 2-form on $\mathrm{Hom}(\pi_1,G)/G$ and therefore defines a symplectic structure on our singular manifold. It is worth mentioning that $H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}(\phi)})$ can be identified with $H^1(\Sigma, \mathfrak{g}_{\mathrm{Ad}(\phi)})$ which is the De Rham cohomology group of $\mathfrak{g}_{\mathrm{Ad}(\phi)}$-valued 1-forms on Σ.

Our symplectic structure ω defines a Poisson structure on $\mathrm{Hom}(\pi_1, G)/G$. The Poisson brackets provide the space of smooth functions with a Lie algebra structure. We are interested in a certain family of functions on $\mathrm{Hom}(\pi_1, G)/G$, namely, the ones constructed from class functions on G. Given a class function $f : G \to \mathbb{R}$ and α a closed curve in Σ, we define

$$f_\alpha : \mathrm{Hom}(\pi_1, G)/G \to \mathbb{R} : [\phi] \to f(\phi([\alpha])). \tag{4.24}$$

Clearly f_α only depends on the homotopy class of α. The main result of [15] is the following theorem.

Theorem 4.1.1. *Let α and β be two oriented closed curves in Σ with transverse double points and let f and $\tilde{f}$ be two class functions on G. Then the Poisson bracket $\{f_\alpha, \tilde{f}_\beta\}$ with respect to the symplectic structure ω is a function $Hom(\pi_1, G)/G \to \mathbb{R}$ given by*

$$[\phi] \to \sum_{p \in \alpha \sharp \beta} \varepsilon(p; \alpha, \beta) B(\xi_f(\phi(\alpha_p)), \xi_{\tilde{f}}(\phi(\beta_p))), \tag{4.25}$$

where $\alpha\sharp\beta$ is the set of intersections, $\varepsilon(p; \alpha, \beta) = \pm 1$ depends on the orientation of the intersection and $\alpha_p, \beta_p \in \pi_1(\Sigma, p)$ are the corresponding curves with base point p.

Two curves α and β have *transverse double points* if they are transverse at their intersections and the intersections are simple points both of α and of β. A point $p \in \Sigma$ is a *simple point* of α if p is attended exactly once by α.

4.1.4 Symplectic structure for cotangent group

Let's apply this to the cotangent group of the previous section. We consider the singular algebraic variety $\mathrm{Hom}(\pi_1, G \ltimes \mathfrak{g}^*)/G \ltimes \mathfrak{g}^*$ with the canonical Weil-Petersson symplectic form ω on it. Let f and $\tilde{f}$ be class functions on G and f',$\tilde{f}'$ and F',$\tilde{F}'$ the corresponding class functions on $G \ltimes \mathfrak{g}^*$ as defined above. Then we find the following Poisson brackets,

$$\begin{aligned}
\{f'_\alpha, \tilde{f}'_\beta\}([\phi]) &= \sum_{p \in \alpha\sharp\beta} \varepsilon(p; \alpha, \beta) H(\xi_{f'}(\phi(\alpha_p)), \xi_{\tilde{f}'}(\phi(\beta_p))) = 0 && (4.26)\\
\{f'_\alpha, \tilde{F}'_\beta\}([\phi]) &= \sum_{p \in \alpha\sharp\beta} \varepsilon(p; \alpha, \beta) H(\xi_{f'}(\phi(\alpha_p)), \xi_{\tilde{F}'}(\phi(\beta_p))) \\
&= \sum_{p \in \alpha\sharp\beta} \varepsilon(p; \alpha, \beta) B(\xi_f(\pi \circ \phi(\alpha_p)), \xi_{\tilde{f}}(\pi \circ \phi(\beta_p))) \\
&= \{f_\alpha, \tilde{f}_\beta\}_G([\pi \circ \phi]) && (4.27)\\
\{F'_\alpha, \tilde{F}'_\beta\}([\phi]) &= \sum_{p \in \alpha\sharp\beta} \varepsilon(p; \alpha, \beta) H(\xi_{F'}(\phi(\alpha_p)), \xi_{\tilde{F}'}(\phi(\beta_p))), && (4.28)
\end{aligned}$$

where $\{\cdot,\cdot\}_G$ is the Poisson bracket corresponding to the Weil-Petersson symplectic structure on $\mathrm{Hom}(\pi_1, G)/G$. The product in the last line can be calculated analogously to (4.23),

$$\begin{aligned} & H(\xi_{F'}(g,X), \xi_{\tilde{F}'}(h,Y)) \\ = \; & B(T_e(\xi_f \circ l_g)(\tilde{B}^{-1}(X)), \mathrm{Ad}(g)\xi_{\tilde{f}}(h)) + B(T_e(\xi_{\tilde{f}} \circ l_h)(\tilde{B}^{-1}(Y)), \mathrm{Ad}(h)\xi_f(g)) \\ = \; & \left.\frac{d}{dt}\right|_{t=0} \hat{\xi}_f(g\exp(t\tilde{B}^{-1}(X)))(\mathrm{Ad}(g)\xi_{\tilde{f}}(h)) + \left.\frac{d}{dt}\right|_{t=0} \hat{\xi}_{\tilde{f}}(h\exp(t\tilde{B}^{-1}(Y)))(\mathrm{Ad}(h)\xi_f(g)) \\ = \; & \left.\frac{d}{dt}\right|_{t=0} B(\xi_f(\exp(t\tilde{B}^{-1}(X))g), \xi_{\tilde{f}}(\exp(t\tilde{B}^{-1}(Y))h)). \end{aligned} \tag{4.29}$$

4.1.5 Variation and Poisson bracket of more general functions

We have found expressions for the Poisson brackets between functions on $\mathrm{Hom}(\pi_1, G)/G$ derived from class functions on G. But we would like to do this for more general functions on $\mathrm{Hom}(\pi_1, G)/G$. In doing this we assume there exists a class function f on G such that any smooth function on $\mathrm{Hom}(\pi_1, G)/G$ can be written as a function of the f_α for some $\alpha \in \pi_1$. As we will see later this assumption is not too wild[1].

Under this assumption, suppose that ψ_k are smooth functions on $\mathrm{Hom}(\pi_1, G)/G$. Then we can write

$$\psi_k = \tilde{\psi}_k \circ (f_{\alpha_{k,1}}, \ldots, f_{\alpha_{k,n_k}}), \tag{4.30}$$

where $\tilde{\psi}_k : \mathbb{R}^{n_k} \to \mathbb{R}$ is a smooth function and $\alpha_{k,i} \in \pi_1$, $i = 1, \ldots, n_k$. It follows immediately from (4.26) that the functions $\psi'_k : \mathrm{Hom}(\pi_1, G \ltimes \mathfrak{g}^*)/G \ltimes \mathfrak{g}^* \to \mathbb{R}$ Poisson commute,

$$\{\psi'_1, \psi'_2\} = 0. \tag{4.31}$$

In paragraph 4.1.2 we associated a function $F' : G \ltimes \mathfrak{g}^* \to \mathbb{R}$ to a class function f by computing its variation. We would like to generalize this construction to our ψ_k. Recalling that a variation was just like taking a derivative, we use the chain rule to associate to ψ_k the function

$$\Psi_k : \mathrm{Hom}(\pi_1, G \ltimes \mathfrak{g}^*)/G \ltimes \mathfrak{g}^* \to \mathbb{R} : [\phi] \to \sum_{i=1}^{n_k} F(\phi(\alpha_{k,n_i})) \cdot \left.\frac{\partial \tilde{\psi}_k}{\partial x_i}\right|_{(f(\pi\circ\phi(\alpha_{k,1})),\ldots,f(\pi\circ\phi(\alpha_{k,n_k})))}. \tag{4.32}$$

Let us decompose $[\phi] \in \mathrm{Hom}(\pi_1, G \ltimes \mathfrak{g}^*)/G \ltimes \mathfrak{g}^*$ into the functions $\phi_G : \pi_1 \to G$ and $\phi_{\mathfrak{g}^*} : \pi_1 \to \mathfrak{g}^*$. Then $(t, [\phi]) \to [\exp(t\tilde{B}^{-1}(\phi_{\mathfrak{g}^*}))\phi_G]$ becomes a map $\mathbb{R} \times \mathrm{Hom}(\pi_1, G \ltimes \mathfrak{g}^*)/G \ltimes \mathfrak{g}^* \to \mathrm{Hom}(\pi_1, G)/G$. And we could equivalently define

$$\Psi_k : [\phi] \to \left.\frac{d}{dt}\right|_{t=0} \psi(\exp(t\tilde{B}^{-1}(\phi_{\mathfrak{g}^*}))\phi_G). \tag{4.33}$$

[1] For $G = SO(2,1)$ we will see that the trace satisfies the assumption. The same holds for $G = SL(2,\mathbb{R})$ and the displacement length l, as follows from theorem 3.12 in [18]

As a side remark, we compare this to (4.29) and notice that $\{F'_\alpha, \tilde{F}'_\beta\}$ is nothing else than the variation of $\{f_\alpha, \tilde{f}_\beta\}_G$.

Let us use (4.32) to calculate the Poisson bracket

$$\begin{aligned}\{\psi'_1, \Psi'_2\} &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \frac{\partial \tilde{\psi}_1}{\partial x_i}' \frac{\partial \tilde{\psi}_2}{\partial x_j}' \left\{f'_{\alpha_{1,i}}, F'_{\alpha_{2,j}}\right\} \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \frac{\partial \tilde{\psi}_1}{\partial x_i}' \frac{\partial \tilde{\psi}_2}{\partial x_j}' \left\{f_{\alpha_{1,i}}, f_{\alpha_{2,j}}\right\}'_G \\ &= \{\psi_1, \psi_2\}'_G \end{aligned} \tag{4.34}$$

and as a consequence $\{\psi'_1, \Psi'_2\} = \{\Psi'_1, \psi'_2\}$. Furthermore

$$\{\Psi'_1, \Psi'_2\} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \frac{\partial \tilde{\psi}_1}{\partial x_i}' \frac{\partial \tilde{\psi}_2}{\partial x_j}' \left\{F'_{\alpha_{1,i}}, F'_{\alpha_{2,j}}\right\} \tag{4.35}$$

which is precisely the variation of

$$\{\psi_1, \psi_2\}_G = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \frac{\partial \tilde{\psi}_1}{\partial x_i} \frac{\partial \tilde{\psi}_2}{\partial x_j} \left\{f_{\alpha_{1,i}}, f_{\alpha_{2,j}}\right\}_G. \tag{4.36}$$

4.2 Application to $SO(2,1)$ and $SL(2,\mathbb{R})$

4.2.1 $SO(2,1)$: Loop variables

Let us take $G = SO(2,1)$ and orthogonal structure $B(X,Y) = \frac{1}{2}\mathrm{Tr}(XY)$. Because the fundamental representation of $SO(2,1)$ coincides with the adjoint representation, the trace form is equal to the Killing form on $SO(2,1)$. The class function under consideration will be half the trace, $f(g) = \frac{1}{2}\mathrm{Tr}(g)$, inspired by the loop variables (2.33). For the variation ξ_f of f we have

$$B(\xi_f(g), X) = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \mathrm{Tr}(g\exp(tX)) = \frac{1}{2}\mathrm{Tr}(gX). \tag{4.37}$$

If we define $\mathrm{pr} : \mathrm{End}(\mathbb{R}^3) \to \mathfrak{so}(2,1)$ to be the B-orthogonal projection onto $\mathfrak{g}$, we can write

$$\xi_f = \mathrm{pr}. \tag{4.38}$$

Now notice that $g\eta g^t = \eta$ where $\eta = \mathrm{diag}(-1,1,1)$. Projection onto $\mathfrak{so}(2,1)$ is then given by $g \to 1/2(g - \eta g^t \eta)$, hence

$$\xi_f(g) = 1/2(g - g^{-1}). \tag{4.39}$$

On G we therefore have that

$$\begin{aligned}\{f_\alpha, f_\beta\}([\phi]) &= \sum_{p\in\alpha\sharp\beta} \varepsilon(p;\alpha,\beta) B(\xi_f(\phi(\alpha_p)), \xi_f(\phi(\beta_p))) \\ &= \frac{1}{2}\sum_{p\in\alpha\sharp\beta} \varepsilon(p;\alpha,\beta)(f_{\alpha_p\cdot\beta_p} - f_{\alpha_p\cdot\beta_p^{-1}})([\phi]), \end{aligned} \tag{4.40}$$

because $B(\xi_f(g),\xi_f(h)) = 1/32\mathrm{Tr}(g-g^{-1})(h-h^{-1}) = 1/8\mathrm{Tr}(gh+g^{-1}h^{-1}-g^{-1}h-gh^{-1}) = 1/4(\mathrm{Tr}(gh)-\mathrm{Tr}(gh^{-1})) = 1/2(f(gh)-f(gh^{-1}))$ (see [15]).

The corresponding Poisson brackets on $G \ltimes \mathfrak{g}^*$ then become

$$\{f'_\alpha, f'_\beta\}([\phi]) = 0 \tag{4.41}$$

$$\begin{aligned}\{f'_\alpha, F'_\beta\}([\phi]) &= \sum_{p\in\alpha\sharp\beta} \varepsilon(p;\alpha,\beta) B(\xi_f(\pi\circ\phi(\alpha_p)),\xi_f(\pi\circ\phi(\beta_p)))\\ &= \frac{1}{2}\sum_{p\in\alpha\sharp\beta}\varepsilon(p;\alpha,\beta)(f'_{\alpha_p\cdot\beta_p} - f'_{\alpha_p\cdot\beta_p^{-1}})([\phi]).\end{aligned} \tag{4.42}$$

To find $\{F'_\alpha, F'_\beta\}$ we first evaluate (4.29),

$$\begin{aligned}&H(\xi_{F'}(g,X),\xi_{F'}(h,Y))\\ =\;& \frac{d}{dt}\Big|_{t=0} \mathrm{Tr}\left(\exp(t\tilde{B}^{-1}(X))g\xi_f(\exp(t\tilde{B}^{-1}(Y))h)\right)\\ =\;& \frac{d}{dt}\Big|_{t=0} \mathrm{Tr}\left(\exp(t\tilde{B}^{-1}(X))g\frac{1}{2}(\exp(t\tilde{B}^{-1}(Y))h - h^{-1}\exp(-t\tilde{B}^{-1}(Y)))\right)\\ =\;& \frac{1}{2}\mathrm{Tr}(\tilde{B}^{-1}(X)gh) + \frac{1}{2}\mathrm{Tr}(g\tilde{B}^{-1}(Y)h) - \frac{1}{2}\mathrm{Tr}(\tilde{B}^{-1}(X)gh^{-1}) + \frac{1}{2}\mathrm{Tr}(gh^{-1}\tilde{B}^{-1}(Y))\\ =\;& \frac{1}{2}\mathrm{Tr}\left((\tilde{B}^{-1}(X)+\mathrm{Ad}(g)\tilde{B}^{-1}(Y))gh\right) - \frac{1}{2}\mathrm{Tr}\left((\tilde{B}^{-1}(X)-\mathrm{Ad}(gh^{-1})\tilde{B}^{-1}(Y))gh^{-1}\right)\\ =\;& \frac{1}{2}(X+\mathrm{Ad}^*(g^{-1})Y)\xi_f(gh) - \frac{1}{2}(X-\mathrm{Ad}^*(g^{-1})\mathrm{Ad}^*(h)Y)\xi_f(gh^{-1})\\ =\;& \frac{1}{2}F'((g,X)\cdot(h,Y)) - \frac{1}{2}F'((g,X)\cdot(h,Y)^{-1}).\end{aligned} \tag{4.43}$$

Hence

$$\{F'_\alpha, F'_\beta\}([\phi]) = \frac{1}{2}\sum_{p\in\alpha\sharp\beta}\varepsilon(p;\alpha,\beta)\left(F'_{\alpha_p\cdot\beta_p} - F'_{\alpha_p\cdot\beta_p^{-1}}\right)([\phi]), \tag{4.44}$$

which is indeed the variation of (4.40).

Notice that we precisely reproduce the Poisson brackets for the loop variables from paragraph 2.4.3, where the gauge group was given by $SO(2,1)\ltimes\mathfrak{so}(2,1)$. Because these variables form a basis for the smooth functions on the phase space, we conclude that the generalized Weil-Petersson symplectic structure on $\mathrm{Hom}(\pi_1, SO(2,1)\ltimes\mathfrak{so}(2,1))/SO(2,1)\ltimes\mathfrak{so}(2,1)$ coincides with the physical symplectic structure of the phase space of flat $2+1$-dimensional gravity.

4.2.2 $PSL(2,\mathbb{R})$: Teichmüller space

Let's choose $G = PSL(2,\mathbb{R})$ with orthogonal structure B given (like in paragraph 1.1) by $B(X,Y) = 2\mathrm{Tr}(XY)$, which is equal to $1/2$ times the Killing form on $PSL(2,\mathbb{R})$. Define the open subset $\mathrm{Hyp} = \{g \in PSL(2,\mathbb{R}) || \mathrm{Tr}(g)| > 2\}$, i.e. the set of all hyperbolic elements in $PSL(2,\mathbb{R})$, which is invariant under conjugation. Invariant functions can equally well be defined on invariant subspaces of Lie groups and what we derived in paragraph 4.1.1 is still valid (if we change the domains appropriately).

Following [15] we define the *displacement length* $l : \mathrm{Hyp} \to \mathbb{R}_{>0}$ by $\mathrm{Tr}(g) = 2\cosh(l(g)/2)$. Clearly l is a class function on Hyp and therefore we can construct its variation $\xi_l : \mathrm{Hyp} \to \mathfrak{g}$. A straightforward calculation shows that for $g \in \mathrm{Hyp}$ diagonal we get $\xi_l(g) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$. Because any $g \in Hyp$ can be diagonalized by some $h \in G$, (4.4) allows us to write $\xi_l(g) = \mathrm{Ad}(h)\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$. We now arrive at the following important identity,

$$\exp\left(l(g)\xi_l(g)\right) = \exp\left(\mathrm{Ad}(h)\begin{pmatrix} \frac{l(g)}{2} & 0 \\ 0 & -\frac{l(g)}{2} \end{pmatrix}\right) = h\begin{pmatrix} e^{\frac{l(g)}{2}} & 0 \\ 0 & e^{-\frac{l(g)}{2}} \end{pmatrix} h^{-1} = g. \tag{4.45}$$

Recall from paragraph 3.1.7 that we identified Teichmüller space of a surface R of genus $g > 1$ with homomorphism from its fundamental group to the isometry group $PSL(2,\mathbb{R})$,

$$\mathcal{T}_g \equiv \mathrm{Hom}_0(\pi(R), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}). \tag{4.46}$$

Now the subscript 0 means that we stay away from the singular points and the image of the non-identity elements of $\pi_1(R)$ lie in Hyp.

In the introduction we already saw that $PSL(2,\mathbb{R})$ is isomorphic to the identity component of $SO(2,1)$. Hence, the space we examined in the previous paragraph actually is the same as Teichmüller space. Furthermore, by the naturality of the Killing form, we conclude that the orthogonal structures we used coincide. We therefore have established a relationship between the phase space of $2+1$ dimensional gravity and Teichmüller space.

A calculation as in [14] yields the following Wolpert formula for the Poisson bracket between l_α and l_β,

$$\{l_\alpha, l_\beta\}([\phi]) = \sum_{p\in\alpha\sharp\beta} \varepsilon(p;\alpha,\beta)B(\xi_l(\phi(\alpha_p)), \xi_l(\phi(\beta_p))) = \sum_{p\in\alpha_\phi\sharp\beta_\phi} \cos\theta_p. \tag{4.47}$$

Here, α_ϕ is the unique closed geodesic homotopic to α on the hyperbolic surface S_ϕ corresponding to $[\phi]$ in Teichmüller space and θ_p is the counterclockwise angle between α_ϕ and β_ϕ at the intersection point p.

From the previous chapter we know that l_α is nothing else than the length of the closed geodesic homotopic to α on the surface R. In the next section we will see that its variation L' has a nice interpretation as well, it corresponds to the length of the unique closed geodesic in the flat $2+1$D space-time homotopic to α.

4.3 Geometric observables in 2+1 gravity

Our starting point is the representation of the phase space in terms of the holonomies as in (2.19),

$$\mathcal{P} = \mathrm{Hom}_0(\pi_1, ISO(2,1))/ISO(2,1). \tag{4.48}$$

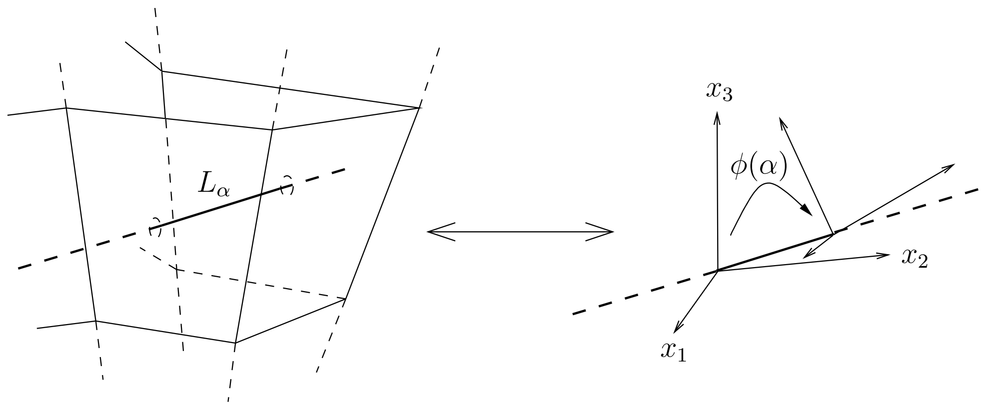


Figure 4.1: *A closed geodesic in the homotopy class α and its lifting to Minkowski space.*

Because we want to establish relations with Teichmüller space we will use $PSL(2,\mathbb{R})\ltimes\mathfrak{sl}(2,\mathbb{R})^*$ instead of $ISO(2,1)$ (see paragraph 2.4.2). Hence we write

$$\mathcal{P} = \mathrm{Hom}_0(\pi_1, PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*)/PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*, \tag{4.49}$$

where the subscript 0 now means that we only consider homomorphisms which restrict to Teichmüller space. The projection of $\mathcal{P}$ onto Teichmüller space $\mathcal{T}_g$

$$\pi_{\mathcal{T}_g} : \mathcal{P} \to \mathcal{T}_g : [\phi] \to [\pi_{SL(2,\mathbb{R})} \circ \phi] \tag{4.50}$$

assigns to a space-time solution with topology $\Sigma \times \mathbb{R}$ a Riemann surface with topology Σ.

In the next subsections we will define several functions on phase space $\mathcal{P}$ which arise from variations of geometric functions on Teichmüller space.

4.3.1 Lengths of closed geodesics

Given a $[\phi] \in \mathcal{P}$ we can construct the corresponding flat space-time M solution by dividing Minkowski space $\mathbb{R}^3$ by the action of $\phi(\alpha)$ for all elements α of the fundamental group π_1. Although such an action does not act properly discontinuously on $\mathbb{R}^3$, there exists a maximal open subset U in $\mathbb{R}^3$ on which all the $\phi(\alpha)$ act properly discontinuously, see Mess [22].

Clearly geodesics on the space-time M originate from projecting down geodesics in Minkowski space, which of course are just the straight lines. In particular a closed geodesic in M homotopic to $\alpha \in \pi_1$ corresponds precisely to a straight line in $\mathbb{R}^3$ which is mapped to itself by $\phi(\alpha)$.

We claim that for any non-identity element $(g, X) \in ISO(2,1) \cong SO(2,1) \ltimes \mathbb{R}^3$ there exists a unique straight line in $\mathbb{R}^3$ which is mapped to itself. To see this we note that g has precisely one eigenvalue equal to 1 [2]. Therefore the straight line is fixed to be in the direction of the corresponding eigenvector v. Now suppose $Y \in \mathbb{R}^3$ lies on the line. By definition Y is mapped to $gY + X$, hence $(g - \mathrm{Id})Y + X$ must be directed along v,

$$g((g - \mathrm{Id})Y + X) + X = (g - \mathrm{Id})Y + X. \tag{4.51}$$

If we write $v_\perp \subset \mathbb{R}^3$ for the two-dimensional subspace orthogonal to v and $P_{v_\perp}$ for the orthogonal projection, this is equivalent to

$$(g - \mathrm{Id})^2 P_{v_\perp}(Y) + (g - \mathrm{Id}) P_{v_\perp}(X) = 0. \tag{4.52}$$

Now $g - \mathrm{Id}$ restricted to $v_\perp$ becomes invertible, such that

$$P_{v_\perp}(Y) = -\left((g - \mathrm{Id})_{v_\perp}\right)^{-1} P_{v_\perp}(X). \tag{4.53}$$

Hence, such a Y exists and it is fixed up to shift in the direction of v along the line. This proves our claim.

Provided that the straight line we found lies in U, we find a closed geodesic in M. Whether or not this is always the case, we do not know. In the case of the torus the closed geodesics we find this way are all located at the initial (or final) singularity of the space-time. But in this thesis we are focusing on genus two or greater, since the torus space-times turn out to be quite degenerate. In general we expect the closed geodesics to contain a lot of information about the geometry of the initial (or final) singularity, but further investigation is necessary. From here on we will stick to the idea that the unfolding of a space-time into Minkowski space, together with a holonomy group acting on it, is a natural construction and in this representation the closed geodesics are perfectly well-defined.

Let us see what this means for $(g, X) \in PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*$. We now identify Minkowski space with $\mathfrak{sl}(2,\mathbb{R})^*$ such that[3] B^* corresponds to the Minkowski metric η. Recall from paragraph 4.1.1 that $\xi_l(g)$ is invariant under $\mathrm{Ad}(g)$. Hence, the eigenvector v in this case is given by $\hat{\xi}_l(g) \in \mathfrak{sl}(2,\mathbb{R})^*$. Furthermore $\hat{\xi}_l(g)$ is automatically normalized, $B^*(\hat{\xi}_l(g), \hat{\xi}_l(g)) = 1$.

It is now easy to write down the length of the closed geodesic in the homotopy class α. If we write $\phi(\alpha) = (g_\alpha, X_\alpha) \in PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*$, then we just have to take the length of X_α in the invariant direction $\hat{\xi}_l(g_\alpha)$,

$$L'_\alpha : \mathcal{M}_g \to \mathbb{R} : [\phi] \to B^*(\hat{\xi}_l(g_\alpha), X_\alpha) = X_\alpha(\xi_l(g_\alpha)). \tag{4.54}$$

The main point is that L' turns out to be precisely the variation of l on $SL(2,\mathbb{R})$.[4] In paragraph 3.2.2 we showed that l_α is the hyperbolic length of the unique closed geodesic on the Riemann surface in the homotopy class α. As a side remark, L' in a slightly different form has been discussed in mathematical literature and is known as the *Margulis invariant* [16].

---

[2] Let v be a (complex) eigenvector of g with eigenvalue λ, then from $g^T \eta g = \eta$ we see that ηv is an eigenvector of g^T with eigenvalue $1/\lambda$. But then g must have an eigenvalue $1/\lambda$ too. Now $\det(g) = 1$ implies that one or all of the eigenvalues equal 1.

[3] We denote by B^* the orthogonal structure on $\mathfrak{g}^*$ corresponding to the orthogonal structure B on $\mathfrak{g}$, i.e. $B^*(X, Y) = X(\tilde{B}^{-1}(Y)) = B(\tilde{B}^{-1}(X), \tilde{B}^{-1}(Y))$.

[4] The single-valuedness of L'_α is now an immediate consequence of $(g, X) \to X(\xi_l(g))$ being a class function on $PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*$, see paragraph 4.1.2.

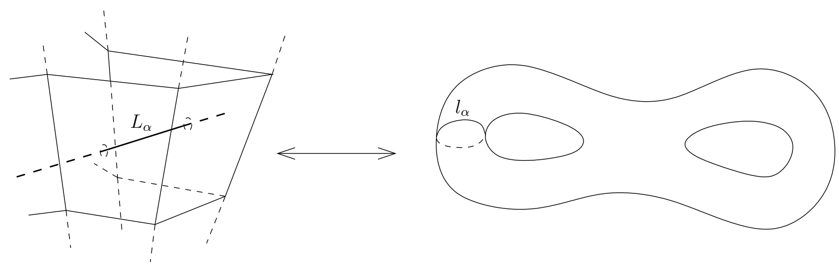


Figure 4.2: *Relation between length of closed geodesic in space-time (left) and in the corresponding Riemann surface (right).*

Because the holonomies must necessarily be boosts, the closed geodesics will all be spacelike, which is also clear from the fact that $\hat{\xi}_l(g)$ always has norm 1. The observables L'_α therefore provide us with a way of probing space-like distances. In the next paragraph we will consider length observables which can be either space-like or time-like.

4.3.2 Distance between closed geodesics

Given two different homotopy classes α_1 and α_2, we can construct the two unique closed space-time geodesics γ_1 and γ_2 belonging to them. A new observable is then given by the distance between.

Let us denote the holonomies corresponding to α_1 and α_2 by $(g_1, X_1), (g_2, X_2) \in PSL(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})^*$ respectively. We try to construct the line-segment c connecting γ_1 and γ_2 at right angles. Since the directions of the geodesics are given by $\xi_l(g_1)$ and $\xi_l(g_2)$, the direction of c will be their cross product which in this case we can write as[5] $\tilde{B}([\xi_l(g_1), \xi_l(g_2)])$. Suppose $Y_1, Y_2 \in \mathfrak{sl}(2,\mathbb{R})^*$ are points on γ_1 and γ_2, then the signed length of c is clearly equal to

$$\frac{(Y_1 - Y_2)([\xi_l(g_1), \xi_l(g_2)])}{\sqrt{|B([\xi_l(g_1), \xi_l(g_2)], [\xi_l(g_1), \xi_l(g_2)])|}}. \tag{4.55}$$

This defines a function $D_{\alpha_1\alpha_2}$ on $\mathcal{M}_g$.

Now let us examine similar functions on Teichmüller space. Given a marked Riemann surface R and two simple closed geodesics γ_1 and γ_2, let Γ be a Fuchsian model of R and denote by $g_1, g_2 \in \Gamma \subset Aut(\mathbb{H})$ the elements corresponding to the homotopy classes of γ_1 and γ_2. Then we distinguish two cases: the liftings of γ_1 and γ_2 intersect in $\mathbb{H}$ or they do not, see figure 4.3.

Lemma 4.3.1. *Using the notation above we have the following:*

[5]To see this we note that B is just a multiple of the Killing form of $SL(2,\mathbb{R})$. We can then use a well-known associativity property of the Killing form, namely, that $B([X,Y],Z) = B(X,[Y,Z])$. Hence $B(X,[X,Y]) = B(Y,[X,Y]) = 0$.

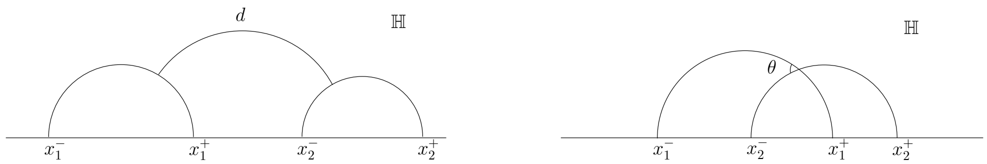


Figure 4.3: *The lifting of γ_1 and γ_2 to $\mathbb{H}$.*

(i) $|B(\xi_l(g_1), \xi_l(g_2))| < 1$ *if and only if* γ_1 *and* γ_2 *intersect. The angle* θ *between* γ_1 *and* γ_2 *at the intersection satisfies*

$$B(\xi_l(g_1), \xi_l(g_2)) = \cos\theta. \tag{4.56}$$

(ii) $|B(\xi_l(g_1), \xi_l(g_2))| > 1$ *if and only if* γ_1 *and* γ_2 *do not intersect. The geodesic distance* d *between* γ_1 *and* γ_2 *satisfies*

$$|B(\xi_l(g_1), \xi_l(g_2))| = \cosh d. \tag{4.57}$$

Proof. Let's denote the points where the geodesics meet the boundary by $x_1^\pm, x_2^\pm \in \mathbb{R}$ as in figure 4.3. Then we have (see [23])

$$1 - 2\frac{(x_2^- - x_1^+)(x_1^- - x_2^+)}{(x_1^- - x_1^+)(x_2^- - x_2^+)} = \begin{cases} \cosh d & \text{no intersection} \\ -\cos\theta & \text{intersection} \end{cases}. \tag{4.58}$$

Let's write $\xi_l(g)^a$ for the components of $\xi_l(g)$ in the basis (1.5). Then we can explicitly write down g in terms of $l(g)$ and $\xi_l(g)$,

$$g = \exp(l(g)\xi_l(g)) = \begin{pmatrix} \cosh\frac{l(g)}{2} + \xi_l(g)^1 \sinh\frac{l(g)}{2} & (\xi_l(g)^2 - \xi_l(g)^0)\sinh\frac{l(g)}{2} \\ (\xi_l(g)^2 + \xi_l(g)^0)\sinh\frac{l(g)}{2} & \cosh\frac{l(g)}{2} - \xi_l(g)^1\sinh\frac{l(g)}{2} \end{pmatrix}. \tag{4.59}$$

Since $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ by (3.4) we find that the fixed points in $\mathbb{R}$ of g_i are given by

$$x_i^\pm = \frac{\xi_l(g_i)^1 \pm 1}{\xi_l(g_i)^2 + \xi_l(g_i)^0}. \tag{4.60}$$

Plugging this into (4.58) we obtain

$$\xi_l(g_1)^0\xi_l(g_2)^0 - \xi_l(g_1)^1\xi_l(g_2)^1 - \xi_l(g_1)^2\xi_l(g_2)^2 = -B(\xi_l(g_1), \xi_l(g_2)). \tag{4.61}$$

Hence, $|B(\xi_l(g_1), \xi_l(g_2))| > 1$ if and only if the geodesics do not intersect and $B(\xi_l(g_1), \xi_l(g_2)) = -\cosh d$; $|B(\xi_l(g_1), \xi_l(g_2))| < 1$ if and only if the geodesics do intersect and $B(\xi_l(g_1), \xi_l(g_2)) = \cos\theta$. □

Notice that the geodesic connecting γ_1 and γ_2 is not uniquely defined by the Riemann surface R alone. Indeed there is an infinite number of homotopy classes of connecting geodesics. However, in case of a marked Riemann surface a preferred connecting geodesic c does exist.

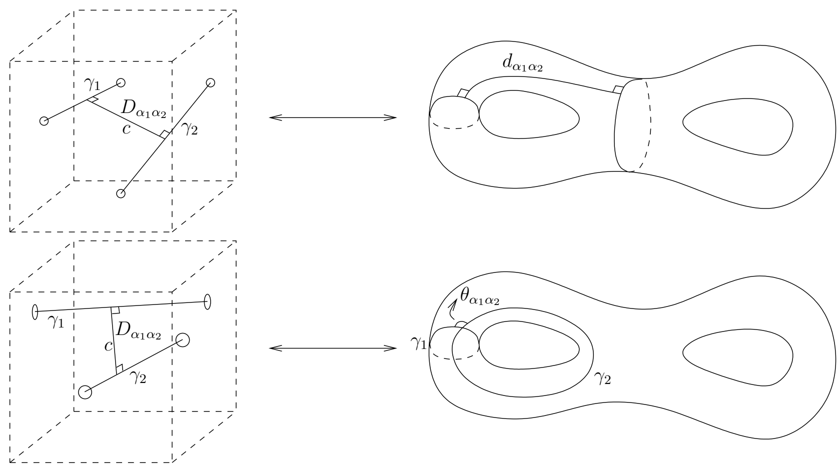


Figure 4.4: *Distance $D_{\alpha_1\alpha_2}$ between space-like (top) and time-like (bottom) separated closed geodesics and their relation to hyperbolic geometry.*

It is fixed by demanding that the composition $c \circ \gamma_1 \circ c^{-1} \circ \gamma_2$ belongs to the homotopy class $\alpha_1 \circ \alpha_2$. Hence, the length of c is a well-defined function on Teichmüller space and it is precisely $d_{\alpha_1\alpha_2}$ which we have calculated above.

A similar remark must be made for the case when γ_1 and γ_2 happen to intersect more than once. The angle $\theta_{\alpha_1\alpha_2}$ then corresponds to the intersection point at which cutting γ_1 and γ_2 and composing them yields an element of $\alpha_1 \circ \alpha_2$.

Theorem 4.3.2. *Let $\alpha_1, \alpha_2 \in \pi_1(R)$ be homotopy classes. Then either one of the following holds:*

(i) For all $[\phi] \in \mathcal{T}_g$ and corresponding Riemann surfaces R, the simple closed geodesics in α_1 and α_2 do not intersect or equivalently $|B(\xi_l(\phi(\alpha_1)), \xi_l(\phi(\alpha_2)))| > 1$. Then $D_{\alpha_1\alpha_2}$ is precisely the variation of the hyperbolic distance $d_{\alpha_1\alpha_2}$ between the closed geodesics on R.

(ii) For all $[\phi] \in \mathcal{T}_g$ and corresponding Riemann surfaces R, the simple closed geodesics in α_1 and α_2 intersect or equivalently $|B(\xi_l(\phi(\alpha_1)), \xi_l(\phi(\alpha_2)))| < 1$. In this case D_{α_1,α_2} is given by the variation of the angle $\theta_{\alpha_1\alpha_2}$ between the closed geodesics at the intersection point.

Proof. We first calculate the variation of the function $(g_1, g_2) \to B(\xi_l(g_1), \xi_l(g_2))$,

$$\begin{aligned} & \frac{d}{dt}\bigg|_{t=0} B\left(\xi_l(\exp(t\tilde{B}^{-1}(X_1))g_1), \xi_l(\exp(t\tilde{B}^{-1}(X_2))g_2)\right) \\ = & \frac{d}{dt}\bigg|_{t=0} B\left(\xi_l(\exp(t\tilde{B}^{-1}(X_1))g_1), \xi_l(g_2)\right) + \frac{d}{dt}\bigg|_{t=0} B\left(\xi_l(g_1), \xi_l(\exp(t\tilde{B}^{-1}(X_2))g_2)\right) \\ = & B\left(T_{g_1}\xi_l \circ T_e r_{g_1}(\tilde{B}^{-1}(X_1)), \xi_l(g_2)\right) + B\left(\xi_l(g_1), T_{g_2}\xi_l \circ T_e r_{g_2}(\tilde{B}^{-1}(X_2))\right). \end{aligned} \tag{4.62}$$

Before we continue we will need some identities. Taking the derivative of (4.6) at $g = g_i$, we obtain $T_{g_i}\xi_l \circ T_e r_{g_i}(\tilde{B}^{-1}(X_i)) = \mathrm{ad}(\tilde{B}^{-1}(X_i)) \circ \mathrm{Ad}(g_i)(\xi_l(g_i)) + \mathrm{Ad}(g_i) \circ T_{g_i}\xi_l \circ T_e r_{g_i}(\tilde{B}^{-1}(X_i))$. Hence

$$(\mathrm{Id} - \mathrm{Ad}(g_i))\,(T_g\xi_l \circ T_e r_g(\tilde{B}^{-1}(X_i))) = [\tilde{B}^{-1}(X_i), \xi_l(g)]. \tag{4.63}$$

Inspired by (4.52) we choose $Y_i \in \mathfrak{sl}(2,\mathbb{R})^*$ such that

$$(\mathrm{Ad}(g^{-1})^* - \mathrm{Id})\left((\mathrm{Ad}(g^{-1})^* - \mathrm{Id})Y_i + X_i\right) = 0, \tag{4.64}$$

or equivalently

$$(\mathrm{Id} - \mathrm{Ad}(g))\left((\mathrm{Id} - \mathrm{Ad}(g))\tilde{B}^{-1}(Y_i) - \tilde{B}^{-1}(X_i)\right) = 0. \tag{4.65}$$

Plugging this into (4.63) we find

$$\begin{aligned}(\mathrm{Id} - \mathrm{Ad}(g_i))^2 T_{g_i}\xi_l \circ T_e r_{g_i}(\tilde{B}^{-1}(X_i)) &= (\mathrm{Id} - \mathrm{Ad}(g_i))[\tilde{B}^{-1}(X_i), \xi_l(g_i)] \\ &= [(\mathrm{Id} - \mathrm{Ad}(g_i))\tilde{B}^{-1}(X_i), \xi_l(g_i)] \\ &= (\mathrm{Id} - \mathrm{Ad}(g_i))^2[\tilde{B}^{-1}(Y_i), \xi_l(g_i)]. \end{aligned} \tag{4.66}$$

To obtain the equality

$$T_{g_i}\xi_l \circ T_e r_{g_i}(\tilde{B}^{-1}(X_i)) = [\tilde{B}^{-1}(Y_i), \xi_l(g_i)] \tag{4.67}$$

we still have to show that the left and right hand side coincide on the kernel of $(\mathrm{Id} - \mathrm{Ad}(g_i))$, which is spanned by $\xi_l(g_i)$. On the one hand we have $B([\tilde{B}^{-1}(Y_i), \xi_l(g_i)], \xi_l(g_i)) = B(\tilde{B}^{-1}(Y_i), [\xi_l(g_i), \xi_l(g_i)]) = 0$ and on the other hand $B(T_{g_i}\xi_l \circ T_e r_{g_i}(\tilde{B}^{-1}(X_i)), \xi_l(g)) = 0$ follows immediately from taking the derivative of $g \to B(\xi_l(g), \xi_l(g)) = 1$.

Plugging (4.67) into (4.62) we obtain for the variation of $(g_1, g_2) \to B(\xi_l(g_1), \xi_l(g_2))$

$$\begin{aligned} B([\tilde{B}^{-1}(Y_1), \xi_l(g_1)], \xi_l(g_2)) + B(\xi_l(g_1), [\tilde{B}^{-1}(Y_2), \xi_l(g_2)]) &= B(\tilde{B}^{-1}(Y_1) - \tilde{B}^{-1}(Y_2), [\xi_l(g_1), \xi_l(g_2)]) \\ &= (Y_1 - Y_2)([\xi_l(g_1), \xi_l(g_2)]). \end{aligned} \tag{4.68}$$

Due to lemma 4.3.3 the normalization factor in (4.55) can be written as

$$\sqrt{|B([\xi_l(g_1), \xi_l(g_2)], [\xi_l(g_1), \xi_l(g_2)])|} = \sqrt{|1 - B(\xi_l(g_1), \xi_l(g_2))^2|}. \tag{4.69}$$

If α_1 and α_2 are non-intersecting, by lemma 4.3.1 $|B(\xi_l(g_1), \xi_l(g_2))| = \cosh(d(g_1, g_2)) > 1$. We conclude that in this case $D_{\alpha_1\alpha_2}$ is the variation of $d_{\alpha_1\alpha_2}$. If α_1 and α_2 do intersect, $B(\xi_l(g_1), \xi_l(g_2)) = \cos(\theta(g_1, g_2)) \in]0, 1[$. And we conclude that $D_{\alpha_1\alpha_2}$ is the variation of $\theta_{\alpha_1\alpha_2}$. □

Lemma 4.3.3. *For $X, Y \in \mathfrak{sl}(2,\mathbb{R})$ we have*

$$B([X, Y], [X, Y]) = B(X, X)B(Y, Y) - B(X, Y)^2. \tag{4.70}$$

Proof. Let's write X and Y in terms of the generators J_a from (1.5),

$$X = x^a J_a, \;\; Y = y^a J_a. \tag{4.71}$$

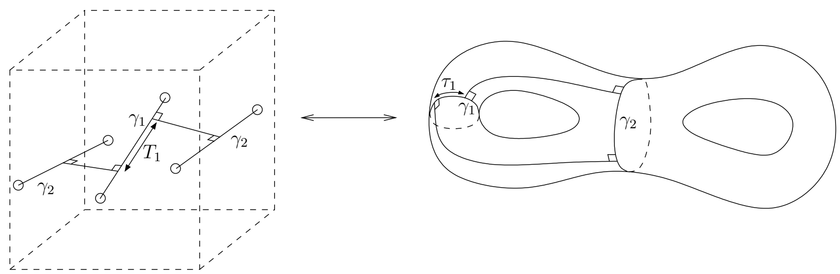


Figure 4.5: *The variation T_1 of the Fenchel-Nielsen twist parameter τ_1 corresponds to the length of a segment of γ_1.*

The J_a satisfy the commutation relations $[J_a, J_b] = \epsilon_{ab}{}^c J_c$ and they form an orthonormal basis, $B(J_a, J_b) = \eta_{ab}$, so

$$\begin{aligned}
B([X,Y],[X,Y]) &= B([x^a J_a, y^b J_b],[x^{a'} J_{a'}, y^{b'} J_{b'}]) \\
&= x^a y^b \epsilon_{ab}{}^c x^{a'} y^{b'} \epsilon_{a'b'}{}^{c'} B(J_c, J_{c'}) \\
&= x^a y^b \epsilon_{ab}{}^c x^{a'} y^{b'} \epsilon_{a'b'}{}^{c'} \eta_{cc'} \\
&= x^a y^b x^{a'} y^{b'} (\eta_{aa'}\eta_{bb'} - \eta_{ab'}\eta_{ba'}) \\
&= (x^a x_a)(y^b y_b) - (x^a y_a)^2 \\
&= B(X,X)B(Y,Y) - B(X,Y)^2.
\end{aligned}$$

□

4.3.3 General picture

In the last two paragraphs we have investigated several different length observables and some pattern is starting to emerge. Especially the difference between space-like and time-like geodesics and their lengths is striking. We have seen that space-like geodesics have a counterpart in the Riemann surface associated to the space-time and that the length of (a segment of) the geodesic in space-time is the variation of the hyperbolic length. Time-like geodesics contrarily have associated a point in the Riemann surface and their length corresponds to variation of angles at that point.

As an example of more complicated constructions we can do, see figure 4.5, which shows the variation of the twist parameter τ_1. In the next paragraph we will see that we can do even other constructions which involve punctures or holes in the Riemann surface.

4.4 Particles

We have seen in the previous chapter that we can extend the notion of Teichmüller space to Riemann surfaces R with punctures and holes of fixed geodesic boundary length. Such a

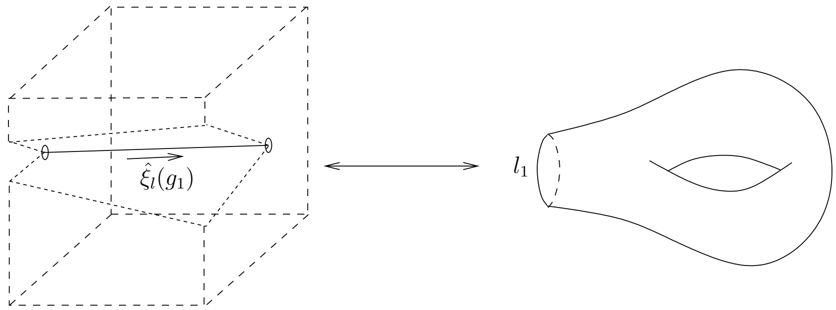


Figure 4.6: *Space-time with tachyonic particle corresponds to Riemann surface with hole.*

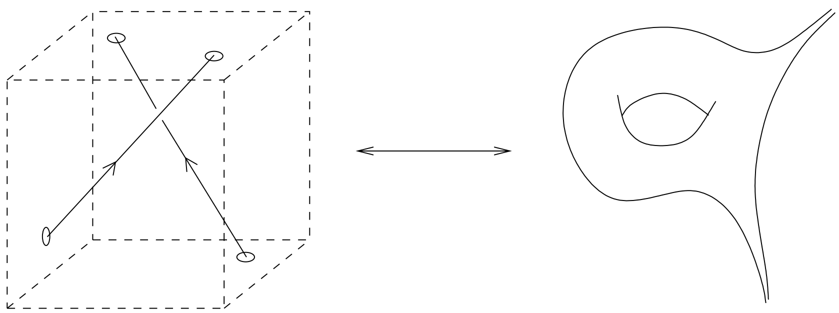


Figure 4.7: *A torus space-time with two massless particles corresponds to a twice-punctured torus.*

Teichmüller space $\mathcal{T}_{g,s}^{l_1,\ldots,l_s}$ will give rise to a phase space describing a space-time with spatial topology equal to the topology of R. We will examine the properties of such space-times.

First we apply the result of paragraph 4.3.1 to the boundary geodesics of R. Since we fixed the boundary lengths to be $l_1, \ldots, l_s$, their variation must vanish. Therefore the lengths of closed space-time geodesics around the singularities will vanish. Furthermore we found that the direction left invariant by a Möbius transformation g is given by $\hat{\xi}_l(g) \in \mathfrak{sl}(2,\mathbb{R})^*$. If the boundary length is non-zero, $\hat{\xi}_l(g)$ is necessarily space-like. We conclude that our space-time has a cone-singularity traveling along a line in the direction of $\hat{\xi}_l(g)$ as in figure 4.6.

In physical terms we would call such a line singularity a tachyon, a particle with negative squared mass. It turns out that the squared mass m_i^2 of the tachyon corresponding to the boundary with hyperbolic length l_i is just equal to $-l_i^2$ (modulo factors of 2π).

Applying the previous considerations to a Riemann surface with zero boundary length, i.e. a surface with a puncture, we obtain a space-time with a singularity which moves in a light-like direction. Hence, we obtain a space-time with a massless particle. We could for example consider the interaction of two massless particles in a torus universe by studying the Teichmüller space of the twice punctured torus (figure 4.7).

4.4.1 Conical singularities

We can even extend our construction to space-time solutions with massive particles. See [3] and [9] for a thorough treatment of point particles coupled to 2+1 dimensional gravity. It can be shown that a space-time solution for a (spin-less) particle of mass m in polar coordinates is locally given by the line-element

$$ds^2 = -dt^2 + dr^2 + r^2 d\alpha^2, \tag{4.72}$$

where the angle coordinate α runs between 0 and $(1-m)2\pi$. Hence, we obtain a space-time with a conical singularity with deficit angle (in the rest-frame) equal to $2\pi m$, where we take the mass in units of the Planck mass.

We claim, without actually proving it, that the phase space of 2+1 dimensional gravity with massive particles can now be constructed from the Teichmüller space of Riemann surfaces with conical singularities with deficit angles equal to 2π times the masses of the particles. Conical singularities in Riemann surfaces arise when we allow the Fuchsian model of the Riemann surface to contain elliptic elements. More precisely a Fuchsian model of a Riemann surface with s punctures and c conical singularities consists precisely of s conjugacy classes of parabolic elements, c conjugacy classes of elliptic elements and further only hyperbolic conjugacy classes (apart from the identity). Analogous to (3.11), the deficit angle ϵ corresponding to an elliptic Möbius transformation γ is related to its trace by

$$\mathrm{Tr}(\gamma) = 2\cos\left(\frac{\epsilon}{2}\right). \tag{4.73}$$

One should be careful when treating such Riemann surfaces, because not all theorems about Riemann surfaces in chapter 3 hold when they contain conical singularities. For instance, when a Riemann surface contains conical singularities with deficit angle smaller than π, not all homotopy classes contain a closed geodesic.

As an example, we could consider a space-time solution of two particles with masses m_1 and m_2 on a torus and we could define the observable D to be the shortest distance between their world lines. The Riemann surface associated to this space-time therefore is a torus with two conical singularities with deficit angles $2\pi m_1$ and $2\pi m_2$. It is not hard to see that the result from paragraph 4.3.2 can be extended to this case and that we obtain that D is the variation of the hyperbolic distance d between the conical singularities, see figure 4.8.

The purpose of this paragraph was to show that it is possible to incorporate matter in the form of point particles in our space-time using the holonomy formulation. We will, however, not work out the details and in the following chapter we will restrict ourselves to space-time solutions corresponding to Riemann surfaces without singularities[6].

[6] In paragraph 5.2.2, however, we will consider a one-holed torus, but we do this to simplify our calculation and we will see in paragraph 5.2.3 that the construction can be extended to Riemann surfaces without holes

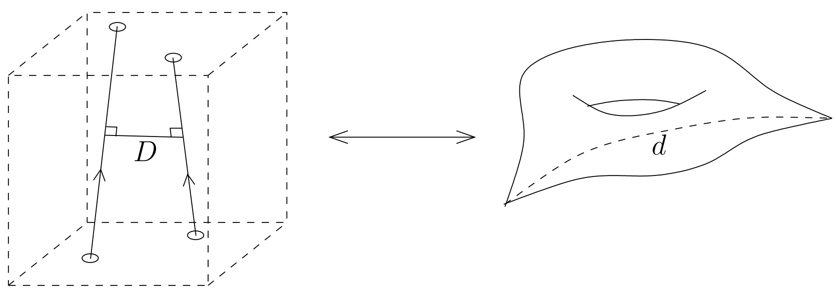


Figure 4.8: *The distance D between two massive particles in a torus space-time is the variation of the hyperbolic distance d between two conical singularities on the Riemann surface.*

Chapter 5

Quantisation of gravity in 2+1 dimensions

5.1 Geometric quantisation

The starting point of geometric quantisation is a manifold M together with a symplectic structure ω. The goal of geometric quantisation is to construct a (separable) Hilbert space $\mathcal{H}$ together with an injective map $\mathcal{O}$ which adds a self-adjoint operator $\mathcal{O}_f$ on $\mathcal{H}$, often denoted by $\hat{f}$, to each real function $f \in \Omega(M)$ on M. We want them to satisfy a number of properties (see also [11]):

(i) $\mathcal{O}$ should be linear, i.e. $\mathcal{O}_{\lambda f+\mu g} = \lambda\mathcal{O}_f + \mu\mathcal{O}_g$ for $f, g \in \Omega(M)$ and $\lambda, \mu \in \mathbb{R}$.

(ii) The constant function 1 should be mapped to the identity operator,

$$\mathcal{O}_1 = \mathrm{Id}_{\mathcal{H}}. \tag{5.1}$$

(iii) The Poisson bracket on M should be mapped to the commutator on $\mathcal{H}$,

$$[\mathcal{O}_f, \mathcal{O}_g] = i\hbar\mathcal{O}_{\{f,g\}}. \tag{5.2}$$

(iv) A complete set of observables $\{f_1, \ldots, f_n\} \subset \Omega(M)$ should be mapped to a complete set of operators[1] $\{\mathcal{O}_{f_1}, \ldots, \mathcal{O}_{f_n}\}$ on $\mathcal{H}$.

Unfortunately in most cases it will not be possible to find a pair $(\mathcal{H}, \mathcal{O})$ satisfying all these properties. Indeed, already in the simplest case of a cotangent bundle $M = T^*\mathbb{R}^n$ with coordinates q_i and p_i, we cannot fulfill requirement (iii) for functions with cubic terms. Often

[1] A set of observables is called *complete* if the only function on M Poisson commuting with all the observables is the zero function. Analogously a set of operators is called *complete* if the only operator commuting with them all is the zero operator.

we have to choose a complete set of observables together maybe with some more physical observables, which we require to satisfy (iii). It is possible that this choice will lead to inequivalent quantisations.

Actually the quantisation procedure involves two major steps. The first step, known as *prequantisation*, concerns the construction of a *prequantum Hilbert space*. This Hilbert space is already near what we want, but it turns out to be too large. Therefore in the second step we have to choose a *polarisation*, which essentially involves identifying configuration and momentum variables. For a thorough mathematical treatment we refer to [29] or [11].

5.1.1 Prequantisation

In some sense the prequantum Hilbert space $\tilde{\mathcal{H}}$ is formed by the wave-functions on the whole phase space, contrarily to what we are used to in standard quantum mechanics where we use wave-function of the position variables alone. To be precise

Definition 5.1.1. *A* prequantisation *of a symplectic manifold* (M, ω) *is a Hermitian line bundle* L *over* M *together with a connection* ∇ *with curvature* $\frac{1}{\hbar}\omega$*. The* prequantum Hilbert space $\tilde{\mathcal{H}}$ *is then the space of square-integrable sections of* L *(with respect to Hermitian form and the volume form defined by* ω*).*

Let us examine the case that M is given by a cotangent bundle $T^*\mathbb{R}^m$, where the symplectic form ω is the canonical one. We can choose a nowhere vanishing section of L, hence we have a trivialisation $L \cong M \times \mathbb{C}$. A connection ∇ is then just given by

$$\nabla = d + \alpha, \tag{5.3}$$

where d is the exterior derivative and α is any one-form on M. The curvature two-form of ∇ now is $id\alpha$. Hence, we have to choose α to be ($-\frac{i}{\hbar}$ times) a *symplectic potential* θ of ω. We are now ready to write down a map $\tilde{\mathcal{O}}$,

$$\tilde{\mathcal{O}}_f = -i\hbar X_f - \theta(X_f) + f, \tag{5.4}$$

where X_f denotes the Hamiltonian vector field of f with respect to ω. It is not hard to check that $\tilde{\mathcal{O}}$ satisfies the properties (i), (ii) and (iii).

5.1.2 Polarisation

To cut down the size[2] of $\tilde{\mathcal{H}}$ we want our wave functions to be constant in certain directions through the phase space. Mathematically this comes down to identifying a *polarisation* of M.

[2] In physical terms, the Hilbert space is too large for instance to obtain uncertainty relations for conjugate variables. Indeed $\tilde{\mathcal{H}}$ contains wave functions which have support on an arbitrary small neighbourhood of any point in the phase space.

Definition 5.1.2. *A* polarisation *of* (M, ω) *is an integrable Lagrangian subbundle of the complexified tangent bundle* $TM_{\mathbb{C}}$ *of* M*.*

Essentially the idea now is that we restrict $\tilde{\mathcal{H}}$ to sections of L whose covariant derivative along the polarisation vanishes. Actually some difficulties arise, but we will not go in to them here, see for instance Woodhouse' book [29].

In case of a cotangent bundle the most obvious choice of a polarisation is the so-called *vertical polarisation* which in local coordinates is spanned by the $\frac{\partial}{\partial p_i}$ where the p_i coordinatise the cotangent spaces.

5.1.3 Application to Teichmüller space

We have seen that Teichmüller space $\mathcal{T}_{g,s}$ has a set of global coordinates $(l_i, \tau_i) \in (\mathbb{R}_{>0} \times \mathbb{R})^n$, where $n = 3g-3+s$. As a matter of fact it follows that l_i and τ_i together with their variations L_i and T_i form a global set of coordinates for the phase space $\mathcal{P} = T^*\mathcal{T}_{g,s}$. Using the results from paragraph 4.1.5 we find

$$\{L_i, l_j\} = \{T_i, \tau_j\} = \{L_i, T_j\} = \{l_i, \tau_j\} = 0 \tag{5.5}$$

and

$$\{L_i, \tau_j\} = \{l_i, T_j\} = \delta_{ij}. \tag{5.6}$$

Hence the physical symplectic structure is

$$\omega_{ph} = \sum_{i=1,\ldots,n} dL_i \wedge d\tau_i + dl_i \wedge dT_i. \tag{5.7}$$

We can even globally choose canonical coordinates[3] $(q_i, p^i)_{i=1,\ldots,2n}$ such that $T^*\mathcal{T}_{g,s}$ gets identified with $T^*\mathbb{R}^{2n}$ with its canonical symplectic structure $\sum_i dp^i \wedge dq_i$ and where the q_i are functions on $\mathcal{T}_{g,s}$ and the p_i are linear in L_i and T_i.

We have written our phase space to the canonical form $T^*\mathbb{R}^{2n}$, so we can use the Stone-von Neumann theorem, which tells us that any representation of the Heisenberg algebra (defined by the standard commutation relations between $\hat{q}_i$ and $\hat{p}^i$) is unitarily equivalent to $L^2(\mathbb{R}^{2n})$ with

$$\hat{q}_i\phi = q_i\phi, \quad \hat{p}^i\phi = -i\hbar\frac{\partial\phi}{\partial q_i}. \tag{5.8}$$

This choice is equivalent to the vertical polarization mentioned above. We can now write down the map $\mathcal{O}$ for functions f with terms which are at most linear in the momenta. Let us write such a function f as $f(q_i, p^i) = f_0(q_i) + f_k(q_i)p^k$. Then (5.4) for f in the vertical polarisation becomes

$$\mathcal{O}_f = f_0(q_i) + f_k(q_i)\frac{\partial}{\partial q_k}. \tag{5.9}$$

[3]For instance put $q_i = \log(l_i)$, $q_{n+i} = l_i\tau_i$, $p^i = -l_iT_i - \tau_iL_i$ and $p^{n+i} = L_i/l_i$ for $i = 1, \ldots, n$

Now let f be a function on Teichmüller space $\mathcal{T}_{g,s}$ and F its variation as function on $\mathcal{P} = T^*\mathcal{T}_{g,s}$. Since they both define maps on $\mathcal{P}$ which are at most linear in momenta, we can apply (5.9) and we get

$$\mathcal{O}_f\phi = f\cdot\phi, \tag{5.10}$$
$$\mathcal{O}_F\phi = i\hbar\{f,\phi\}_{WP}. \tag{5.11}$$

It is easily checked that $\mathcal{O}$ satisfies the properties (i)-(iv) for functions like f and F.

We conclude that our choice of physical observables, namely, functions of the $SO(2,1)$-holonomy together with functions which are linear in the displacement part of the holonomy, has uniquely determined their quantisation (up to unitary equivalence). The quantum Hilbert space $\mathcal{H}$ consists of wave functions on Teichmüller space, which parametrises the $SO(2,1)$-holonomies, and length observables correspond to derivatives acting on these wave functions. To be more precise, $\mathcal{H} = L^2(\mathcal{T}_{g,s}, \omega^n_{WP})$, the space of square-integrable functions with respect to the volume form defined by the Weil-Petersson symplectic structure. We should mention that this is actually a standard choice in several other approaches, like loop quantum gravity[4].

5.1.4 Mapping class group

In the previous paragraph we have constructed a quantisation of the phase space $\mathcal{P}$. The near uniqueness of the construction was due to the fact that Teichmüller space is topologically trivial. However, recall from paragraph 2.4.1 that the space of distinguishable classical space-time solutions is not precisely $\mathcal{P}$. Two space-time solutions which differ by a large diffeomorphism are classically indistinguishable. It seems therefore natural to replace our phase space $\mathcal{P} = T^*\mathcal{T}_{g,s}$ by $\mathcal{P}_0 = T^*\mathcal{M}_{g,s}$, the cotangent bundle of moduli space, which was defined in paragraph 3.1.3. In paragraph 3.4 we saw that moduli space can be obtained by dividing Teichmüller space by the action of the mapping class group. We can naturally extend mapping class transformations to $\mathcal{P}$ and therefore we have

$$\mathcal{P}_0 = \mathcal{P}/MCG. \tag{5.12}$$

Unfortunately the moduli space $\mathcal{M}_{g,s}$ is topologically much more complicated. Moreover, it is quite hard to identify good observables on $\mathcal{P}_0$. For instance, the functions l_α, L_α and $D_{\alpha_1\alpha_2}$ are certainly not invariant under the MCG. An example of a well-defined function on $\mathcal{M}_{g,s}$ is $l_{\min}$, the length of the shortest closed geodesic, which we will investigate at the end of paragraph 5.2.1.

Due to the topological complexity, starting off quantising the phase space $\mathcal{P}_0$ is difficult. In particular, there will be no unique quantisation. Instead, we will use the quantisation we obtained in the previous paragraph and we will impose the mapping class symmetry by

[4] Although in LQG the space of spin-connections is infinite-dimensional, still the Hilbert space consists in some sense of wave functionals of spin-connections and the vielbeins are represented as functional derivatives. See [26] for more details.

constraint operators. Let $g : \mathcal{T}_{g,s} \to \mathcal{T}_{g,s}$ be a mapping class transformation. Obviously g should act on elements of $\mathcal{H}$, which are functions on Teichmüller space, as

$$(\hat{g}\phi)(x) = \phi(g^{-1}x). \tag{5.13}$$

In paragraph 3.4 it was mentioned that the symplectic form ω_{WP} is left invariant by the mapping class transformations. Hence, g leaves the volume form invariant and therefore $\hat{g}$ in (5.13) is a unitary operator on $\mathcal{H}$.

It seems natural to require wave-functions to be invariant under the operator $\hat{g}$. We will investigate the consequences of this requirement for the spectra of certain length observables in the next few paragraphs. Actually we can only require eigenstates of an operator $\hat{A}$ to be invariant under those mapping class transformations g that commute with $\hat{A}$, $[\hat{g}, \hat{A}] = 0$.

5.2 Spectra of length observables

We have defined an (unbounded) operator $\hat{\Psi}$ on $\mathcal{H} = L^2(\mathcal{T}_g, \omega_{WP}^{3g-3+s})$ associated to the variation Ψ of a function ψ on Teichmüller space. In the next paragraphs we will compute the spectra of the geometric observables we have defined in paragraph 4.3.

5.2.1 Length of closed geodesics

In Fenchel-Nielsen coordinates the Weil-Petersson symplectic form is given by (3.39),

$$\omega_{WP} = \sum_{i=1}^{3g-3+s} dl_i \wedge d\tau_i, \tag{5.14}$$

where the l_i are the hyperbolic lengths of the simple closed geodesics $\gamma_i \in \mathcal{L}$ in the pants decomposition and τ_i are the corresponding twisting parameters. Hence, we can write the observable $\hat{L}_i$ acting on a state $\phi \in \mathcal{H}$ as

$$\hat{L}_i\phi = i\hbar\{l_i, \phi\} = \frac{\partial\phi}{\partial\tau_i}. \tag{5.15}$$

As a matter of fact, since the l_i Poisson commute with each other (with respect to ω_{WP}), the operators $\hat{l}_i$ and $\hat{L}_i$ commute and we can write a simultaneous eigenstate of all $\hat{l}_i$ and $\hat{L}_i$, $i = 1, \ldots, 3g-3+s$, with eigenvalues $l_i \in \mathbb{R}_{>0}$ and $L_i \in \mathbb{R}$ respectively,

$$\phi_{l_i,L_i}(l'_i, \tau'_i) = \prod_{i=1}^{3g-3+s} \delta(l'_i - l_i)\exp\left(\frac{i}{\hbar}L_i\tau'_i\right). \tag{5.16}$$

Clearly these eigenstates form an orthogonal basis of $\mathcal{H}$. There are no further restrictions on the eigenvalues l_i and L_i, so we conclude that the geodesic length operator on the phase space $\mathcal{P}$ has spectrum equal to $\mathbb{R}$.

Let us see what happens when we require the eigenstates of $\hat{L}_i$ to be invariant under the mapping class transformations which commute with $\hat{L}_i$. Since the length l_i is invariant under a Dehn twist g along γ_i, the same holds for L_i. Hence, we require the eigenstates of L_i to be invariant under $\hat{g}$. We can easily write down the action of the Dehn twist g in terms of the Fenchel-Nielsen coordinates, namely,

$$\tau_i \to \tau_i + l_i \tag{5.17}$$

while keeping the other coordinates fixed. When we impose (5.16) to be invariant under this transformation, we obtain the following requirement for the eigenvalues,

$$L_i l_i \in 2\pi\hbar\mathbb{Z} \qquad (i = 1, \dots, 3g-3+s). \tag{5.18}$$

In paragraph 5.1.4 we mentioned the function $l_{\min}$ on moduli space $\mathcal{M}_{g,s}$. We can divide Teichmüller space into regions $U_\alpha = \{x \in \mathcal{T}_{g,s} | l_\alpha(x) < l_\beta(x) \text{ for all } \beta \in \pi_1\}$. Obviously $l_{\min}$ coincides with l_α on U_α for all $\alpha \in \pi_1$. Moreover the union of the U_α is dense in $\mathcal{T}_{g,s}$, i.e. $\overline{\bigcup_{\alpha\in\pi_1} U_\alpha} = \mathcal{T}_{g,s}$. Consequently, the variation $L_{\min}$ of $l_{\min}$ is defined on the whole of $\mathcal{P} = T^*\mathcal{T}_{g,s}$ apart from a subset of measure zero, where $l_{\min}$ is not differentiable. An eigenstate ϕ of $\hat{L}_{\min}$ therefore must satisfy

$$i\hbar \left.\frac{\partial\phi}{\partial\tau_\alpha}\right|_{U_\alpha} = L_{\min}\phi|_{U_\alpha}\,, \tag{5.19}$$

where τ_α is the twist parameter with respect to l_α. Fortunately, U_α is invariant under the flow generated by $\frac{\partial}{\partial\tau_\alpha}$ and in particular it is invariant under a Dehn twist $\tau_\alpha \to \tau_\alpha + l_\alpha$. Hence, we can construct an eigenstate of $\hat{L}_{\min}$ by gluing together eigenstates ϕ_α of $\hat{L}_\alpha$ restricted to U_α. Of course one should check if we can do this neatly, but the point is that the eigenstates ϕ_α must at least be invariant under mapping class transformations which leave L_α and l_α, and therefore U_α, invariant. Consequently, the eigenvalues $L_{\min}$ and $l_{\min}$ of a simultaneous eigenstate of $\hat{L}_{\min}$ and $\hat{l}_{\min}$ shall have to obey the same relation as (5.18),

$$L_{\min} l_{\min} \in 2\pi\hbar\mathbb{Z}. \tag{5.20}$$

5.2.2 Distance between geodesics

We would like to probe the spectrum of time-like distances. Therefore we will consider the operator $D_{\alpha_1\alpha_2}$ for intersecting homotopy classes α_1 and α_2. We have seen in paragraph 4.3.2 that $D_{\alpha_1\alpha_2}$ equals the variation of the angle $\theta_{\alpha_1\alpha_2}$ between the closed geodesics $\gamma_1 \in \alpha_1$ and $\gamma_2 \in \alpha_2$ at the intersection point. Hence

$$\hat{D}_{\alpha_1\alpha_2}\phi = i\hbar\{\theta_{\alpha_1\alpha_2}, \phi\} \tag{5.21}$$

for $\phi \in \mathcal{H}$.

Let us start by considering a specific situation, namely, we take our Riemann surface R to be of genus 1 with a hole of geodesic boundary length l_0. We define the lengths l_1 and l_2 of

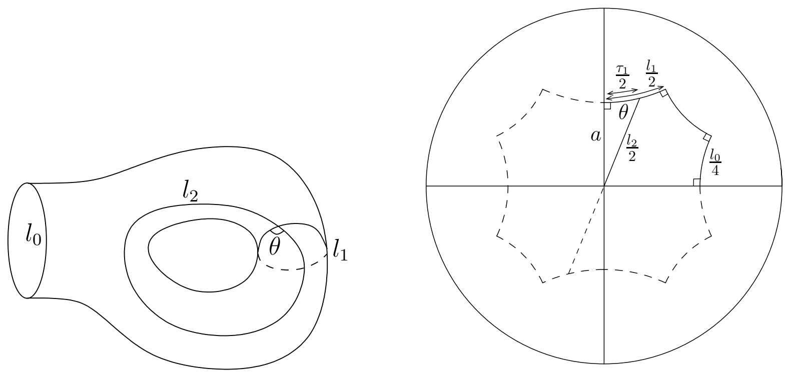


Figure 5.1: *Angle between two geodesics in the one-hole torus.*

γ_1 and γ_2 as in figure 5.1. Analogous to what we did in paragraph 3.3.1 (figure 3.5) we cut R along γ_1 and two geodesics connecting γ_1 with the boundary. The polygon we obtain this way we can draw in the Poincaré disc (considering the symmetry), see the right picture of figure 5.1.

Lemma 5.2.1. *In case of the one-holed torus described above* $\sinh\frac{l_1}{2}\sinh\frac{l_2}{2} \geq \cosh\frac{l_0}{4}$ *and the angle* θ *is given by*

$$\sin\theta = \frac{\cosh\frac{l_0}{4}}{\sinh\frac{l_1}{2}\sinh\frac{l_2}{2}}. \tag{5.22}$$

Proof. Applying lemma 3.2.1 (ii) to the right-angled pentagon in figure 5.1 we find

$$\sinh a = \frac{\cosh\frac{l_0}{4}}{\sinh\frac{l_1}{2}}. \tag{5.23}$$

Furthermore lemma 3.2.1 (iv) implies

$$\frac{\sinh a}{\sin\theta} = \sinh\frac{l_2}{2} \tag{5.24}$$

and

$$\cosh\frac{l_2}{2} = \cosh\frac{\tau_1}{2}\cosh a. \tag{5.25}$$

Combining (5.23) and (5.25) we find

$$\cosh\frac{l_2}{2} = \cosh\frac{\tau_1}{2}\cosh a = \cosh\frac{\tau_1}{2}\sqrt{\frac{\cosh^2\frac{l_0}{4}}{\sinh^2\frac{l_1}{2}}+1} \geq \sqrt{\frac{\cosh^2\frac{l_0}{4}}{\sinh^2\frac{l_1}{2}}+1}, \tag{5.26}$$

hence $\sinh\frac{l_1}{2}\sinh\frac{l_2}{2} \geq \cosh\frac{l_0}{4}$. Combining (5.23) and (5.24) yields

$$\sin\theta = \frac{\cosh\frac{l_0}{4}}{\sinh\frac{l_1}{2}\sinh\frac{l_2}{2}}. \tag{5.27}$$

□

We would like to find a function ρ on Teichmüller space conjugate to this θ,

$$\{\theta, \rho\}_{WP} = 1. \tag{5.28}$$

We keep in mind that we can always add a function of θ to ρ without changing (5.28). Let's for the moment restrict ourselves to half of Teichmüller space corresponding to $0 < \theta < \frac{\pi}{2}$. Then we can use l_1 and l_2 as coordinates (with domain such that $\sinh\frac{l_1}{2}\sinh\frac{l_2}{2} \geq \cosh\frac{l_0}{4}$). We can find an explicit solution for ρ by solving a partial differential equation which we obtain from (5.22) using the Wolpert formula (3.40),

$$\begin{aligned} 1 = \{\theta, \rho\}_{WP} &= \cos\theta\left(\frac{\partial\theta}{\partial l_1}\frac{\partial\rho}{\partial l_2} - \frac{\partial\theta}{\partial l_2}\frac{\partial\rho}{\partial l_1}\right) \\ &= \frac{\partial(\sin\theta)}{\partial l_1}\frac{\partial\rho}{\partial l_2} - \frac{\partial(\sin\theta)}{\partial l_2}\frac{\partial\rho}{\partial l_1} \\ &= \frac{\sin\theta}{2}\left(\coth\frac{l_2}{2}\frac{\partial\rho}{\partial l_1} - \coth\frac{l_1}{2}\frac{\partial\rho}{\partial l_2}\right). \end{aligned} \tag{5.29}$$

We now use standard techniques for solving first-order linear partial differential equations. Let's change coordinates

$$l_i \to \lambda_i = \int^{l_1} \coth\frac{l_1'}{2} dl_1' = 2\log\left(\sinh\frac{l_1}{2}\right), \tag{5.30}$$

such that (5.29) becomes

$$\frac{\partial\rho}{\partial\lambda_1} - \frac{\partial\rho}{\partial\lambda_2} = \frac{2\tanh\frac{l_1}{2}\tanh\frac{l_2}{2}}{\sin\theta} = \frac{2e^{\frac{1}{2}(\lambda_1+\lambda_2)}}{\cosh\frac{l_0}{4}\sqrt{1+e^{\lambda_1}}\sqrt{1+e^{\lambda_2}}}. \tag{5.31}$$

The general solution[5] to this differential equation is $\rho = \rho_1 + \Phi(\lambda_1+\lambda_2) = \rho_1 + \Phi'(\sin(\theta))$ for arbitrary function Φ and ρ_1 is given by

$$\begin{aligned} \rho_1 &= \int_{-\infty}^{\lambda_1} \frac{2e^{\lambda_1+\lambda_2}}{\cosh\frac{l_0}{4}\sqrt{e^t+1}\sqrt{e^{\lambda_1+\lambda_2-t}+1}} dt \\ &= \frac{4}{\sin\theta}\int_0^{\sinh\frac{l_1}{2}} \frac{ds}{\sqrt{s^2+1}\sqrt{\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}s^2+1}} \end{aligned} \tag{5.32}$$

where we changed the integration variable $t \to s = e^{t/2}$ and we used that $e^{\lambda_1+\lambda_2} = \cosh^2\frac{l_0}{4}/\sin^2\theta$. The integral in (5.32) is an elliptic integral[6] so we can express ρ_1 in terms of an inverse Jacobi elliptic function[7],

$$\rho_1 = \frac{4}{\sin\theta}\mathrm{sc}^{-1}\left(\sinh\frac{l_1}{2}\middle| 1 - \frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right). \tag{5.33}$$

[5] The general solution (see [25]) to the differential equation $\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x,y)$ is given by $w = \int_{x_0}^x f(t, y - ax + at)dt + \Phi(y - ax)$ for arbitrary x_0 and arbitrary function Φ.

[6] For definition and properties of elliptic integrals and Jacobi elliptic functions see [2] or *The Wolfram Functions Site* [1].

[7] `http://functions.wolfram.com/09.46.02.0002.01`

We can also express the inverse Jacobi elliptic functions in terms of the incomplete elliptic integral of the first kind by using that[8] $F(i\sinh^{-1}(z)|m) = i\mathrm{sc}^{-1}(z|1-m)$, such that we can write

$$\begin{aligned}\rho_1 &= -\frac{4i}{\sin\theta}F\left(i\frac{l_1}{2}\middle|\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)\\ &= \frac{4}{\sin\theta}F\left(\arctan(\sinh\frac{l_1}{2})\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)\end{aligned} \quad (5.34)$$

Since 5.29 is anti-symmetric under interchange of l_1 and l_2 we can immediately write down a second solution

$$\rho_2 = -\frac{4}{\sin\theta}\mathrm{sc}^{-1}\left(\sinh\frac{l_2}{2}\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right). \quad (5.35)$$

We will first check that the difference $\Delta\rho = \rho_1 - \rho_2$ is indeed a function of θ,

$$\begin{aligned}\Delta\rho &= \frac{4}{\sin\theta}\left(F\left(\arctan(\sinh\frac{l_1}{2})\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)+F\left(\arctan(\sinh\frac{l_2}{2})\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)\right)\\ &= \frac{4}{\sin\theta}F\left(\frac{\pi}{2}\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)\\ &= \frac{4}{\sin\theta}K\left(1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right),\end{aligned} \quad (5.36)$$

where we used [2] (equation 17.4.13) and K is the complete elliptic integral of the first kind.

Let's now choose $\rho = \frac{1}{2}(\rho_1+\rho_2)$. Then a rather uninspiring calculation yields

$$\begin{aligned}\rho &= \frac{-2i}{\sin\theta}F\left(i\log\left(\frac{\cosh\frac{l_1}{2}}{\cosh\frac{l_2}{2}}\right)\middle|\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right)\\ &= \frac{2}{\sin\theta}\mathrm{sc}^{-1}\left(\frac{1}{2}\left(\frac{\cosh\frac{l_1}{2}}{\cosh\frac{l_2}{2}}-\frac{\cosh\frac{l_2}{2}}{\cosh\frac{l_1}{2}}\right)\middle|1-\frac{\sin^2\theta}{\cosh^2\frac{l_0}{4}}\right),\end{aligned} \quad (5.37)$$

such that ρ is anti-symmetric in l_1 and l_2. Figure 5.2 shows Mathematica plots of ρ as function of l_1 and l_2. It is not hard to see that

$$\{(l_1,l_2)\in\mathbb{R}^2|\sinh\frac{l_1}{2}\sinh\frac{l_2}{2}>\cosh\frac{l_0}{4}\}\to]0,\frac{\pi}{2}[\times\mathbb{R}:(l_1,l_2)\to(\theta,\rho) \quad (5.38)$$

is actually smooth and injective. To find the image we note that $x\to\mathrm{sc}^{-1}(x|m)$ is a bounded strictly increasing function for fixed $m\in]-1,1[$. The asymptotic values are[9] $\pm K(m)$ at $x\to\pm\infty$. Hence, for fixed θ we have $-\frac{1}{2}\Delta\rho(\theta)<\rho<\frac{1}{2}\Delta\rho(\theta)$.

[8] http://functions.wolfram.com/08.05.16.0005.01

[9] http://functions.wolfram.com/09.46.03.0013.01, http://functions.wolfram.com/09.46.03.0014.01

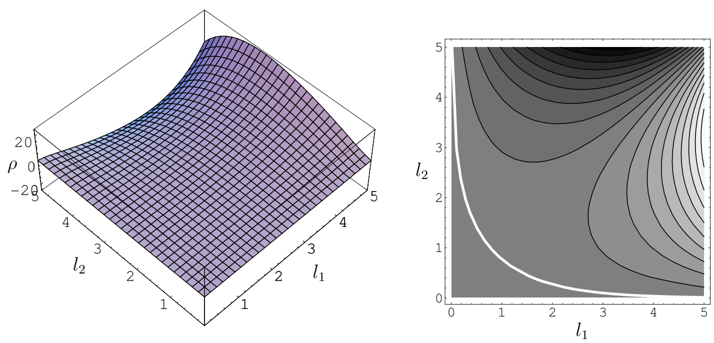


Figure 5.2: *A 3D plot and contour plot of* ρ *as function of* l_1 *and* l_2 *(with* $l_0 = 1$*). The white line in the contour plot corresponds to* $\sinh \frac{l_1}{2} \sinh \frac{l_2}{2} = \cosh \frac{l_0}{4}$.

If we allow θ to take values in $]0, \pi[$ we obtain coordinates for the whole Teichmüller space. It is now straightforward to find the eigenstates of $\hat{D}_{\alpha_1\alpha_2}$. Let ϕ be a smooth function on Teichmüller space written in the (θ, ρ) coordinates, then

$$\hat{D}_{\alpha_1\alpha_2}\phi = i\hbar\frac{\partial\phi}{\partial\rho}. \tag{5.39}$$

Due to the fact that ρ takes values in a bounded range for fixed θ, we run into trouble when we check the symmetry of $\hat{D}$. Indeed we find that boundary terms emerge in the inner-product,

$$\begin{aligned} \left\langle \phi_1 \middle| \hat{D}\phi_2 \right\rangle &= i\hbar\int_0^\pi d\theta \int_{-\Delta\rho(\theta)}^{\Delta\rho(\theta)} d\rho\phi_1(\theta,\rho)^* \frac{\partial}{\partial\rho}\phi_2(\theta,\rho) \\ &= \left\langle \hat{D}\phi_1 \middle| \phi_2 \right\rangle + \\ & i\hbar\int_0^\pi d\theta \left\{\phi_1(\theta,\Delta\rho(\theta))^*\phi_2(\theta,\Delta\rho(\theta)) - \phi_1(\theta,-\Delta\rho(\theta))^*\phi_2(\theta,-\Delta\rho(\theta))\right\}. \end{aligned} \tag{5.40}$$

Since $\hat{D}$ represents a physical observable we certainly want it to be self-adjoint. Since $\hat{D}$ is not a bounded operator we should have defined a domain for it. An obvious choice is

$$\text{dom}(\hat{D}) = \{\phi \in \mathcal{H} | \phi(\theta, \Delta\rho(\theta)) = \phi(\theta, -\Delta\rho(\theta)) = 0\}. \tag{5.41}$$

Clearly $\hat{D}$ is a symmetric operator, since the boundary term in (5.40) vanishes on $\text{dom}(\hat{D})$. However, $\hat{D}$ is not a self-adjoint operator because the domain of its adjoint $\hat{D}^\dagger$ has no restrictions on the boundary values. Hence we need a so-called *self-adjoint extension* of $\hat{D}$. It can be shown [5] that any such extension amounts to extending the domain to

$$\text{dom}_\alpha(\hat{D}) = \{\phi \in \mathcal{H} | \phi(\theta, \Delta\rho(\theta)) = e^{i\alpha(\theta)}\phi(\theta, -\Delta\rho(\theta))\}, \tag{5.42}$$

for some real function α of θ.

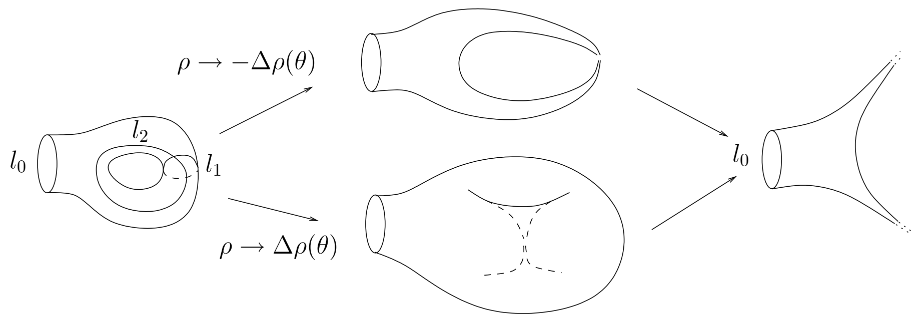


Figure 5.3: *When we let $\rho \to \pm\Delta\rho(\theta)$ our one-holed torus degenerates to a sphere with one hole and two punctures.*

Now we can write down the eigenstates of $\hat{D}$. Because $\hat{D}$ and $\hat{\theta}$ commute, we should be able to find simultaneous eigenstates. Indeed

$$\phi_{\theta,D}(\theta', \rho') = \delta(\theta' - \theta) e^{-i\frac{D}{\hbar}\rho'} \tag{5.43}$$

is an eigenfunction of $\hat{D}$ and $\hat{\theta}$ with eigenvalue D and θ respectively. We have to make sure that this ϕ is in the domain $\mathrm{dom}_\alpha(\hat{D})$ of $\hat{D}$, which implies that

$$e^{-i\frac{D}{\hbar}\Delta\rho(\theta)} = e^{i\alpha(\theta)} e^{i\frac{D}{\hbar}\Delta\rho(\theta)} \tag{5.44}$$

or equivalently

$$D \cdot \Delta\rho(\theta) + \alpha(\theta)\hbar \in 2\pi\hbar\mathbb{Z}. \tag{5.45}$$

It is easily checked that for these values of θ and D we obtain an orthogonal basis of eigenstates $\phi_{\theta,D}$ for $\mathcal{H}$. The discretization of D was actually inevitable, due to necessary orthogonality of eigenstates of a self-adjoint operator.

The ambiguity α in the extension of $\hat{D}$ might be resolved on physical grounds, but the main point is that requiring D to be a physical observable implies that its spectrum is discretized. In figure 5.3 we have illustrated what happens when ρ approaches $\pm\Delta\rho(\theta)$. Hence, independent of the value of θ and whether we let ρ approach $+\Delta\rho(\theta)$ or $-\Delta\rho(\theta)$, we end up with a sphere with one hole and two punctures. The hyperbolic structure of this degenerate state is completely determined by the boundary length l_0.

Let us examine the characteristics of the function $\theta \to \Delta\rho(\theta)$. It has a minimum at $\theta = \pi/2$ equal to

$$\Delta\rho(\frac{\pi}{2}) = 4K(\tanh^2 \frac{l_0}{4}) \tag{5.46}$$

which is plotted in figure 5.4. Apparently, depending on whether θ is measured to be close to $\pi/2$, the time-like distance $\hat{D}$ takes eigenvalues which are integer multiples of a distance slightly smaller than the Planck length l_{Pl}. But since $\Delta\rho$ diverges for $\theta \to 0$ and $\theta \to \pi$, $\hat{D}$ has arbitrarily small eigenvalues if not restricted to some eigenspace of θ.

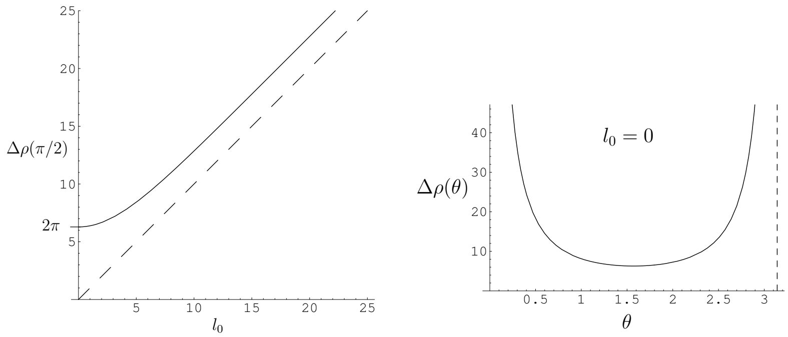


Figure 5.4: $\Delta\rho$ *as function of* l_0 *at* $\theta = \pi/2$ *and as function of* θ *at* $l_0 = 0$.

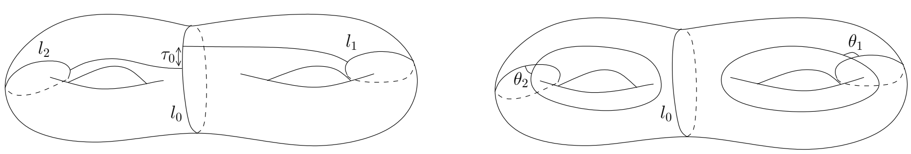


Figure 5.5: *A Riemann surface of genus 2 parametrized by coordinates* $l_0, \tau_0, \theta_1, \rho_1, \theta_2, \rho_2$.

5.2.3 Higher genus

We turn to a more complicated example to show that we can use θ,ρ-coordinates for more general surfaces. Let's glue two handles together to obtain a Riemann surface of genus 2, as in figure 5.5. We know that the Fenchel-Nielsen coordinates $l_0, \tau_0, l_1, \tau_1, l_2, \tau_2$ parametrize Teichmüller space and that the symplectic form is given by (3.39)

$$\omega = dl_0 \wedge d\tau_0 + dl_1 \wedge d\tau_1 + dl_2 \wedge d\tau_2. \tag{5.47}$$

We now replace l_1, τ_1, l_2, τ_2 by $\theta_1, \rho_1, \theta_2, \rho_2$ which we defined in the previous paragraph. From (5.28) we derive that

$$d\theta_1 \wedge d\rho_1 = dl_1 \wedge d\tau_1 + \left(\frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial\tau_1} - \frac{\partial\theta_1}{\partial\tau_1}\frac{\partial\rho_1}{\partial l_0} \right) dl_0 \wedge d\tau_1 + \left(\frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial l_1} - \frac{\partial\theta_1}{\partial l_1}\frac{\partial\rho_1}{\partial l_0} \right) dl_0 \wedge dl_1, \tag{5.48}$$

where we view θ_1 and ρ_1 as functions of l_0, l_1, τ_1. Hence, treating l_0 as a variable causes the symplectic structure to become more complicated than in the case of the one-holed torus.

To obtain a simple form for ω again we have to return to the definition of the Fenchel-Nielsen coordinates. Recall from paragraph 3.3.2 that the twist parameter τ_0 is defined to be the distance between certain distinguished points on the boundary of a pair of pants. As you can see in figure 5.5, in our case the distinguished points are the points on γ_0 closest to γ_1 and γ_2 respectively. Obviously there are other ways of distinguishing points on γ_0. As a matter of fact, distinguishing different points is equivalent to redefining τ_0 by adding functions

$\Delta\tau_0(l_0, l_1, \tau_1)$ and $\Delta\tau_0(l_0, l_2, \tau_2)$. The strategy is now clear: we try to define $\Delta\tau_0$ such that the symplectic structure ω becomes

$$\omega = dl_0 \wedge d\tilde{\tau}_0 + d\theta_1 \wedge d\rho_1 + d\theta_2 \wedge d\rho_2, \tag{5.49}$$

where $\tilde{\tau}_0(l_0, \tau_0, l_1, \tau_1, l_2, \tau_2) = \tau_0 + \Delta\tau_0(l_0, l_1, \tau_1) + \Delta\tau_0(l_0, l_2, \tau_2)$. Due to (5.48) $\Delta\tau_0$ must satisfy

$$\begin{aligned} \frac{\partial \Delta\tau_0}{\partial \tau_1} &= \frac{\partial\theta_1}{\partial\tau_1}\frac{\partial\rho_1}{\partial l_0} - \frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial\tau_1}, \\ \frac{\partial \Delta\tau_0}{\partial l_1} &= \frac{\partial\theta_1}{\partial l_1}\frac{\partial\rho_1}{\partial l_0} - \frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial l_1}. \end{aligned} \tag{5.50}$$

Of course such a solution can only exist if the partial derivatives are consistent, so let's calculate

$$\begin{aligned} \frac{\partial}{\partial l_1}\left(\frac{\partial \Delta\tau_0}{\partial \tau_1}\right) - \frac{\partial}{\partial \tau_1}\left(\frac{\partial \Delta\tau_0}{\partial l_1}\right) &= \frac{\partial}{\partial l_1}\left(\frac{\partial\theta_1}{\partial\tau_1}\frac{\partial\rho_1}{\partial l_0} - \frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial\tau_1}\right) - \frac{\partial}{\partial \tau_1}\left(\frac{\partial\theta_1}{\partial l_1}\frac{\partial\rho_1}{\partial l_0} - \frac{\partial\theta_1}{\partial l_0}\frac{\partial\rho_1}{\partial l_1}\right) \\ &= \frac{\partial}{\partial l_0}\left(\frac{\partial\theta_1}{\partial l_1}\frac{\partial\rho_1}{\partial\tau_1} - \frac{\partial\theta_1}{\partial\tau_1}\frac{\partial\rho_1}{\partial l_1}\right) \\ &= \frac{\partial}{\partial l_0}\{\theta_1, \rho_1\} = 0. \end{aligned} \tag{5.51}$$

Hence, in principle we can solve (5.50), but unfortunately we have not yet found a nice geometric interpretation for the 'new' distinguished points on γ_1. We leave this as an exercise to the reader.

Notice that a Dehn twist along γ_1 is still represented by a shift $\tilde{\tau}_0 \to \tilde{\tau}_0 + l_0$.

Quantisation now again becomes straightforward and the results for the spectra we found in paragraph 5.2.1 and 5.2.2 apply also to $\hat{D}_1$, $\hat{D}_2$ and $\hat{L}_0$.

Chapter 6

Conclusions

6.1 Discussion of results

The main thing we have learned from chapter 4 is that the description of the phase space of vacuum general relativity in $2+1$ dimensions as a cotangent bundle of Teichmüller space allows for an elegant representation of length observables. We have seen a number of cases in which a physical length observable arises as the variation of the hyperbolic counterpart. It led us to the claim that space-like distances in space-time come from variations of hyperbolic distances on the associated Riemann surface. By contrast, time-like distances come from variations of angles on Riemann surfaces. We have therefore classically already established a striking difference between space-like and time-like distances.

In chapter 5 we constructed a geometric quantisation which turned out to be unique up to unitary equivalence under two assumptions. First, we made the assumption that the phase space is given by the cotangent bundle to Teichmüller space. Second, we identified functions of the spin-connection and functions linear in the corresponding momenta as physical observables. It is well-known [7] that inequivalent quantisations exist of $2+1$ dimensional gravity. For instance, as a Chern-Simons theory the classical phase space of gravity only differs from our phase space by a measure zero set. However, it can be shown [21] that physical states are not in a one to one correspondence and their quantisation shall therefore certainly differ.

We computed the spectra of two length observables, the length L_α of a closed geodesics in homotopy class α and the distance $D_{\alpha_1\alpha_2}$ between two particular such geodesics. The first one necessarily measures space-like distances, while we used the second one to measure time-like distances.

We found that the spectrum of L_α consisted of the whole real line. On imposing mapping class invariance we did find a peculiar restriction on the eigenvalues of a combined eigenstate of L_α and its hyperbolic counterpart l_α, namely,

$$L_\alpha \cdot l_\alpha \in 2\pi\hbar\mathbb{Z}. \tag{6.1}$$

Here l_α can be interpreted as the total boost parameter corresponding to the holonomy around α. We were unable to provide a precise physical interpretation of this result.

The spectrum of time-like distance $D_{\alpha_1\alpha_2}$ again turns out to be the whole real line $\mathbb{R}$. But now when we consider the eigenvalues of a combined eigenstate of the distance $D_{\alpha_1\alpha_2}$ and the angle $\theta_{\alpha_1\alpha_2}$ between the closed geodesics, we find the restriction[1]

$$D_{\alpha_1\alpha_2} \cdot \Delta\rho(\theta) \in 2\pi\hbar\mathbb{Z} \tag{6.2}$$

where $\Delta\rho(\theta)$ is a function which diverges for $\theta \to 0$ and $\theta \to \pi$ and which has a minimum of the order of 2π. We are therefore lead to conclude that the distance between closed geodesics which are nearly perpendicular is quantized to a multiple of a minimal distance which is of the order of the Planck length.

Let us compare our findings with similar calculations in literature. Freidel et al. [13] calculated the spectrum of lengths of arbitrary paths in 2+1 dimensional space-time using loop quantum gravity. They find, at the kinematical level, that the eigenvalues are given by square roots of the Casimir invariants of irreducible representations of $SO(2,1)$. The space-like lengths correspond to the continuous series of irreducible representations and the eigenvalues are given by $\sqrt{s^2+1/4}\hbar$ for $s \in \mathbb{R}_{>0}$. The time-like lengths correspond to the discrete series and the eigenvalues are given by $\pm\sqrt{n(n-1)}\hbar$ for $n \in \mathbb{Z}_{>0}$. More precisely, an eigenvalue is given by any sum of these. Hence, in this approach time-like distances get quantized, but the eigenvalues are not spread uniformly. They also mention a slightly different approach in which the eigenvalues of the time-like lengths take the simpler form $\pm(n-1/2)$ for $n \in \mathbb{Z}_{>0}$.

The continuous space-like spectrum agrees with what we find for the lengths of closed geodesics. Indeed, only once we fix our state to be an eigenstate of l_α we find that L_α has discrete eigenvalues. The time-like spectrum however is more difficult to compare. Both our approach and the loop quantum gravity approach predict some kind of discretisation of time-like length spectrum. As long as we restrict ourselves to distances between geodesics which are nearly perpendicular we get similar results as the second approach mentioned by Freidel et al. But our whole spectrum is more subtle. This brings us to an important point we should make.

Since we work in a reduced phase space we are forced to define our observables in a gauge independent way. Although such observables seem more physical than gauge dependent ones, it becomes harder to examine different features of the theory separately. For instance, to examine the quantum geometry we cannot just take a random path in our space-time and measure its length, as is done in the loop quantum gravity approach. Instead, we have to identify geometrically distinguished points or regions to attach a path to, before we can measure its length gauge independently. As a consequence the physical meaning of such a length observable can become obscured, because in some sense it does not anymore measure only the 'amount of space-time along the path', but it has also become dependent of the dynamics of the distinguished points or regions it is attached to[2].

[1] We have ignored the ambiguity which arose in choosing a self-adjoint extension of $D_{\alpha_1\alpha_2}$. It was shown in paragraph 5.2.2 that the only effect of this choice is a shift of $D_{\alpha_1\alpha_2} \cdot \Delta\rho(\theta)$.

[2] Actually this way of defining a gauge invariant observable is analogous to a method proposed by Rovelli (see for instance [26]) to construct a gauge invariant observable from two partial observables (in the case of a

Let us finish this discussion with an argument which may explain why we find that $D_{\alpha_1\alpha_2}$ reproduces the discrete spectrum only where $\theta_{\alpha_1\alpha_2}$ is near $\pi/2$. When the angle $\theta_{\alpha_1\alpha_2}$ approaches zero or π the two closed geodesics will be nearly parallel and the distance becomes highly sensitive to the location of the geodesics. On the other hand, when the geodesics are perpendicular, the distance between them depends at most linearly on the location of the geodesics. Hence, the further you get from $\theta_{\alpha_1\alpha_2} = \pi/2$ the more obscured the length observable becomes, due to the dependence on the geometrical definition of the distinguished points. So we might expect the spectrum of $D_{\alpha_1\alpha_2}$ at $\theta_{\alpha_1\alpha_2} = \pi/2$ to be closest to the spectrum of an elementary gauge dependent length operator.

Let us briefly mention the role of the mapping class group. It is debatable if and how one should implement the symmetry generated by the mapping class group into the quantum theory. A lot of literature (for example [24]) has focused on this question for the case of torus topology. But for the torus the situation is quite different, because the space of $SO(2,1)$ holonomies is not the same as the Teichmüller space of the torus. In particular the mapping class group does not act properly discontinuously on this space. In our case we do not have this problem and the mapping class group acts nicely on our phase space. Hence, we can divide out the mapping class symmetry already classically and we obtain the cotangent bundle to moduli space as new phase space. In paragraph 5.1.4 we argue that working with this phase space would complicate quantisation, in the sense that we cannot ensure uniqueness of the quantum theory. This same ambiguity arises when imposing the mapping class elements as operator on our original Hilbert space under which the elements should transform in some way. Indeed requiring the wave functions to be fully invariant is not the only option. However Carlip argues in [8] that studies have shown that in some cases invariance under the mapping class group is necessary to ensure the right classical limit. Perhaps giving a physical interpretation of the relation (6.1), which was a consequence of mapping class invariance, would shed some light on the matter.

6.2 Possible further investigations

- We mentioned in paragraph 4.3.1 that we have not actually shown that the closed geodesics we are talking about really lie in space-time. Rather we think of the geodesic length function as a geometric function defined on the holonomy group of the space-time. Some more detailed calculations on the construction of a region in Minkowski space on which the holonomy group acts properly discontinuously are needed.

one-dimensional system). Let f and T be two partial observables, then we define $F_{f,T}(\tau)$ to be the value of f when T takes the value τ. Hence, we view T as some sort of time-variable. In the analogy we could consider a system where the phase space is that of space-time solutions together with two distinguished points. We define f to be the distance between the two points and T a function that takes the value τ_0 when the two points are geometrically distinguished. The observable $F_{f,T}(\tau_0)$ is now gauge independent. What we have done is measuring the spectrum of the latter, while in the loop approach the spectrum of f is measured. It is reasonable, however, that these spectra need not coincide, due to the large role of the 'time variable' T. Indeed, Dittrich and Thiemann in [10] provide examples of simple systems that show that all sorts of situations can occur: for instance, the spectrum of f can be discrete while $F_{f,T}$ has continuous spectrum or vice versa. This of course does not mean that their spectra will never be related.

- How about a generalization to higher dimensions? Crucial to our approach was that the theory under consideration turned out to be finite dimensional. More specifically, general relativity in $2+1$ dimensions is a topological field theory in the sense that the only metric degrees of freedom are global. This made it possible for us to use standard quantum mechanical techniques to quantise the theory. Of course, general relativity in $3+1$ dimensions is not particularly topological. But there exist topological field theories in $3+1$ dimensions which resemble general relativity. One of these theories is BF theory with $SO(3,1)$ as a gauge group. Perhaps there exists an analogue to the relation we found between geometrical observables in gravity in $2+1$ dimensions and geometry on Riemann surfaces. Indeed $SO_0(3,1)$ is isomorphic to $PSL(2,\mathbb{C})$ which is the isometry group of the three dimensional hyperbolic space $\mathbb{H}^3$. Hence, associated to a $3+1$ dimensional 'BF-space-time' we have a hyperbolic 3-manifold. Now, however, the Poincaré group $ISO(3,1)$ cannot be written as a cotangent bundle, since there are more rotations than translations in $3+1$ dimensions.

Appendix A

Adding a negative cosmological constant

In this section we will repeat some of the previous constructions for the case we have a negative cosmological constant $\Lambda = -1/\lambda^2$. Now our space-time is not modeled on Minkowski space, but on its counterpart with constant negative curvature, namely anti-de Sitter space (AdS). Actually we will use its universal covering space, which we will denote by $\widetilde{AdS}$.

A.1 Anti-de Sitter space

Following Carlip's book [6] (paragraph 4.4), we can represent AdS as a submanifold of $2+2$ dimensional Minkowski space with coordinates X_1, X_2, T_1 and T_2 and metric

$$ds^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2. \tag{A.1}$$

To obtain AdS we impose the condition

$$X_1^2 + X_2^2 - T_1^2 - T_2^2 = -\lambda^2. \tag{A.2}$$

It is convenient to write the coordinates in matrix form

$$g = \frac{1}{\lambda}\begin{pmatrix} X_1 + T_1 & X_2 + T_2 \\ X_2 - T_2 & -X_1 + T_1 \end{pmatrix}, \tag{A.3}$$

such that AdS is obtained by imposing

$$\det g = 1. \tag{A.4}$$

Hence we can parametrize AdS by $SL(2,\mathbb{R})$. It is easily checked that the metric (A.1) corresponds to the trace form $\frac{1}{4}\lambda^2 B(X,Y) = \frac{1}{2}\lambda^2 \mathrm{Tr}(XY)$ on $\mathfrak{sl}(2,\mathbb{R})$. Indeed, at $g = Id$ we have $dT_1 = 0$, so

$$dg = \frac{1}{\lambda}\begin{pmatrix} dX_1 & dX_2 + dT_2 \\ dX_2 - dT_2 & dX_1 \end{pmatrix}. \tag{A.5}$$

Therefore $\frac{1}{2}\lambda^2\mathrm{Tr}(dgdg) = dX_1^2 + dX_2^2 - dT_2^2$, which is precisely (A.1) with $dT_1 = 0$.

From the definition we immediately see that the (orientation preserving) isometry group of AdS is just $SO(2,2)$, which is isomorphic to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2$. The latter representation of the isometry group is again clear if we identify AdS with $SL(2,\mathbb{R})$, such that the action of $(r^+, r^-) \sim (-r^+, -r^-) \in SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2$ is just given by left and right multiplication, $g \to r^+ g r^-$.

In general, if we have a Lie group G with Lie algebra $\mathfrak{g}$, then we can consider its Killing form $K : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defined by $K(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y))$. Now suppose G is semi-simple. Then the Killing form is non-degenerate and by left translating (or right translating) the Killing form to all tangent spaces of G, we obtain a (pseudo-)Riemannian metric on G, the Killing metric, which we will also denote by K. The left and right translations on G are isometries with respect this metric. We can now consider geodesics on G with respect to the metric compatible connection. They correspond to paths with extremal length (with respect to the Killing metric). It not hard to show that they correspond to translations of one-parameter subgroups of G, i.e. all geodesics on G are of the form

$$t \to x\exp(tX) \tag{A.6}$$

for some $x \in G$ and $X \in \mathfrak{g}$.

In our case, $G = SL(2,\mathbb{R})$, the Killing form is $K(X,Y) = \mathrm{Tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y)) = 4\mathrm{Tr}(XY) = 2B(X,Y)$. Hence, the geodesics corresponding to the Killing form coincide with the geodesics on AdS. We conclude that all geodesics in AdS can be written as

$$t \to x\exp(tX). \tag{A.7}$$

If we choose $X \in \mathfrak{sl}(2,\mathbb{R})$ to be of unit norm with respect to our metric, i.e. $1/4\lambda^2 B(X,X) = \pm 1$, then the parameter t is really a length parameter along the geodesic in AdS.

A.2 Phase space

Let us consider a space-time M with topology $\mathbb{R}\times\Sigma$ which is a solution to the Einstein equations with cosmological constant $\Lambda < 0$ and assume Σ is space-like. Just as in the case of a vanishing cosmological constant, Mess [22] has shown that M is uniquely determined by its collection of holonomies . Moreover any homomorphism H from the fundamental group to the isometry group of AdS, $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2$, determines a (maximal) space-time, provided that the projection of π to both copies of $SL(2,\mathbb{R})$ is Fuchsian. Two such space-times are equivalent if and only if the homomorphisms differ by conjugation with an element of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\mathbb{Z}_2$. Hence, we can write for the collection of physical space-times

$$\mathcal{P} = \mathrm{Hom}_0(\pi_1(\Sigma), \mathrm{Isom}(AdS))/\mathrm{Isom}(AdS), \tag{A.8}$$

where the subscript 0 means that we restrict to Fuchsian homomorphisms. Given an element in $\mathcal{P}$ we can explicitly construct the corresponding space-time by taking the quotient of a certain open subset of the universal covering of AdS by the action of the holonomy group.

A.3 Closed geodesics in space-times with negative cosmological constant

Suppose we are given a space-time M as above and let $H : \pi_1(\Sigma) \to \mathrm{Isom}(AdS)$ be its holonomy homomorphism. To find closed geodesics in M, we consider projections from geodesics in our model space AdS, just as we did in section 4.3.1. Closed geodesics in a homotopy class $[\gamma] \in \pi_1(\Sigma)$ of M correspond precisely to those geodesics in AdS which are left invariant by the action of the holonomy $H_{[\alpha]}$ around the path α. Let's write $H_{[\alpha]} = (r^+, r^-) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$. According to (A.7) a geodesic $\gamma : \mathbb{R} \to SL(2,\mathbb{R})$ is determined by $x \in SL(2,\mathbb{R})$ and $X \in \mathfrak{sl}(2,\mathbb{R})$,

$$\gamma(t) = x \exp(tX). \tag{A.9}$$

Invariance under $H_{[\alpha]}$ then implies that there exists a $t_0 \in \mathbb{R}$ such that

$$\gamma(t_0) = H_{[\alpha]}(\gamma(0)) = r^+\gamma(0)r^- \quad \text{and} \quad \dot\gamma(t_0) = T_{\gamma(0)}H_{[\alpha]}(\dot\gamma(0)). \tag{A.10}$$

Plugging in (A.9) and using that $\dot\gamma(t) = T_e l_{\gamma(t)}(X)$, we get

$$x\exp(t_0X) = r^+xr^- \quad \text{and} \quad T_e l_{r^+xr^-}(X) = T_xH_{[\alpha]} \circ T_e l_x(X). \tag{A.11}$$

Because $T_xH_{[\alpha]} \circ T_e l_x = T_e(l_{r^+} \circ r_{r^-} \circ l_x) = T_e(l_{r^+x} \circ r_{r^-})$, the latter condition is just equivalent to

$$\mathrm{Ad}(r^-)X = X. \tag{A.12}$$

Now we use some identities for $SL(2,\mathbb{R})$ which we derived earlier. We can choose X to be of unit norm and therefore according to (4.45) we can write

$$X = \frac{1}{\lambda}\xi_l(r^-). \tag{A.13}$$

The first condition from (A.11) now implies that $x^{-1}r^+x$ and r^- must be in the same one-parameter subgroup of $SL(2,\mathbb{R})$. In particular $\xi_l(r^-) = \xi_l(x^{-1}r^+x)$ and by the equivariance (4.4) of ξ_l we have

$$\mathrm{Ad}(x)\xi_l(r^-) = \xi_l(r^+). \tag{A.14}$$

This fixes x up to a translation $\exp(tX)$, i.e. a translation along the geodesic. We conclude that there exists a unique closed geodesic homotopic to α in M.

It is now easy to compute its length, which according to a previous remark is just our t_0 from (A.11). Plugging (A.13) into (A.11) and comparing with (4.45) we see that

$$\frac{t_0}{\lambda}\xi_l(r^-) = \frac{1}{2}l(x^{-1}r^+xr^-)\xi_l(x^{-1}r^+xr^-), \tag{A.15}$$

but $\xi_l(x^{-1}r^+xr^-) = \xi_l(r^-)$, so $t_0 = \lambda/2l(x^{-1}r^+xr^-) = \lambda/2l(r^+xr^-x^{-1})$. Now

$$\begin{aligned} r^+xr^-x^{-1} &= \exp(\frac{1}{2}l(r^+)\xi_l(r^+))x\exp(\frac{1}{2}l(r^-)\xi_l(r^-))x^{-1} \\ &= \exp(\frac{1}{2}l(r^+)\xi_l(r^+))\exp(\frac{1}{2}l(r^-)\xi_l(r^+)) \\ &= \exp(\frac{1}{2}(l(r^-) + l(r^+)\xi_l(r^+)) \end{aligned} \tag{A.16}$$

and we conclude that the length of the geodesic is given by a simple formula,

$$t_0 = \frac{\lambda}{2}(l(r^+) + l(r^-)). \tag{A.17}$$

We therefore find a well-defined function L_α on phase space $\mathcal{P}$ which is the length of the unique closed geodesic in the homotopy class α and if we write $H_{[\alpha]} = (r^+_\alpha, r^-_\alpha)$, then it can be expressed as

$$L_\alpha = \frac{\lambda}{2}\left(l(r^+_\alpha) + l(r^-_\alpha)\right). \tag{A.18}$$

A.4 Comparison to the $\Lambda = 0$ case

We have seen that the isometry group Isom(AdS) of anti-de Sitter space AdS can be characterized as the product of two copies of $SL(2,\mathbb{R})$ (forgetting the minus-sign ambiguity for the moment). As a consequence, the phase space $\mathcal{P}_{\Lambda<0}$ decomposes into the product of two copies of Teichmüller space,

$$\mathcal{P}_{\Lambda<0} \cong \mathcal{T}_g \times \mathcal{T}_g. \tag{A.19}$$

In the $\Lambda = 0$ case, however, the isometry group of Isom($\mathbb{R}^{2,1}$) could be characterized as the cotangent group $T^*SL(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ and the phase space $\mathcal{P}_{\Lambda=0}$ essentially became the cotangent bundle to Teichmüller space

$$\mathcal{P}_{\Lambda=0} \cong T^*\mathcal{T}_g. \tag{A.20}$$

We also saw in chapter 4 that the physical symplectic structure on phase space $\mathcal{P}_{\Lambda=0}$ was closely related to the canonical Weil-Petersson structure on Teichmüller space. Something similar happens for the $\Lambda < 0$ case, but now the physical symplectic structure is related in an even simpler way, namely, it is just the sum of the Weil-Petersson symplectic structures originating from both copies of Teichmüller space[1],

$$\omega_{ph} = \omega^+_{WP} + \omega^-_{WP}. \tag{A.21}$$

In particular, both copies of Teichmüller space are physically independent of eachother.

In the previous paragraph we constructed a length observable L_α on the phase space $\mathcal{P}_{\Lambda<0}$ and now we see that it decomposes into a sum of l_α, which is the length of the unique closed geodesic on the Riemann surface, as a function of the first and the second copy of Teichmüller space. In view of the $\Lambda = 0$ case where we found that L_α was the variation of l_α, this is perhaps not totally unexpected. However, it becomes very difficult to do more complicated constructions like we did in paragraph 4.3.2 for the $\Lambda = 0$ case.

Finally let us make some remarks about quantisation of $\Lambda < 0$ in absence of matter. Since phase space $\mathcal{P}_{\Lambda<0}$ is not of the form of a cotangent bundle, quantisation is less straightforward

[1] We will not prove this here, but the reader will probably be convinced when comparing the Poisson algebra of formula 4.73 in [6] to theorem 3.15 in [15].

than in $\Lambda = 0$ case. However, since phase space can be charaterized as a product of two symplectic manifolds $\mathcal{T}_g$, it is expected that our quantum Hilbert space will be a tensor product of two quantum Hilbert spaces appearing in a geometric quantisation of $\mathcal{T}_g$. Hence, we have to find a geometric quantisation of Teichmüller space $\mathcal{T}_g$ with its Weil-Petersson symplectic structure. Actually the construction of a quantum Teichmüller space has received quite some attention from mathematicians, see for instance [27]. We can now represent our observable L_α on our Hilbert space as a sum of the operator representations of l_α on both quantum Teichmüller spaces. It is suggested in [27] (paragraph 4.3) that geodesic length functions are represented as self-adjoint operators on quantum Teichmüller space and that their spectrum is the positive real line $\mathbb{R}_{>0}$. Hence, in this case L_α will be represented as self-adjoint operator and its spectrum will be $\mathbb{R}_{>0}$ as well. However, further investigation is needed here.

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