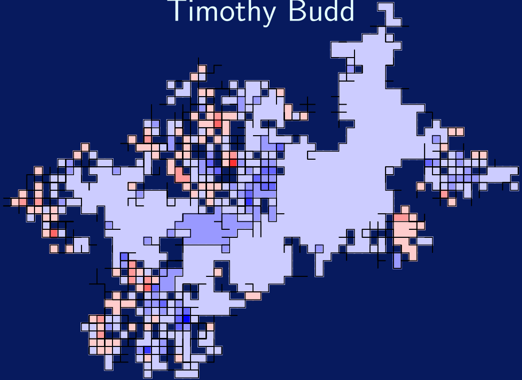
Lattice Paths, Combinatorics and Interactions
CIRM - June 22, 2021

Winding of simple walks on the square lattice

Timothy Budd



Based on arXiv:1709.04042

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Combinatorial problem involving winding angles

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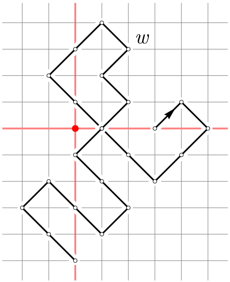

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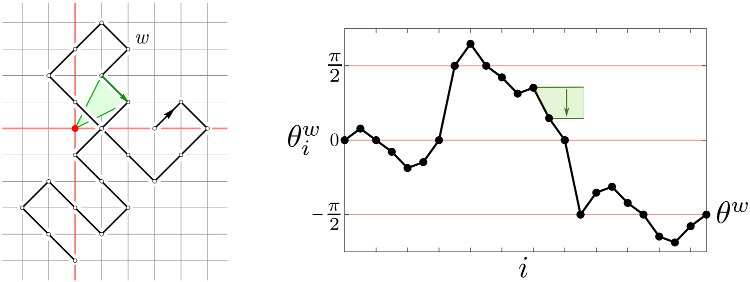

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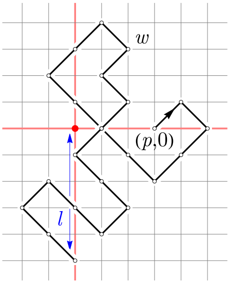
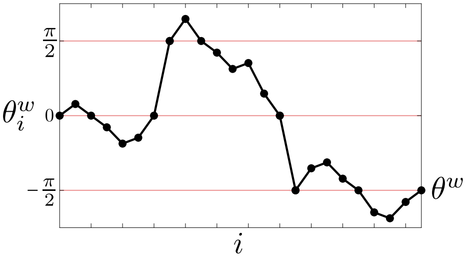
- Can we compute the following generating function?

$$W_{\ell,p}^{(\alpha)}(t) := \sum_w t^{|w|} \mathbf{1}_{\{w_0=(p,0),\ |w_{|w|}|=\ell,\ \theta^w=\alpha\}}. \qquad (p, \ell \geq 1, \alpha \in \tfrac{\pi}{2}\mathbb{Z})$$





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Theorem (TB '17)

There exist formal power series

$$\hat{f}_m(z;t) \quad \in \mathbb{R}[\![z,t]\!], \qquad \text{“eigenvectors”}$$

$$\mathrm{W}_m^{(\alpha)}(t) = \frac{2K(4t)}{\pi\, m} q_{4t}^{m|\alpha|/\pi} \quad \in \mathbb{R}[\![t]\!], \qquad \text{“eigenvalues”}$$

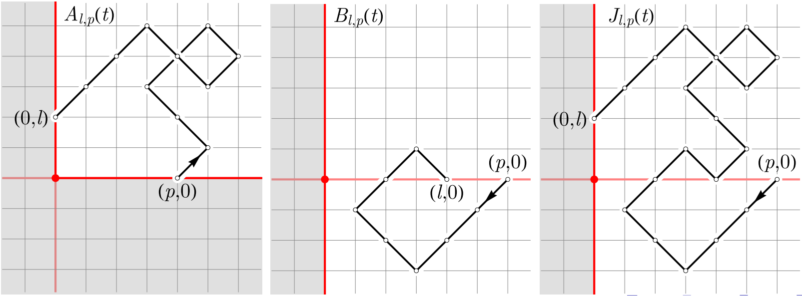
providing the eigendecomposition

$$\frac{1}{\ell p} W_{\ell,p}^{(\alpha)}(t) = \sum_{m=1}^{\infty} [z^\ell]\hat{f}_m(z;t)\ \ \mathrm{W}_m^{(\alpha)}(t)\ \ [z^p]\hat{f}_m(z;t).$$

Building blocks

- Three types of building blocks: type A, B, J.

$$\sum_{m=1}^{\infty} A_{\ell,m}(t)B_{m,p}(t) = J_{\ell,p}(t).$$



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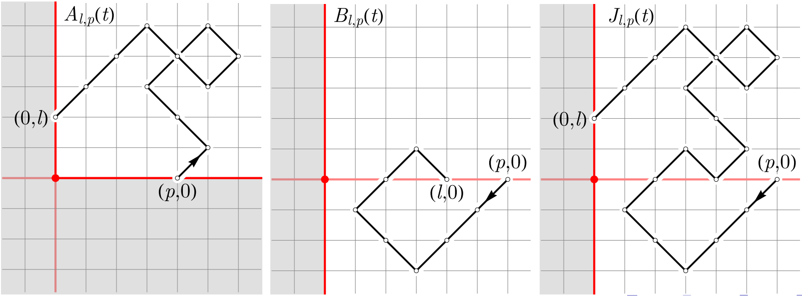

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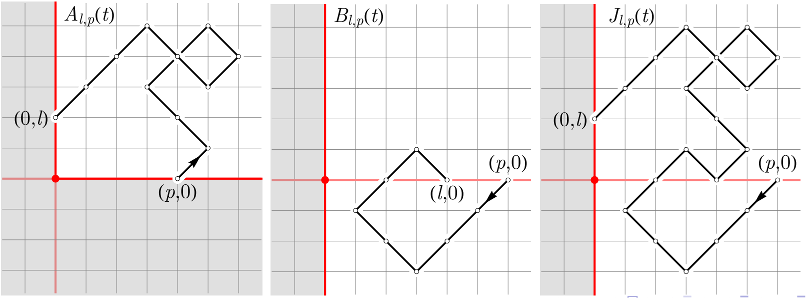
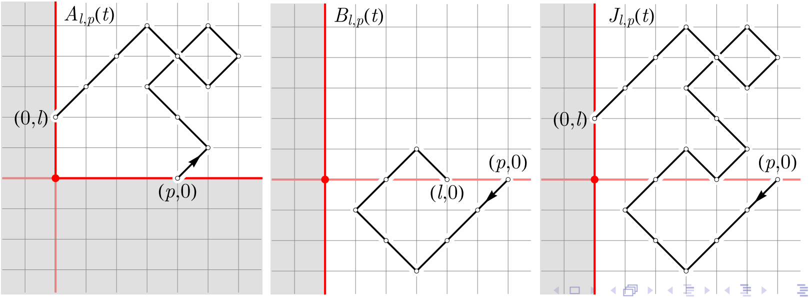
- To formalize this: fix $k = 4t \in (0,1)$ and choose convenient Hilbert space + basis.



Building blocks (operators)

- Let basis $(e_p)_{p=1}^{\infty}$ of $\ell^2(\mathbb{C})$ be such that $\langle e_\ell, e_p \rangle = p\,1_{\{\ell=p\}}$ and let

$$\langle e_\ell, \mathsf{A}_k e_p \rangle = \ell p\, A_{\ell,p}(t), \quad \langle e_\ell, \mathsf{B}_k e_p \rangle = B_{\ell,p}(t), \quad \langle e_\ell, \mathsf{J}_k e_p \rangle = \ell\, J_{\ell,p}(t).$$



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- Then indeed $\mathsf{J}_k = \mathsf{A}_k\mathsf{B}_k$:

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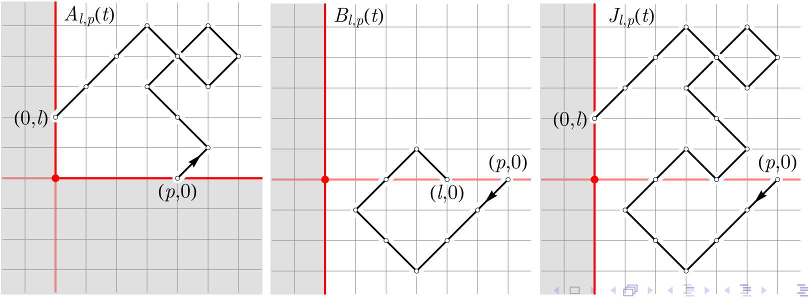

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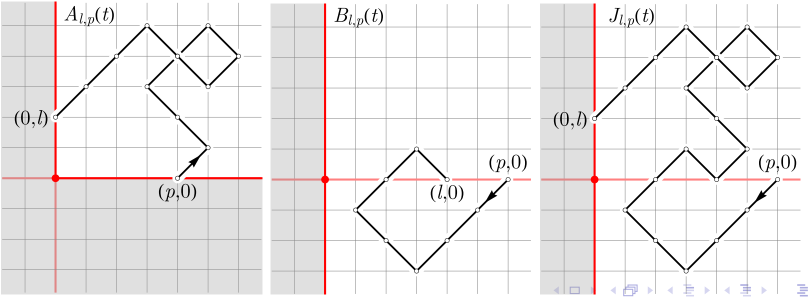
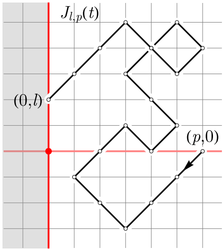
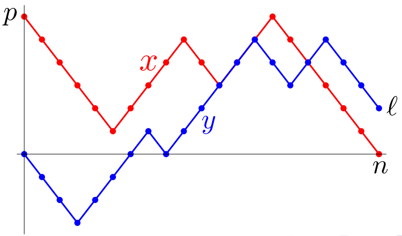
- A_k, B_k, J_k are self-adjoint compact operators that commute: admit simultaneous eigendecomposition!



The operator J_k

$$J_{\ell,p}(t) = \sum_{n=1}^{\infty} t^n \frac{p}{n} \binom{n}{\frac{n-p}{2}} \binom{n}{\frac{n-\ell}{2}} 1_{\{n-p \text{ and } n-\ell \text{ nonnegative and even}\}}$$





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- Not only is J_k self-adjoint, $\langle e_\ell, \mathrm{J}_k e_p \rangle = \ell \, J_{\ell,p}(t)$, but also $\mathrm{J}_k = \mathrm{R}_k^\dagger \mathrm{R}_k$ with (recall $k = 4t$)

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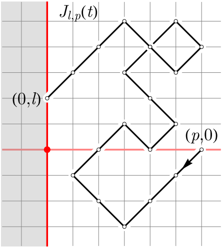

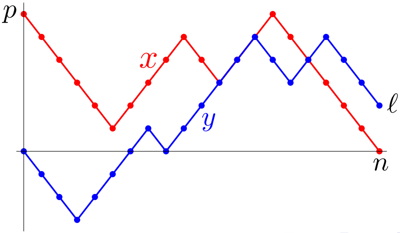

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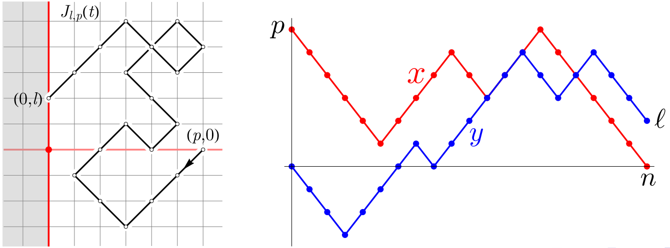
$$= \sum_{n=1}^{\infty} e_n\ [z^n]\psi_k(z)^p, \qquad \psi_k(z) := \frac{1-\sqrt{1-k\,z^2}}{\sqrt{k}\,z}.$$



Dirichlet space $\mathcal{D}$

- $\mathcal{D} = \mathcal{D}(\mathbb{D})$ is Hilbert space of analytic functions f on the unit disk $\mathbb{D} \subset \mathbb{C}$ with $f(0) = 0$ and finite norm w.r.t. $(\mathrm{d}A(x+iy) := \frac{1}{\pi}\mathrm{d}x\mathrm{d}y)$

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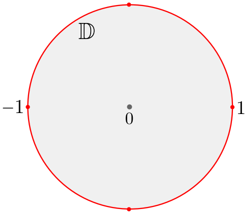

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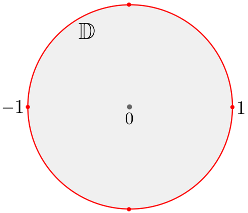

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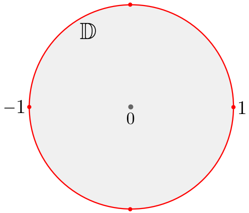

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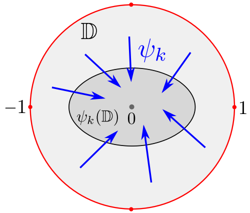

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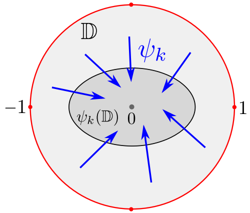
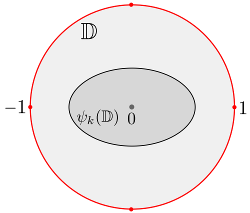
- By conformal invariance of the Dirichlet inner product,

$$\langle f, \mathrm{J}_k g\rangle_{\mathcal{D}} = \left\langle f, \mathrm{R}_k^{\dagger}\mathrm{R}_k g\right\rangle_{\mathcal{D}} = \langle f\circ\psi_k, g\circ\psi_k\rangle_{\mathcal{D}} = \langle f, g\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}.$$



- $\langle f, \mathrm{J}_k g\rangle_{\mathcal{D}(\mathbb{D})} = \langle f, g\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}$: To diagonalize J_k it suffices to find a basis (f_m) that is orthogonal w.r.t. both $\langle\cdot,\cdot\rangle_{\mathcal{D}(\mathbb{D})}$ and $\langle\cdot,\cdot\rangle_{\mathcal{D}(\Psi_k(\mathbb{D}))}$.



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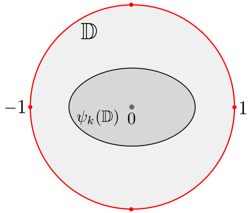
- Look for a nice conformal mapping!



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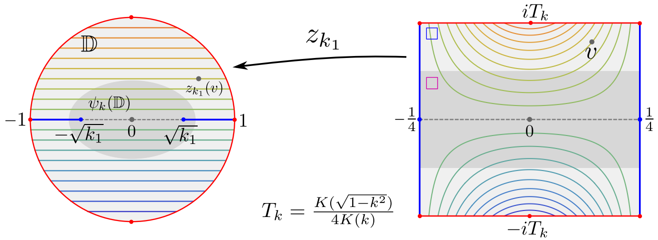
- The elliptic function

$$z_{k_1}(v) = \sqrt{k_1}\,\mathrm{sn}(4K(k_1)v, k_1) \qquad \left(k_1 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)$$

determines isomorphisms $\mathcal{D}(\mathbb{D}) \to \mathcal{D}(\textcolor{blue}{\square})$ and $\mathcal{D}(\psi_k(\mathbb{D})) \to \mathcal{D}(\textcolor{magenta}{\square})$:

$$\begin{aligned} \langle f, g\rangle_{\mathcal{D}(\mathbb{D})} &= \langle f\circ z_{k_1}, g\circ z_{k_1}\rangle_{\mathcal{D}(\textcolor{blue}{\square})} \\ \langle f, g\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))} &= \langle f\circ z_{k_1}, g\circ z_{k_1}\rangle_{\mathcal{D}(\textcolor{magenta}{\square})} \end{aligned} \qquad \mathcal{D}(\textcolor{blue}{\square}) = \left\{ \begin{array}{l} \mathbb{R} + i(-T_k, T_k) \xrightarrow{h} \mathbb{C} \;:\; \|h\|_{\mathcal{D}(\textcolor{blue}{\square})} < \infty \\ h(v+1) = h(v) = h(\tfrac{1}{2} - v),\; h(0) = 0 \end{array} \right\}.$$



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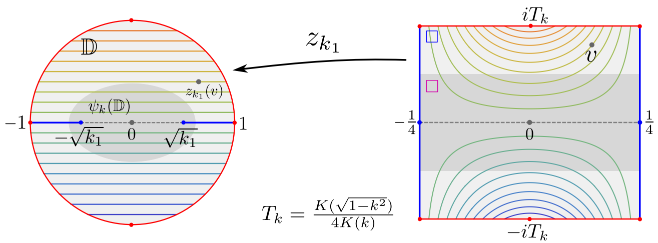
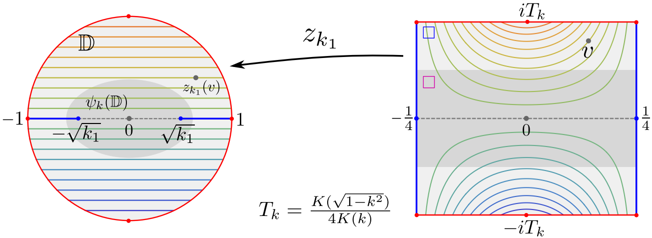
- Orthogonal basis for $\mathcal{D}({\color{blue}\square})$ and $\mathcal{D}({\color{magenta}\square})$:
  $v \mapsto \cos(2\pi m(v + \frac{1}{4})) - \cos(\frac{1}{2}\pi m), \qquad m = 1, 2, \ldots$



- Orthonormal basis $(\hat{f}_m)_{m=1}^{\infty}$ of $\mathcal{D}(\mathbb{D})$ given by

$$\hat{f}_m(z) = \frac{f_m(z)}{\|f_m\|_{\mathcal{D}(\mathbb{D})}}, \quad f_m(z) = \cos(2\pi m(z_{k_1}^{-1}(z) + \tfrac{1}{4})) - \cos(\tfrac{1}{2}\pi m).$$



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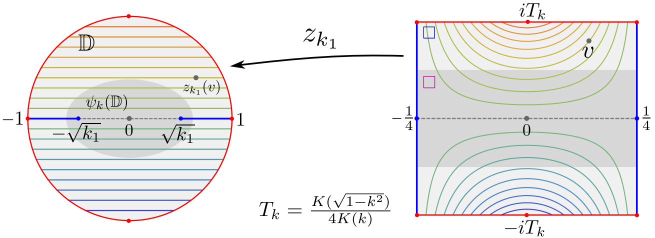
- So J_k has eigenvectors $(\hat{f}_m)_{m\geq 1}$ and eigenvalues

$$\frac{\langle f_m, f_m\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}}{\langle f_m, f_m\rangle_{\mathcal{D}(\mathbb{D})}} = \frac{\sinh(2m\pi T_k)}{\sinh(4m\pi T_k)} = \frac{1}{q_k^{m/2} + q_k^{-m/2}}, \qquad \underset{\text{“nome”}}{q_k = e^{-4\pi T_k}}.$$



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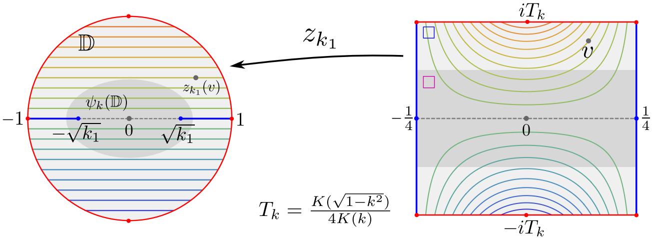
$$\hat{f}_m(z) = \frac{f_m(z)}{\|f_m\|_{\mathcal{D}(\mathbb{D})}}, \qquad f_m(z) = \cos(2\pi m(z_{k_1}^{-1}(z) + \tfrac{1}{4})) - \cos(\tfrac{1}{2}\pi m).$$

- So J_k has eigenvectors $(\hat{f}_m)_{m\geq 1}$ and eigenvalues

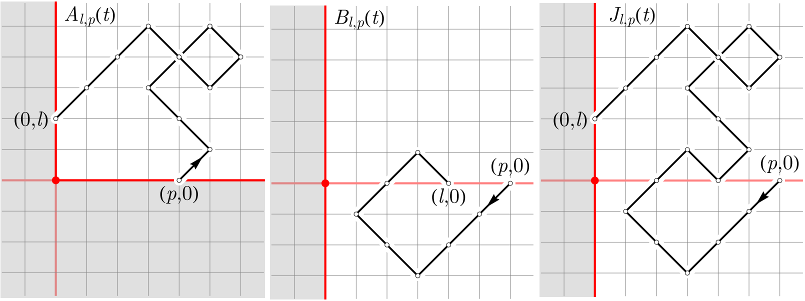
$$\frac{\langle f_m, f_m\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}}{\langle f_m, f_m\rangle_{\mathcal{D}(\mathbb{D})}} = \frac{\sinh(2m\pi T_k)}{\sinh(4m\pi T_k)} = \frac{1}{q_k^{m/2} + q_k^{-m/2}}, \qquad \begin{array}{c} q_k = e^{-4\pi T_k} \\ \text{“nome”} \end{array}.$$

- The generating function for J-type walks has eigendecomposition

$$\tfrac{1}{p} J_{\ell,p}(t) = \sum_{m=1}^{\infty} [z^{\ell}]\hat{f}_m(z;t)\ \frac{1}{q_k^{m/2} + q_k^{-m/2}}\ [z^{p}]\hat{f}_m(z;t).$$



- May work out eigenvalues of A_k and B_k too (same eigenvectors $\hat{f}_m$):



$A_k: \quad \frac{\pi}{2K(k)} \frac{m}{q_k^{-m/2} - q_k^{m/2}}$ $\qquad$ $B_k: \quad \frac{2K(k)}{\pi} \frac{1}{m} \frac{1 - q_k^m}{1 + q_k^m}$ $\qquad$ $J_k: \quad \frac{1}{q_k^{m/2} + q_k^{-m/2}}$

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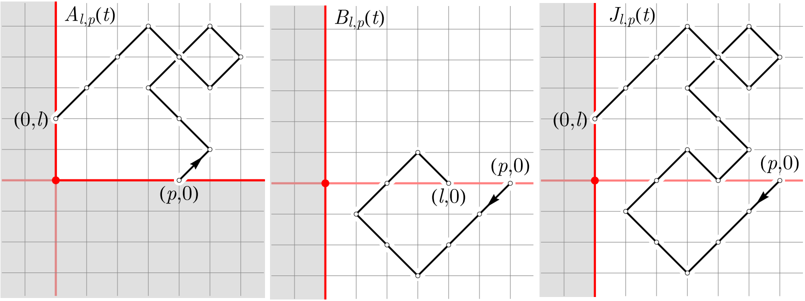


$\mathsf{A}_k : \quad \frac{\pi}{2K(k)} \frac{m}{q_k^{-m/2} - q_k^{m/2}}$ $\qquad$ $\mathsf{B}_k : \quad \frac{2K(k)}{\pi} \frac{1}{m} \frac{1-q_k^m}{1+q_k^m}$ $\qquad$ $\mathsf{J}_k : \quad \frac{1}{q_k^{m/2} + q_k^{-m/2}}$

- Recall $W_{\ell,p}^{(\alpha)}(t) = \langle e_\ell, \mathsf{Y}_k^{(\alpha)} e_p \rangle$, $\alpha \in \frac{\pi}{2}\mathbb{Z}$, where

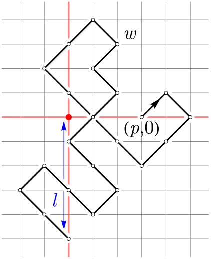
$$\mathsf{Y}_k^{(\alpha)} = \sum_{N=0}^{\infty} \#\{\text{simple walks from } 0 \text{ to } \alpha \text{ of length } N\} \cdot \mathsf{J}_k^N \, \mathsf{B}_k.$$

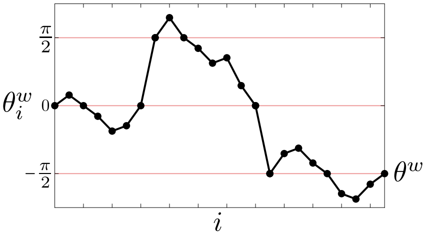
It has eigenvalues

$$\mathsf{Y}_k^{(\alpha)} f_m = \frac{2K(k)}{\pi} \frac{1}{m} q_k^{m|\alpha|/\pi} \, f_m.$$

Putting the building blocks together

$$w \text{ is encoded by } \begin{cases} \text{a simple walk } (\alpha_j)_{j=0}^{N} \text{ on } \frac{\pi}{2}\mathbb{Z} \text{ from } 0 \text{ to } \alpha \\ \text{a sequence } \underbrace{w^{(0)},\dots,w^{(N-1)}}_{\text{type } J}, \underbrace{w^{(N)}}_{\text{type } B} \text{ of ``matching'' walks} \end{cases}$$





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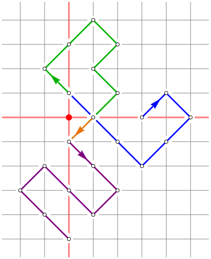

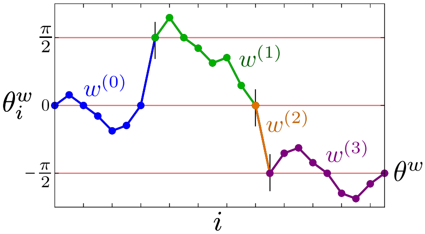

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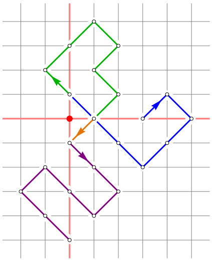

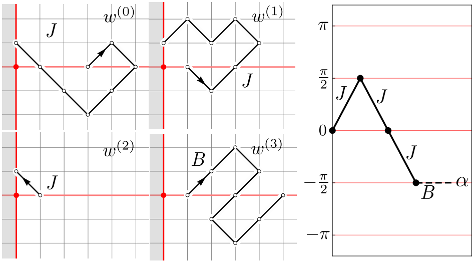

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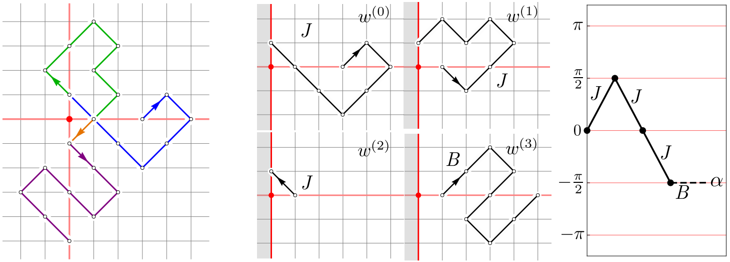

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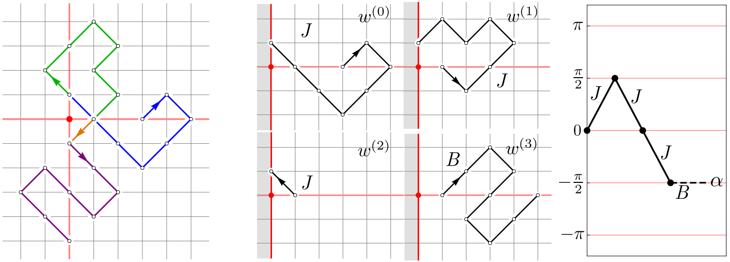
- It has eigenvalues $\mathrm{Y}_k^{(\alpha)} \hat{f}_m = \frac{2K(k)}{\pi m} q_k^{m|\alpha|/\pi} \, \hat{f}_m$, proving

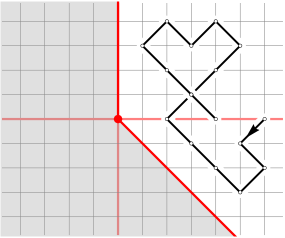
$$\frac{1}{\ell p} W_{\ell,p}^{(\alpha)}(t) = \sum_{m=1}^{\infty} [z^\ell] \hat{f}_m(z;t) \, \frac{2K(4t)}{\pi m} q_{4t}^{m|\alpha|/\pi} \, [z^p] \hat{f}_m(z;t).$$



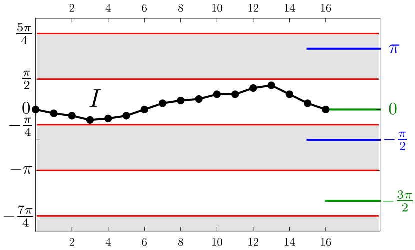
Reflection principle

- For $I = (\beta_-, \beta_+)$, $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$ and p, ℓ even, let

$$W_{\ell,p}^{(\alpha,I)}(t) = \sum_w t^{|w|} 1_{\{w_0=(p,0),\, |w_{|w|}|=\ell,\, \theta^w=\alpha,\, \theta_i^w \in I \text{ for } 1\le i<|w|\}}.$$



$\alpha = 0, I = (-\frac{\pi}{4}, \frac{\pi}{2})$



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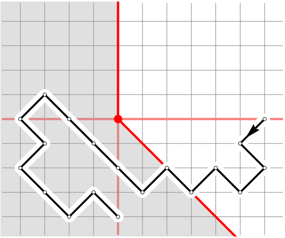

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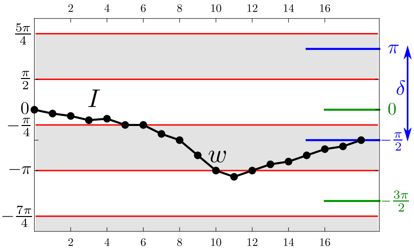

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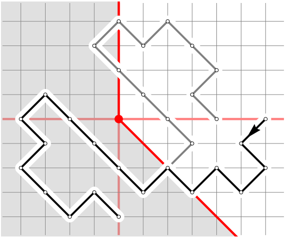

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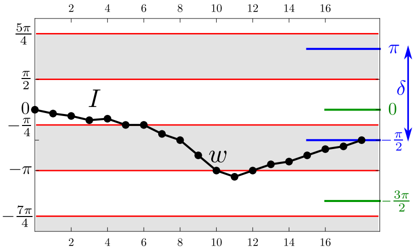

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- If $\theta^w \in 2\beta_+ - \alpha + \delta\mathbb{Z}$ then $\theta^{w'} \in \alpha + \delta\mathbb{Z}$, $\delta = 2(\beta_+ - \beta_-)$.

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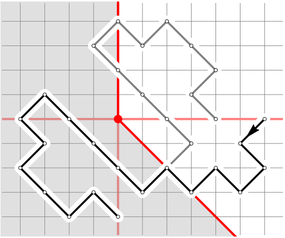

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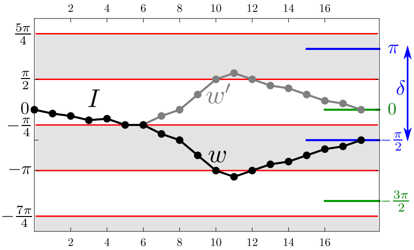

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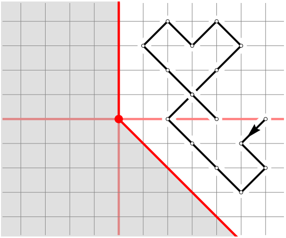
- $W_{\ell,p}^{(0,(-\pi/4,\pi/2))}(t) = \langle e_\ell, Xe_p \rangle_{\mathcal{D}}$ and X has e.v. $\frac{2K(k)}{\pi m} \frac{1-q_k^m}{1+q_k^{m/2}+q_k^m}$.



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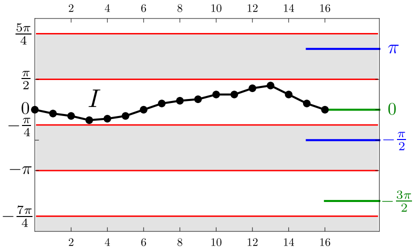
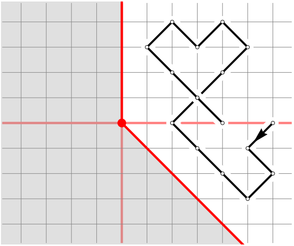
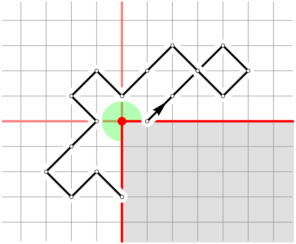


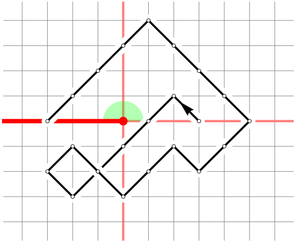
More examples
See [TB,'17, Theorem 1] for the general case.



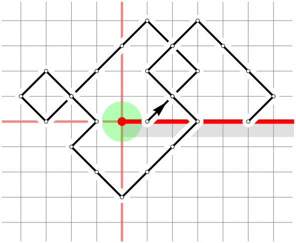
$\langle e_\ell, \bullet\, e_p \rangle_{\mathcal{D}}$, $\dfrac{2K(k)}{\pi m} \dfrac{1-q_k^m}{1+q_k^{m/2}+q_k^m}$



$\dfrac{1}{\ell p} \langle e_\ell, \bullet\, e_p \rangle_{\mathcal{D}}$, $\dfrac{\pi m}{2K(k)} \dfrac{1}{q_k^{-m\alpha/\pi}-q_k^{m\alpha/\pi}}$



$\dfrac{1}{l} \langle e_\ell, \bullet\, e_p \rangle_{\mathcal{D}}$, $\dfrac{1}{q_k^{m\alpha/\pi}+q_k^{-m\alpha/\pi}}$

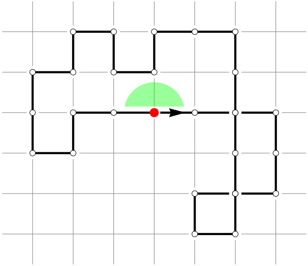


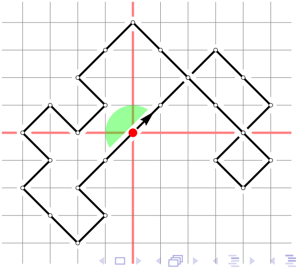
$\dfrac{1}{p} \langle e_\ell, \bullet\, e_p \rangle_{\mathcal{D}}$, $q_k^{m\alpha/\pi}$

Application: Excursions

- Consider set $\mathcal{E}$ of excursions from the origin (rectilinear or diagonal).

$$F^{(\alpha)}(t) := \sum_{w \in \mathcal{E}} t^{|w|} 1_{\{\theta^w = \alpha\}}, \qquad \alpha \in \frac{\pi}{2}\mathbb{Z}.$$





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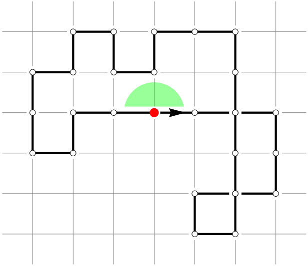
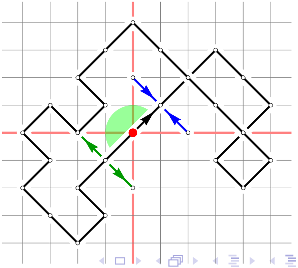
$$F^{(\alpha)}(t) \coloneqq \sum_{w \in \mathcal{E}} t^{|w|} 1_{\{\theta^w = \alpha\}}, \qquad \alpha \in \frac{\pi}{2}\mathbb{Z}.$$

- By flipping the first and last step:

$$F^{(\alpha)}(t) = 4 \sum_{m,l,p=1}^{\infty} (-1)^{l+p+m+1} m \, W_{2l,2p}^{(|\alpha|+m\pi/2)}(t)$$





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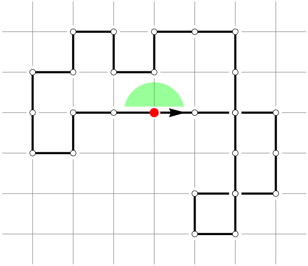
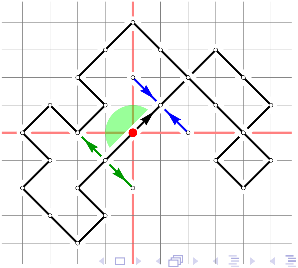
$$= \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{q_k^n(1-q_k^n)^2}{1-q_k^{4n}} q_k^{2n|\alpha|/\pi}$$



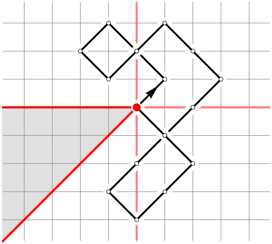


Excursions in cones

- For $I = (\beta_-, \beta_+)$, $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$, a reflection principle shows

$$F^{(\alpha,I)}(t) := \sum_{w\in\mathcal{E}} t^{|w|} 1_{\{w_1=(1,1),\, \theta^w=\alpha,\, \theta_i^w \in I \text{ for all } i\}}$$

$$= \frac{1}{4}\sum_{n\in\mathbb{Z}} \left(F^{(\alpha+n\delta)}(t) - F^{(2\beta_+-\alpha+n\delta)}(t) \right), \quad \delta := 2(\beta_+ - \beta_-)$$



$$\alpha = -\pi/2$$
$$\beta_- = -\pi$$
$$\beta_+ = 3\pi/4$$

Excursions in cones

- For $I=(\beta_-,\beta_+)$, $\beta_\pm\in\frac{\pi}{4}\mathbb{Z}$, $\alpha\in I\cap\frac{\pi}{2}\mathbb{Z}$, a reflection principle shows

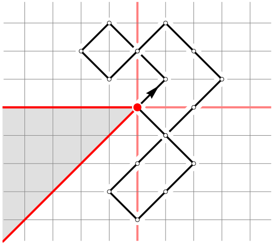
$$F^{(\alpha,I)}(t):=\sum_{w\in\mathcal{E}}t^{|w|}1_{\{w_1=(1,1),\,\theta^w=\alpha,\,\theta^w_i\in I\text{ for all }i\}}$$

$$=\frac{1}{4}\sum_{n\in\mathbb{Z}}\left(F^{(\alpha+n\delta)}(t)-F^{(2\beta_+-\alpha+n\delta)}(t)\right),\qquad \delta:=2(\beta_+-\beta_-)$$

$$=\frac{\pi}{8\delta}\sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}}\left(\cos\left(\frac{4\sigma\alpha}{\delta}\right)-\cos\left(\frac{4\sigma(2\beta_+-\alpha)}{\delta}\right)\right)F\left(t,\frac{4\sigma}{\delta}\right),$$

where

$$F(t,b):=\sum_{\alpha\in\frac{\pi}{2}\mathbb{Z}}F^{(\alpha)}(t)e^{ib\alpha}=\frac{1}{\cos\left(\frac{\pi b}{2}\right)}\left[1-\frac{\pi\tan\left(\frac{\pi b}{4}\right)}{2K(k)}\frac{\theta_1'\left(\frac{\pi b}{4},\sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4},\sqrt{q_k}\right)}\right]$$



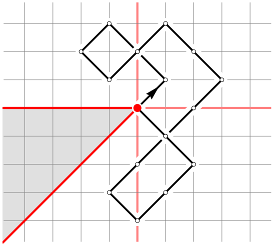
$$\alpha=-\pi/2$$
$$\beta_-=-\pi$$
$$\beta_+=3\pi/4$$

Theorem (TB, '17)

Excursions of winding angle α in a cone $I = (\beta_-, \beta_+)$ with $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$, are enumerated by the finite sum

$$F^{(\alpha,I)}(t) = \frac{\pi}{8\delta} \sum_{\sigma \in (0,\delta)\cap\frac{\pi}{2}\mathbb{Z}} \left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+ - \alpha)}{\delta}\right)\right) F\left(t, \frac{4\sigma}{\delta}\right).$$

- *$F(t,b)$ is algebraic if $b \in \mathbb{Q} \setminus \mathbb{Z}$ and transcendental if $b \in \mathbb{Z}$;*
- *$F^{(\alpha,I)}(t)$ is transcendental only in the cases*
  - $\beta_\pm \in \frac{\pi}{2}\mathbb{Z} + \frac{\pi}{4}$,
  - $\beta_\pm \in \pi\mathbb{Z} + \frac{\pi}{2}$ *and* $\alpha \in \pi\mathbb{Z}$.

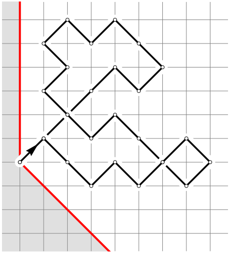


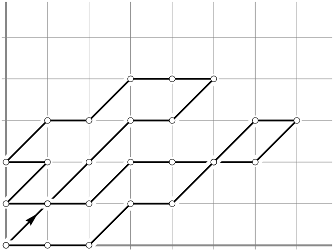
$$\alpha = -\pi/2$$
$$\beta_- = -\pi$$
$$\beta_+ = 3\pi/4$$

Example: Gessel's sequence

- Special algebraic case $\alpha = 0$, $I = (-\pi/2, \pi/4)$:

$$F^{(0,I)}(t) = \frac{1}{4} F\left(t, \frac{4}{3}\right) = \frac{1}{2}\left[\frac{\sqrt{3}\pi}{2K(4t)} \frac{\theta_1'\left(\frac{\pi}{3}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi}{3}, \sqrt{q_k}\right)} - 1\right]$$





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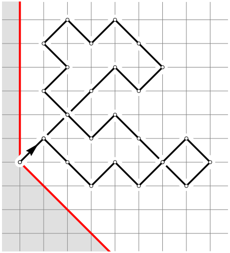
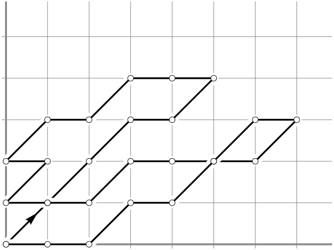
- Agrees with Gessel's conjecture [Kauers, Koutschan, Zeilberger, '09], [Bostan, Kurkova, Raschel, '13], [Bousquet-Mélou, '16], [Bernardi, Bousquet-Mélou, Raschel, '17]:

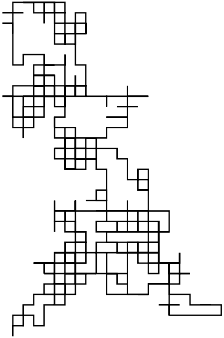
$$F^{(0,I)}(t) = \sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n}.$$





Application: winding of a random loop

- Consider a uniform loop of length 2ℓ on $\mathbb{Z}^2$.



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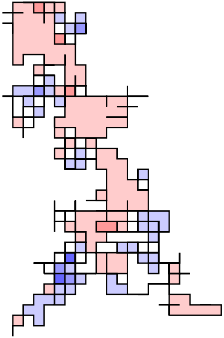

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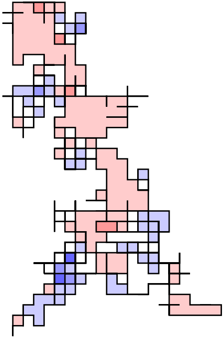

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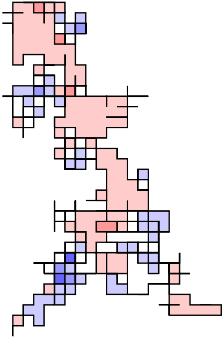

Theorem (TB, '17)

$$\mathbb{E}_{2\ell}[Area_{2\pi n}] = \frac{1}{\binom{2\ell}{\ell}^2} \frac{2\ell}{n} [t^{2\ell}] \frac{q_{4t}^{2n}}{1 - q_{4t}^{4n}}$$

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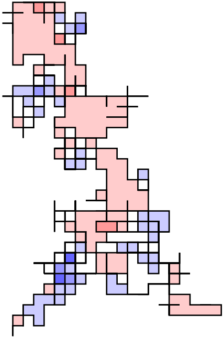

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- Reproduces the analogous result $\frac{1}{2\pi n^2}$ for Brownian motion. [Comtet, Desbois, Ouvry, '90] [Yor, '80] [Garban, Trujillo Ferreras, '06]

- Proof: the expected area is

$$\mathbb{E}_{2\ell}[\text{Area}_{2\pi n}] = \frac{\left|\{w : w_0 = w_{2\ell} \in \mathbb{Z}^2_{\text{odd}},\ \theta^w = 2\pi n\}\right|}{\binom{2\ell}{\ell}^2}$$

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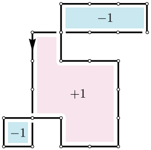

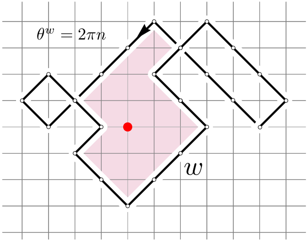

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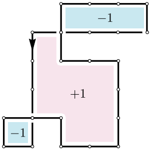

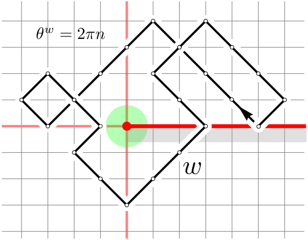

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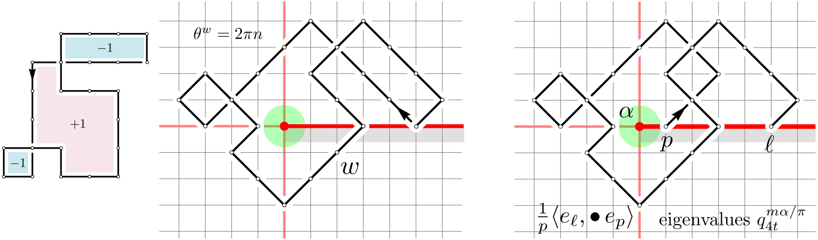

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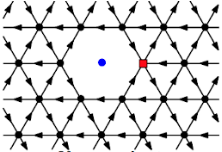
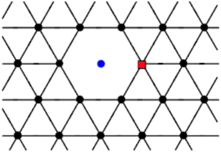
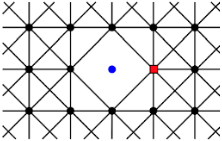


Kreweras lattice



Triangular Lattice



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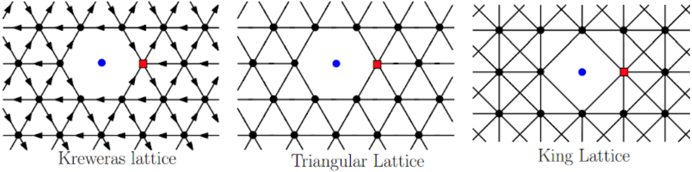

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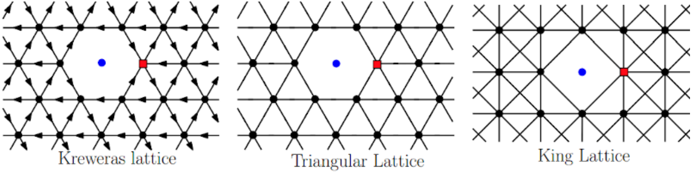

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- Closely related to lattice path counting: generating function of paths ending at $\boldsymbol{w}$ with weight $t_{\boldsymbol{z}}$ per step $\boldsymbol{z}$ is harmonic (on suitable domain).

A curious operator

$$(\Delta h)(\boldsymbol{w}) = h(\boldsymbol{w}) - \sum_{\boldsymbol{z}} t_{\boldsymbol{z}} h(\boldsymbol{w} - \boldsymbol{z}).$$

- Introduce another operator

$$(Lh)(\boldsymbol{w}) = \sum_{\boldsymbol{z}} (\boldsymbol{z} \times \boldsymbol{w})\, t_{\boldsymbol{z}} h(\boldsymbol{w} - \boldsymbol{z}), \qquad \boldsymbol{z} \times \boldsymbol{w} := z_1 w_2 - z_2 w_1.$$

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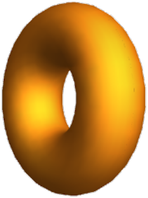
- This curve carries a natural Hamiltonian vector field

$$X_S := xy \left(\frac{\partial S}{\partial x} \frac{\partial}{\partial y} - \frac{\partial S}{\partial y} \frac{\partial}{\partial x} \right) \quad \text{such that} \quad (Lf_{x,y})(\boldsymbol{w}) = X_S f_{x,y}(\boldsymbol{w}).$$

The case of genus 1: rotations?

- Focus on small steps:
  $t_{\nwarrow}, t_{\uparrow}, t_{\nearrow}, t_{\rightarrow}, t_{\searrow}, t_{\downarrow}, t_{\swarrow}, t_{\leftarrow}$ and non-degenerate cases where $S(x, y) = 1$ has genus 1.

$x, y \in \mathbb{C}$



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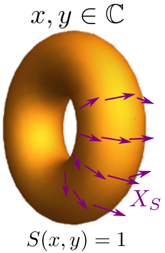

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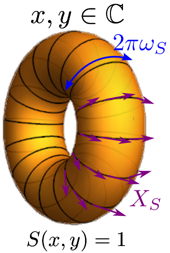

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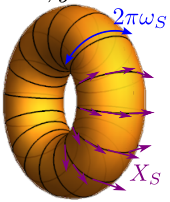

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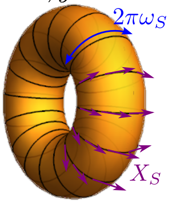

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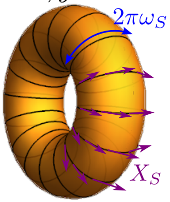

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The family of operators $e^{\alpha\omega_S L}$, $\alpha \in \mathbb{R}$, implement continuous "rotation" by angle α interpolating discrete symmetries!

Polar decomposition

- If we can rotate harmonic functions, we can describe them in polar coordinates:

$$h(\boldsymbol{w}) \quad \leftrightarrow \quad \hat{h}_\alpha(r) := (e^{\alpha \omega_S L} h)(r, 0), \quad r \in \mathbb{Z}_{\geq 0},\ \alpha \in \mathbb{R}$$

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- Harmonic function in polar coordinates satisfies

$$\frac{\partial^2}{\partial\alpha^2}\hat{h}_\alpha(r) = \omega_s^2 D\hat{h}_\alpha(r). \qquad \left[\text{compare Laplace eq.: } \tfrac{\partial^2 f}{\partial\alpha^2} = -r\tfrac{\partial}{\partial r}\left(r\tfrac{\partial f}{\partial r}\right)\right]$$

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- Harmonic function in polar coordinates satisfies

$$\frac{\partial^2}{\partial\alpha^2}\hat{h}_\alpha(r) = \omega_s^2 D\hat{h}_\alpha(r). \qquad \left[\text{compare Laplace eq.: } \frac{\partial^2 f}{\partial\alpha^2} = -r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right]$$

- Separation of variables (analogue of Bessel functions): $Dg_m(r) = m^2 g_m(r)$.

Polar decomposition

- If we can rotate harmonic functions, we can describe them in polar coordinates:

$$h(\boldsymbol{w}) \quad \leftrightarrow \quad \hat{h}_\alpha(r) := (e^{\alpha\omega_s L}h)(r,0), \quad r \in \mathbb{Z}_{\geq 0},\ \alpha \in \mathbb{R}$$

- If $\Delta h = 0$, $(\, h(r,0)\,)_{r\geq 0}$ determines $(\,(L^2 h)(r,0)\,)_{r\geq 0}$:

$$\begin{aligned}(L^2 h)(r,0) &= t_{\searrow}^2 r^2\, h(r-2,2) + t_{\searrow} t_{\downarrow} r(r-1)h(r-1,2) + \dots \\ &= -c_{-2}\, r(r-1)h(r-2,0) - c_{-1}\, r(2r-1)h(r-1,0) + c_0 r^2 h(r,0) \\ &\quad - c_1 r(2r+1)h(r+1,0) - c_2 r(r+1)h(r+2,0) =: (Dh(\cdot,0))(r).\end{aligned}$$

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- Separation of variables (analogue of Bessel functions): $Dg_m(r) = m^2 g_m(r)$.

In the diagonal simple walk case ($t_{\nearrow} = t_{\searrow} = t_{\swarrow} = t_{\nwarrow} = t$), the "eigenvectors" $\hat{f}_m(z)$ provide the Bessel modes: $g_m(r) = r[z^r]\hat{f}_m(z)$!

Thanks for you attention!
Comments?