Lie Algebras in Particle Physics 2021/2022 (NWI-NM101B)[1]

Timothy Budd

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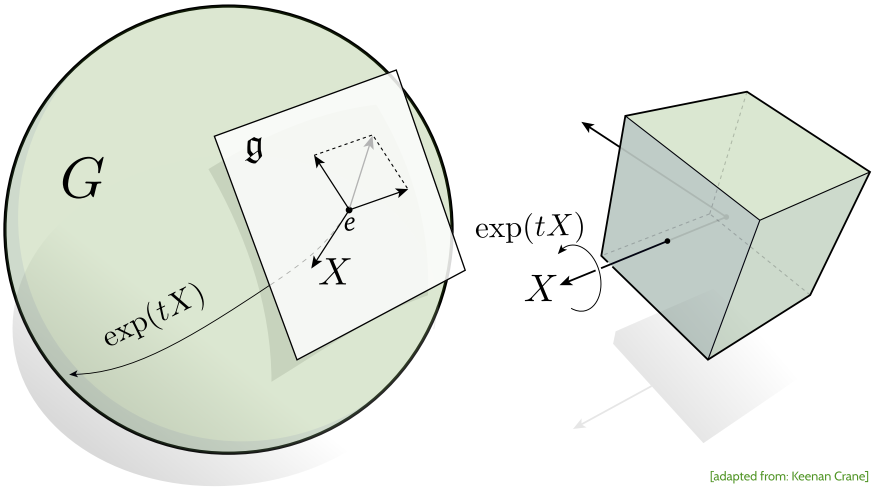


Contents

[1]These lecture notes are an extended version of notes by F. Saueressig and are loosely based on the book *Lie Algebras in Particle Physics* by H. Georgi.

1 Introduction to groups and representations

Group theory is the mathematical theory underlying the notion of symmetry. Understanding the symmetries of a system is of great importance in physics. In classical mechanics, Noether's theorem relates symmetries to conserved quantities of the system. Invariance under time-translations implies the conservation of energy. Invariance with respect to spatial translations leads to the conservation of momentum, and rotational invariance to the conservation of angular momentum. These conserved quantities play a crucial role in solving the equations of motion of the system, since they can be used to simplify the dynamics. In quantum mechanics, symmetries allow one to classify eigenfunctions of, say, the Hamiltonian. The rotational symmetry of the hydrogen atom implies that its eigenstates can be grouped by their total angular momentum and angular momentum in the z-direction, which completely fixes the angular part of the corresponding wave-functions.

In general a symmetry is a transformation of an object that preserves some properties of the object. The two-dimensional rotational symmetries SO(2) correspond to the linear transformations of the Euclidean plane that preserve lengths. A permutation of n particles preserves local particle number. Since a symmetry is a transformation, the composition of two symmetries of an object is always another symmetry. The concept of a group is introduced exactly to capture the way in which such symmetries compose.

Definition 1.1 (Group). A group $(G, \cdot)$ is a set G together with a binary operation $\cdot$ that sends f, g to an element $f \cdot g$. It is required to satisfy

(i) *closure*: if $f, g \in G$ then also $f \cdot g \in G$;

(ii) *associativity*: if $f, g, h \in G$ then $f \cdot (g \cdot h) = (f \cdot g) \cdot h$;

(iii) *identity*: there exists an element $e \in G$ such that $e \cdot f = f \cdot e = f$ for all $f \in G$;

(iv) *inverse*: for every $f \in G$ there is an element $f^{-1} \in G$ such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

Let us start by looking at a few abstract examples (check that these indeed satisfy the group axioms!):

Example 1.2. $G = \{1, -1\}$ with the product given by multiplication.

Example 1.3. The integers under addition $(\mathbb{Z}, +)$.

Example 1.4. The permutation group $(S_X, \circ)$ of a set X: S_X is the set of bijections $\{\psi : X \to X\}$ and $\circ$ is given by composition, i.e.

$$\text{if} \quad \psi : X \to X \quad \text{and} \quad \psi' : X \to X \quad \text{then} \quad \psi \circ \psi' : X \to X : x \mapsto \psi(\psi'(x)).$$

In particular, if n is a positive integer we write $S_n \coloneqq S_{\{1,2,\ldots,n\}}$ for the permutation group on $\{1, 2, \ldots, n\}$.

Example 1.5. The general linear group $G = \mathrm{GL}(N, \mathbb{C})$ consisting of all complex $N \times N$ matrices with non-zero determinant with product given by matrix multiplication.

Based on these examples several remarks are in order. First of all, the binary operation of a group is not necessarily implemented as a multiplication, as should be clear from examples 2 and 3. One is free to choose a notation different from $f = g \cdot h$ to reflect the product structure, e.g. $f = g \circ h$, $f = g + h$, $f = gh$. Often we will write G instead of $(G, \cdot)$ to indicate a group if the product is clear from the context.

Notice that there is no requirement that $g \cdot h = h \cdot g$ in the definition of a group, and indeed examples 1.4 and 1.5 contain elements g and h such that $g \cdot h \neq h \cdot g$.

Definition 1.6 (Abelian group). A group G is abelian when

$$g \cdot h = h \cdot g \quad \text{for all} \quad g, h \in G.$$

Otherwise the group is called non-abelian.

The groups in the examples are of different size:

Definition 1.7 (Order of a group). The *order* $|G|$ of a group G is the number of elements in its set. A group of finite order is called a *finite group*, and a group of infinite order is an *infinite group*.

For example, S_n is a finite group of order $n!$, while $\mathrm{GL}(N, \mathbb{C})$ is an infinite group.

Although one may use different operations to define a group (multiplication, addition, composition, ...) this does not mean that the resulting abstract groups are necessarily distinct. In fact, we will identify any two groups that are related through a relabeling of their elements, i.e. any two groups that are isomorphic in the following sense.

Definition 1.8 (Isomorphic groups). Two groups G and G' are called *isomorphic*, $G \cong G'$, if there exists a bijective map $\phi : G \to G'$ that preserves the group structure, i.e.

$$\phi(g \cdot h) = \phi(g) \cdot \phi(h) \quad \text{for all} \quad g, h \in G. \tag{1.1}$$

Example 1.9 (Cyclic group). For positive integer n the three groups defined below are all isomorphic (see exercises). Any one of them may be taken as the definition of the *cyclic group* of order n.

- the set $\mathbb{Z}_n \coloneqq \{0, 1, 2, \ldots, n-1\}$ equipped with addition modulo n;
- the set $C_n \subset S_n$ of cyclic permutations on $\{1, 2, \ldots, n\}$ equipped with composition;
- the nth roots of unity $U_n \coloneqq \{z \in \mathbb{C} : z^n = 1\}$ equipped with multiplication.

Abstract group theory is a vast topic of which we will only cover a few aspects that are most relevant for physics. Central in this direction are representations.

Definition 1.10 (Representation). A *representation* D of a group G on a vector space V is a mapping $D : G \to \mathrm{GL}(V)$ of the elements of G to invertible linear transformations of V, i.e. elements of the general linear group $\mathrm{GL}(V)$, such that the product in G agrees with composition in $\mathrm{GL}(V)$,

$$D(g)D(h) = D(g \cdot h) \quad \text{for all} \quad g, h \in G. \tag{1.2}$$

Often $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$ meaning that $D(g)$ is given by a $n \times n$ matrix and the product in $\mathrm{GL}(V)$ is simply matrix multiplication. The *dimension* $\dim(D)$ of a representation is the dimension of V.

Similarly to the isomorphism of groups, we consider two representations to be equivalent if they are related by a similarity transformation.

Definition 1.11 (Equivalent representations). The representations $D : G \to \mathrm{GL}(V)$ and $D' : G \to \mathrm{GL}(V')$ of G are *equivalent* (or *isomorphic*), denoted $D \cong D'$, if there exists an invertible linear map $S : V \to V'$ such that

$$D'(g) = SD(g)S^{-1} \quad \text{for any} \quad g \in G. \tag{1.3}$$

Having introduced abstract groups and their representations, we disentangle two aspects of the symmetry of a physical system: the abstract group captures how symmetry transformations compose, while the representation describes how the symmetry transformations act on the system.[2] This separation is very useful in practice, thanks to the fact that there are far fewer abstract groups than conceivable physical systems with symmetries. Understanding the properties of some abstract group teaches us something about all possible systems that share that symmetry group. As we will see later in the setting of Lie groups, one can to a certain extent classify all possible abstract groups with certain properties. Furthermore, for a given abstract group one can then try to classify all its (inequivalent) representations, i.e. all the ways in which the group can be realized as a symmetry group in the system.

For instance, the isospin symmetry of pions in subatomic physics and the rotational symmetry of the hydrogen atom share (almost) the same abstract group. This means that their representations are classified by the same set of quantum numbers (the total (iso)spin and (iso)spin in the z-direction).

First we will focus on representations of finite groups, while the remainder of the course deals with Lie groups, i.e. infinite groups with the structure of a finite-dimensional manifold (like $U(1), \mathrm{SO}(3), \mathrm{SU}(2), \ldots$).

1.1 Finite groups and their representations

When dealing with a finite group $G = \{e, g_1, g_2, \ldots\}$ one may conveniently summarize its structure in a group multiplication table, also known as its *Cayley table*:

$$\begin{array}{c|ccccc} \cdot & e & g_1 & g_2 & \cdots \\ \hline e & e & g_1 & g_2 & \cdots \\ g_1 & g_1 & g_1 \cdot g_1 & g_1 \cdot g_2 & \cdots \\ g_2 & g_2 & g_2 \cdot g_1 & g_2 \cdot g_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{1.4}$$

Note that the closure and inverse axioms of G (axioms (i) and (iv) in Definition 1.1) precisely state that each column and each row contains all elements of the group exactly once. Associativity on the other hand is not so easily visualized in the table.

We can easily deduce using the Cayley table that there exists a unique abstract group of order 3. Indeed, if $G = \{e, a, b\}$ with e the identity, then we know that the Cayley table is of the form

$$\begin{array}{c|ccc} \cdot & e & a & b \\ \hline e & e & a & b \\ a & a & ? & ? \\ b & b & ? & ? \end{array}$$

There is only one way to fill in the question marks such that each column and each row contains

[2] By definition the representations we are considering act as linear transformations on a vector space. So we only cover physical systems with a linear structure in which the symmetries are linear transformations. This is not much of a restriction, since all quantum mechanical systems are of this form.

the three elements e, a, b (à la sudoku!):

$$\begin{array}{c|c|c|c} \cdot & e & a & b \\ \hline e & e & a & b \\ \hline a & a & b & e \\ \hline b & b & e & a \end{array} \tag{1.5}$$

It is easily checked that this multiplication is associative and thus defines a group. In fact, we have already encountered this group as a special case of the cyclic group of Example 1.9 when $n = 3$, i.e. G is isomorphic to the group $\mathbb{Z}_3$ of addition modulo 3 (with isomorphism given by $e \to 0$, $a \to 1$, $b \to 2$ for instance).

Given the abstract group, one can construct representations. Let us start with one-dimensional representations and therefore specify the vector space on which the representation acts to be $V = \mathbb{C}$. Just like any group, $\mathbb{Z}_3$ has a *trivial representation*

$$D_{\text{triv}}(e) = 1\,, \qquad D_{\text{triv}}(a) = 1\,, \qquad D_{\text{triv}}(b) = 1\,. \tag{1.6}$$

A non-trivial one-dimensional representation is given by rotations of the complex plane by $2\pi/3$,

$$D_1(e) = 1\,, \qquad D_1(a) = e^{2\pi i/3}\,, \qquad D_1(b) = e^{4\pi i/3}\,. \tag{1.7}$$

A straightforward computation verifies that these representations indeed obey the multiplication laws indicated in the group multiplication table.

How about higher-dimensional representations? One such representation that can be constructed naturally for any finite group is the regular representation, which we introduce now.

Definition 1.12 (Regular representation). The *regular representation* of a finite group $G = \{g_1, g_2, \ldots, g_n\}$ is obtained by choosing the vector space V to be the one spanned by its group elements,

$$V = \left\{ \sum_{i=1}^{n} \lambda_i |g_i\rangle : \lambda_i \in \mathbb{C} \right\}. \tag{1.8}$$

For $h, g \in G$ the relation

$$D_{\text{reg}}(h)|g\rangle \equiv |h \cdot g\rangle \tag{1.9}$$

defines a representation $D_{\text{reg}} : G \to \text{GL}(V)$, by extending to linear combinations,

$$D_{\text{reg}}(h) \sum_{i=1}^{n} \lambda_i |g_i\rangle = \sum_{i=1}^{n} \lambda_i |h \cdot g_i\rangle. \tag{1.10}$$

The dimension of the regular representation thus equals the order of the group, $\dim(D_{\text{reg}}) = |G|$. Note that in the basis $|g_1\rangle, \ldots, |g_n\rangle$ these vectors correspond to the column vectors

$$|g_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots \qquad , |g_n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

so if the group multiplication table is known, it is easy to construct explicit matrices for the regular representation. Indeed $D(h)$ corresponds to the matrix that has $|h \cdot g_1\rangle, \ldots, |h \cdot g_n\rangle$ as its columns.

In the example of $\mathbb{Z}_3$ in which we order the basis elements as $|e\rangle$, $|a\rangle$, $|b\rangle$ a quick look at the Cayley table (1.5) leads to

$$D_{\text{reg}}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_{\text{reg}}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D_{\text{reg}}(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

You may check explicitly by matrix multiplication that these matrices indeed constitute a representation $D_{\mathrm{reg}} : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$.

Besides constructing the special regular representation, there are other ways to produce higher-dimensional representations. Given two representations D_1 and D_2 of a group one may naturally construct two new representations of the group with larger dimension: their direct sum $D_1 \oplus D_2$ and their tensor product $D_1 \otimes D_2$. To understand their definition, we need to recall the direct sum and tensor product of vector spaces.

Definition 1.13 (Direct sum and tensor product of vector spaces). If V is a vector space with basis $|v_1\rangle, \ldots, |v_n\rangle$ and W is a vector space with basis $|w_1\rangle, \ldots, |w_m\rangle$, then

- the *direct sum* $V \oplus W$ is the vector space with basis

$$|v_1\rangle, \ldots, |v_n\rangle, |w_1\rangle, \ldots, |w_m\rangle;$$

- the *tensor product* $V \otimes W$ is the vector space with basis

$$|v_1, w_1\rangle, \ldots, |v_1, w_m\rangle, |v_2, w_1\rangle, \ldots, |v_2, w_m\rangle, \ \ldots\ , |v_n, w_1\rangle, \ldots, |v_n, w_m\rangle.$$

As you can check (and the notation already suggests), we have that the dimensions of these vector spaces are related by

$$\dim(V \oplus W) = \dim(V) + \dim(W), \qquad \dim(V \otimes W) = \dim(V) \times \dim(W). \tag{1.12}$$

The representations $D_1 \oplus D_2$ and their tensor product $D_1 \otimes D_2$ act precisely on these vector spaces.

Definition 1.14. If $D_1 : G \to GL(V)$ and $D_2 : G \to GL(W)$ are representations of the group G and the bases of V and W are chosen as in Definition 1.13, then

- the *direct sum representation* $D_1 \oplus D_2 : G \to GL(V \oplus W)$ is the representation determined by

$$(D_1 \oplus D_2)(g)\,|v_i\rangle = D_1(g)|v_i\rangle, \qquad (D_1 \oplus D_2)(g)|w_i\rangle = D_2(g)|w_i\rangle; \tag{1.13}$$

- the *tensor representation* $D_1 \otimes D_2 : G \to GL(V \otimes W)$ is the representation determined by

$$(D_1 \otimes D_2)(g)\,|v_i, w_j\rangle = (D_1(g)|v_i\rangle) \otimes (D_2(g)|w_j\rangle). \tag{1.14}$$

In the exercises you will check explicitly that these abstract definitions satisfy the axioms of a representation. The direct sum representation has a less abstract interpretation when examined at the level of the matrices. Since $(D_1 \oplus D_2)(g)$ sends vectors in $V \subset V \oplus W$ to vectors in $V \subset V \oplus W$ and similarly for $W \subset V \oplus W$, the corresponding matrix has the block-diagonal form

$$(D_1 \oplus D_2)(g) = \begin{pmatrix} D_1(g) & \mathbf{0} \\ \mathbf{0} & D_2(g) \end{pmatrix}, \tag{1.15}$$

where $\mathbf{0}$ represents a block of zeroes of appropriate size.

1.2 Irreducible representations

A key role in the classification of representations of a group is played by the so-called irreducible representations, whose definition relies on the notion of an invariant subspace.

Definition 1.15 (Invariant subspace). An *invariant subspace* W of a representation $D : G \to \mathrm{GL}(V)$ is a linear subspace $W \subset V$ such that

$$D(g)w \in W \quad \text{for every} \quad w \in W \text{ and } g \in G. \tag{1.16}$$

Trivially $W = V$ and $W = \{\mathbf{0}\}$ are invariant subspaces, so we are usually more interested in *proper* invariant subspaces W, meaning that $W \subset V$ is invariant and $W \neq \{\mathbf{0}\}$ and $W \neq V$.

Definition 1.16 (Irreducible representation). A representation is *reducible* if it has a proper invariant subspace, and *irreducible* otherwise.

Note in particular, that any one-dimensional representation is irreducible, because a one-dimensional vector space does not have any proper subspaces. The importance of irreducible representations (regularly abbreviated to *irreps*) is that they form the smallest building blocks from which all other representations can be constructed by taking direct sums (see Definition 1.14).

Definition 1.17 (Completely reducible representation). A representation $D : G \to \mathrm{GL}(V)$ is called *completely reducible* if it is equivalent to the direct sum $D_1 \oplus D_2 \oplus \cdots \oplus D_k$ of (not necessarily distinct) irreducible representations $D_1, \ldots, D_k$ of G.

Equivalently this is saying that there exists a change of basis $S : V \to V$ such that $SD(g)S^{-1}$ is block-diagonal for all $g \in G$ and such that the blocks are matrices of irreducible representations of G.

Let us return to the example of the regular representation D_{reg} of $\mathbb{Z}_3$ given in (1.11) and convince ourselves that it is reducible and completely reducible. To see that it is reducible, we should identify a proper invariant subspace. An example is given by the 1-dimensional subspace $W \subset \mathbb{C}^3$ spanned by the vector $(1,1,1)^T$. This vector is mapped to itself by all three matrices of the representation, so W is indeed invariant. In order to see that D_{reg} is completely reducible we observe that the three matrices can be simultaneously diagonalized. Setting $\omega \equiv e^{2\pi i/3}$ the appropriate similarity transform is given by

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \qquad S^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{1.17}$$

because evaluating $D'(g) = SD_{\mathrm{reg}}(g)S^{-1}$ yields

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad D'(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{1.18}$$

Since D' is diagonal, we conclude that the regular representation of $\mathbb{Z}_3$ is equivalent to a direct sum of three one-dimensional irreducible representations, $D_{\mathrm{reg}} \cong D' = D_{\mathrm{triv}} \oplus D_1 \oplus \overline{D_1}$, where D_{reg} is the trivial representation (1.6), D_1 is the complex 1-dimensional representation (1.7) and $\overline{D_1}$ is its complex conjugate.

That this regular representation is completely reducible is not a coincidence, but something that is universally true for representations of finite groups:

Theorem 1.18. Every representation of a finite group is completely reducible.

The proof, given below, relies on examining unitary representations.

Definition 1.19 (Unitary representation). A representation $D : G \to \mathrm{GL}(\mathbb{C}^N)$ is *unitary* if $D(g)^\dagger D(g) = \mathbb{1}$ for all $g \in G$.

In the exercises you will prove the following result, which shows that we may safely restrict ourselves this special family of representations.

Lemma 1.20. Every representation of a finite group is equivalent to a unitary representation.

Proof of Theorem 1.18. Let $D : G \to \mathrm{GL}(V)$ be a representation. By Theorem 1.20 it is equivalent to a unitary representation $D' : G \to \mathrm{GL}(\mathbb{C}^N)$. Let us denote by $\langle v|w\rangle = \sum_{i=1}^N \overline{v}_i w_i$ the standard Hermitian inner product on $\mathbb{C}^N$. If D' is irreducible we are done, so suppose D' is reducible. In particular, it is at least two-dimensional. By Definition 1.16 it has a proper invariant subspace $W \subset V$.

We claim that its orthogonal complement

$$W^\perp = \{v \in \mathbb{C} : \langle v|w\rangle = 0 \text{ for all } w \in W\} \tag{1.19}$$

is invariant as well. We thus need to show that for every $g \in G$ and $v \in W^\perp$ that $D'(g)v \in W^\perp$ as well, or that $\langle D'(g)v|w\rangle = 0$ for all $w \in W$. This we may checked using the unitarity of $D'(g)$:

$$\langle D'(g)v|w\rangle = \langle v|D'(g)^\dagger w\rangle \overset{\text{unitary}}{=} \langle v|D'(g)^{-1}w\rangle = \langle v|D'(g^{-1})w\rangle = 0,$$

where in the last equality we used that $w' = D'(g^{-1})w \in W$ because W is invariant and $\langle v|w'\rangle = 0$ because $v \in W^\perp$.

Let us now choose a basis $w_1, \ldots, w_n$ of W and a basis $v_1, \ldots, v_{N-n}$ of $W^\perp$, which combined form a new basis of $\mathbb{C}^N$. In this new basis $D'(g)$ is block-diagonal, because $\langle v_i|D'(g)w_j\rangle = 0$ and $\langle w_j|D'(g)v_i\rangle = 0$ for all $g \in G$. We conclude that D', and therefore also D, is equivalent to a direct sum of two representations D_1 and D_2 that both have smaller dimension.

We may iterate this procedure for D_1 and D_2 separately, until we are left with only irreducible representations. Since the dimensions of the representations decrease at every step, this stops quickly. $\square$

It is good to keep in mind that Lemma 1.20 and Theorem 1.18 may fail for infinite groups. For instance, the group $(\mathbb{Z}, +)$ of Example 1.3 has the two-dimensional representation D given by

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{1.20}$$

It is reducible, because the subspace spanned by $(1,0)^T$ is invariant. However, it is not completely reducible. If it were, D would be equivalent to the direct-sum of two one-dimensional representations or, equivalently, the matrices $D(x)$ would be simultaneously diagonalizable for all x. But none of the $D(x)$ with $x \neq 0$ are diagonalizable (let alone simultaneously).

1.3 Schur's lemmas

The decomposition into irreducible representations is important for many problems in physics, because the decomposition imposes a strong regidity to the system. We will illustrate this below in a quantum mechanical example, but the abstract formulation is captured by the important Schur's lemmas.

Theorem 1.21 (Schur's lemma - Part I). Let $D_1 : G \to \mathrm{GL}(V_1)$ and $D_2 : G \to \mathrm{GL}(V_2)$ be irreducible representations of a group G, and $A : V_2 \to V_1$ a linear map. If A *intertwines*

D_1 and D_2, meaning that $D_1(g)A = AD_2(g)$ for all $g \in G$, then $A = 0$ or D_1 and D_2 are equivalent.

Proof. The proof is done in two steps.

A is injective or $A = 0$: Let us consider the *kernel* of A, i.e. the subspace $\ker(A) = \{v \in V_2 : Av = 0\} \subset V_2$. For every $g \in G$ we have that

$$w \in \ker(A) \implies D_1(g)Aw = 0 \overset{\text{intertwines}}{\implies} AD_2(g)w = 0 \implies D_2(g)w \in \ker(A), \tag{1.21}$$

which shows that $\ker(A)$ is a (not necessarily proper) invariant subspace of D_2. Since D_2 is irreducible, $\ker(A) = \{0\}$ or $\ker(A) = V_2$, meaning that A is injective or $A = 0$.

A is surjective or $A = 0$: Let us consider the image $\mathrm{im}(A) = \{Av : v \in V_2\} \subset V_1$. For every $g \in G$ we have

$$v \in \mathrm{im}(A) \implies v = Aw \text{ for some } w \in V_2 \tag{1.22}$$
$$\implies D_1(g)v = D_1(g)Aw = AD_2(g)w \tag{1.23}$$
$$\implies D_1(g)v \in \mathrm{im}(A), \tag{1.24}$$

which shows that $\mathrm{im}(A)$ is an invariant subspace of D_1. Since D_1 is irreducible, $\mathrm{im}(A) = V_1$ or $\mathrm{im}(A) = \{0\}$, meaning that A is surjective or $A = 0$.

Together this implies that $A = 0$ or A is invertible. The latter case means that D_1 and D_2 are equivalent, since then $D_1(g) = AD_2(g)A^{-1}$ for all $g \in G$. □

The results is useful if you want to determine whether two different-looking irreducible representations are equivalent: if you can find any non-zero linear map A that intertwines them, then they are equivalent. The second Schur's lemma specializes this result in the case that both representations act on the same linear space $V_1 = V_2$.

Theorem 1.22 (Schur's lemma - Part II). Let $D : G \to \mathrm{GL}(V)$ be a finite-dimensional irreducible representation. If a linear map $A : V \to V$ commutes with all elements of D, i.e. $[D(g), A] = 0$ for all $g \in G$, then A is proportional to the identity.

Proof. If A commutes with all elements of D, then for any $x \in \mathbb{C}$ the same is true for $A - x\mathbb{1}$. According to Theorem 1.21, for every $x \in \mathbb{C}$ we have either that $A - x\mathbb{1} = 0$ or that $A - x\mathbb{1}$ is invertible. The latter condition is equivalent to $\det(A - x\mathbb{1}) \neq 0$. Since $\det(A - x\mathbb{1})$ is a non-constant polynomial (namely the characteristic polynomial of the matrix A), like any non-constant polynomial it must have a root at some $\lambda \in \mathbb{C}$. Hence for $x = \lambda$ the first condition must be satisfied, namely that $A = \lambda\mathbb{1}$. This is precisely what we intended to prove. □

In other words, if you can find any linear map A that is not proportional to the identity and that commutes with a representation D, then D must be reducible.

Let us summarize the restrictions stated in Shur's lemmas for a linear map $A : V \to V$ that commutes with all elements of a completely reducible representation $D : G \to \mathrm{GL}(V)$, so this applies to any representation of a finite group G. Let us consider the matrix A in the basis in which D is block-diagonal. More precisely, let D_1, D_2, D_3 be the inequivalent irreducible representations of G, then we may assume that $D(g)$ takes the form

$$D(g) = \begin{pmatrix} D_{i_1}(g) & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & D_{i_2}(g) & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & D_{i_3}(g) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.25}$$

where the indices $i_1, i_2, \ldots \in \{1, 2, \ldots\}$ may repeat, since an irrep may in general be included more than once in the decomposition. Let us examine the matrix A, whose blocks we accordingly label as

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{31} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $AD(g) = D(g)A$ is equivalent to

$$A_{k\ell} D_{i_\ell}(g) = D_{i_k}(g) A_{k\ell} \qquad \text{for all} g \in G, \tag{1.26}$$

meaning that $A_{k\ell}$ intertwines D_{i_k} and D_{i_ℓ}. If $i_k \neq i_\ell$ then D_{i_k} and D_{i_ℓ} are inequivalent and the lemma implies that $A_{k\ell} = 0$. If $i_k = i_\ell$ then the second lemma implies that there is a complex number $a_{k\ell}$ such that $A_{k\ell} = a_{k\ell}\mathbb{1}$. In other words, there is only very littly freedom left in A, even though the dimension of the vector space involved can be really large.

For example, if $D = D_1 \oplus D_2 \oplus D_3$ decomposes into the three inequivalent irreducible representations, then A is diagonal and determined by three complex numbers a_{11}, a_{22}, a_{33},

$$D(g) = \begin{pmatrix} D_1(g) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_2(g) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_3(g) \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} a_{11}\mathbb{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a_{22}\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & a_{33}\mathbb{1} \end{pmatrix}. \tag{1.27}$$

Or if $D = D_1 \oplus D_1 \oplus D_2 \oplus D_2$ decomposes into two pairs or irreps, then A is determined by eight complex numbers,

$$D(g) = \begin{pmatrix} D_1(g) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1(g) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_2(g) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_2(g) \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} a_{11}\mathbb{1} & a_{12}\mathbb{1} & \mathbf{0} & \mathbf{0} \\ a_{21}\mathbb{1} & a_{22}\mathbb{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & a_{33}\mathbb{1} & a_{34}\mathbb{1} \\ \mathbf{0} & \mathbf{0} & a_{43}\mathbb{1} & a_{44}\mathbb{1} \end{pmatrix}. \tag{1.28}$$

This situation naturally arises in quantum mechanical systems, where the vector space V is the Hilbert space of states of the system. If the system possesses a symmetry, then the symmetry group G acts on the Hilbert space V via a unitary representation $D : G \to GL(V)$. Observables, like the Hamiltonian, are encoded in hermitian operators $\hat{O} : V \to V$ that commute with the action of the symmetry group, i.e. $[D(g), \hat{O}] = 0$. In other words, $\hat{O}$ is precisely of the form of the operator A described above. So if we choose a basis for the Hilbert space such that the representation D is block-diagonal as in (1.25), then the conclusions of Shur's lemmas apply to the matrix elements of $\hat{O}$.

In particular, in the presence of symmetries time evolution of a quantum mechanical system, encoded in the Hamiltonian $\hat{H}$, is heavily constrained. Most of the matrix elements of $\hat{H}$ are required to vanish, as can be seen from the examples (1.27) and (1.28). If an irreducible representation occurs only once in the decomposition of the Hilbert space like in (1.27), then any state transforming under that irrep will necessarily be an eigenstate of $\hat{H}$. Another consequence is that the energy spectrum will necessarily come with multiplicities depending on the dimensions of the irreducible representations.

2 Lie groups and Lie algebras

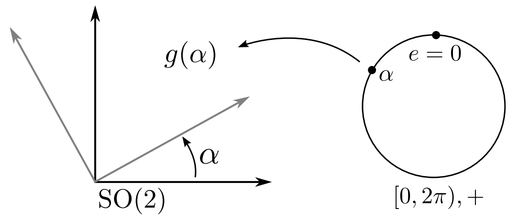
Informally, if the elements of a group G depend smoothly, $g = g(\alpha)$, on a set of real parameters α_a, $a = 1, \dots, n$, then G is called a Lie group of dimensions n. A prototypical example is the rotation group

$$\mathrm{SO}(2) = \left\{ g \in \mathbb{R}^{2\times 2} : g^T g = \mathbb{1},\, \det g = 1 \right\}$$

of the two-dimensional plane, consisting of orthogonal 2×2 matrices with determinant 1. The group elements $g \in \mathrm{SO}(2)$ can be parametrized by a single real parameter, the (counterclockwise) rotation angle α,

$$g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{2.1}$$

Note that this parametrization at the same time provides an isomorphism $\alpha \mapsto g(\alpha)$ between the group $[0, 2\pi)$ with addition modulo 2π and SO(2), meaning that they represent the same abstract group. The set has the topology of a circle (since we identify angles 0 and 2π), and is thus a 1-dimensional smooth manifold.



Formally, a Lie group is defined as follows:

> **Definition 2.1 (Lie group).** A Lie group of dimension n is a group $(G, \cdot)$ whose set G possesses the additional structure of a real n-dimensional smooth manifold, such that the group structures, composition $(g, h) \to g \cdot h$ and inversion $g \to g^{-1}$, are smooth maps.

Recall that, informally speaking, a set G is an n-dimensional smooth manifold if the neighbourhood of any element can be smoothly parametrized by n parameters $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. In terms of such a parametrization the composition and inversion correspond locally to maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^n$ respectively, which we require to be smooth in the usual sense.

In practice, we will exclusively work with a special class of Lie groups, the matrix Lie groups.

> **Definition 2.2 (Matrix Lie group).** A *matrix Lie group* is a Lie group whose elements are (complex or real) matrices and the group operation is matrix multiplication.

This is not really a restriction since almost all Lie groups that are of relevance in physics are (isomorphic to) matrix Lie groups[3]. Note that the dimension of the matrices is in general unrelated to the dimension of the Lie group (for instance, the 2×2 matrices of $SO(2)$ comprise a 1-dimensional Lie group).

2.1 Generators of a Lie group

Let us parametrize the elements $g(\alpha)$ in a neighbourhood of the identity of the Lie group with the convention that $\alpha_a = 0$ corresponds to the identity, $g(0) = e = \mathbb{1}$. It turns out that a lot of information on the Lie group can be recovered by studying the infinitesimal vicinity of the identity. The generators play an important role in this.

[3]For the mathematically inclined reader: it follows from the Peter-Weyl theorem that any compact Lie group is isomorphic to a matrix Lie group. On the level of the Lie Algebra, one has an even stronger result (Ado's theorem) stating that every Lie algebra of a Lie group is isomorphic to a matrix Lie algebra.

> **Definition 2.3 (Generators of a Lie group).** The n *generators* of an n-dimensional Lie group are defined as
>
> $$\xi_a = \frac{\partial}{\partial \alpha_a} g(\alpha)\big|_{\alpha=0}, \qquad a = 1, \ldots, n. \tag{2.2}$$

Of course the generators depend on the chosen parametrization, but the generators of one parametrization are necessarily linear combinations of those of another parametrization.[4] Put differently, the vector space spanned by the generators is independent of the parametrization. Since the parametrization is smooth, we may approximate elements in the close vicinity of the identity via the Taylor expansion

$$g(\alpha) = \mathbb{1} + \alpha_a \, \xi_a + O(\alpha^2), \tag{2.3}$$

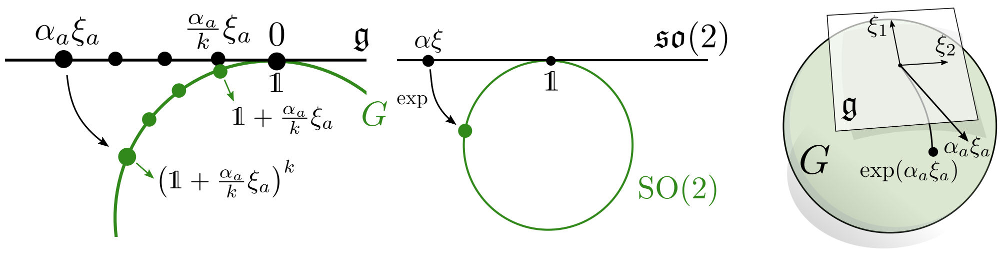
where the Einstein summation convention to sum over repeated indices is understood: $x_a y_a := \sum_{a=1}^{n} x_a y_a$.

Applying (2.2) to (2.1) shows that SO(2) possesses a single generator given by

$$\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

There is a natural way to recover group elements from linear combinations of the generators. Notice that if the vector α_a is not particularly small, then the matrix $\mathbb{1} + \alpha_a \xi_a$ does *not* give an element of the group, because we have neglected the contributions from the higher order terms in (2.3). However, we can get a better approximation of a group element by choosing a large integer k and considering the matrix $\mathbb{1} + \frac{\alpha_a}{k}\xi_a$, which we can then compose with itself k times. In the limit of $k \to \infty$ this necessarily results in a group element. This motivates the introduction of the exponential map.

> **Definition 2.4 (The exponential map).** For any linear combination $\alpha_a \xi_a$ of generators, the exponential $e^{\alpha_a \xi_a}$ is the group element given by
>
> $$e^{\alpha_a \xi_a} = \lim_{k \to \infty} \left(\mathbb{1} + \alpha_a \frac{\xi_a}{k} \right)^k = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\alpha_a \xi_a \right)^m. \tag{2.5}$$



We may check that the exponential map of multiples of (2.4) reproduces the parametrization (2.1). To see this, note that for any integer n

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n} = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n+1} = (-1)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

[4] Note that in the physics literature one often uses a convention with an additional imaginary unit in the definition of the generator, i.e. $g(\epsilon) = \mathbb{1} + i\,\epsilon_a\,\tilde{\xi}_a + \ldots$ and $\tilde{\xi}_a \equiv -iX_a$. The reason is that most Lie groups encountered in physics are unitary, in which case the generators $\tilde{\xi}_a$ are Hermitian while the ξ_a are antihermitian (see section 2.4 below). We will stick to the mathematical convention without the i, which is more convenient when considering the algebraic structure.

Then we can compute $e^{\alpha X}$ via the series expansion (2.5),

$$e^{\alpha\xi} = \sum_{n=0}^{\infty} \frac{1}{n!}(\alpha\xi)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\alpha^{2n}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\alpha^{2n+1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$
$$= \cos\alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin\alpha\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}. \tag{2.6}$$

Note that it is a bit of a coincidence that the parametrization $e^{\alpha\xi}$ reproduces our original parametrization $g(\alpha)$, as we could have started with a different parametrization $g(\alpha)$. In any case we see that the exponential map of SO(2) covers the whole group. This turns out to be true for almost all the Lie groups that we will consider in this course.

Definition 2.5 (compactness). A matrix Lie group is *compact* if the matrix entries of all group elements are bounded, i.e. smaller in absolute value than some fixed $C > 0$.

Definition 2.6 (connectedness). A Lie group is *connected* if every element can be reached from the identity by a continuous path.

Note that $SO(2)$ is connected and compact. The group $O(2)$ of 2×2 orthogonal real matrices is compact but not connected: these matrices can have determinant ±1, but there is no way to reach a matrix with determinant -1 via a continuous path from the identity (which has determinant 1).

Fact 2.7. If G is a connected and compact matrix Lie group, then any element $g \in G$ can be obtained as the exponential of a linear combination of generators, $g = e^{\alpha_a\xi_a}$.

If all generators mutually commute, i.e. $[\xi_a, \xi_b] = 0$ for all $a, b = 1, \ldots n$, then it is easily seen from the definition of the exponential map that $A = \alpha_a\xi_a$ and $B = \beta_a\xi_a$ satisfy

$$e^A \cdot e^B = e^B \cdot e^A = e^{A+B}. \qquad \text{(if } [\xi_a,\xi_b]=0) \tag{2.7}$$

In other words the Lie group is necessarily Abelian and the group multiplication is equivalent to addition of the corresponding generators.

For general non-Abelian Lie groups however we know that there are $A = \alpha_a\xi_a$ and $B = \beta_a\xi_a$ that fail this relation, i.e.

$$e^A e^B \neq e^{(A+B)} \neq e^A e^B. \tag{2.8}$$

Note that we do have

$$e^{\lambda A} e^{\mu A} = e^{(\lambda+\mu)A} \quad \text{for } \lambda, \mu \in \mathbb{R}. \tag{2.9}$$

In particular, setting $\lambda = 1$ and $\mu = -1$ we find for the group inverse of e^A,

$$e^A e^{-A} = e^{0\,A} = \mathbb{1}, \qquad \Longrightarrow \qquad (e^A)^{-1} = e^{-A}. \tag{2.10}$$

Another important property of the exponential map is that it relates the determinant in the group to the trace of the generators,[5]

$$\det e^A = e^{\operatorname{tr} A}. \tag{2.11}$$

[5]If you would like to see this proved, recall that every (complex) matrix A can be put in Jordan normal form J by an invertible matrix S, $A = SJS^{-1}$. Then $e^A = e^{SJS^{-1}} = Se^JS^{-1}$, so that $\det e^A = \det e^J$ while also $\operatorname{tr} A = \operatorname{tr} J$. It remains to show that $\det e^J = e^{\operatorname{tr} J}$. Since J is upper triangular, by the definition of the exponential the same is true for e^J, which has its diagonal entries equal to the exponential of the diagonal entries of J. So $\det e^J$ is equal to the product of the diagonal entries of e^J, which is $e^{\operatorname{tr} J}$.

2.2 Lie algebras

As we will see below, the generators ξ_a span a vector space $\mathfrak{g} = \{\alpha_a \xi_a : \alpha_a \in \mathbb{R}\}$ with a particular structure, known as a Lie algebra.

Definition 2.8 (Lie algebra). A Lie algebra $\mathfrak{g}$ is a vector space together with a "Lie bracket" $[\cdot,\cdot]$ that satisfies

(i) *closure*: if $X, Y \in \mathfrak{g}$ then $[X, Y] \in \mathfrak{g}$;

(ii) *linearity*: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{C}$;

(iii) *antisymmetry*: $[X, Y] = -[Y, X]$ for $X, Y \in \mathfrak{g}$;

(iv) *Jacobi identity*: $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ for $X, Y, Z \in \mathfrak{g}$.

When working with matrices, like in our case, we can take the Lie bracket to be defined by the commutator of the matrix multiplication

$$[X, Y] \coloneqq XY - YX. \tag{2.12}$$

It is then easy to see that this bracket satisfies linearity and antisymmetry. In the exercises you will verify that it also satisfies the Jacobi identity by expanding the commutators. Properties (ii)-(iv) thus follow directly from working with the generators of a matrix Lie group. We will now see that the closure (i) of the generators X_a follows from the closure of the group multiplication.

Let $A, B \in \mathfrak{g}$, i.e. A and B are linear combinations of the generators ξ_a, and let $\epsilon \in \mathbb{R}$, then typically

$$e^{\epsilon A} e^{\epsilon B} \neq e^{\epsilon B} e^{\epsilon A}, \quad \text{or equivalently} \quad e^{-\epsilon A} e^{-\epsilon B} e^{\epsilon A} e^{\epsilon B} \neq \mathbb{1}. \tag{2.13}$$

However, by the closure of the group $e^{-\epsilon A} e^{-\epsilon B} e^{\epsilon A} e^{\epsilon B}$ must be some group element. Let us determine this group element for small ϵ. Expanding to second order in ϵ,

$$\begin{aligned} &e^{-\epsilon A} e^{-\epsilon B} e^{\epsilon A} e^{\epsilon B} \\ &\quad = \left(1 - \epsilon A + \tfrac{1}{2}\epsilon^2 A^2\right)\left(1 - \epsilon B + \tfrac{1}{2}\epsilon^2 B^2\right)\left(1 + \epsilon A + \tfrac{1}{2}\epsilon^2 A^2\right)\left(1 + \epsilon B + \tfrac{1}{2}\epsilon^2 B^2\right) + O(\epsilon^3) \end{aligned}$$

The terms proportional to ϵA, ϵB, $\epsilon^2 A^2$, $\epsilon^2 B^2$ all cancel, leaving

$$e^{-\epsilon A} e^{-\epsilon B} e^{\epsilon A} e^{\epsilon B} = \mathbb{1} + \epsilon^2 (AB - BA) + O(\epsilon^3). \tag{2.14}$$

Setting $\epsilon = \sqrt{t}$ we have found a familiy of group elements $g(t) = e^{-\sqrt{t}A} e^{-\sqrt{t}B} e^{\sqrt{t}A} e^{\sqrt{t}B}$ parametrized by $t \geq 0$ that is differentiable at $t = 0$ where $g(0) = \mathbb{1}$, so its derivative

$$\frac{\partial}{\partial t} g(t)\big|_{t=0} = [A, B]$$

is a generator. This shows that the closure property (i) holds.

We conclude that to any Lie group G we can associate a Lie algebra $\mathfrak{g}$ spanned by the generators of G. In mathematical terms the linear space spanned by the generators is the tangent space of the manifold G at the identity $e \in G$, so we find that the tangent space of a Lie group G naturally possesses a Lie algebra structure $\mathfrak{g}$. By convention, we will use Fraktur font to denote Lie algebras: the Lie algebras of the Lie groups $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, ..., are denoted by $\mathfrak{so}(N)$, $\mathfrak{su}(N)$, ...

Since the generators ξ_a form a basis of the Lie algebra, the commutator of two generators $[\xi_a, \xi_b]$ is a real linear combination of generators again. The real coefficients are called the structure constants.

Definition 2.9 (Structure constants). Given a Lie group G with generators ξ_a, $a = 1, \ldots, n$. The *structure constants* f_{abc} of the Lie algebra $\mathfrak{g}$ spanned by ξ_a are defined through the relation

$$[\xi_a, \xi_b] = f_{abc}\xi_c. \tag{2.15}$$

The properties of the structure constants are summarized as follows:

1) The structure constants depend on the basis of generators chosen (we will investigate how the structure constants change under a basis transformation in Section 4.2).

2) The anti-symmetry of the commutator implies that the structure constants are anti-symmetric in the first two indices

$$f_{abc} = -f_{bac}\,. \tag{2.16}$$

3) The Jacobi identity

$$[\,\xi_a\,,\,[\,\xi_b\,,\,\xi_c\,]\,] + [\,\xi_b\,,\,[\,\xi_c\,,\,\xi_a\,]\,] + [\,\xi_c\,,\,[\,\xi_a\,,\,\xi_b\,]\,] = 0 \tag{2.17}$$

can be equivalently stated as a relation among the structure constants:

$$f_{bcd}\, f_{ade} + f_{abd}\, f_{cde} + f_{cad}\, f_{bde} = 0\,. \tag{2.18}$$

4) The structure constants are sufficient to compute the product of the group elements exactly. This follows from the Baker-Campbell-Hausdorff formula for exponentials of matrices

$$e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\text{more multi-commutators}}\,. \tag{2.19}$$

The multi-commutators can again be expressed in terms of the structure constants.

2.3 Representations of Lie groups and Lie algebras

Recall that a representation D of a group G on $\mathbb{C}^N$ is a mapping $G \to \mathrm{GL}(\mathbb{C}^N)$ that is compatible with the group structure of G, namely $D(g \cdot h) = D(g)D(h)$ for all $g, h \in G$. In the case of a Lie group G one requires in addition that D is smooth, meaning that if the group elements $g(\alpha)$ are smoothly parametrized by α_a then the matrix $D(g(\alpha))$ should depend smoothly on α_a as well. Analogously we can introduce the concept of a representation of a Lie algebras.

Definition 2.10 (Representation of a Lie algebra). An N-dimensional representation $\mathfrak{D}$ of a lie algebra $\mathfrak{g}$ is a linear map $\mathfrak{D} : \mathfrak{g} \to \mathbb{C}^{N\times N}$ that assigns an $N \times N$ matrix $\mathfrak{D}(X)$ to every $X \in \mathfrak{g}$ and preserves the Lie bracket, i.e. $\mathfrak{D}([X, Y]) = [\mathfrak{D}(X), \mathfrak{D}(Y)]$ for all $X, Y \in \mathfrak{g}$.

Representations of a Lie group and a Lie algebra are closely related. Indeed, if D is an N-dimensional representation of G we can consider the generators of G in the representation D given by

$$X_a = \frac{\partial}{\partial \alpha_a} D(e^{\alpha_a \xi_a})\big|_{\alpha=0}, \qquad a = 1, \ldots, n, \tag{2.20}$$

or, equivalently,

$$D(\mathbb{1} + \alpha_a\xi_a + O(\alpha^2)) = \mathbb{1} + \alpha_a X_a + O(\alpha^2). \tag{2.21}$$

We may then introduce a corresponding N-dimensional representation of $\mathfrak{g}$ by setting $\mathfrak{D}(\xi_a) = X_a$ for $a = 1, \ldots, n$, and more generally $\mathfrak{D}(\alpha_a\xi_a) = \alpha_a X_a$. It satisfies

$$D(e^A) = e^{\mathfrak{D}(A)} \qquad \text{for every } A \in \mathfrak{g}, \tag{2.22}$$

because for any $\alpha_a \in \mathbb{R}^n$ we have

$$D(e^{\alpha_a \xi_a}) \overset{(2.5)}{=} \lim_{k\to\infty} D\left(\mathbb{1} + \frac{\alpha_a}{k}\xi_a\right)^k \overset{(2.21)}{=} \lim_{k\to\infty} \left(\mathbb{1} + \frac{\alpha_a}{k}X_a\right)^k \overset{(2.5)}{=} e^{\alpha_a X_a}.$$

We claim that then indeed $\mathfrak{D}([A,B]) = [\mathfrak{D}(A), \mathfrak{D}(B)]$. On the one hand we have for $\epsilon \in \mathbb{R}$ that

$$\begin{aligned} e^{-\epsilon\mathfrak{D}(A)}e^{-\epsilon\mathfrak{D}(B)}e^{\epsilon\mathfrak{D}(A)}e^{\epsilon\mathfrak{D}(B)} &\overset{(2.22)}{=} D(e^{-\epsilon A})D(e^{-\epsilon B})D(e^{\epsilon A})D(e^{\epsilon B}) \\ &= D(e^{-\epsilon A}e^{-\epsilon B}e^{\epsilon A}e^{\epsilon B}) \\ &\overset{(2.14)}{=} D\left(\mathbb{1} + \epsilon^2[A,B] + O(\epsilon^3)\right) \\ &\overset{(2.21)}{=} \mathbb{1} + \epsilon^2\mathfrak{D}([A,B]) + O(\epsilon^3). \end{aligned}$$

On the other hand, by applying (2.14) directly we also have

$$e^{-\epsilon\mathfrak{D}(A)}e^{-\epsilon\mathfrak{D}(B)}e^{\epsilon\mathfrak{D}(A)}e^{\epsilon\mathfrak{D}(B)} = \mathbb{1} + \epsilon^2[\mathfrak{D}(A), \mathfrak{D}(B)] + O(\epsilon^3).$$

We conclude that $\mathfrak{D}$ indeed satisfies the requirements of a representation of $\mathfrak{g}$.

Importantly, the corresponding generators $X_1, \ldots, X_n$ satisfy the commutation relations $[X_a, X_b] = f_{abc}X_c$ in terms of the structure constants f_{abc} of the Lie group, because

$$[X_a, X_b] = [\mathfrak{D}(\xi_a), \mathfrak{D}(\xi_b)] = \mathfrak{D}([\xi_a, \xi_b]) \overset{(2.15)}{=} \mathfrak{D}(f_{abc}\xi_c) = f_{abc}X_c. \tag{2.23}$$

Conversely, any collection of $N \times N$ matrices $X_1, \ldots, X_n$ satisfying

$$[X_a, X_b] = f_{abc}X_c \tag{2.24}$$

determines a representation of $\mathfrak{g}$. And one can find a corresponding representation of the Lie group by considering the exponential (2.22).[6] The possibility to go back an forth between Lie groups and Lie algebras via the exponential map is very powerful, because Lie algebras are often much easier to work with.

We finish this chapter with a discussion of several explicit families of Lie groups and their Lie algebras. Two families of Lie groups, the *special unitary groups* $\mathrm{SU}(N)$ and *special orthogonal groups* $\mathrm{SO}(N)$, play an important role in physics. We also include the definition of the less familiar family of *compact symplectic groups* $\mathrm{Sp}(N)$, because together $\mathrm{SU}(N)$, $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ form the only infinite families of the (yet to be introduced) simple compact Lie groups, as we will see in the last chapter of these lecture notes.

2.4 The special unitary group $\mathrm{SU}(N)$

For $N \geq 2$, the special unitary group in N dimensions is

$$\mathrm{SU}(N) = \left\{U \in \mathbb{C}^{N\times N} : U^\dagger U = \mathbb{1}, \det U = 1\right\}. \tag{2.25}$$

Its importance as a symmetry group stems from the fact that it leaves invariant the N-dimensional scalar product familiar from quantum mechanics,

$$\langle\, v \,|\, w \,\rangle \equiv v_a^* w_a, \qquad v, w \in \mathbb{C}^N, \tag{2.26}$$

[6]Warning: This does not always work as we will see later. Often a group element g can be written as $g = e^A$ in multiple ways (e.g. $e^{2\pi\xi} = \mathbb{1} = e^{0\xi}$ in our example of SO(2)), so $D(e^A) := e^{\mathfrak{D}(A)}$ is only well-defined if $e^{\mathfrak{D}(A)}$ is independent of the choice of A.

where we used the summation convention. Indeed, we observe that for any $U \in \mathrm{SU}(N)$,

$$\langle Uv|Uw\rangle = (Uv)^*_a(Uw)_a = U^*_{ab}v^*_b\, U_{ac}w_c = v^*_b(U^\dagger)_{ba}U_{ac}w_c \overset{U^\dagger U=\mathbb{1}}{=} v^*_b\,\delta_{bc}\,w_c = \langle v|w\rangle. \tag{2.27}$$

Note that we have not used that $\det U = 1$. The most general linear transformations preserving the inner product thus form the larger unitary group

$$\mathrm{U}(N) = \left\{U \in \mathbb{C}^{N\times N} : U^\dagger U = \mathbb{1}\right\}. \tag{2.28}$$

The determinant of a unitary matrix $U \in \mathrm{U}(N)$ necessarily satisfies $|\det U| = 1$. Therefore $U(N)$ and $SU(N)$ differ only by a phase: any unitary matrix $U \in \mathrm{U}(N)$ can be written as

$$U = e^{i\alpha}U', \quad e^{iN\alpha} = \det U, \quad U' \in \mathrm{SU}(N). \tag{2.29}$$

Since this phase commutes with all elements of the group it does not play a role in understanding the algebraic structure of unitary matrices.[7] For this reason we concentrate on $\mathrm{SU}(N)$ rather than $\mathrm{U}(N)$.

In order to determine the corresponding Lie algebra we need to translate the conditions (2.25) for the Lie group elements into conditions on the generators. Writing an arbitrary group element in terms of the exponential map, $U(\alpha) \equiv e^{\alpha_a\xi_a}$, one has $U^\dagger = e^{\alpha_a\xi_a^\dagger}$ and $U^{-1} = e^{-\alpha_a\xi_a}$. The condition $U^\dagger = U^{-1}$ then implies that the generators are antihermitian $\xi_a^\dagger = -\xi_a$. In addition, $\det U = 1$ imposes that the generators must be traceless, $\mathrm{tr}\,\xi_a$=0. This property follows from

$$\log\,\det U = 0 \overset{!}{=} \mathrm{tr}\,\log\, e^{\alpha_a\xi_a} = \alpha_a\,\mathrm{tr}\,\xi_a\,. \tag{2.30}$$

Since this must hold for any value of the α_a the generators must be traceless. Thus the Lie algebra $\mathfrak{su}(N)$ consists of all antihermitian, trace-free matrices,

$$\mathfrak{su}(N) = \left\{X \in \mathbb{C}^{N\times N} : X^\dagger = -X,\, \mathrm{tr}\,X = 0\right\}. \tag{2.31}$$

The number of generators can be deduced from the following counting argument: a $N \times N$ matrix with complex entries has $2N^2$ parameters. The condition that the matrix is antihermitian imposes N^2 constraints. Requiring that the matrices are trace-free removes an additional generator (the one corresponding to global phase rotations). Hence

$$\dim \mathfrak{su}(N) = N^2 - 1\,. \tag{2.32}$$

2.5 The special orthogonal group $\mathrm{SO}(N)$

For $N \geq 2$, the special orthogonal group $\mathrm{SO}(N)$ is the real analogue of the special unitary group,

$$\mathrm{SO}(N) = \left\{\Lambda \in \mathbb{R}^{N\times N} : \Lambda^T\Lambda = \mathbb{1},\, \det\Lambda = 1\right\}. \tag{2.33}$$

This time it is the Euclidean scalar product on $\mathbb{R}^N$ that is preserved,

$$\langle\, v\,,\, w\,\rangle \equiv v_a\, w_a, \tag{2.34}$$

since for any $\Lambda \in \mathrm{SO}(N)$ we have

$$\langle \Lambda v, \Lambda w\rangle = (\Lambda v)_a(\Lambda w)_a = v_b(\Lambda^T)_{ba}\Lambda_{ac}w_c \overset{\Lambda^T\Lambda=\mathbb{1}}{=} v_b\,\delta_{bc}\,w_c = \langle v, w\rangle. \tag{2.35}$$

Again $\mathrm{SO}(N)$ is not the largest group of matrices preserving the inner product, because we have not used that $\det\Lambda = 1$. The largest group $\mathrm{O}(N) = \left\{\Lambda \in \mathbb{R}^{N\times N} : \Lambda^T\Lambda = \mathbb{1}\right\}$ is that of all orthogonal $N \times N$ matrices including reflections. Since any orthogonal matrix has determinant ± 1,

[7] We will see later that the absence of this phase in $SU(N)$ entails that the Lie algebra $\mathfrak{su}(N)$ is *simple* (in a precise sense), while $\mathfrak{u}(N)$ is not.

O(N) really consists of two disjoint components, one for each sign of the determinant. The component containing the identity matrix is precisely SO(N). We will focus on the latter because it is connected (and compact), and therefore any element can be obtained via the exponential map.

For any $\Lambda \in \mathrm{SO}(N)$ we may write $\Lambda = e^{\alpha_a \xi_a}$ with ξ_a real $N \times N$ matrices. Then $\Lambda^{\mathrm{T}} = e^{\alpha_a \xi_a^{\mathrm{T}}}$ and $\Lambda^{-1} = e^{-\alpha_a \xi_a}$. Thus $\Lambda^{\mathrm{T}} = \Lambda^{-1}$ implies that the generators of the Lie algebra $\mathfrak{so}(N)$ are antisymmetric martrices $\xi_a^{\mathrm{T}} = -\xi_a$. The restriction on the determinant implies that the generators are traceless, but this already follows from antisymmetry. Hence we find that

$$\mathfrak{so}(N) = \left\{ X \in \mathbb{R}^{N \times N} : X^{\mathrm{T}} = -X \right\}. \tag{2.36}$$

The number of generators follows from counting the independent entries in an antisymmetric $N \times N$ matrix and is given by

$$\dim \mathfrak{so}(N) = \frac{1}{2} N(N-1). \tag{2.37}$$

2.6 The compact symplectic group Sp(N)

For $N \geq 1$, the compact symplectic group $\mathrm{Sp}(N) \subset \mathrm{SU}(2N)$ consists of the unitary $2N \times 2N$ matrices that leave invariant the skew-symmetric scalar product

$$(\, v \,,\, w \,) \equiv \sum_{a=1}^{2N} v_a \, J_{ab} \, w_b \,, \qquad J = \begin{bmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{bmatrix}. \tag{2.38}$$

Hence it satisfies

$$\mathrm{Sp}(N) = \left\{ U \in \mathbb{C}^{2N \times 2N} : U^\dagger U = \mathbb{1},\, U^T J U = J \right\}. \tag{2.39}$$

The last condition can be equivalently written as $U^{-1} = J^{-1} U^{\mathrm{T}} J$. In terms of the exponential map this means that

$$e^{-\alpha_a \xi_a} = J^{-1} e^{\alpha_a \xi_a^{\mathrm{T}}} J = e^{\alpha_a J^{-1} \xi_a^{\mathrm{T}} J}, \tag{2.40}$$

where the last equality follows easily from the power series expansion of the exponential. Since $J^{-1} = -J$, it follows that the generators of the Lie algebra satisfy

$$\xi_a^{\mathrm{T}} = J \xi_a J. \tag{2.41}$$

Together with the fact that the generators have to be antihermitian for U to be unitary, this leads us to identify the Lie algebra as

$$\mathfrak{sp}(N) = \left\{ X \in \mathbb{C}^{2N \times 2N} : X^\dagger = -X,\, X^T = JXJ \right\}. \tag{2.42}$$

With some work one may deduce from these conditions that $\mathfrak{sp}(N)$ has dimensions $N(2N+1)$.

3 Irreducible representations of $\mathfrak{su}(2)$

The exponential map provides a connection between Lie groups and the underlying algebra. As a consequence we can construct representations of the Lie group from representations of the corresponding Lie algebra. The goal of this section is to construct all finite-dimensional irreducible representations of the Lie algebra $\mathfrak{su}(2)$.

3.1 The defining representation of $\mathfrak{su}(2)$

Whenever we have a matrix Lie group G (recall Definition 2.2) and the corresponding Lie algebra $\mathfrak{g}$, we get one representation for free. The *defining representation* $D(g) = g$ for $g \in G$ obviously satisfies the requirements of a Lie group representation. The corresponding defining representation of the Lie algebra $\mathfrak{g}$ is $\mathfrak{D}(X) = X$ for $X \in \mathfrak{g}$.

Let us examine the defining representation of $\mathfrak{su}(2)$, also called the *fundamental representation*[8], which we introduced in Section 2.4,

$$\mathfrak{su}(2) = \left\{ X \in \mathbb{C}^{2\times 2} : X^\dagger = -X, \, \mathrm{tr}\, X = 0 \right\}. \tag{3.1}$$

A basis of traceless Hermitian matrices, well-known from quantum mechanics, is given by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2}$$

satisfying

$$\sigma_a^\dagger = \sigma_a, \qquad \mathrm{tr}\, \sigma_a = 0, \qquad \sigma_a \, \sigma_b = \delta_{ab} \, \mathbb{1} + i\epsilon_{abc} \, \sigma_c. \tag{3.3}$$

Here ϵ_{abc} is the completely anti-symmetric ϵ-tensor with $\epsilon_{123} = 1$. To obtain a basis of antihermitian traceless matrices, we then only have to multiply the Pauli matrices by i. In particular, if we set

$$\xi_a = -\tfrac{i}{2}\sigma_a \tag{3.4}$$

then

$$[\, \xi_a \, , \, \xi_b \,] = \epsilon_{abc} \, \xi_c \, , \qquad (a, b, c = 1, 2, 3). \tag{3.5}$$

It follows that (with this choice of basis) the structure constants of $\mathfrak{su}(2)$ are $f_{abc} = \epsilon_{abc}$.

3.2 Irreducible representations of $\mathfrak{su}(2)$ from the highest-weight construction

Recall from (2.24) that every set of three matrices X_1, X_2, X_3 satisfying

$$[X_a, X_b] = \epsilon_{abc} X_c \qquad (a, b, c = 1, 2, 3) \tag{3.6}$$

constitutes another representation of $\mathfrak{su}(2)$.

> **Fact 3.1.** Just like in the case of finite groups, every representation of a compact Lie group is equivalent to a unitary representation.

We have seen in Section 2.4 that the generators in a unitary representation are antihermitian. Since $\mathrm{SU}(2)$ is compact, it is thus sufficient to look for antihermitian matrices X_1, X_2, X_3 satisfying (3.6).

Recall that a representation $D : G \to \mathrm{GL}(V)$ is irreducible if it has no proper invariant subspace $W \subset V$, i.e. a subspace such that $D(g)w \in W$ for all $w \in W$ and $g \in G$. An invariant subspace W can be equivalently characterized in terms of the Lie algebra $\mathfrak{g}$.

[8] Although there is no confusion possible in the case of $\mathfrak{su}(2)$, where everyone agrees that the fundamental representation is the two-dimensional defining representation, there are some inconsistencies in the literature in the terminology "fundamental representation" for other Lie groups. For this reason we will stick with using defining representation.

Definition 3.2 (Invariant subspace of a Lie algebra representation). A linear subspace $W \subset \mathbb{C}^N$ is an invariant subspace of a Lie algebra representation $\mathfrak{D} : \mathfrak{g} \to \mathbb{C}^{N\times N}$ if $Xw \in W$ for all $X \in \mathfrak{g}$ and $w \in W$.

Of course, it is sufficient to check that $X_a w \in W$ for all $w \in W$ and the n generators X_a of the representation. An irreducible representation of $\mathfrak{su}(2)$ therefore corresponds to a set of $N \times N$ antihermitian matrices satisfying (3.6) that do not leave any proper subspace $W \subset \mathbb{C}^N$ invariant.

Such representations can be constructed systematically via the *highest-weight construction*. The first step in the construction diagonalizes as many generators as possible. Recall the following fundamental fact in linear algebra.

Fact 3.3. If $A_1, \ldots, A_k$ are hermitian, antihermitian or unitary matrices, then they are simultaneously diagonalizable if and only if they all commute, i.e. $[A_i, A_j] = 0$ for all $i, j = 1, \ldots, k$.

For $\mathfrak{su}(2)$ this implies that only one generator may be taken diagonal, which without loss of generality we take to be X_3. Since X_3 is antihermitian,

$$J_3 \equiv i X_3 \tag{3.7}$$

is a real diagonal $N \times N$ matrix. The remaining generators X_1 and X_2 are recast into raising and lowering operators

$$J_\pm \equiv \frac{1}{\sqrt{2}} (J_1 \pm i\, J_2) = \frac{i}{\sqrt{2}} (X_1 \pm i\, X_2)\,, \qquad J_a \equiv i X_a. \tag{3.8}$$

Since $X_a^\dagger = -X_a$, one has $(J_-) = (J_+)^\dagger$. Furthermore, the commutator (3.6) implies

$$[\, J_3\,,\, J_\pm\,] = \pm J_\pm\,, \qquad [\, J_+\,,\, J_-\,] = J_3\,. \tag{3.9}$$

The key property of the raising and lowering operators is that they raise and lower the eigenvalues of J_3 by one unit. To be precise, suppose $|m\rangle$ is an eigenvector of J_3 with eigenvalue m, which is necessarily real because J_3 is Hermitian. Then $J_\pm|m\rangle$ is an eigenvector of J_3 with eigenvalue $m \pm 1$ (unless $J_\pm|m\rangle = 0$), because

$$J_3\, J_\pm\, |m\rangle = J_\pm\, J_3\, |m\rangle \pm J_\pm\, |m\rangle = (m \pm 1)\, J_\pm\, |m\rangle\,, \tag{3.10}$$

where we used the commutator (3.9) in the first step.

The states transforming within a given irreducible representation are found from the highest weight construction. Since the representation is finite-dimensional and J_3 is hermitian, J_3 has a largest eigenvalue $j \in \mathbb{R}$. Let $|j\rangle$ be one of the corresponding (normalized) eigenvectors, which we call a *highest-weight state*,

$$J_3|j\rangle = j\,|j\rangle\,, \qquad \langle j|j\rangle = 1. \tag{3.11}$$

The property that it is the highest-weight state implies that it is annihilated by the raising operator

$$J_+|j\rangle = 0, \tag{3.12}$$

since by construction there cannot be any state with J_3-eigenvalue $j+1$. Applying J_- lowers the J_3-eigenvalue by one unit, and we denote the corresponding eigenvector by $|j-1\rangle$,

$$J_-|j\rangle = N_j\,|j-1\rangle\,, \tag{3.13}$$

where we introduced a real normalization constant $N_j > 0$ to ensure $\langle j-1|j-1\rangle = 1$. The value of N_j can be computed from the normalization of the highest weight state and commutator.

Employing the definition (3.13)

$$\begin{aligned} N_j^2 \langle j-1|j-1\rangle &= \langle j|\, J_+\, J_-\, |j\rangle \\ &= \langle j|\, [\, J_+\, ,\, J_-\,]\, |j\rangle \\ &= \langle j|\, J_3\, |j\rangle = j. \end{aligned} \tag{3.14}$$

Thus we conclude that $N_j = \sqrt{j}$. Conversely, applying J_+ to $|j-1\rangle$ leads back to the state $|j\rangle$:

$$J_+\,|j-1\rangle = J_+ \frac{1}{N_j}\, J_-\,|j\rangle = \frac{1}{N_j}\,[\, J_+\, ,\, J_-\,]\,|j\rangle = \frac{1}{N_j} J_3\,|j\rangle = \frac{j}{N_j}\,|j\rangle = N_j\,|j\rangle\,. \tag{3.15}$$

We can continue this process and define $|j-k\rangle$ to be the properly normalized state obtained after k applications of J_- on $|j\rangle$, assuming it does not vanish. In particular we introduce the normalization coefficients $N_{j-k} > 0$ via

$$J_-\,|j-k\rangle \equiv N_{j-k}\,|j-k-1\rangle\ . \tag{3.16}$$

We claim that the states $|j\rangle, |j-1\rangle, \ldots$ span an invariant subspace W of the representation. It is clear from the construction that W is invariant under J_3 and J_-, so it remains to verify that for $J_+|j-k\rangle \in W$ for all k. This can be checked by induction: we have seen that always $J_+|j-1\rangle \in W$. Now suppose $J_+|j-k\rangle \in W$ for some $k \geq 1$. Then

$$J_+|j-k-1\rangle = \frac{1}{N_{j-k}} J_+J_-|j-k\rangle = \frac{1}{N_{j-k}}\left(J_-J_+|j-k\rangle + J_3|j-k\rangle\right), \tag{3.17}$$

but both terms are in W: $J_-J_+|j-k\rangle \in W$ because $J_+|j-k\rangle \in W$ by the induction hypothesis and $J_3|j-k\rangle = (j-k)|j-k\rangle \in W$. Hence W is an invariant subspace of the representation determined by X_1, X_2, X_3. Since we assumed this representation to be irreducible and $W \neq \{0\}$, it follows that W must be the full vector space on which the representation acts. This implies in particular that our choice of higher-weight state $|j\rangle$ was unique (up to phase).

The definition (3.16) allows to derive the recursion relation

$$|N_{j-k}|^2 - |N_{j-k+1}|^2 = j-k\,. \tag{3.18}$$

This follows from

$$\begin{aligned} |N_{j-k}|^2 &= |N_{j-k}|^2 \,\langle\, j-k-1\,|\, j-k-1\,\rangle \\ &= \langle\, j-k\,|\, J_+\, J_-\, |j-k\rangle \\ &= \langle\, j-k\,|\, [J_+\, ,\, J_-\,]\, |j-k\rangle + \langle\, j-k\,|\, J_-\, J_+\, |j-k\rangle \\ &= j-k+|N_{j-k+1}|^2\,. \end{aligned} \tag{3.19}$$

Noticing that the recursion relation (3.18) is a telescopic sum, the solution for the normalization coefficients is obtained as

$$|N_{j-k}|^2 = |N_j|^2 + \sum_{l=1}^{k}(j-l) = \tfrac{1}{2}(k+1)(2j-k)\,. \tag{3.20}$$

In order for the number of states transforming within the representation to be finite, there must also be a lowest-weight state with J_3 eigenvalue $j-l$, $l \in \mathbb{N}$. By definition this state is annihilated by J_-,

$$J_-\,|j-l\rangle = 0, \tag{3.21}$$

since there is no state with J_3-eigenvalue $j-l-1$. This implies that the norm N_{j-l} introduced in (3.16) and computed in (3.20) has to vanish

$$N_{j-l} = \frac{1}{\sqrt{2}} \sqrt{(2j-l)(l+1)} \stackrel{!}{=} 0\,. \tag{3.22}$$

This happens if and only if $j = 0, 1/2, 1, 3/2, \ldots$ and $l = 2j$. In other words, one obtains a finite-dimensional representation if j is either integer or half-integer. The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ can then be labeled by their j-value, and are called the *spin-j representations*. The spin-j representation is $2j+1$-dimensional and the J_3-eigenvalues of the states cover the range $-j \leq m \leq j$ with unit steps.

We have proved that any N-dimensional irreducible representation of $\mathfrak{su}(2)$ is equivalent to the spin-j representation with $2j+1 = N$, meaning that we can find a basis $|m\rangle \equiv |j,m\rangle$, $m = -j, -j+1, \ldots, j$ such that

$$J_3|j,m\rangle = m\,|j,m\rangle, \quad J_-|j,m\rangle = N_m\,|j,m-1\rangle, \quad J_+|j,m\rangle = N_{m+1}\,|j,m+1\rangle, \tag{3.23}$$

$$N_m = \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)}. \tag{3.24}$$

In particular we can read off explicit matrices for the generators in the spin-j representation

$$[J_3]_{m'm} = \langle j,m' \,|\, J_3\,|j,m\rangle = m\,\delta_{m',m}\,. \tag{3.25}$$

and

$$\begin{aligned}[J_+]_{m'm} &= \langle j,m' \,|\, J_+\,|j,m\rangle = N_{m+1}\,\langle j,m' \,|j,m+1\rangle \\ &= \frac{1}{\sqrt{2}} \sqrt{(j+m+1)(j-m)}\,\delta_{m',m+1}\,.\end{aligned} \tag{3.26}$$

The matrix representation of the lowering operator is obtained via $J_- = (J_+)^\dagger$. It remains to check that this spin-j representation is indeed an irreducible representation. It is easy to verify that the matrices J_3, J_+, J_- satisfy the commutation relations (3.9). Solving (3.7) and (3.8) then gives explicit matrices X_a satisfying the commutation relations $[X_a, X_b] = \epsilon_{abc} X_c$. To see that the representation is irreducible let's consider an arbitrary (non-vanishing) vector

$$|\psi\rangle = \sum_{m=-j}^{j} a_m |j,m\rangle. \tag{3.27}$$

Note that we can always find a $k \geq 0$ such that

$$J_+^k|\psi\rangle \propto |j,j\rangle, \tag{3.28}$$

namely $k = 2j$ if $a_{-j} \neq 0$ or $k = 2j-1$ if $a_{-j} = 0$ and $a_{-j+1} \neq 0$, etcetera. But then we can obtain all basis vectors from $|\psi\rangle$ by repeated application of J_- and J_+ via

$$J_-^l J_+^k |\psi\rangle \propto |j, j-l\rangle. \tag{3.29}$$

This means that if $|\psi\rangle$ is in some invariant subspace then all basis vectors are in this subspace, so the spin-j representation has no proper invariant subspaces! This completes the proof of the following theorem.

Theorem 3.4. A complete list of inequivalent irreducible representations of $\mathfrak{su}(2)$ is given by the $(2j+1)$-dimensional spin-j representations (3.23) for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

We close this section by checking that for $j = 1/2$ we reproduce the two-dimensional fundamental representation of $\mathfrak{su}(2)$. Using the basis $|1/2, 1/2\rangle, |1/2, -1/2\rangle$ we evaluate (3.25) to

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3 = iX_3. \tag{3.30}$$

From (3.26) we obtain

$$J_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.31}$$

Converting back to X_1 and X_2 via

$$X_1 = -\frac{i}{\sqrt{2}}(J_+ + J_-) = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = -\frac{1}{\sqrt{2}}(J_+ - J_-) = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{3.32}$$

one recovers $X_a = -\frac{i}{2}\sigma_a$ with σ_a the Pauli matrices (3.2).

4 The adjoint representation

4.1 Definition

Let G be a matrix Lie group defined in terms of N-dimensional matrices and $\mathfrak{g}$ the corresponding Lie algebra. If the dimension of G, i.e. the number of generators of the Lie algebra $\mathfrak{g}$, is n, then G possesses a natural n-dimensional representation with underlying vector space given by the Lie algebra $\mathfrak{g}$ itself.

Definition 4.1 (Adjoint representation of a Lie group). The adjoint representation $D_{\text{ad}} : G \to \text{GL}(\mathfrak{g})$ of G is given by

$$D_{\text{ad}}(g)X = gXg^{-1} \qquad \text{for } g \in G, X \in \mathfrak{g}. \tag{4.1}$$

This definition makes sense because the elements of G and $\mathfrak{g}$ are both $N \times N$ matrices. To see that $gXg^{-1} \in \mathfrak{g}$, notice that

$$t \mapsto g\, e^{tX} g^{-1} \tag{4.2}$$

determines a smooth path in G that visits the identity at $t = 0$. Hence,

$$\partial_t \left(g\, e^{tX} g^{-1}\right)|_{t=0} = gXg^{-1} \in \mathfrak{g}. \tag{4.3}$$

Let us look at the corresponding representation $\mathfrak{D}_{\text{ad}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of the Lie algebra, where $\mathfrak{gl}(\mathfrak{g})$ is the space of linear mappings from $\mathfrak{g}$ to itself[9]. According to (2.21) we should consider a generator $Y \in \mathfrak{g}$ and expand the action of $D_{\text{ad}}(\mathbb{1} + \epsilon Y)$ to first order in ϵ,

$$\begin{aligned} D_{\text{ad}}(\mathbb{1} + \epsilon Y)X &= (\mathbb{1} + \epsilon Y)X(\mathbb{1} - \epsilon Y + \ldots) \\ &= X + \epsilon[Y, X] + \ldots \\ &= (\mathbb{1} + \epsilon[Y, \cdot\,] + \ldots)X \\ &= (\mathbb{1} + \epsilon\mathfrak{D}_{\text{ad}}(Y) + \ldots)X. \end{aligned}$$

Hence $\mathfrak{D}_{\text{ad}}(Y)$ acts on the Lie algebra $\mathfrak{g}$ by taking the Lie bracket $[Y, \cdot\,]$. Note that the Lie algebra plays a double role here: to each element $Y \in \mathfrak{g}$ we associate a linear transformation $\mathfrak{D}_{\text{ad}}(Y)$ of $\mathfrak{g}$ given by $\mathfrak{D}_{\text{ad}}(Y)X = [Y, X]$. Since the notation $\mathfrak{D}_{\text{ad}}(Y)X$ can be a bit awkward, it is convenient to introduce the ket notation $|X\rangle$ for the elements $X \in \mathfrak{g}$ on which the adjoint representation acts. Then $\mathfrak{D}_{\text{ad}}(Y)(X) = [Y, X]$ can be more succinctly summarized as

$$Y|X\rangle = |[Y, X]\rangle, \qquad X, Y \in \mathfrak{g}. \tag{4.4}$$

To obtain explicit matrices $[T_a]_{bc}$ for this representation, we choose a basis of generators ξ_a spanning the Lie algebra $\mathfrak{g}$ and consider an inner product[10] such that $\langle \xi_a | \xi_b \rangle = \delta_{ab}$. Recall that the commutator of the generators is given in terms of the structure constants by

$$[\,\xi_a\,,\,\xi_b\,] = f_{abc}\,\xi_c\,. \tag{4.5}$$

To find the matrix element $[T_a]_{bc}$ corresponding to the action of $T_a = \mathfrak{D}_{\text{ad}}(\xi_a)$ on $\mathfrak{g}$ we use the inner product to find

$$[T_a]_{bc} = \langle \xi_b |\, \xi_a \,| \xi_c \rangle \overset{(4.4)}{=} \langle \xi_b | [\xi_a, \xi_c] \rangle \overset{(4.5)}{=} f_{acd} \langle \xi_b | \xi_d \rangle = f_{acb}. \tag{4.6}$$

Let us summarize this in a definition.

[9] The notation $\mathfrak{gl}(V)$ for linear mappings from V to V stems from the fact that it is precisely the Lie algebra of the Lie group $\text{GL}(V)$ of invertible linear mappings from V to V. Note also that in the mathematical literature $\mathfrak{D}_{\text{ad}}$ is simply denoted by ad.

[10] We will see in a minute that the algebra has a canonical inner-product given by the Cartan-Killing metric. Here the inner product is arbitrarily defined by our choice of basis.

Definition 4.2 (Adjoint representation of a Lie algebra). The adjoint representation $\mathfrak{D}_{\text{ad}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is given by $Y|X\rangle = |[Y, X]\rangle$. Once a basis of $\mathfrak{g}$ has been chosen with structure constants f_{abc}, the generators of $\mathfrak{g}$ in the adjoint representation are given by the matrices

$$[T_a]_{bc} \equiv f_{acb}\,, \qquad a, b, c = 1, \dots, n. \tag{4.7}$$

The generators T_a satisfy the commutator relation $[T_a, T_b] = f_{abc}T_c$. This can be verified by the Jacobi identity (2.18) (see exercises).

The dimension of the adjoint representation equals the dimension of the Lie group. This is similar to the regular representation encountered in the context of finite groups (definition 1.12), where the elements of the group served as a basis of the linear space on which the representation acts. Now it is not the group elements but the generators that span the linear space.

4.2 Cartan-Killing metric

There is a natural metric on the space of generators:

Definition 4.3 (Cartan-Killing metric). The Cartan-Killing metric

$$\gamma_{ab} := \text{tr}\,[T_a\,T_b] = [T_a]_{cd}\,[T_b]_{dc} = f_{adc}f_{bcd} \tag{4.8}$$

provides a scalar product on the space of generators of the adjoint representation. (Note that the trace tr is with respect to the matrix-indices of the generators $[T_a]_{bc}$.)

Fact 4.4. As a side-remark let us mentions that if $\mathfrak{g}$ is a simple[a] Lie algebra and one is not worried over the precise normalization, one may equally compute the Cartan-Killing metric by taking the trace $\gamma_{ab} = c\ \text{tr}[X_a X_b]$ in any other irreducible representation than the adjoint one, because they are all equal up to a positive real factor c.

[a] A Lie algebra is simple if its adjoint representation is irreducible. A more useful definition of a simple Lie algebra will be given in Chapter 7. For now it is sufficient to know that the Lie algebras $\mathfrak{su}(N)$ for $N \geq 2$ and $\mathfrak{so}(N)$ for $N \geq 3$ are all simple.

A natural question concerns the freedom in writing down the generators T_a. Since the generators ξ_a constitute a basis of a linear space, there is the freedom of transforming to a new basis X'_a by performing an invertible, linear transformation L:

$$\xi'_a = L_{ba}\,\xi_b\,. \tag{4.9}$$

The change of basis induces a change in the structure constants

$$f_{abc} \mapsto f'_{abc} = L_{da}\,L_{eb}\,f_{def}\,\left[L^{-1}\right]_{cf}\,. \tag{4.10}$$

This is seen by applying the transformation (4.9) to the commutator

$$\begin{aligned}[\xi'_a, \xi'_b] &\overset{(4.5)}{=} f'_{abc}\,\xi'_c \overset{(4.9)}{=} f'_{abc}\,L_{fc}\,\xi_f \\ &\overset{!}{=} L_{da}\,L_{eb}\,[\,\xi_d\,,\,\xi_e\,] \overset{(4.5)}{=} L_{da}\,L_{eb}\,(f_{def}\xi_f)\,.\end{aligned} \tag{4.11}$$

As a consequence the generators of the adjoint representation transform according to

$$\begin{aligned}[T_a]_{bc} \mapsto \left[T'_a\right]_{bc} &\overset{(4.7)}{=} f'_{acb} \overset{(4.10)}{=} L_{da}\,L_{fc}\,f_{dfe}\,[L^{-1}]_{be} \\ &\overset{(4.7)}{=} L_{da}\,L_{fc}\,[T_d]_{ef}\,[L^{-1}]_{be} = L_{da}\,[L^{-1}T_dL]_{bc}.\end{aligned} \tag{4.12}$$

This, in turn, implies that the Cartan-Killing metric transforms as

$$\gamma_{ab} = \text{tr}\,[T_a\,T_b] \mapsto \gamma'_{ab} = \text{tr}\,\left[T'_a\,T'_b\right] = L_{ca}\,L_{db}\,\text{tr}\left[L^{-1}T_c L\,L^{-1}T_d L\right] = L_{ca}\,L_{db}\,\gamma_{cd}, \tag{4.13}$$

where we used the cyclicity of the trace.

Since the Cartan-Killing metric is symmetric in the generator indices a, b, one may choose the basis transformation L such that that after the transformation γ_{ab} is of the diagonal form

$$\gamma_{ab} = \text{tr}\,[T_a\,T_b] = \begin{bmatrix} -\mathbb{1}_{n_-} & 0 & 0 \\ 0 & +\mathbb{1}_{n_+} & 0 \\ 0 & 0 & 0\,\mathbb{1}_{n_0} \end{bmatrix} \tag{4.14}$$

for some integers n_-, n_+ and n_0 that add up to n. This fact about symmetric matrices may be familiar to you from general relativity (where the spacetime metric can be brought by a local basis transformation to Minkowski form, i.e. $n_- = 1$, $n_+ = 3$) and is known under the name of Sylvester's law of inertia.

The block containing the negative eigenvalues is generated by the n_- "compact" (rotation-type) generators, the positive eigenvalues belong to the n_+ "non-compact" (boost-type) generators and the zero-entries originate from nilpotent generators where $(X_a)^p = 0$ for some $p > 1$. Thus the eigenvalues of the Cartan-Killing metric provide non-trivial information on the compactness of the Lie algebra.

Definition 4.5 (Compact Lie algebra). A Lie algebra $\mathfrak{g}$ is *compact* if the Cartan-Killing metric is negative definite, meaning that it admits a basis such that $\gamma_{ab} = -\delta_{ab}$.

Remark 4.1. It is important to remark here that compactness of the Lie algebras is a slightly stronger condition than compactness of the corresponding Lie group (definition 2.5). For example $U(1) = \{u \in \mathbb{C} : |u| = 1\}$ is a compact Lie group, because its (single) matrix element has bounded absolute value. The corresponding Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$ is not compact, because it has a single basis generator $\xi_1 = i$ with structure constant $f_{111} = 0$, implying that the Cartan-Killing metric on $\mathfrak{u}(1)$ vanishes. However, it can be shown that compactness of the Lie algebra always implies compactness of the Lie group. Moreover, for simple Lie groups of dimension larger than one the two notions of compactness are equivalent.

A consequence of working in the basis where $\gamma_{ab} = -\delta_{ab}$ is that the structure constants of the compact Lie algebra $\mathfrak{g}$ become completely antisymmetric[11]

$$f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{acb} = -f_{cba}\,. \tag{4.15}$$

The proof of this property is rather instructive. It expresses the structure constants in terms of the Cartan-Killing metric via

$$\text{tr}\,(\,[\,T_a\,,\,T_b\,]\;T_c) \overset{(4.5)}{=} f_{abd}\;\text{tr}\,(\,T_d\,T_c) \overset{(4.8)}{=} f_{abd}\gamma_{dc} \overset{\gamma_{dc}=-\delta_{dc}}{=} -f_{abc}\,. \tag{4.16}$$

The cyclicity of the trace then shows that

$$\begin{aligned} f_{abc} &= -\,\text{tr}\,(\,[\,T_a\,,\,T_b\,]\;T_c) \\ &= -\,\text{tr}\,(\,T_a\,T_b\,T_c - T_b\,T_a\,T_c) \\ &= -\,\text{tr}\,(\,T_b\,T_c\,T_a - T_c\,T_b\,T_a) \\ &= -\,\text{tr}\,(\,[\,T_b\,,\,T_c\,]\;T_a) \\ &= f_{bca}. \end{aligned} \tag{4.17}$$

Together with the anti-symmetry of the structure constants in the first two indices, this establishes the relation (4.15).

[11]This is the basis which is typically adopted when studying non-abelian gauge groups in quantum field theory.

4.3 Casimir operator

Suppose $\mathfrak{g}$ is compact with generators X_a in any irreducible representation. Then we can introduce an important quadratic operator.

Definition 4.6 (Casimir operator). The (quadratic) Casimir operator of a compact Lie algebra in this representation is given by

$$C := \gamma^{ab} X_a X_b, \tag{4.18}$$

where $\gamma^{ab} = [\gamma^{-1}]^{ab}$ is the inverse of the Cartan-Killing metric γ_{ab}.

Note that generally $C \notin \mathfrak{g}$ since it is not built from commutators of the generators, but it makes perfect sense as a matrix acting on the same vectors as X_a.

Theorem 4.7. The Casimir operator is independent of the choice of basis X_a and commutes with all generators, i.e. $[C, X_a] = 0$.

Proof. To see that it is independent of the basis let us look at a transformation $X_a \mapsto X'_a = L_{ba} X_b$ as before. Then

$$C \mapsto C' \overset{(4.18)}{=} (\gamma')^{ab} X'_a X'_b = (\gamma')^{ab} L_{ca} X_c L_{db} X_d \tag{4.19}$$

$$\overset{(4.13)}{=} \left[L(L^t \gamma L)^{-1} L^t\right]_{cd} X_c X_d = \gamma^{cd} X_c X_d = C. \tag{4.20}$$

Without loss of generality we may assume thus assume that we have chosen generators such that $\gamma_{ab} = -\delta_{ab}$. Then we find for any c,

$$[C, X_b] \overset{(4.18)}{=} [X_a X_a, X_b] = X_a [X_a, X_b] + [X_a, X_b] X_a \tag{4.21}$$

$$\overset{(4.5)}{=} X_a f_{abc} X_c + f_{abc} X_c X_a \overset{(4.15)}{=} 0. \tag{4.22}$$

□

Note that the Casimir operator does not just commute with all generators, but also with the group elements of G in the corresponding representation D. Indeed, using the exponential map (Definition 2.4) any element $g \in G$ can be written as $D(g) = \exp(\alpha_a X_a)$ and therefore satisfies

$$[C, D(g)] = [C, \exp(\alpha_a X_a)] \overset{\text{Thm. 4.7}}{=} 0. \tag{4.23}$$

This puts us precisely in the setting of Schur's Lemma, Theorem 1.22, which allows us to conclude that if D is irreducible the Casimir operator is proportional to the identity,

$$C = C_D \, \mathbb{1}, \tag{4.24}$$

where C_D is a positive real number that only depends on the representation D. It can therefore be used to label or distinguish irreducible representations.

As an example let us look at $\mathfrak{su}(2)$ with the standard basis

$$[X_a, X_b] = \epsilon_{abc} X_c. \tag{4.25}$$

In the exercises you will determine that the corresponding Cartan-Killing metric is

$$\gamma_{ab} = -2\delta_{ab}. \tag{4.26}$$

The Casimir operator is therefore

$$C = \gamma^{ab} X_a X_b = -\frac{1}{2}(X_1^2 + X_2^2 + X_3^2) \overset{J_a = iX_a}{=} \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) \tag{4.27}$$

Given what we know about the total angular momentum (or spin) in quantum mechanics it should not come as a surprise that it commutes the generators J_a. To compute the value C_j in the spin-j representation it is sufficient to focus on a single state, say the highest-weight state $|j\rangle = |j, m = j\rangle$. Using that

$$J^+J^- + J^-J^+ \overset{(3.8)}{=} \frac{1}{2}(J_1 + iJ_2)(J_1 - iJ_2) + \frac{1}{2}(J_1 - iJ_2)(J_1 + iJ_2) = J_1^2 + J_2^2, \tag{4.28}$$

we find

$$C|j\rangle \overset{(4.27)}{=} \frac{1}{2}(J_3^2 + J^+J^- + J^-J^+)|j\rangle = \frac{1}{2}(J_3^2 + [J^+, J^-] + 2J^-J^+)|j\rangle \tag{4.29}$$

$$\overset{(3.9)}{=} \frac{1}{2}(J_3^2 + J_3 + 2J^-J^+)|j\rangle \overset{(3.12)}{=} \frac{1}{2}(J_3^2 + J_3)|j\rangle, \tag{4.30}$$

and therefore $C_j = \frac{1}{2}j(j+1)$. We see that the Casimir operator indeed distinguishes between the different irreducible representations of $\mathfrak{su}(2)$.

For larger Lie algebras, like $\mathfrak{su}(N)$ with $N \geq 3$, this is no longer the case since irreducible representations are typically labeled by more than one parameter. Luckily in general the quadratic Casimir operator C is not the only operator that commutes with all group elements. Further *Casimir invariants* that are polynomials in X_a of order higher than two can then be constructed. It turns out that all Casimir invariants together do distinguish the irreducible representations.

5 Root systems, simple roots and the Cartan matrix

5.1 Cartan subalgebra

Central to the classification of Lie algebras and their representations is the selection of a maximal set of commuting generators. Mathematically this is described by the Cartan subalgebra of a Lie algebra. First we need to know what a subalgebra is.

Definition 5.1 (Subalgebra). A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ on which the Lie bracket closes: $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. This can also be denoted

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \tag{5.1}$$

If $H_i \in \mathfrak{g}$ for $i = 1, \ldots m$ form a collection of commuting Hermitian generators, $[H_i, H_j] = 0$ for all i and j, then the H_i are called *Cartan generators*. Trivially they span a subalgebra of $\mathfrak{g}$.

Remark 5.1. Here you may rightfully object that the Lie algebras $\mathfrak{g}$ we have encountered so far do not contain any Hermitian generators (except the trivial $0 \in \mathfrak{g}$). Indeed, in the case of the special unitary and orthogonal groups, the generators ξ_a were antihermitian $\xi_a^\dagger = -\xi_a$. In order to find Hermitian generators we have to allow taking complex linear combinations $\beta_a \xi_a$, $\beta_a \in \mathbb{C}$, instead of just real linear combinations $\alpha_a \xi_a \in \mathfrak{g}$, $\alpha_a \in \mathbb{R}$. In mathematical terms such complex linear combinations take value in the *complexified* Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$. Note that we have already implicitly used such generators in Section 3.2 when introducing the lowering and raising operators $J_\pm = \frac{-i}{\sqrt{2}}(X_1 \pm iX_2)$ which are elements of $\mathfrak{D}(\mathfrak{g}_\mathbb{C})$ but not of $\mathfrak{D}(\mathfrak{g})$ (they are not antihermitian). When dealing with representation theory and classification of Lie algebras it is typically a lot easier to work with $\mathfrak{g}_\mathbb{C}$. We will do so implicitly in these lectures even when we write just $\mathfrak{g}$!

Definition 5.2 (Cartan subalgebra). A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *Cartan subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] = 0$ and it is maximal, in the sense that there every element $X \in \mathfrak{g}$ that commutes with all of $\mathfrak{h}$, i.e. $[X, \mathfrak{h}] = 0$, is contained in $\mathfrak{h}$.

Fact 5.3. Although a Lie algebra $\mathfrak{g}$ can have many Cartan subalgebras, they are all equivalent in the sense that they can be related by a basis transformation of $\mathfrak{g}$. Therefore we often talk about *the* Cartan subalgebra of $\mathfrak{g}$.

Definition 5.4 (Rank). The *rank* of $\mathfrak{g}$ is the dimension of its Cartan subalgebra.

If the rank of $\mathfrak{g}$ is r, the Cartan generators $H_1, \ldots, H_r$ form a maximal set of generators that can be simultaneously diagonalized in any representation $\mathfrak{D} : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^N)$. Hence, we can find a basis $\{|\mu, x\rangle\}$ of $\mathbb{C}^N$ such that

$$\mathfrak{D}(H_i)|\mu, x\rangle = \mu_i|\mu, x\rangle, \tag{5.2}$$

where x is some additional label to specify the state (in case of multiplicities).

Definition 5.5 (Weights). The vector of eigenvalues $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$ of a state is called the *weight* of the vector $|\mu, x\rangle$.

Example 5.6 ($\mathfrak{su}(2)$). The rank of $\mathfrak{su}(2)$ is 1 since no two generators commute. The Cartan generator (spanning the Cartan subalgebra of $\mathfrak{su}(2)$) is typically chosen to be $J_3 = iX_3$. The weights of the spin-j representation of $\mathfrak{su}(2)$ are the eigenvalues of J_3, which are $j, j-1, \ldots, -j$.

5.2 Roots

For the remainder of this chapter we focus on the adjoint representation $\mathfrak{D}_{\text{ad}}$ of a compact Lie algebra $\mathfrak{g}$ on itself, given by $\mathfrak{D}_{\text{ad}}(X)|Y\rangle \equiv X|Y\rangle = |[X,Y]\rangle$ for $X, Y \in \mathfrak{g}$. Suppose $\mathfrak{g}$ has rank r. The vectors $|H_i\rangle$ have weight $(0,\ldots,0) \in \mathbb{R}^r$, since for all $i,j = 1,\ldots,r$ we have

$$H_i|H_j\rangle \overset{(4.4)}{=} |[H_i,H_j]\rangle \overset{\text{Cartan}}{=} 0. \tag{5.3}$$

Conversely, by the maximality criterion (Definition 5.2), any simultaneous eigenvector $|X\rangle$ of the Cartan generators with zero weight is a Cartan generator.

Definition 5.7 (Roots). A nonvanishing weight $\alpha = (\alpha_1,\ldots,\alpha_r)$ of the adjoint representation is called a *root*.

For each root there is an associated joint eigenvector $|E_\alpha\rangle$ satisfying

$$H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle \qquad \text{for all } i = 1,\ldots,r. \tag{5.4}$$

We will see later that this vector is uniquely specified (up to scalar multiplication) by the weight $\alpha = (\alpha_1,\ldots,\alpha_r)$, meaning that there are no multiplicities in the joint spectrum of the Cartan generators.

A basis of $\mathfrak{g}$ is thus given by

$$\{H_1,\ldots,H_r\} \cup \{E_\alpha : \alpha \text{ is a root}\}, \tag{5.5}$$

implying in particular that a Lie algebra $\mathfrak{g}$ of dimension n and rank r must have exactly $n-r$ distinct roots $\alpha \in \mathbb{R}^r$. At the level of $\mathfrak{g}$ (5.4) implies that

$$[H_i, E_\alpha] = \alpha_i\, E_\alpha. \tag{5.6}$$

This looks very similar to the commutation relation $[J_3, J_\pm] = J_\pm$ of (3.9) in the case of $\mathfrak{su}(2)$, suggesting that E_α plays a role of raising or lowering operator. This we can check easily. Let us look at a root α' and corresponding eigenvector $|E_{\alpha'}\rangle$ satisfying $H_i|E_{\alpha'}\rangle = \alpha'_i|E_{\alpha'}\rangle$ and act on it with the operator $E_{\alpha'}$, then

$$\begin{aligned} H_i|[E_\alpha, E_{\alpha'}]\rangle &= H_i E_\alpha|E_{\alpha'}\rangle = (E_\alpha H_i + \alpha_i E_\alpha)|E_{\alpha'}\rangle \\ &= (\alpha'_i + \alpha_i) E_\alpha |E_{-\alpha}\rangle = (\alpha'_i + \alpha_i)|[E_\alpha, E_{\alpha'}]\rangle. \end{aligned} \tag{5.7}$$

There are now three options:

- $[E_\alpha, E_{\alpha'}] = 0$;
- $[E_\alpha, E_{\alpha'}] \neq 0$ and $\alpha + \alpha' \neq 0$, implying that $|[E_\alpha, E_{\alpha'}]\rangle \propto |E_{\alpha+\alpha'}\rangle$ is the eigenvector corresponding to root $\alpha + \alpha'$;
- $[E_\alpha, E_{\alpha'}] \neq 0$ and $\alpha + \alpha' = 0$, implying that $[E_\alpha, E_{\alpha'}]$ is a Cartan generator in $\mathfrak{h}$.

Notice that, contrary to the Cartan generators, the generators E_α cannot be Hermitian. Indeed, taking the conjugate of (5.6) we find

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger. \tag{5.8}$$

Instead we can choose the normalization such that $E_\alpha^\dagger = E_{-\alpha}$, similarly to the relation $J_+^\dagger = J_-$ between the raising and lowering operators in $\mathfrak{su}(2)$.

We equip $\mathfrak{g}$ with a complex scalar product, closely related to the Cartan-Killing metric[12], given by

$$\langle Y|X\rangle = \operatorname{tr}\left(Y^\dagger X\right). \tag{5.9}$$

[12] We are really working with the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ here, explaining the need for a Hermitian conjugate in the scalar product. Note that if X and Y are anti-Hermitian, then $\langle Y|X\rangle = -\operatorname{tr}(YX)$ is a negative multiple of the Cartan-Killing metric. In particular, it is positive-definite when $\mathfrak{g}$ is compact.

The eigenvectors $|H_i\rangle$ and $|E_\alpha\rangle$ can then be normalized such that

$$\langle H_i|H_j\rangle = \delta_{ij}, \qquad \langle E_\alpha|E_\beta\rangle = \delta_{\alpha\beta}, \tag{5.10}$$

where $\delta_{\alpha\beta} \coloneqq \prod_{i=1}^{r} \delta_{\alpha_i\beta_i}$. The set $\{H_i\} \cup \{E_\alpha\}$ then forms an orthonormal basis of $\mathfrak{g}$.

Example 5.8 (Roots of $\mathfrak{su}(2)$). Since $\mathfrak{su}(2)$ is three-dimensional and of rank 1 it must have two roots. These are precisely ± 1, corresponding to the generators $E_{\pm 1} = J_\pm$.

5.3 Root system

It turns out that the collection of roots α satisfies very rigid conditions, that we capture in the following definition. Here we use the notation $|\alpha|^2 = \alpha \cdot \alpha$ and $\alpha \cdot \beta = \alpha_i \beta_i$.

Definition 5.9 (Root system). A *root system* Φ or rank r is a finite subset of non-zero vectors in $\mathbb{R}^r$ satisfying

(i) Φ spans $\mathbb{R}^r$.

(ii) The only scalar multiples of $\alpha \in \Phi$ that also belong to Φ are $\pm\alpha$.

(iii) For any $\alpha, \beta \in \Phi$ the ratio $\frac{\alpha \cdot \beta}{\alpha \cdot \alpha}$ is an integer or a half-integer.

(iv) For every $\alpha, \beta \in \Phi$ the Weyl reflection $\beta - 2\frac{\alpha \cdot \beta}{\alpha \cdot \alpha}\alpha$ of β in the hyperplane perpendicular to α is also in Φ.

We will now show that

Theorem 5.10. The set of roots of a compact Lie algebra $\mathfrak{g}$ forms a root system.

We will verify the properties one by one. Property (i) follows from the compactness of $\mathfrak{g}$:

Proof of property (i). Suppose the roots do not span $\mathbb{R}^r$. Then there exists a nonzero vector $\beta \in \mathbb{R}^r$ such that $\beta \cdot \alpha = 0$ for all $\alpha \in \Phi$. But that means that all vectors $|E_\alpha\rangle$ are annihilated by $\beta_i H_i$, $\beta_i H_i |E_\alpha\rangle = \beta \cdot \alpha |E_\alpha\rangle = 0$, and the same is true by definition for the states $|H_j\rangle$. Since those states together form a basis of $\mathfrak{g}$, it follows that $[\beta_i H_i, \mathfrak{g}] = 0$. Since $\beta_i H_i$ is represented by the zero matrix in the adjoint representation, we have $\langle \beta_i H_i | \beta_i H_i\rangle = 0$. But this is in contradiction with the compactness of $\mathfrak{g}$, which requires that the Cartan-Killing metric is negative definite. □

The next steps rely on the fact that to each pair $\pm\alpha$ of roots one may associate an $\mathfrak{su}(2)$-subalgebra.

Lemma 5.11. If α is a root, then

$$E_\pm \coloneqq \frac{1}{|\alpha|} E_{\pm\alpha}, \qquad E_3 \coloneqq \frac{1}{|\alpha|^2}\alpha_i H_i. \tag{5.11}$$

satisfy the commutation relations of $\mathfrak{su}(2)$,

$$[E_+, E_-] = E_3, \qquad [E_3, E_\pm] = \pm E_\pm. \tag{5.12}$$

Proof. If α is a root, then $[E_\alpha, E_{-\alpha}] \in \mathfrak{h}$ has to be in the Cartan subalgebra $\mathfrak{h}$. This follows from (5.7) with $\alpha' = -\alpha$,

$$H_i |[E_\alpha, E_{-\alpha}]\rangle = (\alpha_i - \alpha_i)|[E_\alpha, E_{-\alpha}]\rangle = 0. \tag{5.13}$$

On the other hand using (5.10),

$$\langle H_i|[E_\alpha, E_{-\alpha}]\rangle = \mathrm{tr}(H_i[E_\alpha, E_{-\alpha}])$$

$$
\begin{aligned}
&= \text{tr}(H_i E_\alpha E_{-\alpha}) - \text{tr}(H_i E_{-\alpha} E_\alpha) \\
&= \text{tr}(E_{-\alpha}[H_i, E_\alpha]) \\
&= \alpha_i \ \text{tr}(E_{-\alpha} E_\alpha) \\
&= \alpha_i \langle E_\alpha | E_\alpha \rangle = \alpha_i .
\end{aligned}
\tag{5.14}
$$

Since the H_i form an orthonormal basis of the Cartan subalgebra we conclude that

$$
[E_\alpha, E_{-\alpha}] = \alpha_i H_i. \tag{5.15}
$$

This verifies the first relation of (5.12), while the second follows from (5.6). □

Let us use this structure to verify that the root α specifies the generator E_α uniquely. Suppose that there exists another eigenvector $|E'_\alpha\rangle$ orthogonal to $|E_\alpha\rangle$ with the same root α . Then the vector $E_{-\alpha}|E'_\alpha\rangle$ has zero weight and is thus a Cartan generator, which can be identified following a computation similar to (5.14),

$$
\langle H_i |\, E_{-\alpha}\, |E'_\alpha\rangle = -\alpha_i \ \text{tr}\left(E_{-\alpha}\, E'_\alpha\right) = 0\,. \tag{5.16}
$$

Thus the projection on the Cartan subalgebra vanishes. As a consequence $|E'_\alpha\rangle$ must be a lowest weight state, $E_-|E'_\alpha\rangle = 0$. But

$$
E_3 |E'_\alpha\rangle = |\alpha|^{-2}\, \alpha_i H_i\, |E'_\alpha\rangle = |E'_\alpha\rangle\,. \tag{5.17}
$$

Thus the state $|E'_\alpha\rangle$ has an E_3-eigenvalue $m = +1$. This is in contradiction with the fact shown in Chapter 4 that the lowest weight states of an $\mathfrak{su}(2)$-representation must have a negative or zero E_3-eigenvalue (namely $-j$ in the spin-j representation). Thus we conclude that one cannot have two generators E_α and E'_α corresponding to the same root. Similar arguments lead to the following.

Proof of property (iii). Let us consider a root β different from α. From (5.11) is follows that

$$
E_3 |E_\beta\rangle = |\alpha|^{-2} \alpha_i H_i |E_\beta\rangle = \frac{\alpha \cdot \beta}{\alpha \cdot \alpha} |E_\beta\rangle. \tag{5.18}
$$

But from the $\mathfrak{su}(2)$ representation theory the eigenvalue $(\alpha \cdot \beta)/(\alpha \cdot \alpha)$ has to be half-integer. □

Proof of property (ii). Notice that it is sufficient to show that if α is a root that $t\alpha$ for $t > 1$ cannot be a root. To this end, suppose to the contrary that $\alpha' = t\alpha$ is a root. Applying property (iii) to α and $t\alpha$ we find that both t and $1/t$ must be half-integers, so the only option is $t = 2$.

Let us consider the $\mathfrak{su}(2)$-subalgebra (5.11) associated to root α. The vector $|E_{\alpha'}\rangle$ has E_3-eigenvalue 2 and therefore it has a component in a spin-j representation for some integer $j \geq 2$. Acting with the lowering operator E_- on this component we construct a vector with root α. But this one is not a multiple of $|E_\alpha\rangle$, since $|E_\alpha\rangle$ lives in the spin-1 representation (because $E_+|E_\alpha\rangle = 0$ and $E_3|E_\alpha\rangle = |E_\alpha\rangle$). We have shown that this is impossible, finishing our proof by contradiction. □

Proof of property (iv). Suppose α and β are two distinct roots. The state $|E_\beta\rangle$ has E_3-eigenvalue $m = (\alpha\cdot\beta)/(\alpha\cdot\alpha)$ with respect to the $\mathfrak{su}(2)$-representation (5.11) with m half-integer. Hence $|E_\beta\rangle$ can be decomposed into components transforming in various spin-j representations for $j \geq |m|$. If $m \geq 0$ then acting $2m$ times with the lowering operator produces a non-vanishing state $E_-^{2m}|E_\beta\rangle$ satisfying

$$
H_i E_-^{2m} |E_\beta\rangle = -2m\alpha_i E_-^{2m} |E_\beta\rangle + E_-^{2m} H_i |E_\beta\rangle = (\beta_i - 2m\alpha_i)\, E_-^{2m} |E_\beta\rangle. \tag{5.19}
$$

It therefore has root equal to $\beta - 2m\alpha = \beta - 2(\alpha\cdot\beta)/(\alpha\cdot\alpha)\alpha$. Similarly, if $m < 0$ one may act $2|m|$ times with the raising operator to produce a non-vanishing state $E_+^{2|m|}|E_\beta\rangle$ with root $\beta + 2|m|\alpha = \beta - 2(\alpha\cdot\beta)/(\alpha\cdot\alpha)\alpha$. □

We conclude that the roots of any compact Lie algebra $\mathfrak{g}$ form a root system Φ with rank r equal to the rank of $\mathfrak{g}$, i.e. the dimension of its Cartan subalgebra. Although we will not prove this in detail, any root system Φ characterizes a unique compact Lie algebra.

Fact 5.12. Compact Lie algebras $\mathfrak{g}$ are in 1-to-1 correspondence with root systems Φ.

5.4 Angles between roots

Property (iii) has a direct consequence for the angle $\theta_{\alpha\beta}$ between any two roots $\alpha, \beta \in \Phi$ for which $\alpha \neq \pm\beta$. Namely, both $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ and $2\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ must be integers. Hence, by the cosine rule

$$4\cos^2\theta_{\alpha\beta} = 4\frac{(\alpha\cdot\beta)^2}{|\alpha|^2|\beta|^2} = \frac{2(\alpha\cdot\beta)}{\alpha\cdot\alpha}\,\frac{2(\alpha\cdot\beta)}{\beta\cdot\beta} \in \mathbb{Z} \tag{5.20}$$

is an integer. Since $0 < \theta_{\alpha\beta} < \pi$, this quantity must be an integer smaller than 4. Therefore there are only a couple of possibilities for the angle $\theta_{\alpha\beta}$ and the length ratio

$$\frac{|\alpha|}{|\beta|} = \sqrt{\frac{2(\alpha\cdot\beta)}{\beta\cdot\beta}\Big/\frac{2(\alpha\cdot\beta)}{\alpha\cdot\alpha}} \tag{5.21}$$

of the roots, which are summarized in the following table.

$4\cos^2\theta_{\alpha\beta}$	$\theta_{\alpha\beta}$	$\lvert\alpha\rvert/\lvert\beta\rvert$
0	$\pi/2$	arbitrary
1	$\pi/3$ or $2\pi/3$	1
2	$\pi/4$ or $3\pi/4$	$\sqrt{2}$ or $\frac{1}{\sqrt{2}}$
3	$\pi/6$ or $5\pi/6$	$\sqrt{3}$ or $\frac{1}{\sqrt{3}}$

(5.22)

5.5 Simple roots

In order to proceed to classify root systems it is useful to adopt a convention as to which of the two roots $\pm\alpha$ should be interpreted as a raising operator (the other then being a lowering operator).

Definition 5.13 (Positive root). A root $\alpha = (\alpha_1, \ldots, \alpha_r)$ is called *positive* if its first non-zero component is positive, and *negative* otherwise.

If α is positive we call $|\alpha|^{-1}E_\alpha$ a raising operator, while if α is negative $|\alpha|^{-1}E_\alpha$ is a lowering operator.

Definition 5.14 (Simple root). A *simple root* of $\mathfrak{g}$ is a positive root that cannot be obtained as the sum of two positive roots.

Simple roots will be highlighted with a hat on top of the root ($\hat{\alpha}$) in the sequel.

Lemma 5.15. The simple roots satisfy the following important properties:

(i) A root system of rank r has precisely r simple roots, which form a basis of $\mathbb{R}^r$.

(ii) All other roots can be obtained from successive Weyl reflections of simple roots (see property (iv) of the root system).

(iii) If $\hat{\alpha}$ and $\hat{\beta}$ are simple roots then $\hat{\alpha} - \hat{\beta}$ is not a root.

(iv) If $\hat{\alpha}$ and $\hat{\beta}$ are different simple roots, then their scalar product is nonpositive, $\hat{\alpha}\cdot\hat{\beta} \leq 0$.

Proof of Lemma 5.15(i) . We first prove that the simple roots are linearly independent, i.e. we need to show that the only solution to $a_i\hat{\alpha}^i = 0$ occurs for $a_i = 0$. To this end suppose that another solution exists. Let us write $a_i = b_i - c_i$ in such a way that $b_i = a_i$ and $c_i = 0$ when $a_i > 0$, while $b_i = 0$ and $c_i = -a_i$ when $a_i < 0$. Then $b_i\hat{\alpha}^i = c_i\hat{\alpha}^i$ and therefore $(b_i\hat{\alpha}^i)\cdot(c_i\hat{\alpha}^i) = |b_i\hat{\alpha}^i|^2 = |b_i\hat{\alpha}^i|^2 > 0$. On the other hand, by expanding the inner product

$$(b_i\hat{\alpha}^i)\cdot(c_i\hat{\alpha}^i) = b_i c_j\, \hat{\alpha}^i\cdot\hat{\alpha}^j \leq 0,$$

because $b_i c_j = 0$ when $i = j$ by definition, while $b_i c_j \geq 0$ and $\hat{\alpha}_i\cdot\hat{\alpha}_j \leq 0$ when $i \neq j$. This is a contradiction, implying that the simple roots are linearly independent. In particular, there can be at most r roots.

Next we show that any root α can be written as a linear combination of simple roots. Assume α is a positive root (if it is negative $-\alpha$ is a positive root). If α is simple, then we are done. Otherwise $\alpha = \alpha_1 + \alpha_2$ is a sum of positive roots α_1 and α_2. We can repeatedly apply this decomposition of non-simple roots into sums of positive roots, until we are left only with simple roots. This is the desired linear combination. Since the roots span $\mathbb{R}^r$ the same is true for the simple roots, which thus form a basis of $\mathbb{R}^r$. □

We skip the proof of (ii). The two remaining parts are easily deduced as follows.

Proof of Lemma 5.15(iii) . Assume that $\hat{\alpha}$ and $\hat{\beta}$ are simple roots and $\hat{\alpha}-\hat{\beta}$ is a root. Then either $\hat{\alpha}-\hat{\beta}$ or $\hat{\beta}-\hat{\alpha}$ is a positive root. But then either $\hat{\alpha} = \left(\hat{\alpha}-\hat{\beta}\right)+\hat{\beta}$ or $\hat{\beta} = \left(\hat{\beta}-\hat{\alpha}\right)+\hat{\alpha}$ can be written as the sum of two positive roots. This contradicts the assumption that $\hat{\alpha}$ and $\hat{\beta}$ are simple. □

Proof of Lemma 5.15(iv) . From (iii) it follows that $E_{-\hat{\alpha}}|E_{\hat{\beta}}\rangle = |[E_{-\hat{\alpha}}, E_{\hat{\beta}}]\rangle = 0$ since $\hat{\beta}-\hat{\alpha}$ cannot be a root. Hence $|\hat{\beta}\rangle$ is a lowest-weight state in the $\mathfrak{su}(2)$-representation of $\hat{\alpha}$, but then its E_3-eigenvalue $\hat{\alpha}\cdot\hat{\beta}/(\hat{\alpha}\cdot\hat{\alpha})$ cannot be positive. □

The information about the r simple roots $\hat{\alpha}^1,\ldots,\hat{\alpha}^r$ is conveniently stored in the Cartan matrix.

Definition 5.16 (Cartan matrix). Given the r simple roots of a simple, compact Lie algebra, the $r\times r$ *Cartan matrix* A is defined by the matrix elements

$$A_{ij} = \frac{2\hat{\alpha}^i\cdot\hat{\alpha}^j}{|\hat{\alpha}^j|^2}\,. \tag{5.23}$$

From the definition it follows that

- the diagonal elements of the Cartan matrix are always 2.
- the off-diagonal elements encode the angles and relative lengths of the simple roots. Assuming $A_{ij} \leq A_{ji}$ the possible values are

$$\begin{array}{c|c|c|c|c} (A_{ij}, A_{ji}) & (0,0) & (-1,-1) & (-2,-1) & (-3,-1) \\ \text{angle} & \frac{\pi}{2} & \frac{2\pi}{3} & \frac{3\pi}{4} & \frac{5\pi}{6} \end{array} \tag{5.24}$$

Example 5.17 (Cartan matrix of $\mathfrak{su}(3)$). The two simple roots are

$$\hat{\alpha}^1 = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \qquad \hat{\alpha}^2 = \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right). \tag{5.25}$$

Evaluating the scalar products of the roots gives

$$|\hat{\alpha}^1|^2 = |\hat{\alpha}^2|^2 = 1\,, \qquad \hat{\alpha}^1 \cdot \hat{\alpha}^2 = -\tfrac{1}{2}\,. \tag{5.26}$$

Substituting these values into the definition of the Cartan matrix (5.23) gives

$$A = \left[\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right]. \tag{5.27}$$

6 Irreducible representations of $\mathfrak{su}(3)$ and the quark model

After SU(2), whose representations we have already studied, SU(3) is arguably the most important (simple) compact Lie group in particle physics. The simplest application is the quark model that describes mesons and baryons in terms of their constituent quarks. The main idea is that the three lightest quarks, the up, down and strange quark, are related by SU(3) flavour symmetry, spanning a 3-dimensional vector space that transforms under the defining representation (or *fundamental representation*). Their anti-particles transform accoding to the complex conjugate of this representation (or *anti-fundamental representation*) to be defined below. Any composite states of quarks and anti-quarks live in tensor products of these representations, which necessarily decompose into irreducible representations of SU(3). The similarities between the masses of certain mesons and baryons can be explained by recognizing the particles with similar masses as living in the same irreducible representation. To understand this grouping let us start by classifying the irreducible representations.

Remark 6.1. This approximate flavour-SU(3) symmetry is unrelated to the exact color-SU(3) symmetry of the Standard Model gauge group SU(3) × SU(2) × U(1) that was discovered later.

6.1 The algebra

As we have seen in Section 2.4, the fundamental representation of the Lie algebra $\mathfrak{su}(3)$ is given by the traceless antihermitian 3 × 3 matrices,

$$\mathfrak{su}(3) = \left\{ X \in \mathbb{C}^{3\times 3} : X^\dagger = -X,\ \operatorname{tr} X = 0 \right\}.$$

A basis of Hermitian traceless 3 × 3 matrices, analogous to the Pauli matrices of $\mathfrak{su}(2)$, is given by the *Gell-Mann matrices*

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{6.1}$$

They are normalized such that $\operatorname{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. It is customary to take the generators in the fundamental representation to be

$$\xi_a = -\tfrac{i}{2}\lambda_a, \qquad \operatorname{tr}(\xi_a \xi_b) = -\tfrac{1}{2}\delta_{ab}, \quad a, b = 1, \ldots, 8. \tag{6.2}$$

The structure constants f_{abc} are completely antisymmetric and the only nonvanishing constants (with indices in ascending order) are

$$f_{123} = 1, \quad f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = -f_{367} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \tag{6.3}$$

Any other set of Hermitian matrices $X_1, \ldots, X_8$ satisfying

$$[X_a, X_b] = f_{abc} X_c \tag{6.4}$$

therefore determines a representation of $\mathfrak{su}(3)$ (and of SU(3) via the exponential map).

6.2 Cartan subalgebra and roots

By examining the structure constants (6.3) one may observe that at most two generators can be found to commute, meaning that the Lie algebra $\mathfrak{su}(3)$ is of rank 2. The Cartan subalgebra is conventionally taken to be spanned by the generators $H_1 = i\xi_3 = \frac{1}{2}\lambda_3$ and $H_2 = i\xi_8 = \frac{1}{2}\lambda_8$, which in the defining representation are diagonal. These generators have clear physical interpretations in the quark model, since

$$\mathcal{I}_3 = H_1 = \frac{1}{2}\lambda_3, \quad Y = \frac{2}{\sqrt{3}}H_2 = \frac{1}{\sqrt{3}}\lambda_8, \quad Q = \mathcal{I}_3 + \frac{1}{2}Y \tag{6.5}$$

are respectively the *isospin*, *hypercharge* and *electrical charge* generators of the quarks.

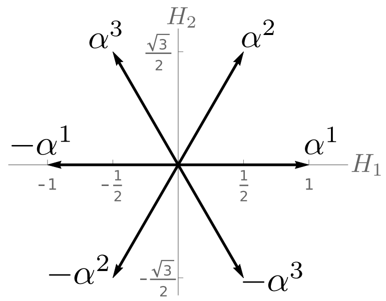
The remaining generators can be combined into raising and lowering operators E_α,

$$E_{\pm\alpha^1} = \frac{i}{\sqrt{2}}(\xi_1 \pm i\xi_2), \quad E_{\pm\alpha^2} = \frac{i}{\sqrt{2}}(\xi_4 \pm i\xi_5), \quad E_{\pm\alpha^3} = \frac{i}{\sqrt{2}}(\xi_6 \pm i\xi_7) \tag{6.6}$$

with corresponding roots

$$\alpha^1 = (1,0), \quad \alpha^2 = (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}), \quad \alpha^3 = (-\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}). \tag{6.7}$$

The root diagram therefore looks as follows.



6.3 Representations

Let us assume $X_1, \ldots, X_8$ are N-dimensional skew-Hermitian matrices satisfying (6.4) and let us assume they determine an irreducible representation of $\mathfrak{su}(3)$. The corresponding matrices $T_a = iX_a$ are Hermitian, and $H_1 = iX_3 = T_3$ and $H_2 = iX_8 = T_8$ are the Cartan generators.

Recall from (5.12) that for each root α^k, the operators $E_{\pm\alpha^k}$ and $\alpha^k_i H_i$ form an $\mathfrak{su}(2)$-subalgebra inside $\mathfrak{su}(3)$,

$$[E_{\alpha^k}, E_{-\alpha^k}] = \alpha^k_i H_i, \qquad [\alpha^k_i H_i, E_{\pm\alpha^k}] = \pm E_{\pm\alpha^k}, \tag{6.8}$$

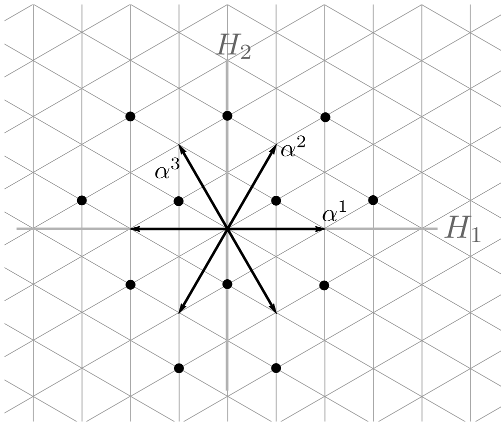
which may or may not be reducible (note that $|\alpha^k| = 1$). In any case our knowledge about $\mathfrak{su}(2)$-representations implies that the eigenvalues of $\alpha^k_i H_i$ are half-integers. These eigenvalues are precisely the inner products $\alpha^k \cdot \mu$ of the roots with the weights $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ of the representation. Hence, the weights must lie on the vertices of the triangular lattice depicted in the figure below.

If $\mu \in \mathbb{R}^2$ is a weight, meaning that $H_i|\Psi\rangle = \mu_i|\Psi\rangle$ for some state $|\Psi\rangle$, then

$$H_i E_{\pm\alpha^k}|\Psi\rangle = (E_{\pm\alpha^k}H_i \pm \alpha^k_i E_{\pm\alpha^k})|\Psi\rangle = (\mu_i \pm \alpha^k_i)E_{\pm\alpha^k}|\Psi\rangle, \tag{6.9}$$

so $E_{\pm\alpha^k}|\Psi\rangle$ is a state with weight $\mu \pm \alpha^k$ provided $E_{\pm\alpha^k}|\Psi\rangle \neq 0$. Using a similar reasoning as for the roots in Section 5.3 it follows that if $\alpha^k \cdot \mu > 0$ then all of μ, $\mu - \alpha^k$, $\mu - 2\alpha^k$, $\ldots$, $\mu - 2(\alpha^k \cdot \mu)\alpha^k$ are also weights. In particular, the set of weights is invariant under the Weyl reflections $\mu \to \mu - 2(\alpha^k \cdot \mu)\alpha^k$, i.e. reflections in the planes orthogonal to the roots α^1, α^2 and α^3.

Moreover, any two states with different weights can be obtained from each other by actions of the operators $E_{\pm\alpha^k}$, because the collection of all states accessible from some initial state span an invariant subspace, which has to be the full vector space because of irreducibility. Therefore the weights must differ by integer multiples of α^k. Combining these observations we conclude that the set of weights necessarily occupy a convex polygon that is invariant under Weyl reflections (thus either a hexagon or a triangle), like in the following figure.



6.4 Highest weight states

In Chapter 3 we have seen that irreducible representations of $\mathfrak{su}(2)$ are characterized by their highest weights (j in the spin-j representation). Something similar is true for $\mathfrak{su}(3)$, but now the weights are two-dimensional so we need to specify what it means for a weight to be highest. To this end we make use of the simple roots of $\mathfrak{su}(3)$ which we have determined to be

$$\hat{\alpha}^1 = \alpha^2 = (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}), \qquad \hat{\alpha}^2 = -\alpha^3 = (\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{3}) \tag{6.10}$$

in Example 5.17.

Definition 6.1 (Highest weight). A state $|\Psi\rangle$ that is annihilated by the raising operators $E_{\hat{\alpha}^1}$ and $E_{\hat{\alpha}^2}$ is called a *highest weight state*. Its weight is called the *highest weight*.

Any irreducible representation can be shown to have a unique highest weight with multiplicity one. In the weight diagram above it corresponds to the right-most weight (i.e. largest H_1-eigenvalue).

Definition 6.2 (Label). The label $(p, q) \in \mathbb{Z}^2$ of an irreducible representation is given in terms of the highest weight μ by

$$p = 2\,\hat{\alpha}^1 \cdot \mu, \quad q = 2\,\hat{\alpha}^2 \cdot \mu. \tag{6.11}$$

The irreducible representations of $\mathfrak{su}(3)$ satisfy the following properties, which we do not prove here.

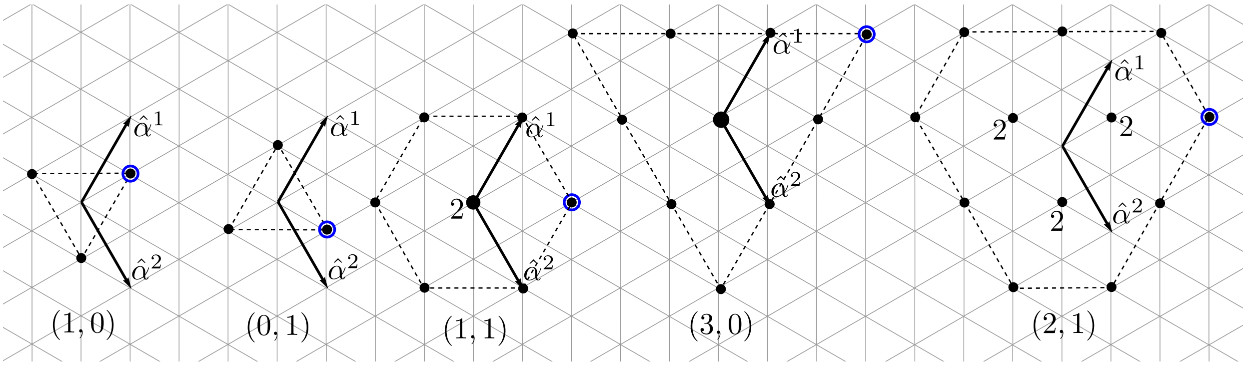
- Each label (p, q), $p, q = 0, 1, 2, \ldots$, corresponds to a unique irreducible representation of $\mathfrak{su}(3)$.
- The set of weights of the representation with label (p, q) can be determined as follows. By inverting (6.11) the highest weight is

$$\mu = \Big(\tfrac{1}{2}(p+q), \tfrac{1}{2\sqrt{3}}(p-q)\Big). \tag{6.12}$$

  The Weyl reflections of μ determine the corners of a convex polygon P, that is either a hexagon or a triangle. The complete set of weights then corresponds to those vectors of the form $\mu + n_1\hat{\alpha}^1 + n_2\hat{\alpha}^2$ with $n_1, n_2 \in \mathbb{Z}$ that are contained in P.

- The multiplicities of the weights (i.e. the number of linearly independent states with that weight) can be deduced from the diagram. The weights occur in nested "rings" that are either hexagonal (the outer ones) or triangular (the inner ones). The weights on the outer ring have multiplicity 1. Each time one moves inwards from a hexagonal ring to the next ring the multiplicity increases by 1. Inside triangular rings the multiplicities do not change anymore.

Examples of weight diagrams with small labels are shown below. The highest weights are circled (in blue), and multiplicities are indicated on the weights when they are larger than 1.



As one may verify for these examples, the dimension of the representation is generally given by

$$N = \frac{1}{2}(p+1)(q+1)(p+q+2). \tag{6.13}$$

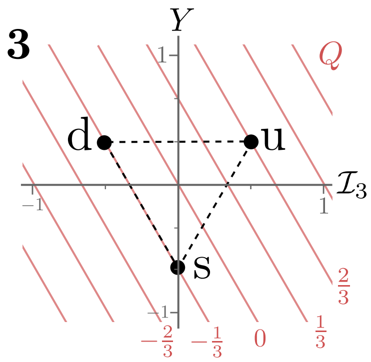
For $p \neq q$ the two irreducible representations (p, q) and (q, p) are not equivalent but related by complex conjugation. To see this, let T_a be the generators of the irreducible (p, q)-representation. Since $[T_a, T_b] = i f_{abc} T_c$ with real structure constants f_{abc} we have that

$$[-T_a^*, -T_b^*] = [T_a, T_b]^* = i f_{abc}(-T_c^*), \tag{6.14}$$

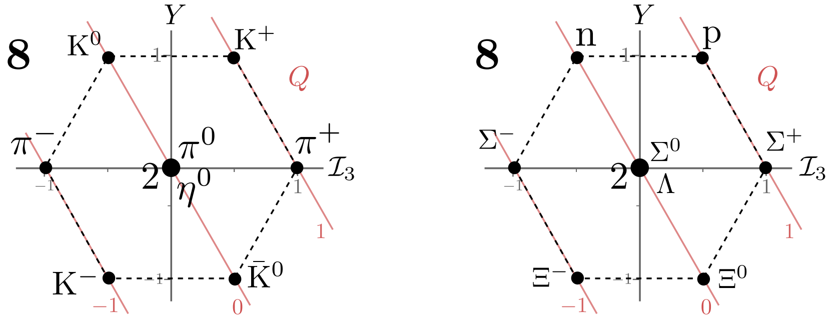
showing that the matrices $-T_a^*$ also generate a representation, which is automatically irreducible too. All eigenvalues of $H_1 = T_3$ and $H_2 = T_8$ have changed sign, thus changing the set of weights to those of the (q, p) representation. The irreducible representations are often denoted by their dimension (6.13) in bold face $\mathbf{N}$ instead of their label (p, q), where if $p > q$ the representation $\mathbf{N}$ corresponds to (p, q) and $\bar{\mathbf{N}}$ to (q, p).

Let us discuss the interpretation of some of these representations in the quark model.

- $\mathbf{3}$, $(p, q) = (1, 0)$: This is the defining representation or *fundamental representation* (6.2) of $\mathfrak{su}(3)$ as can be easily checked by determining the eigenvalues of $H_1 = \frac{1}{2}\lambda_3$ and $H_2 = \frac{1}{2}\lambda_8$. The three basis states, $|\mathrm{u}\rangle, |\mathrm{d}\rangle, |\mathrm{s}\rangle$ correspond to the up, down and strange quark respectively. Their arrangement determines the values of the isospin and charge via (6.5).
- $\bar{\mathbf{3}}$, $(p, q) = (0, 1)$: the *complex conjugate representation* or *anti-fundamental representation*. Since the corresponding states have opposite quantum numbers as compared to the fundamental representation, it stands to reason that they correspond to the anti-quarks $|\bar{\mathrm{u}}\rangle, |\bar{\mathrm{d}}\rangle, |\bar{\mathrm{s}}\rangle$.



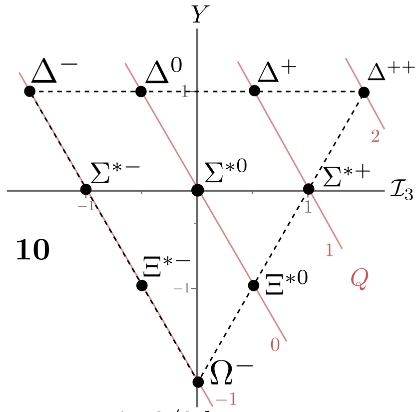
- **8**, $(p, q) = (1, 1)$: this is the *octet* or *adjoint representation*, since the non-zero weights are exactly the roots of $\mathfrak{su}(3)$. It appears in the decomposition $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ of the tensor product of the fundamental and anti-fundamental representation, and therefore describes bound states of a quark with an anti-quark, which are *mesons*. It also appears in the decomposition $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ of the tensor product of three fundamental representations, describing bound states of three quarks, which are *baryons*. The baryons in the adjoint representation have spin 1/2.



mesons

spin-1/2 baryons

- **10**, $(p, q) = (3, 0)$: the *decuplet* representation also appears in the decomposition of $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ and describes the spin-3/2 baryons.



spin-3/2 baryons

We close this section with several remarks concerning the quark model.

- The existence of quarks was unknown in the early sixties. It is on the basis of the nice grouping (the *Eightfold way*) of the discovered mesons and baryons into octets and decuplets that Gell-Mann and Zweig (independently in 1964) postulated the existence of more fundamental particles living in the fundamental representation of $\mathfrak{su}(3)$ (dubbed “quarks” by Gell-Mann). At that time the spin-3/2 baryon Ω^- had not been detected yet, but on the basis of symmetry its approximate mass and quantum numbers were predicted and soon confirmed.

- The $\mathfrak{su}(3)$ flavor symmetry is only an approximate symmetry of the standard model (contrary to the $\mathfrak{su}(3)$ color symmetry of QCD) because, among other reasons, the quark masses are quite different ($m_\mathrm{u} = 2.3\,\mathrm{MeV}$, $m_\mathrm{d} = 4.8\,\mathrm{MeV}$, $m_\mathrm{s} = 95\,\mathrm{MeV}$) although a lot lighter than the three other quarks ($m_\mathrm{c} = 1270\,\mathrm{MeV}$, $m_\mathrm{b} = 4\,\mathrm{GeV}$, $m_\mathrm{t} = 172\,\mathrm{GeV}$). Since the difference in masses between the up and down quark is much smaller than between the up and strange quark, one would expect that the symmetry relating the up and down quarks is more accurate. This smaller symmetry corresponds precisely to $\mathfrak{su}(2)$-subrepresentation associated to the root $\alpha^1 = (1, 0)$, which has the isospin $\mathcal{I}_3$ as Cartan generator. Indeed, the masses of the mesons and baryons in the corresponding $\mathfrak{su}(2)$-multiplets (horizontal lines in the diagrams) are quite close, e.g. the proton and neutron among the spin-1/2 baryons.

7 Classification of compact, simple Lie algebras

7.1 Simple Lie algebras

Recall (Definition 4.5) that a Lie algebra is called *compact* if its Cartan-Killing metric is negative definite. An important result, that we will not prove, is that any compact Lie algebra $\mathfrak{g}$ decomposes as a direct sum of compact, *simple* Lie algebras

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_2. \tag{7.1}$$

Hence, in order to classify compact Lie algebras it is sufficient to classify compact, simple Lie algebras and then consider all possible compositions. In order to introduce the concept of simple Lie algebras, we need a few definitions.

Recall that a subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a subalgebra of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A stronger condition is the following.

Definition 7.1 (Invariant subalgebra). A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called *invariant* if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, i.e.

$$[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}. \tag{7.2}$$

For a Lie algebra $\mathfrak{g}$ with a basis $X_1, \ldots, X_n$ of generators and corresponding structure constants f_{abc} the k-dimensional subspace spanned by $X_1, \ldots, X_k$ is a subalgebra precisely when

$$f_{ij\alpha} = 0 \qquad \text{for } 1 \le i, j \le k,\ \ k < \alpha \le n, \tag{7.3}$$

while it is invariant if in addition

$$f_{i\beta\alpha} = 0 \qquad \text{for } 1 \le i \le k,\ \ k < \alpha, \beta \le n. \tag{7.4}$$

Definition 7.2 (Simple Lie algebra). A Lie algebra $\mathfrak{g}$ is *simple* if it has no invariant subalgebras (other than $\{0\}$ and $\mathfrak{g}$ itself).

Equivalently, a Lie algebra is simple precisely when its adjoint representation is irreducible. Indeed, $\mathfrak{h}$ is an invariant subalgebra precisely when $X|Y\rangle = |[X, Y]\rangle \in \mathfrak{h}$ for any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. This is precisely the condition for $\mathfrak{h} \subset \mathfrak{g}$ to be an invariant subspace for the adjoint representation.

7.2 Dynkin diagrams

In Chapter 5 we have seen that compact Lie algebras can be characterized in various equivalent ways (with increasing simplicity) by

- the root system $\Phi = \{\alpha^i\}$;
- the collection of simple roots $\{\hat{\alpha}^i\}$;
- the Cartan matrix $A_{ij} = \frac{2\hat{\alpha}^i \cdot \hat{\alpha}^j}{\hat{\alpha}^j \cdot \hat{\alpha}^j}$.

The Cartan matrix is often visualized via its Dynkin diagram.

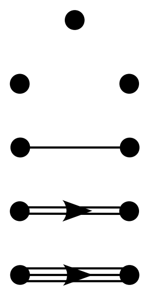
Definition 7.3 (Dynkin diagram). The Dynkin diagram describing a Lie algebra is constructed as follows:

- simple roots are represented by nodes •
- $\max\{|A_{ij}|, |A_{ji}|\}$ gives the number of lines connecting the circles representing the roots $\hat{\alpha}^i$ and $\hat{\alpha}^j$.
- if the roots connected by lines have different length, one adds an arrow pointing towards

the shorter root, i.e. if $A_{ij} = -2$ or $A_{ij} = -3$ then the arrow points from $\hat{\alpha}^i$ to $\hat{\alpha}^j$ (hint: think of the arrow as a "greater than" symbol $>$)

In the exercises it is shown that the Dynkin diagram of a simple compact Lie algebra is connected. Otherwise the Dynkin diagram consists of a number of connected components corresponding to the simple compact Lie algebras in to which it decomposes as a direct sum.

From the definition it follows that there is only one Dynkin diagram of rank 1 and that there are four diagrams of rank 2:



$\mathfrak{su}(2)$	$\mathrm{SU}(2)$
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
$\mathfrak{su}(3)$	$\mathrm{SU}(3)$
$\mathfrak{so}(5)$	$\mathrm{SO}(5)$
$\mathfrak{g}_2$	G_2

Four of these correspond to Lie algebras that are familiar from Chapter 2, while G_2 is the first *exceptional* Lie group that we encounter.

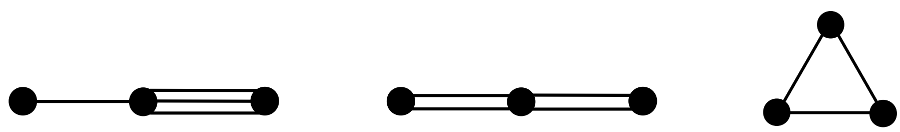
If we go to higher rank, then not every graph built from three or more nodes connected by (multiple) links corresponds to the Dynkin diagram of a compact Lie algebra. Indeed, there have to exist r linearly independent vectors $\hat{\alpha}^1, \ldots, \hat{\alpha}^r$, whose inner products give rise to the links in the diagram. Luckily this is the only criterion that a diagram has to satisfy to qualify as a Dynkin diagram. As we will see now it puts severe restrictions on the types of graphs that can appear.

7.3 Classification of compact, simple Lie algebras

In this section we focus only on **connected** Dynkin diagrams, since we want to classify simple Lie algebras. For the moment we also forget about the arrows on the Dynkin diagrams, since we are just going to consider what angles between the simple roots are possible.

Lemma 7.4 (Dynkin diagrams of rank 3). The only Dynkin diagrams of rank 3 are

Proof. This results from the fact that the sum of angles between 3 linearly independent vectors must be less than 360°. Computing the angle in the first diagram, one has $120°+120°+90° = 330°$. The analogous computation for the second diagram yields $120° + 135° + 90° = 345°$. This also implies that the diagrams shown below are not valid, since the angles add up to 360°:



Adding even more lines of course even increases the sum of angles. □

This has important consequences for diagrams of higher rank, because of the following.

Lemma 7.5 (Subsets of a Dynkin diagram). A connected subset of nodes from a Dynkin diagram together with the links connecting them is again a Dynkin diagram.

Proof. Taking a subset of simple roots preserves the corresponding lengths and angles, and the vectors are still linearly independent. □

In particular, a triple line cannot occur in diagrams with three or more nodes, because otherwise there would be a subset of rank 3 with a triple line. Hence, we have:

Lemma 7.6 (No triple lines). G_2 is the only Dynkin diagram with a triple line.

Another way to shrink a diagram is by shrinking single lines.

Lemma 7.7 (Contraction of a single line). If a Dynkin diagram contains two nodes connected by a single line, then shrinking the line and merging the two nodes results again in a Dynkin diagram.

Proof. Suppose that $\hat{\alpha}$ and $\hat{\beta}$ are connected by a single line. We claim that replacing the pair of vectors $\hat{\alpha}$ and $\hat{\beta}$ by the single vector $\hat{\alpha} + \hat{\beta}$ preserves the necessary properties. By Lemma 7.4 and Lemma 7.5 each other node $\hat{\gamma}$ is connected to at most one of $\hat{\alpha}$ or $\hat{\beta}$ (otherwise there would be a triangle subdiagram). If $\hat{\gamma}$ was connected to $\hat{\alpha}$ then $\hat{\gamma} \cdot (\hat{\alpha} + \hat{\beta}) = \hat{\gamma} \cdot \hat{\alpha}$, while if $\hat{\gamma}$ was connected to $\hat{\beta}$ then $\hat{\gamma} \cdot (\hat{\alpha} + \hat{\beta}) = \hat{\gamma} \cdot \hat{\beta}$. Since also $|\hat{\alpha} + \hat{\beta}|^2 = |\hat{\alpha}|^2 = |\hat{\beta}|^2$, all lines other than the contracted one are preserved. The resulting set of vectors is also still linearly independent. □

As a consequence we have that

Lemma 7.8 (No loops). A Dynkin diagram cannot contain loops.

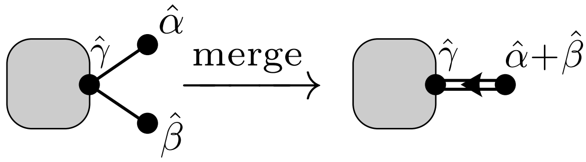
Proof. By Lemma 7.4 a loop must always contain a single edge, which can be contracted by Lemma 7.7. Hence, every loop can be contracted to a triangle, in contradiction with Lemma 7.4. □

Lemma 7.9 (At most one double line). A Dynkin diagram can contain at most one double line.

Proof. If there are two or more double lines, then one can always contract single lines until two double lines becomes adjacent. But this is not allowed by Lemma 7.4. □

At this point we know that Dynkin diagrams always have the shape of a tree with at most one of its links being a double line. To further restrict branching of the tree the following result is quite useful.

Lemma 7.10 (Merging two single lines). If a Dynkin diagram has a node with (at least) two dangling single lines, then merging them into a double line gives another Dynkin diagram. In other words, the following is a valid operation on Dynkin diagrams (with the gray blob representing the remaining part of the diagram):



Proof. Replacing $\hat{\alpha}$ and $\hat{\beta}$ by $\hat{\alpha}+\hat{\beta}$, linear independence of the resulting set of vectors is automatic. Hence it remains to check the inner products between $\hat{\gamma}$ and $\hat{\alpha} + \hat{\beta}$. Using that $|\hat{\alpha}|^2 = |\hat{\beta}|^2 = |\hat{\gamma}|^2$,

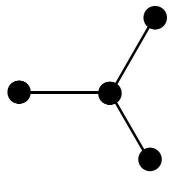
$\hat{\alpha} \cdot \hat{\beta} = 0$, and $2\hat{\alpha} \cdot \hat{\gamma}/|\hat{\gamma}|^2 = 2\hat{\beta} \cdot \hat{\gamma}/|\hat{\gamma}|^2 = -1$, it follows indeed that

$$2\frac{\hat{\gamma} \cdot (\hat{\alpha} + \hat{\beta})}{|\hat{\gamma}|^2} = -2 \qquad \text{and} \qquad 2\frac{\hat{\gamma} \cdot (\hat{\alpha} + \hat{\beta})}{|\hat{\alpha} + \hat{\beta}|^2} = -1. \tag{7.5}$$

This corresponds precisely with the diagram on the right. □

As a consequence a Dynkin diagram can only contain one particular junction.

Lemma 7.11 (At most one junction or double line). A Dynkin diagram contains one double line, or one junction, or neither. A junction is a node with more than two neighbours. Any junction must have the form



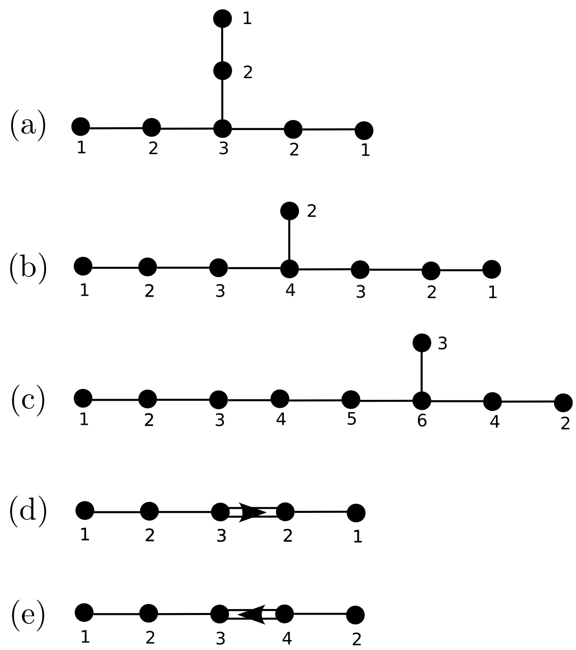
,

i.e. three neighbours connected by single lines.

Proof. By repeated contraction (Lemma 7.7) and merging (Lemma 7.10) any diagram with a different type of junction (with more than three neighbours or involving double lines) can be reduced to a diagram with two adjacent double lines, which is not allowed. The same is true if a diagram has two junctions, or both a junction and a double line. □

The remaining step is to get rid of some remaining diagrams on a case-by-case basis, by showing that they cannot satisfy linear independence.

Lemma 7.12 (Diagrams failing linear independence). The following diagrams cannot be realized by linear independent vectors:

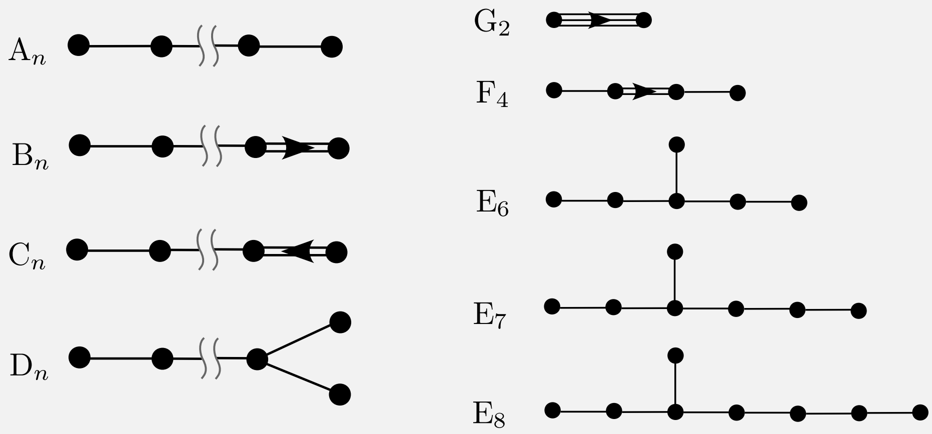


Proof. The labels $\mu_1, \ldots, \mu_r$ indicated on the diagram are such that $\sum_{j=1}^r \mu_j \hat{\alpha}^j = 0$. To see this one may straightforwardly compute $|\sum_{j=1}^r \mu_j \hat{\alpha}^j|^2 = \sum_{i,j=1}^r \mu_i \mu_j \hat{\alpha}^i \cdot \hat{\alpha}^j = 0$ using the inner products

$\hat{\alpha}^i \cdot \hat{\alpha}^j$ as prescribed by the diagram. Such a linear relation implies that the vectors $\hat{\alpha}^j$ are not linearly independent. □

This is all we need to deduce the final classification theorem. In general, connected Dynkin diagrams are denoted by a capital letter denoting the family and a subscript denoting the rank, e.g. B_2 for the Dynkin diagram of the rank-2 Lie algebra $\mathfrak{so}(5)$. The same notation is also sometimes confusingly used to specify the corresponding Lie algebra and/or Lie group.

Theorem 7.13 (Classification of simple, compact Lie algebras). Any simple, compact Lie algebra has a Dynkin diagram equal to a member of one of the four infinite families A_n, B_n, C_n, D_n, $n \geq 1$, or to one of the exceptional Lie algebras G_2, F_4, E_6, E_7, or E_8. In other words its Dynkin diagram appears in the following list:



Proof. Let's have a look at the three cases or Lemma 7.11.

No junction or double line: Then the diagram is clearly of the form of A_n for some $n \geq 1$.

One double line: The diagram has no junction. If the double line occurs at an extremity then it is of the form of B_n of C_n. If it sits in the middle, then by Lemma 7.12(d) and (e) it can have only one single edge on both sides, i.e. of the form F_4.

One junction: Let's denote the lengths of the paths emanating from the junction by $\ell_1 \leq \ell_2 \leq \ell_3$. Using Lemma 7.12: (a) implies $\ell_1 = 1$; (b) implies $\ell_2 \leq 2$; and (c) implies $\ell_2 = 1$ or $\ell_3 \leq 4$. It is then easy to see that D_n ($\ell_2 = 1$), E_6 ($\ell_2 = \ell_3 = 2$), E_7 ($\ell_2 = 2$, $\ell_3 = 3$), and E_8 ($\ell_2 = 2$, $\ell_3 = 4$) are the only possibilities. □

As one may note by looking at the infinite families for small n, there are some overlaps between the families. Since there is only one rank-1 diagram it makes sense to label it by A_1, and require $n \geq 2$ for the remaining families. However, $B_2 = C_2$ and $D_3 = A_3$, while D_2 consisting of two disconnected nodes does not correspond to a simple Lie algebra. Hence, to make the classification unique one may require $n \geq 1$ for A_n, $n \geq 2$ for B_n, $n \geq 3$ for C_n, and $n \geq 4$ for D_n.

Finally, going back to the common Lie groups that we identified in Chapter 6 it is possible to obtain the following identification:

Cartan label	Lie algebra	dimension	rank
A_n	$\mathfrak{su}(n+1)$	$(n+1)^2-1$	$n\geq 1$
B_n	$\mathfrak{so}(2n+1)$	$(2n+1)n$	$n\geq 2$
C_n	$\mathfrak{sp}(n)$	$(2n+1)n$	$n\geq 3$
D_n	$\mathfrak{so}(2n)$	$(2n-1)n$	$n\geq 4$
G_2		14	2
F_4		52	4
E_6		78	6
E_7		133	7
E_8		248	8

From the overlaps described above it follows that we have the following isomorphisms between Lie algebras,

Cartan label	Lie algebra
A_1, B_1, C_1	$\mathfrak{so}(3)\simeq\mathfrak{su}(2)\simeq\mathfrak{sp}(1)$
B_2, C_2	$\mathfrak{so}(5)\simeq\mathfrak{sp}(2)$
A_3, D_3	$\mathfrak{so}(6)\simeq\mathfrak{su}(4)$

, (7.6)

as well as the nonsimple $\mathfrak{so}(4)=\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ (see exercises).

7.4 Concluding remark on unified gauge theories

The idea behind *Grand Unified Theories* (GUT) is to embed the gauge group of the standard model of particle physics, $\mathrm{SU}(3)\times\mathrm{SU}(2)\times U(1)$ in a simple, compact Lie group. In other words we want to recognize $\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ as a subalgebra in one of the simple, compact Lie algebras of Theorem 7.13. Based on the Dynkin diagrams, one can determine which Lie algebras are suited for this purpose. The simplest possibility turns out to be A_4, i.e. $\mathfrak{su}(5)$.

Let us demonstrate more generally how one can obtain a subalgebra from a rank-r Lie algebra $\mathfrak{g}$ by removing a node $\hat{\alpha}^0$ from its Dynkin diagram. We label the simple roots as $\hat{\alpha}^0,\ldots\hat{\alpha}^{r-1}$ and choose a basis of Cartan generators $H_0,\ldots,H_{r-1}$ such that $\hat{\alpha}^j_0=0$ for $j\geq 1$. Then the Cartan generators $H_1,\ldots,H_{r-1}$ together with the roots arising from Weyl reflections of $\hat{\alpha}^1,\ldots,\hat{\alpha}^{r-1}$ span a rank-$(r-1)$ subalgebra $\mathfrak{g}_1\subset\mathfrak{g}$, whose Dynkin diagram is the original one with the node $\hat{\alpha}^0$ removed. However, one can extend the subalgebra to $\mathfrak{g}_1\oplus\mathfrak{u}(1)$ by adding the Cartan generator H_0, since by construction it commutes with all of $\mathfrak{g}_1$.

In particular, if we remove one of the middle nodes of the diagram A_4 of $\mathfrak{su}(5)$ then the remaining diagram decomposes in an A_2 and an A_1, which together form the Dynkin diagram for $\mathfrak{su}(3)\oplus\mathfrak{su}(2)$. Hence we find the explicit subalgebra

$$\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)\subset\mathfrak{su}(5), \tag{7.7}$$

which forms the basis of the famous Georgi-Glashow GUT based on SU(5).