Lie Algebras in Particle Physics (NWI-NM101B)

Problem sheet #1: Introduction to groups and representations

Tutorial on Thursday 14 April 2022

Exercise 1.1: Isomorphic definitions of the cyclic group

Different instances of a group can give rise to the same abstract group structure, meaning that they are isomorphic as groups. In this exercise you will verify this by constructing some explicit isomorphisms.

a) Show that for any positive integer n the following three groups are isomorphic by identifying explicit isomorphisms.

- the set $\mathbb{Z}_n \coloneqq \{0, 1, 2, \dots, n-1\}$ equipped with addition modulo n;
- the set C_n of cyclic permutations on $\{0, 1, 2, \dots, n-1\}$ equipped with composition;
- the nth roots of unity $U_n \coloneqq \{z \in \mathbb{C} : z^n = 1\}$ equipped with multiplication.

b) Are these isomorphisms unique?

Exercise 1.2: Direct sum and tensor product representations

An important tool in constructing new representations of a group is via the direct sum and the tensor product of two (or more) representations. Here you will recall their construction and check the representation axioms.

Let V be an N-dimensional vector space with basis $|v_1\rangle, \dots, |v_N\rangle$ and W an M-dimensional vector space with basis $|w_1\rangle, \dots, |w_M\rangle$. By definition, the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are the vector spaces defined by specifying their basis as follows:

$$\textbf{Direct sum } V \oplus W \text{ has basis } |v_1\rangle, \dots, |v_N\rangle, |w_1\rangle, \dots, |w_M\rangle,$$
$$\textbf{Tensor product } V \otimes W \text{ has basis } |v_1, w_1\rangle, |v_1, w_2\rangle, \dots, |v_N, w_{M-1}\rangle, |v_N, w_M\rangle.$$

Given two representations $D_1 : G \to \mathrm{GL}(V)$ and $D_2 : G \to \mathrm{GL}(W)$ of the same group G, the direct sum representation $D_1 \oplus D_2 : G \to \mathrm{GL}(V_1 \oplus V_2)$ is defined on its basis elements via

$$(D_1 \oplus D_2)(g)|v_i\rangle = D_1(g)|v_i\rangle, \qquad (D_1 \oplus D_2)(g)|w_i\rangle = D_2(g)|w_i\rangle.$$

The tensor product representation $D_1 \otimes D_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$ is defined on its basis elements $|v_i, w_j\rangle \equiv |v_i\rangle \otimes |w_j\rangle$ via

$$(D_1 \otimes D_2)(g)|v_i, w_j\rangle = (D_1(g)|v_i\rangle) \otimes (D_2(g)|w_j\rangle).$$

a) Show that these are indeed valid representations.

b) What are the dimensions of these representations?

Let us consider the group $G = \{e, a, b\}$ of order 3 with 1-dimensional representation $D_1 : G \to \mathrm{GL}(\mathbb{C})$ given by

$$D_1(e) = 1, \qquad D_1(a) = e^{2\pi i/3}, \qquad D_1(a) = e^{4\pi i/3}$$

and $D_2 : G \to \mathrm{GL}(\mathbb{C}^3)$ the 3-dimensional regular representation

$$D_2(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_2(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D_2(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

c) Determine explicit matrices for the representations $D_1 \oplus D_2$ and $D_1 \otimes D_2$.

Exercise 1.3: Unitary representations

Unitary representations are often easier to work with than arbitrary representations. Luckily, any representation of a finite group can be turned into a unitary one by a similarity transformation. At least, this was claimed so in the lecture. Here you will prove this fact.

Recall that the standard Hermitian inner product on $\mathbb{C}^N$ is given by $\langle v|w\rangle = \sum_{i=1}^N \overline{v}_i w_i$ and that for any $N \times N$ matrix A, $\langle v|Aw\rangle = \langle A^\dagger v|w\rangle$. Let us verify that every representation $D(g) : G \to \mathrm{GL}(\mathbb{C}^N)$ of a finite group G is equivalent to a unitary representation. To do this we introduce the $N \times N$ matrix

$$S = \sum_{g \in G} D(g)^\dagger D(g). \tag{3.1}$$

a) Show that S is Hermitian, $S^\dagger = S$.

This implies that S is diagonalizable with real eigenvalues λ_i: $S = U^{-1}\Lambda U$ with U unitary ($U^\dagger U = \mathbb{1}$) and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ a real diagonal matrix.

b) Show that all eigenvalues of S are positive.
Hint: Let v_i be a normalized eigenvector of S with eigenvalue λ_i and compute $\langle v_i|Sv_i\rangle$.

c) Using the diagonal form of S, construct the square-root $X = S^{1/2}$ and show that X is Hermitian.

d) Verify that the similarity transformation $D'(g) = XD(g)X^{-1}$ defines a unitary representation $D' : G \to \mathrm{GL}(\mathbb{C}^N)$.

Lie Algebras in Particle Physics (NWI-NM101B)

Problem sheet #2(graded): Lie groups and Lie algebras

Tutorial on Thursday 21 April 2022.

This is one of **3 graded assignments** contributing to a bonus point on the exam.
Solutions should be submitted digitally before 13:30 on 28 April via Brightspace Assignments.

Exercise 2.1: Lie algebras and the Jacobi identity (3 points)

Verifying the Jacobi identity and a first encounter with the adjoint representation, which will be the subject of Chapter 5.

Consider the Lie algebra $\mathfrak{g}$ of an n-dimensional matrix Lie group G with generators $\xi_1, \ldots, \xi_n$ that satisfy

$$[\xi_a, \xi_b] = f_{abc}\xi_c \,, \tag{1.1}$$

where f_{abc} are the structure constants and we use Einstein's summation convention over repeated indices.

a) Use that the Lie bracket is implemented as a commutator $[X, Y] = XY - YX$ to prove the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{for } X, Y, Z \in \mathfrak{g}.$$

b) Setting $X = \xi_a$, $Y = \xi_b$ and $Z = \xi_c$ in the Jacobi identity as above, show that

$$f_{bcd}f_{ade} + f_{cad}f_{bde} + f_{abd}f_{cde} = 0\,. \tag{1.2}$$

c) The structure constants allow one to define the *adjoint representation* $\mathfrak{D}_{\text{adj}} : \mathfrak{g} \to \mathbb{C}^{n\times n}$ of $\mathfrak{g}$ with generators $\mathfrak{D}_{\text{adj}}(\xi_a) = T_a$ where T_a are the matrices with entries given by

$$[T_a]_{bc} = f_{acb} \qquad (a, b, c = 1, \ldots, n). \tag{1.3}$$

Show with the help of (1.2) that this indeed satisfies the requirement of a representation, namely that

$$[T_a, T_b] = f_{abc}T_c\,.$$

Exercise 2.2: Non-abelian gauge theories (4 points)

In this exercise we will see how Lie groups and Lie algebras feature in non-abelian gauge theories and you will prove that the gauge field action is gauge invariant.

The vector potential $A_\mu(x)$ in gauge theory is a covariant 4-vector field that takes values in the Lie algebra $\mathfrak{su}(N)$. This means that $A_\mu(x) = A^a_\mu(x)\xi_a$ where ξ_a, $a = 1, \dots, N^2 - 1$, is a basis of generators of $\mathrm{SU}(N)$ and the components $A^a_\mu(x)$ are real fields[1]. The generators satisfy commutation relations $[\xi_a, \xi_b] = f_{abc}\xi_c$ where the basis is chosen such that structure constants f_{abc} are completely antisymmetric. A *gauge transformation* is a smooth mapping $U : \mathbb{R}^4 \to \mathrm{SU}(N)$, i.e. a space-dependent unitary matrix $U(x) \in \mathrm{SU}(N)$. Under such a gauge transformation the vector potential transforms as

$$A_\mu(x) \;\to\; A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + U(x)\partial_\mu U^\dagger(x)\,. \tag{2.1}$$

Suppose $\psi(x) = (\psi_1(x), \dots, \psi_n(x))$ is an N-component field that transforms as

$$\psi(x) \;\to\; \psi'(x) = U(x)\psi(x). \tag{2.2}$$

For any quantity that transforms like this, we introduce the gauge-covariant derivative via

$$D_\mu \coloneqq \partial_\mu + A_\mu(x). \tag{2.3}$$

a) Show that the covariant derivative of $\psi(x)$ transforms in the same way as $\psi(x)$ itself, i.e.

$$D_\mu\psi(x) \;\to\; D'_\mu\psi'(x) = U(x)D_\mu\psi(x). \tag{2.4}$$

Hint: use $U^\dagger U = \mathbb{1}$ to determine how $\partial_\mu U(x)^\dagger$ and $\partial_\mu U(x)$ are related.

From this it follows that also higher-order covariant derivates transform the same way as (2.2), e.g.

$$D'_\mu D'_\nu\psi'(x) = U(x)D_\mu D_\nu\psi(x). \tag{2.5}$$

b) Defining the field strength $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu, A_\nu]$, prove that

$$[D_\mu, D_\nu]\,\psi(x) = F_{\mu\nu}(x)\psi(x)\,. \tag{2.6}$$

c) The components $F^a_{\mu\nu}$ of the field strength are defined via $F_{\mu\nu} = F^a_{\mu\nu}\xi_a$. Prove that

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + f_{abc}A^b_\mu(x)A^c_\nu(x)\,. \tag{2.7}$$

d) Use (2.6) to show that the field strength transforms as

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x)\,. \tag{2.8}$$

Hint: Combine $\left[D'_\mu, D'_\nu\right]\psi'(x) = F'_{\mu\nu}(x)\psi'(x)$ with (2.5).

e) The action of the gauge field in flat Minkowski space is defined by

$$S[A] = -\frac{1}{2}\int \mathrm{d}^4x\,\eta^{\alpha\mu}\eta^{\beta\nu}\,\mathrm{Tr}[F_{\alpha\beta}F_{\mu\nu}]\,, \tag{2.9}$$

where the trace is over the Lie algebra matrix elements. Show that it is invariant under gauge transformations, i.e. $S[A] = S[A']$.

[1] Note that in the gauge theory literature the notation T^a is often used instead of ξ_a and the structure constants are denoted with upper indices f^{abc} instead of lower indices f_{abc}, but this makes no difference.

Exercise 2.3: The Lie group SU(2) (3 points)

For us this is perhaps the most important Lie group, the representations of which will be subject of the next lecture. Here you will get to know the underlying manifold and determine the structure constants.

Recall from the lecture notes that the Lie group SU(2) and the Lie algebra $\mathfrak{su}(2)$ are given by

$$\mathrm{SU}(2) = \left\{ U \in \mathbb{C}^{2\times 2} : U^\dagger U = \mathbb{1},\, \det U = 1 \right\}. \tag{3.1}$$

$$\mathfrak{su}(2) = \left\{ X \in \mathbb{C}^{2\times 2} : X^\dagger = -X,\, \mathrm{tr} X = 0 \right\}. \tag{3.2}$$

a) Show that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{3.3}$$

Hint: first use $\det U = 1$ to show that $U^{-1} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$ if $U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$.

b) Argue that the manifold underlying SU(2) is (diffeomorphic to) the 3-sphere

$$S^3 = \{\vec{x} \in \mathbb{R}^4 : \|\vec{x}\| = 1\}.$$

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.4}$$

satisfy

$$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i\epsilon_{abc} \sigma_c , \tag{3.5}$$

where ϵ_{abc} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

c) Argue that $i\sigma_a$, $a = 1, 2, 3$, forms a basis of $\mathfrak{su}(2)$.

d) Using the conventional normalization $\xi_a = -\frac{i}{2}\sigma_a$, show that

$$[\xi_a, \xi_b] = \epsilon_{abc} \xi_c, \qquad a, b, c = 1, 2, 3. \tag{3.6}$$

Hence, the structure constants of $\mathfrak{su}(2)$ are $f_{abc} = \epsilon_{abc}$.

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Problem sheet #3: Representations of $\mathfrak{su}(2)$

Tutorial on Thursday 28 April 2021.

Exercise 3.1: Matrix representations of $\mathfrak{su}(2)$

Practice obtaining explicit matrices from the highest-weight construction of representations of $\mathfrak{su}(2)$.

Recall that the generators X_1, X_2, X_3 of SU(2) in any representation obey the commutation relations

$$[X_a, X_b] = \epsilon_{abc} X_c \,. \tag{1.1}$$

Here ϵ_{abc} is the totally antisymmetric symbol with $\epsilon_{123} = 1$. The irreducible representations of $\mathfrak{su}(2)$ are labeled by the spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Each representation acts on a $(2j+1)$-dimensional orthonormal basis $|j,m\rangle$ with $m = -j, -j+1, \ldots, j-1, j$. In the lectures we derived that J_3 and $J_\pm = (J_1 \pm i J_2)/\sqrt{2}$, $J_a \equiv i X_a$, act on this basis via

$$J_3|j,m\rangle = m|j,m\rangle, \quad J_+|j,m\rangle = \sqrt{\tfrac{1}{2}(j+m+1)(j-m)}\,|j,m+1\rangle, \qquad J_- = (J_+)^\dagger. \tag{1.2}$$

The matrices corresponding to the generators X_a in the spin-j representation are given by

$$[X_a]_{m',m} = \langle j,m' |\, X_a \,|j,m\rangle \,. \tag{1.3}$$

a) Construct explicitly the matrix representation X_a for $j = 1$.

b) Construct explicitly the matrix representation X_a for $j = \frac{3}{2}$.

c) Demonstrate in both cases that (1.1) is satisfied in the case $a = 1$, $b = 2$.

Exercise 3.2: Tensor representations of $\mathfrak{su}(2)$

Tensor representations are ubiquitous in particle physics: if one particle transforms in one representation and a second particle in another representation, then the state describing the pair of particles naturally transforms in the tensor representation of both. Here you will investigate in the case of $\mathfrak{su}(2)$ how such representations decompose into irreducible components.

The tensor product of two irreducible representations of $\mathfrak{su}(2)$ with spins j and j' determines a new representation that acts on the linear space spanned by the $(2j+1)\cdot(2j'+1)$ basis elements

$$|j,m\rangle \otimes |j',m'\rangle\,, \quad m=-j,-j+1,\ldots,j, \quad m'=-j',-j'+1,\ldots,j'. \tag{2.1}$$

The generators (using the Hermitian $J_a = iX_a$ instead of skew-Hermitian generators X_a) act on these basis states via

$$J_a\left(|j,m\rangle \otimes |j',m'\rangle\right) = \left(J_a\,|j,m\rangle\right)\otimes|j',m'\rangle + |j,m\rangle\otimes\left(J_a\,|j',m'\rangle\right)\,. \tag{2.2}$$

Typically this representation is not irreducible, but can be decomposed as a direct sum of irreducible representations. Denoting by D_j the spin-j representation of $\mathfrak{su}(2)$, it can be shown that the tensor product representation decomposes as

$$D_j \otimes D_{j'} \cong D_{|j-j'|} \oplus D_{|j-j'|+1} \oplus \cdots \oplus D_{j+j'}, \tag{2.3}$$

meaning that each irrep with spin $|j-j'|,\ldots,j+j'$ occurs exactly once. For example the tensor product of $D_{\frac{1}{2}}$ and D_1 decomposes as

$$D_{\frac{1}{2}} \otimes D_1 \cong D_{\frac{3}{2}} \oplus D_{\frac{1}{2}}\,. \tag{2.4}$$

To find the subspaces on which the irreducible representations act, it is convenient to consider the highest weights states ($|j,j\rangle$ in the spin-j representation) and to use the lowering operator J_- to construct the remaining states.

a) Demonstrate that the dimensions of the representations on both sides of (2.3) agree.

b) Using (2.2), show that for the basis (2.1) the third component of the spin satisfies the sum rule

$$J_3\left(|j,m\rangle\otimes|j',m'\rangle\right) = (m+m')\left(|j,m\rangle\otimes|j',m'\rangle\right)\,. \tag{2.5}$$

c) The state $\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,1\rangle$ is said to be the state with the highest weight, because it has the highest value of the third component of the spin. We therefore claim it belongs to the spin-$\frac{3}{2}$ representation of (2.4) and we denote the state by

$$\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle \equiv \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\otimes|1,1\rangle\,. \tag{2.6}$$

Using (2.2), compute the states $\left|\frac{3}{2},\frac{1}{2}\right\rangle$, $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$ and $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ in terms of the product states by applying the operator J^- respectively one, two and three times to (2.6).

d) The four states $\left|\frac{3}{2},m\right\rangle$ form a basis for the spin-$\frac{3}{2}$ representation in (2.4), corresponding to an invariant subspace of the tensor product basis (2.1). We are left with only a two-dimensional subspace which we want to arrange in the $j=\frac{1}{2}$ representation of (2.4). Define

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\otimes|1,1\rangle - \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\otimes|1,0\rangle\,. \tag{2.7}$$

Show that (2.7) satisfies

$$J_3\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \tfrac{1}{2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\,, \tag{2.8}$$

and that is it is perpendicular to the states of the spin-$\frac{3}{2}$ representation.

e) Using (2.2), apply the operator J^- to (2.7) to obtain the last state $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$.

Exercise 3.3: Isospin and branching ratios

Exploring consequences of SU(2)*-isospin symmetry in baryon decay.*

Particles with the same spin and similar mass may be grouped into isospin multiplets which transform in irreducible representations of $\mathfrak{su}(2)$. Examples are the:

$$\begin{aligned} &\text{nucleon-doublet:} && (p, n) \\ &\text{pion-triplet:} && (\pi^+, \pi^0, \pi^-) \\ &\text{delta-quadruplet:} && (\Delta^{++}, \Delta^+, \Delta^0, \Delta^-) \end{aligned}$$

a) Consider the decay

$$|\Delta^{++}\rangle \to |p, \pi^+\rangle \equiv |p\rangle \otimes |\pi^+\rangle. \tag{3.1}$$

The isospin quantum numbers associated with the highest-weight states are

$$J_3|\Delta^{++}\rangle = \frac{3}{2}|\Delta^{++}\rangle\,, \quad J_3|p\rangle = \frac{1}{2}|p\rangle\,, \quad J_3|\pi^+\rangle = |\pi^+\rangle\,. \tag{3.2}$$

Use the action of the generator J_3 on product states to show that the decay (3.1) preserves the isospin quantum number.

b) Consider the 6 different nucleon-pion states

$$\begin{array}{lll} |p, \pi^+\rangle\,, & |p, \pi^0\rangle\,, & |p, \pi^-\rangle\,, \\ |n, \pi^+\rangle\,, & |n, \pi^0\rangle\,, & |n, \pi^-\rangle\,. \end{array} \tag{3.3}$$

which are all eigenstates of J_3. Compute the corresponding J_3-eigenvalues. Use these eigenvalues to determine the irreducible $\mathfrak{su}(2)$ representations which are contained in the nucleon-pion states (3.3).

c) Use your results from Exercise 4.2(c) to determine the result of the decay of $|\Delta^+\rangle$, $|\Delta^0\rangle$, and $|\Delta^-\rangle$.

d) The *branching ratios* for the $|\Delta^+\rangle$-decay and $|\Delta^0\rangle$-decay channels are defined as

$$\frac{\Gamma(\Delta^+ \to n, \pi^+)}{\Gamma(\Delta^+ \to p, \pi^0)} \approx \frac{|\langle n, \pi^+|\Delta^+\rangle|^2}{|\langle p, \pi^0|\Delta^+\rangle|^2}\,, \tag{3.4}$$

$$\frac{\Gamma(\Delta^0 \to p, \pi^-)}{\Gamma(\Delta^0 \to n, \pi^0)} \approx \frac{|\langle p, \pi^-|\Delta^0\rangle|^2}{|\langle n, \pi^0|\Delta^0\rangle|^2}\,. \tag{3.5}$$

Use your result in c) to compute these branching ratios based on the isospin algebra. You may assume that the particle states are orthonormal.

Lie Algebras in Particle Physics (NWI-NM101B)

Problem sheet #4(graded): Adjoint representation

Tutorial on Thursday 12 May 2022.

This is one of **3 graded assignments** contributing to a bonus point on the exam.
Solutions should be submitted digitally before 13:30 on 19 May via Brightspace Assignments.

Exercise 4.1: Compact Lie algebras of small dimension (2 points)

Why is the Lie algebra $\mathfrak{su}(2)$ *ubiquitous? In this exercise you will find out that it is the only compact Lie algebra of dimension at most three.*

Suppose $\mathfrak{g}$ is an n-dimensional matrix Lie algebra with basis $\xi_1, \ldots, \xi_n$ and structure constants f_{abc}. Recall from the lecture that $\mathfrak{g}$ is compact if and only if the Cartan-Killing metric $\gamma_{ab} = f_{adc}f_{bcd}$ is negative-definite. In that case the basis can be chosen such that $\gamma_{ab} = -c\delta_{ab}$ for any desired positive real number c, and as a consequence that f_{abc} is completely antisymmetric.

a) Argue that the dimension n of a compact Lie algebra $\mathfrak{g}$ must be at least 3.
Hint: Explain that the structure constants would vanish for $n < 3$.

b) Show that every three-dimensional compact Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$, meaning that one can find a basis ξ_1, ξ_2, ξ_3 of $\mathfrak{g}$ such that $[\xi_a, \xi_b] = \epsilon_{abc}\xi_c$.
Hint: parametrize the possible structure constants f_{abc} and plug them into $\gamma_{ab} = f_{adc}f_{bcd}$.

Exercise 4.2: The adjoint representation of $\mathfrak{su}(2)$ is $\mathfrak{so}(3)$ (4 points)

In the previous exercise you argued that there is only one compact 3-dimensional Lie algebra, so $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ which are both compact and 3-dimensional must be equivalent. Here you will discover that $\mathfrak{so}(3)$ arises as the adjoint representation of $\mathfrak{su}(2)$. On the level of Lie groups we will see that $SU(2)$ and $SO(3)$ are not quite isomorphic.

Recall the structure constants $f_{abc} = \epsilon_{abc}$, $a = 1, 2, 3$, for the Lie algebra $\mathfrak{su}(2)$.

a) Determine the explicit matrices T_a of the adjoint representation ($[T_a]_{bc} = f_{acb}$).

b) Recall from the lecture notes that in the spin-j representation $C = \frac{1}{2}j(j+1)\mathbb{1}$. Compute the Casimir operator $C = \gamma^{ab}T_aT_b$ of the adjoint representation to check that the adjoint representation is equivalent to the spin-1 representation of $\mathfrak{su}(2)$.

c) Show that the generators T_a are also the generators of $\mathfrak{so}(3)$ in its defining representation.

Recall that at the level of the Lie group $SU(2)$ the adjoint representation $D : SU(2) \to GL(\mathfrak{su}(2))$ is given by $D(g)|X\rangle = |gXg^{-1}\rangle$ with $X \in \mathfrak{su}(2)$ and $g \in SU(2)$.

d) Show that D preserves the scalar product $\langle X|X'\rangle \equiv -2\,\mathrm{tr}(XX')$, i.e. $\langle D(g)X|D(g)X'\rangle = \langle X|X'\rangle$.

The standard generators $X_a = \frac{-i}{2}\sigma_a$ with σ_a the Pauli matrices form and orthonormal basis for this scalar product, $\langle X_a|X_b\rangle = \delta_{ab}$.

e) Show that the matrix elements $[D(g)]_{ab} = \langle X_a|D(g)|X_b\rangle$ of $D(g)$ in this basis determine an orthogonal matrix in $SO(3)$. Hint: use $D(g)|X_b\rangle = [D(g)]_{ab}|X_a\rangle$ and e).

We may therefore interpret the adjoint representation as a mapping $D : SU(2) \to SO(3)$.

f) Show that $D : SU(2) \to SO(3)$ is not quite an isomorphism by computing $D(-\mathbb{1})$.

The adjoint representation $D : SU(2) \to SO(3)$ takes any value in $SO(3)$ exactly twice: $SU(2)$ is a "double cover" of $SO(3)$. The fact that $SO(3)$ is slightly smaller than $SU(2)$ has the consequence that it admits only a subset of the irreps of $SU(2)$: the spin-j representations for $j = 0, 1, 2, \ldots$.

Exercise 4.3: The Lorentz group $\mathrm{SO}(1,3)$ and its Lie algebra (4 points)

The Lorentz group and its representations are or great importance in quantum field theory. It is non-compact and therefore not part of the families of Lie groups we focus on in this course, but should not go untouched! Hence this exercise.

In class we have seen that the orthogonal group $\mathrm{SO}(N)$ corresponds to the subset of real linear maps $\mathrm{GL}(N,\mathbb{R})$ that leave the N-dimensional Euclidean inner product invariant (and excluding reflections). In this exercise we generalize this to the Lorentz group $\mathrm{SO}(1,3)$ that leaves the Minkowski metric invariant, i.e. if v^μ and w^ν are 4-vectors we consider the inner product

$$\langle v,w\rangle := v^\mu\,\eta_{\mu\nu}\,w^\nu, \qquad \eta=\mathrm{diag}(-1,1,1,1), \qquad \mu,\nu=0,1,2,3. \tag{3.1}$$

The Lorentz group $\mathrm{SO}(1,3)$ is defined to be the set of matrices Λ (with matrix elements $[\Lambda]^\alpha{}_\beta$) in $\mathrm{GL}(\mathbb{R}^4)$ such that $\langle v',w'\rangle=\langle v,w\rangle$ where $v'^\mu=[\Lambda]^\mu{}_\nu v^\nu$. Pay attention that placement of indices in this exercise matters: $\eta_{\mu\nu}$ and its inverse are used to raise and lower indices.

a) Show that the Lie algebra $\mathfrak{so}(1,3)$ of the Lorentz group is given by

$$\mathfrak{so}(1,3)=\left\{J\in\mathbb{R}^{4\times4} : [J]^\mu{}_\nu=-\eta_{\nu\beta}[J]^\beta{}_\alpha\eta^{\alpha\mu}\right\} \tag{3.2}$$

and that by lowering indices the latter condition is equivalent to $[J]_{\alpha\beta}=-[J]_{\beta\alpha}$.

b) Argue that a basis of the Lie algebra $\mathfrak{so}(1,3)$ is given by the generators $J^{\mu\nu}$, $0\le\mu<\nu\le3$, defined by $[J^{\mu\nu}]_{\alpha\beta}=\delta^\mu_\alpha\,\delta^\nu_\beta-\delta^\mu_\beta\delta^\nu_\alpha$. What is the dimension of $\mathfrak{so}(3,1)$ based on this?

c) Compute explicitly the Lorentz transformations $[\Lambda]^\mu{}_\nu$ generated by J^{01} and J^{12}, i.e. $\Lambda=e^{tJ^{01}}$ and $\Lambda=e^{tJ^{12}}$ respectively. Do they correspond to a rotation or a boost?

d) Verify that these generators satisfy

$$\left[J^{\mu\nu}\,,\,J^{\alpha\beta}\right]=\eta^{\nu\alpha}J^{\mu\beta}-\eta^{\mu\alpha}J^{\nu\beta}-\eta^{\nu\beta}J^{\mu\alpha}+\eta^{\mu\beta}J^{\nu\alpha}. \tag{3.3}$$

e) Suppose you have a set of 4×4 matrices γ^μ, $\mu=0,1,2,3$ satisfying the Clifford algebra

$$\{\gamma^\mu\,,\,\gamma^\nu\}:=\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\,\eta^{\mu\nu}\,\mathbb{1}. \tag{3.4}$$

Show that the generators

$$S^{\mu\nu}:=\tfrac14\,[\gamma^\mu\,,\,\gamma^\nu] \tag{3.5}$$

also satisfies (3.3) and thus also furnish a representation of the Lorentz group. (Remark: This is the representation which acts on Dirac spinors, as you may recall from Quantum Mechanics 3)

f) Let us introduce the generators $\tilde{J}_i$ and K_i of spatial rotations and boosts as

$$\tilde{J}_i:=\tfrac12\epsilon_{ijk}J_{jk}, \quad K_i:=J_{0i}, \quad i,j,k=1,2,3. \tag{3.6}$$

These can be reorganized in the complex linear combinations

$$A_i^\pm=\tfrac12(\tilde{J}_i\pm i\,K_i), \tag{3.7}$$

which may be interpreted as the generators of the complexified Lie Algebra $\mathfrak{so}(1,3)_\mathbb{C}$ (see Remark 5.1 in the lecture notes). Show using (3.3) that A_i^+ and A_i^- both satisfy the $\mathfrak{su}(2)$ commutation relations,

$$[A_i^\pm,A_j^\pm]=\epsilon_{ijk}A_k^\pm, \qquad [A_i^+,A_j^-]=0. \tag{3.8}$$

This shows that $\mathfrak{so}(1,3)_\mathbb{C}$ decomposes as a double copy $\mathfrak{so}(1,3)_\mathbb{C}=\mathfrak{su}(2)_\mathbb{C}\oplus\mathfrak{su}(2)_\mathbb{C}$, which conveniently allows one to construct irreps of the Lorentz group in terms of those of $\mathfrak{su}(2)$.

Lie Algebras in Particle Physics (NWI-NM101B)

Problem sheet #5: Root systems

Tutorial on Tuesday 17 May 2022, 15:30 - 17:15.

Exercise 5.1: Reconstructing the $\mathfrak{su}(3)$ algebra from its roots

In the lecture we have seen how to associate a root system to a Lie algebra, and we claimed without proof that this root system contains all information of the Lie algebra. In this exercise we will explore how to reconstruct the Lie algebra $\mathfrak{su}(3)$ from its (simple) roots.

The simple roots of $\mathfrak{su}(3)$ are given by

$$\hat{\alpha}^1 = (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}), \qquad \hat{\alpha}^2 = (\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{3}). \tag{1.1}$$

a) Use successive Weyl reflections $(\alpha, \beta) \to \beta - 2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha$ to determine a full set of roots $\{\alpha^1, \alpha^2, \ldots\}$. Draw the root system.

b) Based on the root system, what is the dimension of $\mathfrak{su}(3)$?

By construction the commutators involving one or more Cartan generators H_i are given by

$$[H_i, E_{\alpha^k}] = \alpha_i^k E_{\alpha^k}, \qquad [H_i, H_j] = 0. \tag{1.2}$$

In the lecture we have seen that to a root α^k we may associate an $\mathfrak{su}(2)$-subalgebra generated by $E_3^k = |\alpha^k|^{-2}\alpha_i^k H_i$ and $E_\pm^k = |\alpha^k|^{-1}E_{\pm\alpha^k}$, meaning that they satisfy

$$[E_+^k, E_-^k] = E_3^k, \qquad [E_3^k, E_\pm^k] = \pm E_\pm^k. \tag{1.3}$$

To reconstruct the full Lie algebra it thus remains to determine the commutators $[E_{\alpha^k}, E_{\alpha^l}]$ for linearly independent roots α^k and α^l.

c) Show that in the adjoint representation $H_i E_+^k|E_{\alpha^l}\rangle = (\alpha_i^k + \alpha_i^l)E_+^k|E_{\alpha^l}\rangle$ meaning that if $[E_{\alpha_k}, E_{\alpha_l}] \neq 0$ then $|[E_{\alpha_k}, E_{\alpha_l}]\rangle$ has root $\alpha^k + \alpha^l$.

d) For fixed k the adjoint action of E_3^k and $E_\pm^k$ on $\mathfrak{su}(3)$ determines a representation of the $\mathfrak{su}(2)$-algebra (1.3) which is reducible. Argue on the basis of the root diagram that the representation decomposes into a direct sum of one spin-0, one spin-1, and two spin-1/2 irreps. (It is sufficient to explain this for a single, conveniently chosen root index k.)

e) Use your knowledge of $\mathfrak{su}(2)$ representations to determine the commutator of E_{α^k} (again for a convenient choice of k) with all other generators $E_{\alpha^1}, E_{\alpha^2}, \ldots$ up to phase factors ($|\eta| = 1$). Use that the states $|E_{\alpha^l}\rangle$ are normalized, $\langle E_{\alpha^l}|E_{\alpha^l}\rangle = 1$.

Exercise 5.2: From rotations in a Hilbert space to spin-j irreps

The way we derived the irreps of $\mathfrak{su}(2)$ *is quite similar to how the spectrum of the hydrogen atom is derived in a quantum mechanics course. In this exercise you will find out how these problems are connected.*

Let's consider the quantum-mechanical wave function $\psi(x)$ of a (spinless) particle in a spherically symmetric potential. To simplify matters we ignore the radial dependence and consider the Hilbert space $\mathcal{H}$ of wave functions $\psi : S^2 \to \mathbb{C}$ where $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ is the unit 2-sphere. Note that $\mathcal{H}$ is a vector space since if $\psi_1, \psi_2 \in \mathcal{H}$ then $\psi(x) = \lambda_1\psi_1(x) + \lambda_2\psi_2(x)$ for $\lambda_1, \lambda_2 \in \mathbb{C}$ determines another wave function $\psi \in \mathcal{H}$ by the "superposition principle". It carries a natural representation $D : SO(3) \to GL(\mathcal{H})$ of the 3-dimensional rotation group $SO(3)$ via

$$(D(g)\psi)(\vec{x}) = \psi(g^{\mathrm{T}}\vec{x}), \tag{2.1}$$

implementing the rotational symmetry of the system.

a) Check that D indeed satisfies the requirements of a representation.

b) A basis of generators of $SO(3)$ is given by $[\xi_a]_{bc} = -\epsilon_{abc}$, $a, b, c = 1, 2, 3$. Show that the corresponding generators X_a in representation D are the differential operators

$$\hat{X}_a = \epsilon_{abc}\, x_c \,\frac{\partial}{\partial x_b}. \tag{2.2}$$

c) Why do we sometimes say in quantum mechanics that angular momentum ($\vec{L} = \vec{x} \times \vec{p}$) generates rotations?

d) Show that (2.2) indeed satisfies $[\hat{X}_a, \hat{X}_b] = \epsilon_{abc}\hat{X}_c$. It is sufficient to show this for the particular case $a = 1, b = 2$.

e) Explain that the Casimir operator is given by $C = -\frac{1}{2}(\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2)$.

The spherical Laplacian $\Delta_{S^2} : \mathcal{H} \to \mathcal{H}$ is the operator given by $(\Delta_{S^2}\psi)(x) = \Delta\psi\left(\frac{x}{|x|}\right)$ for $x \in S^2$ in terms of the standard three-dimensional Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

f) Show that the Casimir operator in this representation is given by $C = \frac{1}{2}\Delta_{S^2}$.
Hint: Verify first that $\partial_a \frac{x_b}{|x|} = \frac{|x|^2\delta_{ab} - x_a x_b}{|x|^3}$ and $\Delta\frac{x_b}{|x|} = -2\frac{x_b}{|x|^3}$.

g) As you may know the spherical harmonics $Y_l^m(x)$ for integers l, m satisfying $l \geq 0$ and $-l \leq m \leq l$ form a basis of $\mathcal{H}$ and $\Delta_{S^2} Y_l^m = \ell(\ell + 1)$. Explain based on this fact how the representation D decomposes into irreducible representations of $SO(3)$.

Lie Algebras in Particle Physics (NWI-NM101B)

Problem sheet #6(graded): flavor-SU(3) and Dynkin diagrams

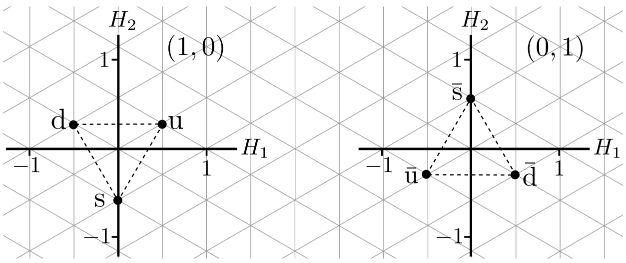
Tutorial on Thursday 2 & 9 June 2022.

This is one of **3 graded assignments** contributing to a bonus point on the exam.
Solutions should be submitted digitally before 15:00 on Tuesday 14 June 2022
via Brightspace Assignments.

Exercise 6.1: Mesons and $\mathfrak{su}(3)$ flavor symmetry (5 Points)

In this exercise you will put to practice your tensor representation skills in order to construct the meson states from the constituent quarks in the quark model.

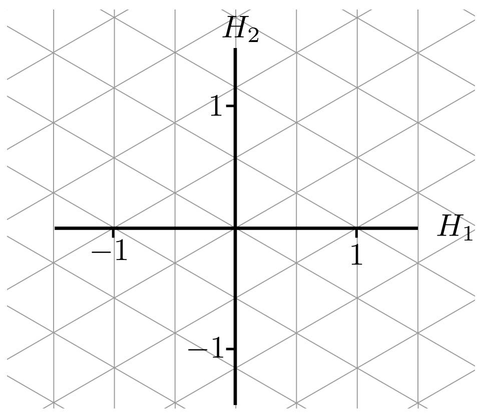
The $\mathfrak{su}(3)$ flavor symmetry is an approximate symmetry in particle physics that mixes the states of three lightest quarks (up, down and strange). The single-quark states $|\mathrm{u}\rangle, |\mathrm{d}\rangle, |\mathrm{s}\rangle$ transform under the fundamental representation $\mathbf{3} = (1,0)$, while the anti-quark states $|\bar{\mathrm{u}}\rangle, |\bar{\mathrm{d}}\rangle, |\bar{\mathrm{s}}\rangle$ transform under the anti-fundamental representation $\bar{\mathbf{3}} = (0,1)$. The weights of these states can be read off from the corresponding weight diagrams.



A basis for the tensor representation $\mathbf{3} \otimes \bar{\mathbf{3}}$, describing bound states of a quark and an anti-quark, is given by the states $|q\bar{q}'\rangle \equiv |q\rangle \otimes |\bar{q}'\rangle$ for $q, q' = \mathrm{u}, \mathrm{d}, \mathrm{s}$. Recall that a generator $X \in \mathfrak{su}(3)$ acts on such a tensor state via

$$X|q\bar{q}'\rangle = (X|q\rangle) \otimes |\bar{q}'\rangle + |q\rangle \otimes (X|\bar{q}'\rangle). \tag{1.1}$$

a) Use (1.1) to determine the weights of the nine basis states of $\mathbf{3}\otimes\bar{\mathbf{3}}$ and draw them in a weight diagram:



b) Recall that the charge Q is related to H_1 and H_2 via $Q = \mathcal{I}_3 + \frac{1}{2}Y = H_1 + \frac{1}{\sqrt{3}}H_2$. Draw the lines of constant integer charge in the above diagram. Identify the charged pion states $|\pi^+\rangle, |\pi^-\rangle$ (with no strange constituent) and kaon states $|K^+\rangle, |K^-\rangle$.

The roots of $\mathfrak{su}(3)$ are $\alpha^1 = (1, 0)$, $\alpha^2 = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $\alpha^3 = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$ (as well as their negatives) and the corresponding raising/lowering generators are

$$E_{\pm\alpha^1} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2), \quad E_{\pm\alpha^2} = \frac{1}{\sqrt{2}}(T_4 \pm iT_5), \quad E_{\pm\alpha^3} = \frac{1}{\sqrt{2}}(T_6 \pm iT_7). \tag{1.2}$$

The fundamental and anti-fundamental representations are given by $T_a = \frac{1}{2}\lambda_a$ respectively $T_a = -\frac{1}{2}\lambda_a^*$, where λ_a are the Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{1.3}$$

c) Show that

$$E_{-\alpha^1}|\mathrm{u}\rangle = \frac{1}{\sqrt{2}}|\mathrm{d}\rangle, \quad E_{-\alpha^1}|\bar{\mathrm{d}}\rangle = -\frac{1}{\sqrt{2}}|\bar{\mathrm{u}}\rangle, \quad E_{-\alpha^2}|\mathrm{u}\rangle = \frac{1}{\sqrt{2}}|\mathrm{s}\rangle, \quad E_{-\alpha^2}|\bar{\mathrm{s}}\rangle = -\frac{1}{\sqrt{2}}|\bar{\mathrm{u}}\rangle. \tag{1.4}$$

d) How can we understand these relations in terms of $\mathfrak{su}(2)$-subalgebras?

e) Use (1.1) to determine $E_{-\alpha^1}|\pi^+\rangle$ and $E_{-\alpha^2}|K^+\rangle$. Show that the following orthonormal linear combinations

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{d}\bar{\mathrm{d}}\rangle - |\mathrm{u}\bar{\mathrm{u}}\rangle\right), \quad |\eta^0\rangle = \frac{1}{\sqrt{6}}\left(|\mathrm{u}\bar{\mathrm{u}}\rangle + |\mathrm{d}\bar{\mathrm{d}}\rangle - 2|\mathrm{s}\bar{\mathrm{s}}\rangle\right) \tag{1.5}$$

therefore belong to the octet representation $\mathbf{8}$ in the decomposition $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

f) Show that the only remaining normalized state $|\eta'\rangle = \frac{1}{\sqrt{3}}(|\mathrm{u}\bar{\mathrm{u}}\rangle + |\mathrm{d}\bar{\mathrm{d}}\rangle + |\mathrm{s}\bar{\mathrm{s}}\rangle)$ is $\mathfrak{su}(3)$-invariant and therefore spans the trivial representation $\mathbf{1}$.
Hint: For an arbitrary generator X of $\mathfrak{su}(3)$, $X|q_a\rangle = [X]_{ca}|q_c\rangle$ and $X|\bar{q}_a\rangle = [X]^*_{ca}|\bar{q}_c\rangle$ with $[X]_{ca}$ the matrix elements of X in the fundamental representation ($q_1 = \mathrm{u}, q_2 = \mathrm{d}, q_3 = \mathrm{s}$). Use (1.1) to show that $X|\eta'\rangle = 0$.

Exercise 6.2: The group G_2 (2 Points)

Reconstructing the root system from a Dynkin diagram.

The Dynkin diagram for the group G_2 is

a) Determine $\hat{\alpha}^1$ if we use the convention that $\hat{\alpha}^2 = (0,1)$ (recall that a simple root has to be positive).

b) Consecutively apply Weyl reflections to the simple roots to obtain the complete root system. Draw the corresponding root diagram.

c) Deduce the dimension of G_2 from b).

Exercise 6.3: Disconnected Dynkin diagram (3 Points)

In this exercise you will verify that the decomposition of a Dynkin diagram into connected components corresponds to the decomposition of a compact Lie algebra into a direct sum of simple compact Lie algebras.

In the lectures we have seen that a compact Lie algebra $\mathfrak{g}$ is uniquely encoded in its Dynkin diagram. Now suppose the Dynkin diagram is not connected.

a) Show that the roots of $\mathfrak{g}$ decompose into two orthogonal subsets $\{\alpha^i\} \cup \{\beta^j\}$, $\alpha^i \cdot \beta^j = 0$.
Hint: First argue that such a decomposition happens for the simple roots and then use Weyl reflections to explain that the same is true for all roots.

b) Show that $\{E_{\alpha^i}, \alpha^i_k H_k\}$ and $\{E_{\beta^j}, \beta^j_k H_k\}$ both span invariant subalgebras of $\mathfrak{g}$.
Hint: What commutators do you need to check? Make use of the known commutation relations among the operators E_{α^i} and H_k.

Hence $\mathfrak{g}$ is not simple, but decomposes as a direct sum of two compact Lie algebras. Recall that $\mathfrak{so}(4)$ is a compact Lie algebra of rank 2 and that its root system is given by

$$\Phi = \{(1,1), (1,-1), (-1,1), (-1,-1)\}. \tag{3.1}$$

c) Draw the Dynkin diagram of $\mathfrak{so}(4)$.

d) Argue that $\mathfrak{so}(4)$ is not simple, but decomposes as a direct sum $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.