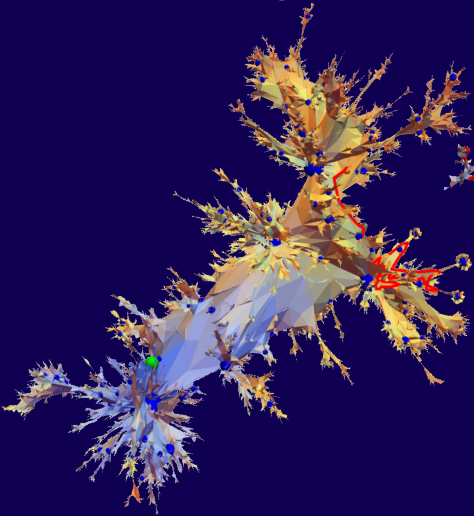
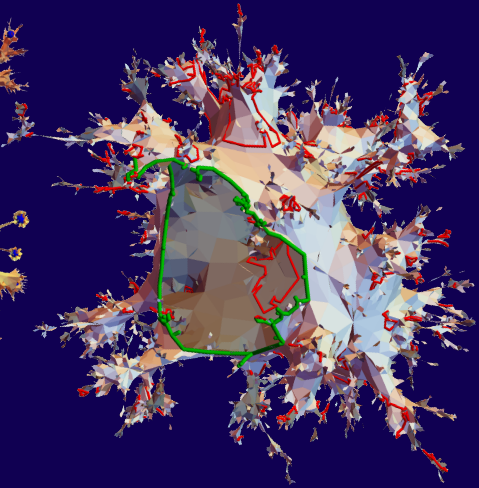
Mark Kac seminar @ Utrecht, The Netherlands - 07-12-2018

Geometry of random planar maps

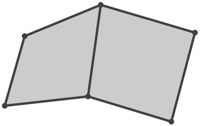
Timothy Budd

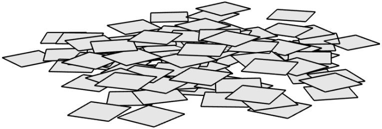


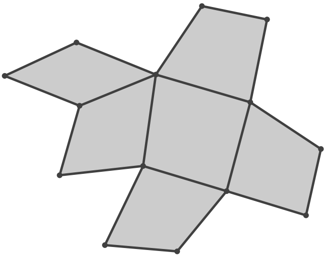


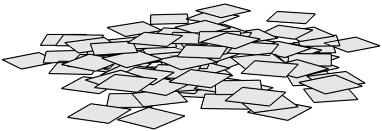
Based partly on joint work with
N. Curien, C. Marzouk

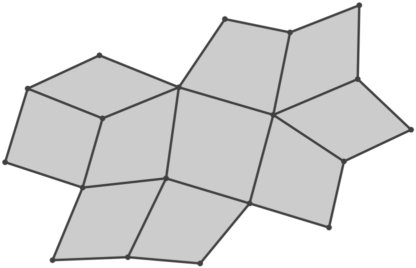
t.budd@science.ru.nl
http://hef.ru.nl/~tbudd/

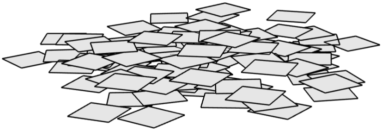


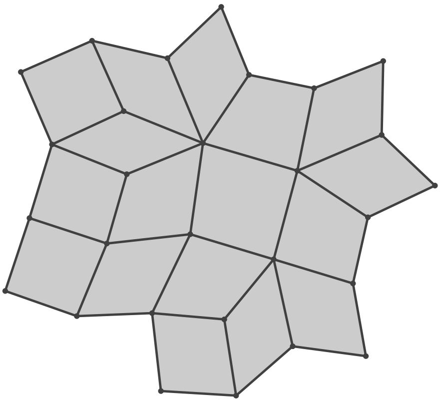


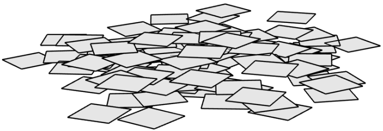


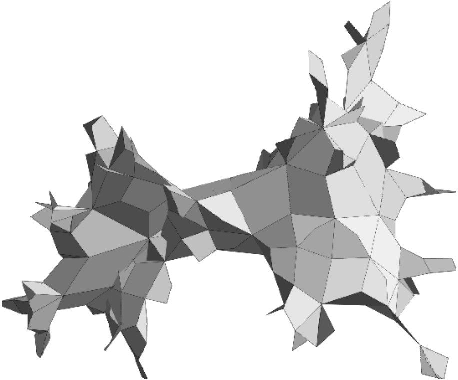


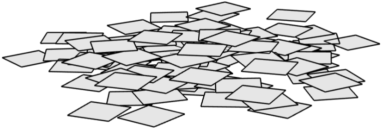




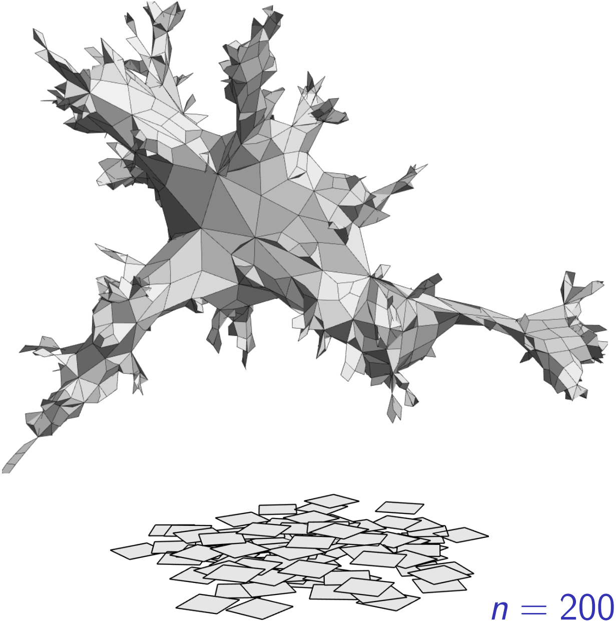




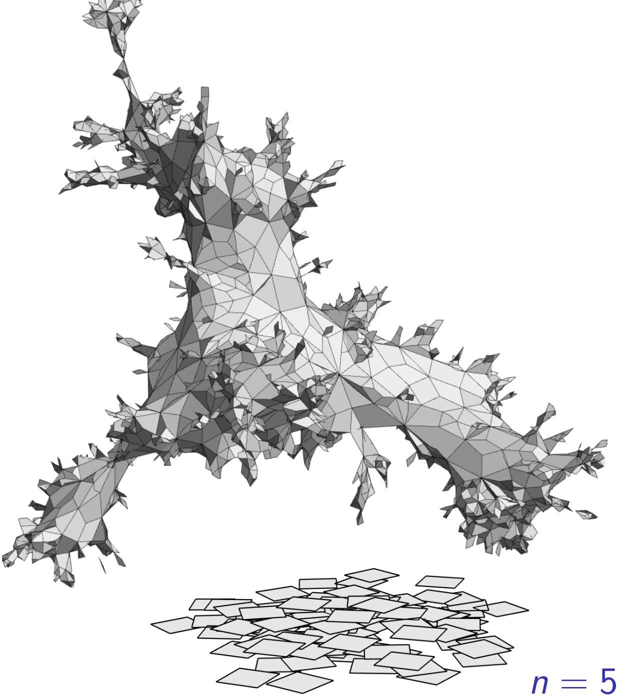




$n = 200$



$n = 2000$



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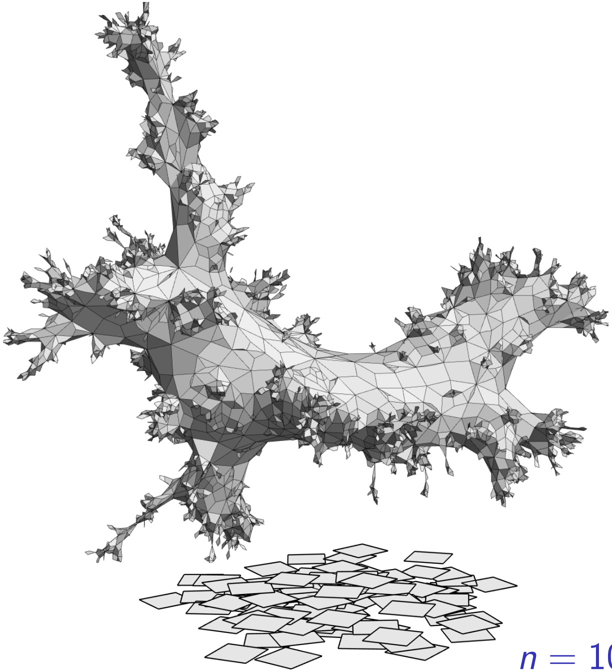

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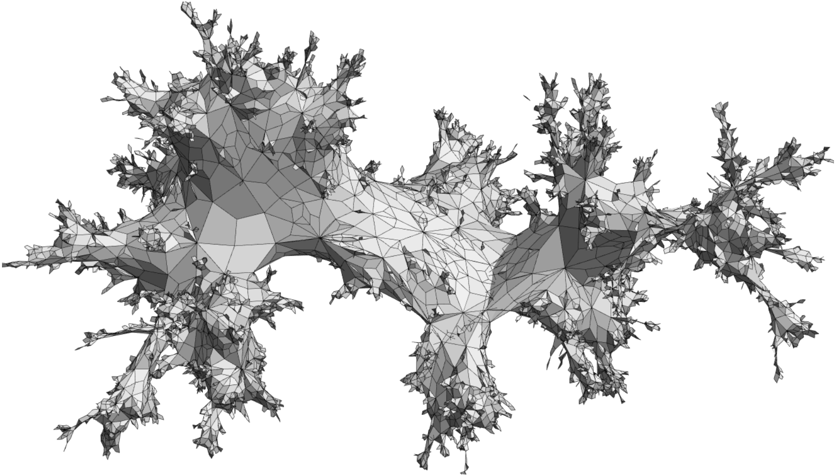

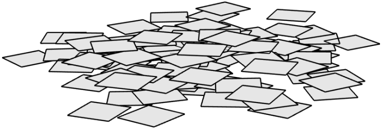

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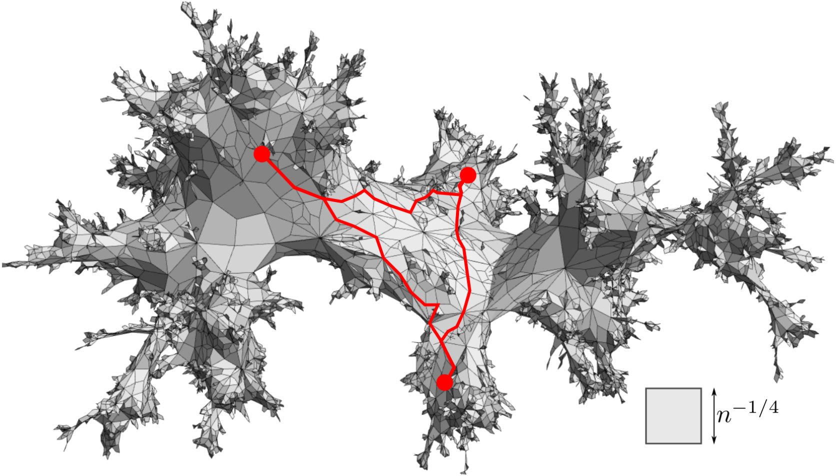

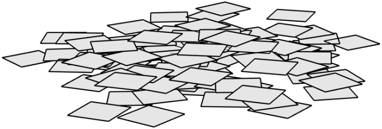

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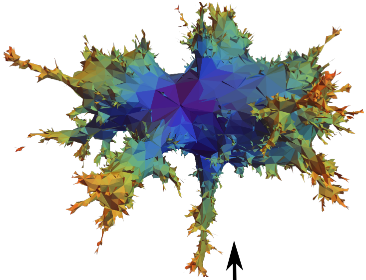
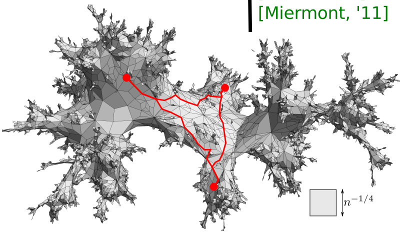
Brownian sphere



$n \to \infty$

Convergence in distribution
w.r.t. Gromov-Hausdorff
topology on metric spaces

[Le Gall, '11]
[Miermont, '11]



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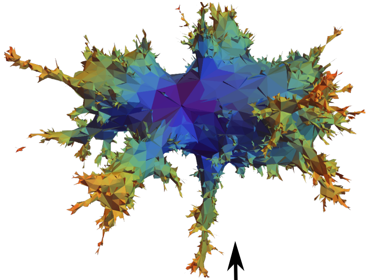

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Convergence in distribution w.r.t. Gromov-Hausdorff topology on metric spaces

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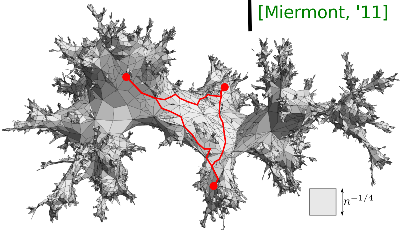
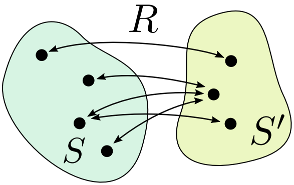


Given metric spaces (S, d) and (S', d'), their Gromov-Hausdorff distance is

$$d_{\mathrm{GH}}((S,d),(S',d')) = \frac{1}{2} \inf_{R} \sup_{\substack{xRx' \\ yRy'}} |d(x,y) - d'(x',y')|,$$

with the inf over correspondences R, i.e. $R \subset S \times S'$ with each element of S and S' appearing at least once in the relation.



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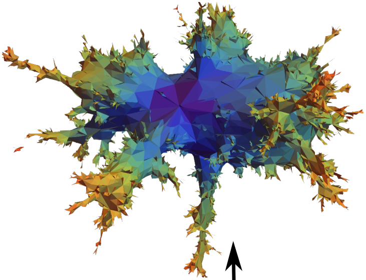

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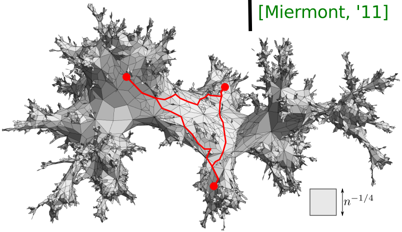

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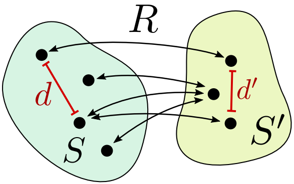
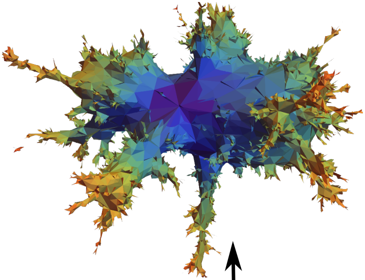


Brownian sphere



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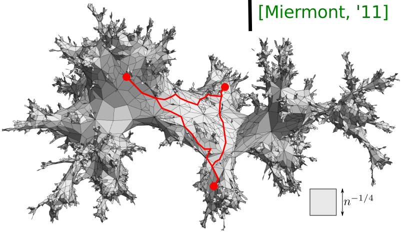
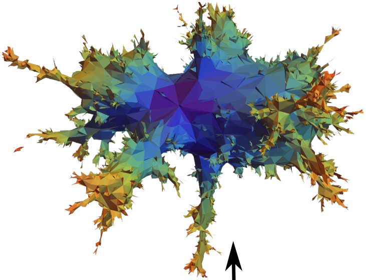


Properties of the Brownian sphere:

- Topology of S^2 a.s.

  [Le Gall, Paulin, '08;, Miermont, '08]

Brownian sphere



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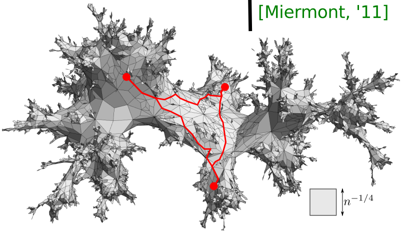

Properties of the Brownian sphere:

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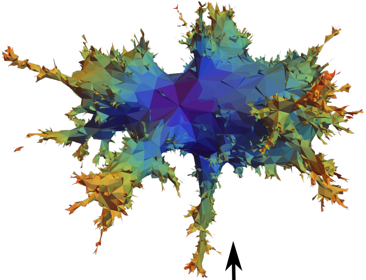
- Hausdorff dimension 4 a.s.

  [Le Gall, '07]
- Spectral dimension 2 a.s

  [Gwynne, Miller, '18]

Brownian sphere



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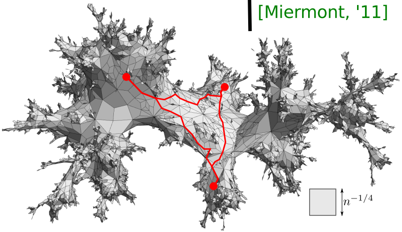

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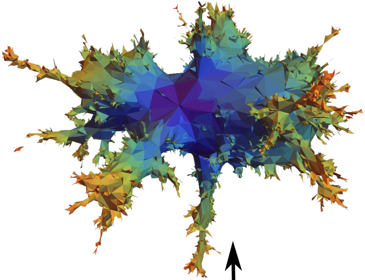
- Hausdorff dimension 4 a.s.
  [Le Gall, '07]
- Spectral dimension 2 a.s
  [Gwynne, Miller, '18]
- Universal scaling limit of
  - p-angulations, $p \geq 3$ [Le Gall, '11; Albenque, '18+]
  - general maps with fixed number of edges
    [Bettinelli, Jacob, Miermont, '14]
  - bipartite maps with prescribed degrees
    [Marzouk, '17]
  - etc. . .

Brownian sphere



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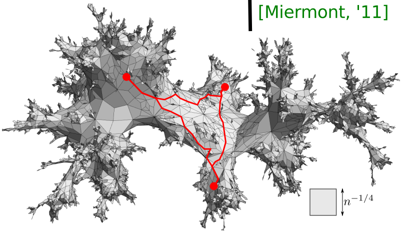
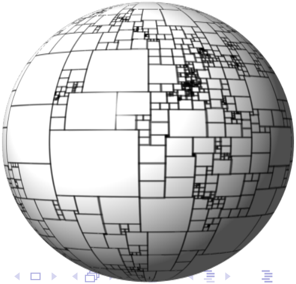


Uniformization of
Riemannian metric



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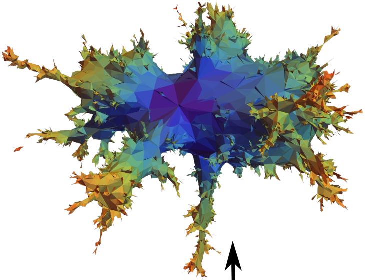
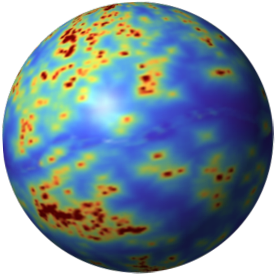


$\mathrm{LQG}_{\gamma=\sqrt{8/3}}$



$n \to \infty$

Convergence in distribution
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Conjectured
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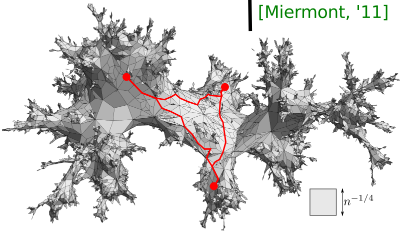
measure on S^2



Uniformization of
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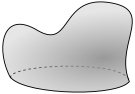
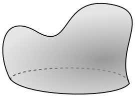
Liouville Quantum Gravity

- Any natural random Riemannian metric on a disk (or sphere)?



Liouville Quantum Gravity

- Any natural random Riemannian metric on a disk (or sphere)?
- Addressed already in '80s in the context of string theory [Polyakov, Knizhnik, Zamolodchikov, David, Distler, Kawai, . . . , '80s].



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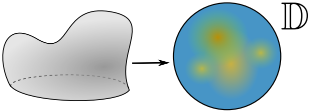
$$e^{\gamma h(z)}(\mathrm{d}x^2 + \mathrm{d}y^2) \qquad (z=x+iy)$$



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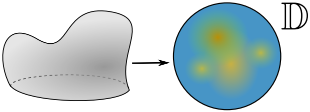
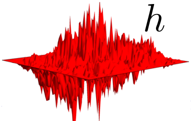
- Take h to be a Gaussian Free Field (GFF), i.e. h is Gaussian with covariance given by the Dirichlet Green's function $G(z_1, z_2) = \log|z_1 - z_1| + O(1)$. More precisely

$$\mathrm{Cov}((h, f_1)_\nabla, (h, f_2)_\nabla) = (f_1, f_2)_\nabla := \frac{1}{2\pi}\int_{\mathbb{D}} \nabla f_1(z) \cdot \nabla f_2(z)\mathrm{d}z.$$





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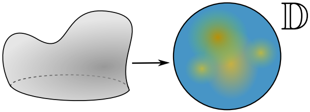

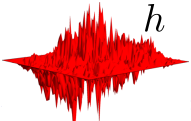

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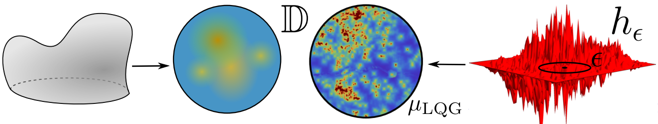
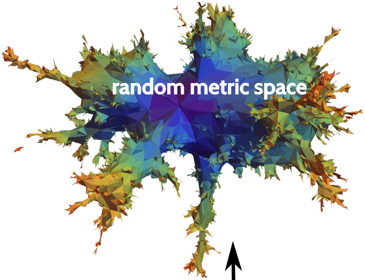
- h is a distribution! How to exponentiate?
- Regularize: let $h_\epsilon(z)$ be average of h on circle of radius ϵ, then

$$\epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} \mathrm{d}x\mathrm{d}y \xrightarrow[\epsilon \to 0]{\text{weak}} \mu_{\mathrm{LQG}} \qquad \text{“Liouville measure”}$$

[Høegh-Krohn '71; Kahane, '85, Duplantier, Sheffield, '11]



Brownian sphere



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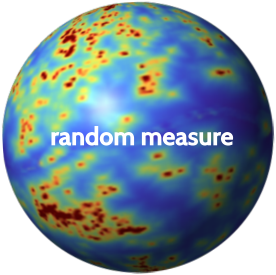

$n \to \infty$

Convergence in distribution w.r.t. Gromov-Hausdorff topology on metric spaces

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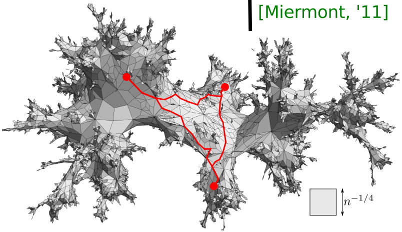
Conjectured convergence of measure on S^2



Uniformization of Riemannian metric

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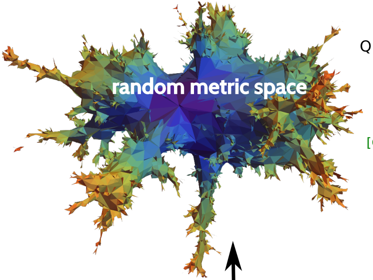
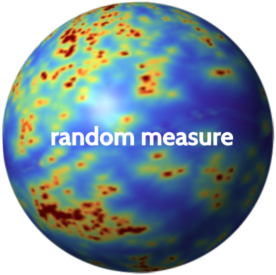


Quantum Loewner Evolution
[Miller, Sheffield, '16]

Uniformization
[Gwynne, Miller, Sheffield, '18]



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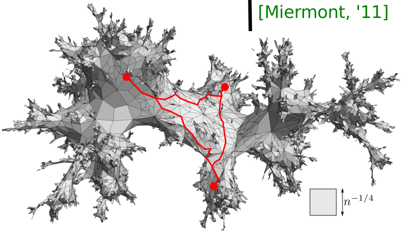
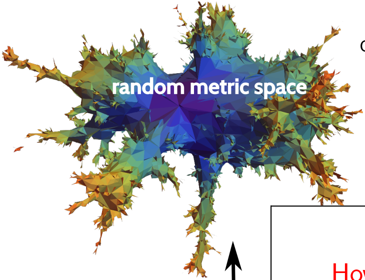
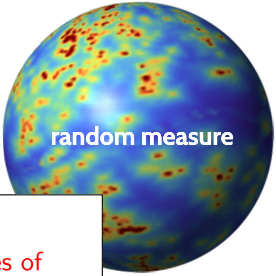


Uniformization of Riemannian metric

Brownian sphere

LQG$_{\gamma=\sqrt{8/3}}$





Quantum Loewner Evolution
[Miller, Sheffield, '16]

Uniformization
[Gwynne, Miller, Sheffield, '18]

$n \to \infty$

Cor
w.r.
top

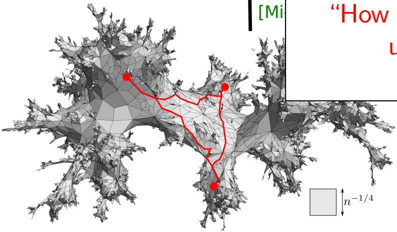
[Le
[Mi

How about other values of
$\gamma \neq \sqrt{8/3}$?

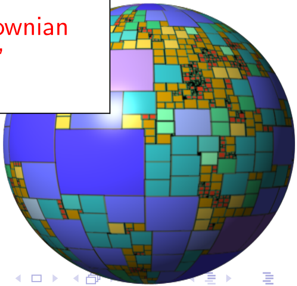
or

"How to escape the Brownian universality class?"

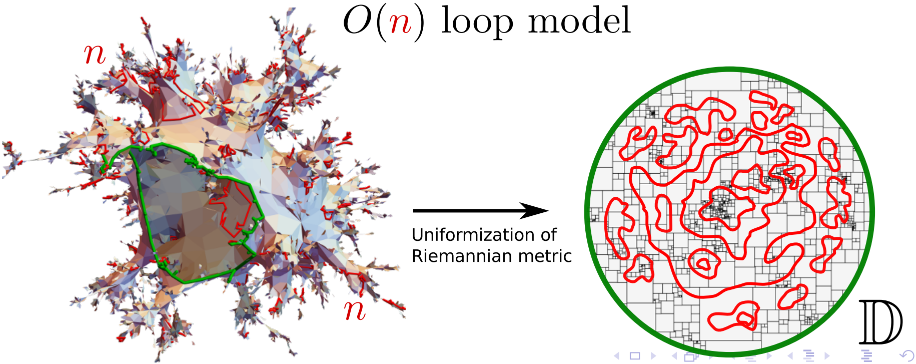
Conjectured convergence of measure on S^2

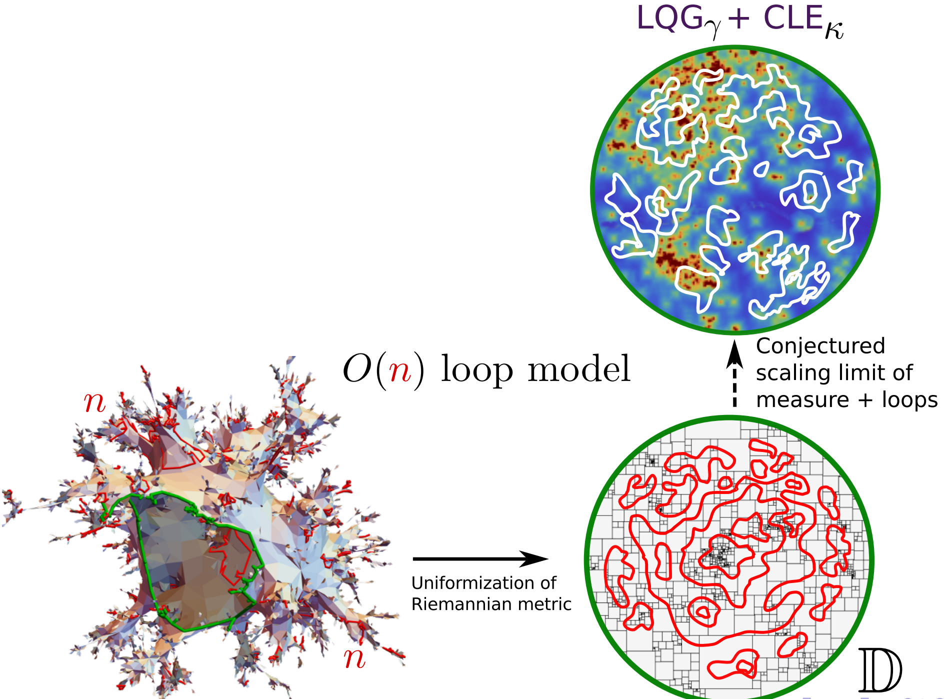


Uniformization of Riemannian metric

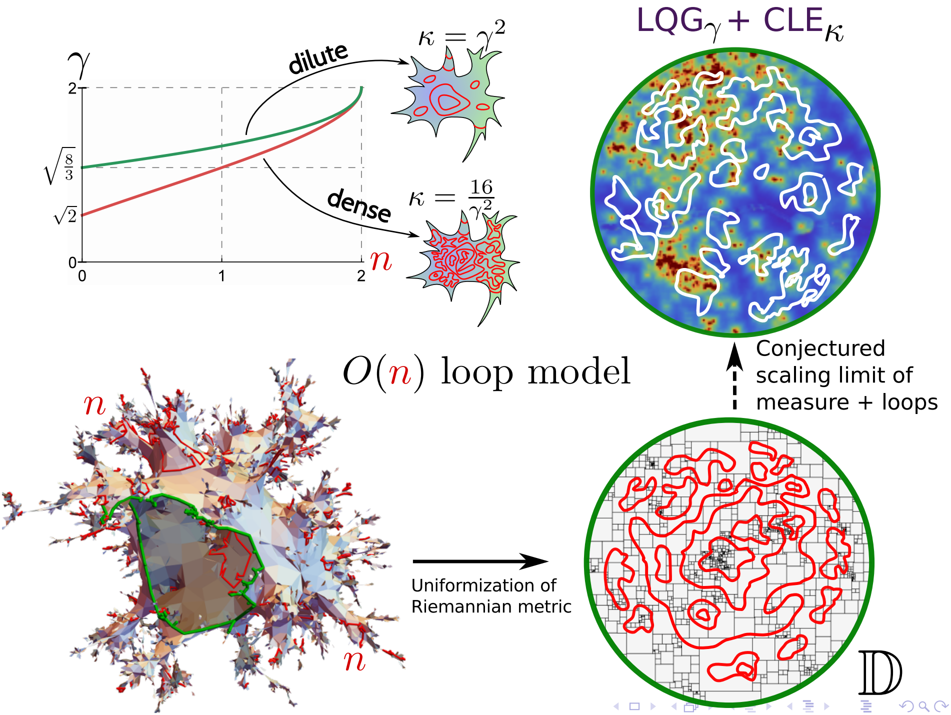


Couple the geometry to a critical statistical system!





$\mathsf{LQG}_\gamma + \mathsf{CLE}_\kappa$
Conjectured scaling limit of measure + loops
$O(n)$ loop model
n
n
Uniformization of Riemannian metric
$\mathbb{D}$



$\mathrm{LQG}_\gamma + \mathrm{CLE}_\kappa$
γ
2
$\sqrt{\frac{8}{3}}$
$\sqrt{2}$
0
0
1
2
n
dilute
$\kappa = \gamma^2$
dense
$\kappa = \frac{16}{\gamma^2}$
$O(n)$ loop model
Conjectured
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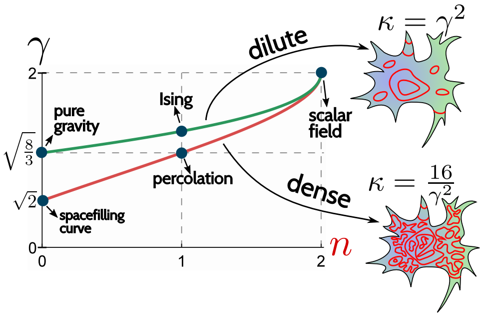
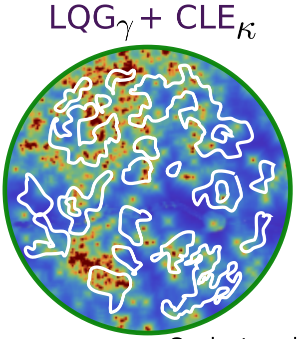
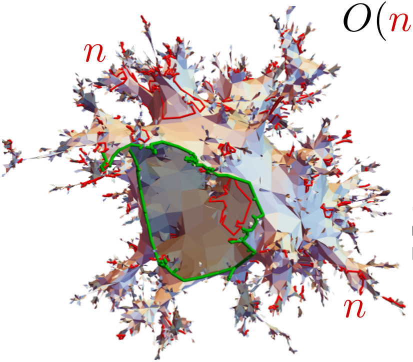
$O(n)$ loop model





Conjectured scaling limit of measure + loops



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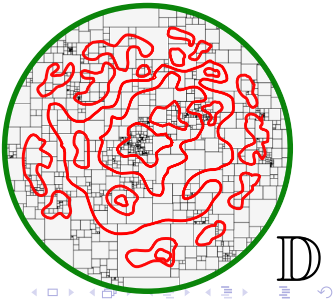

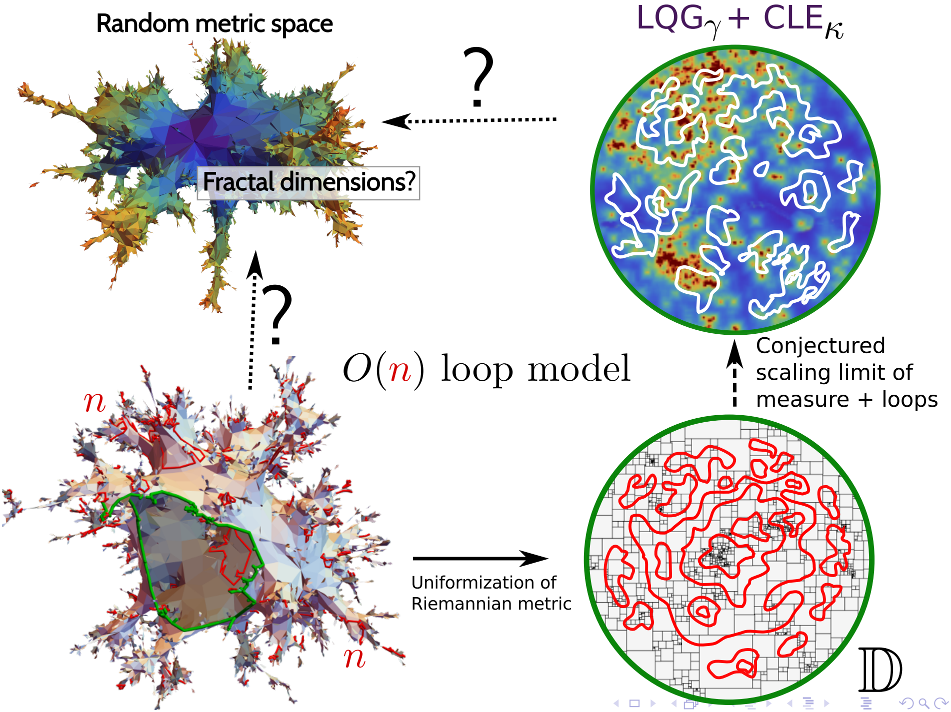
Random metric space
$\mathrm{LQG}_\gamma + \mathrm{CLE}_\kappa$
?
Fractal dimensions?
?
Conjectured
scaling limit of
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n
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Uniformization of
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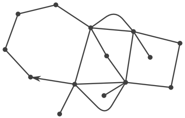
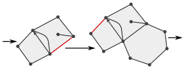
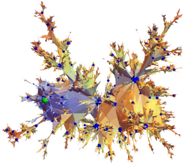
Outline

- Intro to planar maps and Boltzmann distributions

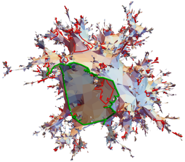




- Peeling exploration



- Geometrical properties (beyond Brownian)



- Towards geometry of maps with an $O(n)$ loop model

Planar maps

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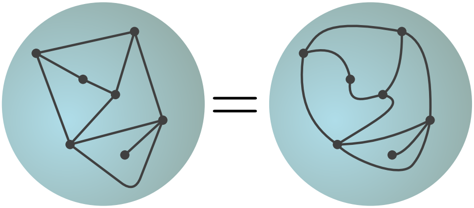

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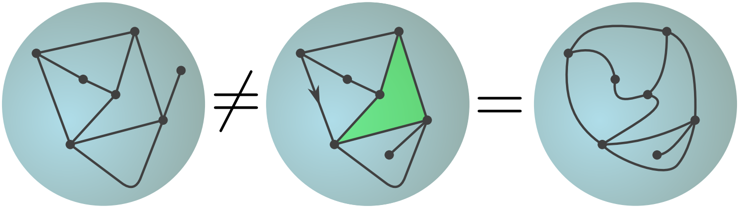

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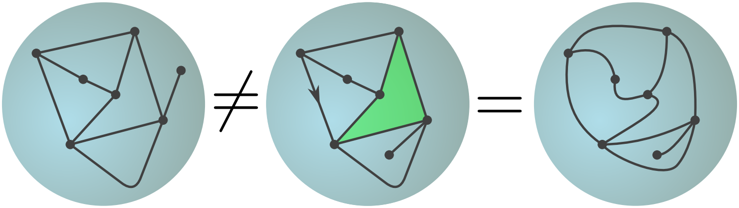

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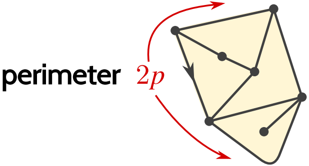

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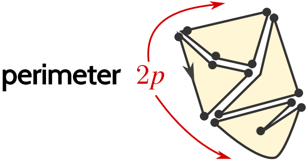
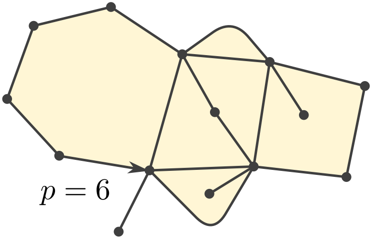
- Think of $\mathfrak{m}$ as a tessellation of the $2p$-gon by (equilateral) polygons.



Boltzmann planar maps

- $\mathcal{M}_p = \{$rooted, bipartite planar maps of perimeter $2p\}$.



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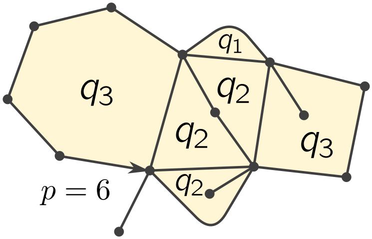
- For $\mathbf{q} = (q_1, q_2, \ldots) \geq 0$ define measure $w_{\mathbf{q}}(\mathfrak{m}) = \prod_{\text{faces } f} q_{\frac{\deg(f)}{2}}$.



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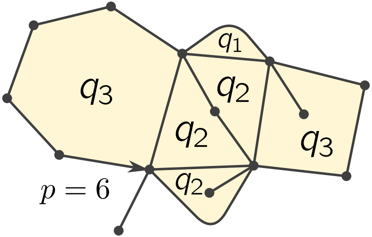

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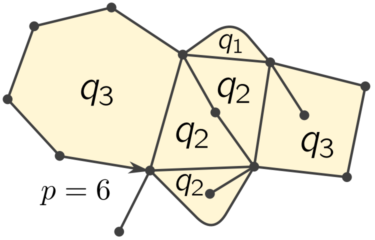

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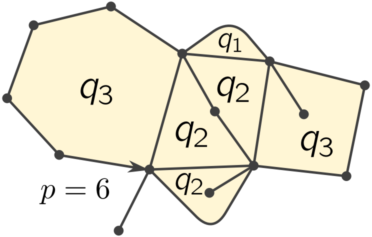
- $\mathbf{q}$ is critical if it admissible and $\mathrm{Var}|\mathfrak{m}^{(p)}| = \infty$.



Scaling limit of the graph distance

- Using bijections with trees, the Gromov-Hausdorff scaling limits of the graph distance are well-understood. Distinguish:
  - generic critical $\mathbf{q}$: Var(degree of typical face) $< \infty$
  - non-generic critical $\mathbf{q}$ of index $\alpha \in (1,2)$: fine-tuning $q_k \sim C\,\kappa^k\,k^{-\alpha-1/2}$ such that $\mathbb{P}$(degree of typical face $> k$) $\sim k^{-\alpha}$.

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Theorem (Bettinelli, Miermont, '15)

If $\mathbf{q}$ is generic critical (and degrees exponentially bounded), then

$$\left(\mathfrak{m}^{(p)}, p^{-1/2} d_{graph}(\cdot,\cdot)\right) \xrightarrow[p\to\infty]{(\mathrm{d})} \textit{Brownian disk of perimeter 1}$$

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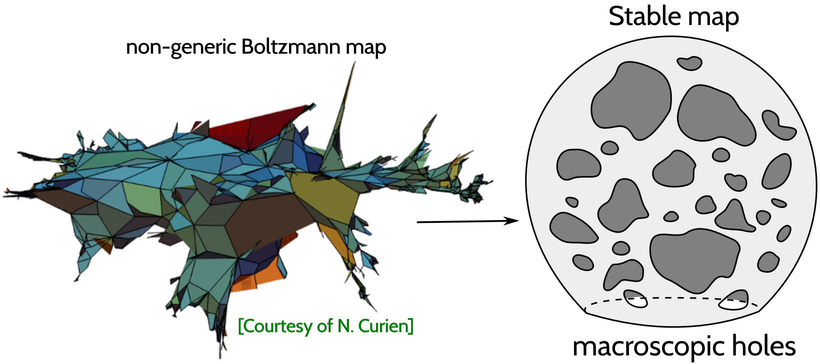
Theorem (Le Gall, Miermont, '11; Marzouk, '18)

If $\mathbf{q}$ is non-generic critical of index $\alpha \in (1,2)$, then

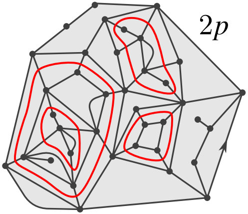
$$\left(\mathfrak{m}^{(p)}, |\mathfrak{m}^{(p)}|^{-\frac{1}{2\alpha}} d_{graph}(\cdot,\cdot)\right) \xrightarrow[|\mathfrak{m}^{(p)}|\to\infty]{(\mathrm{d})} \alpha\textit{-stable map}$$

with a.s. Hausdorff dimension $2\alpha \in (2,4)$.


non-generic Boltzmann map
[Courtesy of N. Curien]
Stable map
macroscopic holes

Relation to the $O(n)$ loop model

- Rigid $O(n)$ loop model: Draw disjoint loops that intersect quadrangles through opposite sides. [Borot, Bouttier, Guitter, '11]



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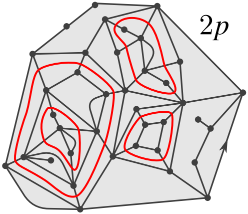
$$w_{n,g,\mathbf{q}}(\mathfrak{m}) = n^{\#} g^{\#} \prod_{\text{reg. faces } f} q_{\frac{\deg(f)}{2}}$$

for $n, g, \mathbf{q} > 0$ fixed.



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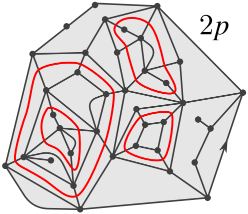

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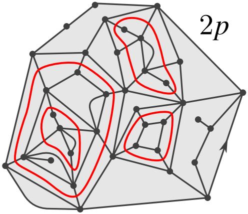
- For $n \in (0, 2)$, the system exhibits four phases as $p \to \infty$: [Borot, Bouttier, Guitter, '11] [TB, Chen, '18]
  - subcritical: treelike/only see boundary
  - pure gravity: microscopic loops
  - dilute critical: self-avoiding loops
  - dense critical: self-touching loops



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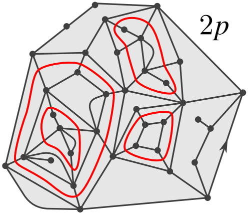

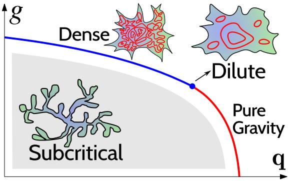

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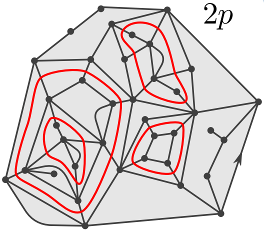

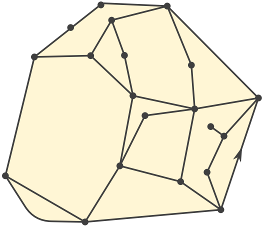

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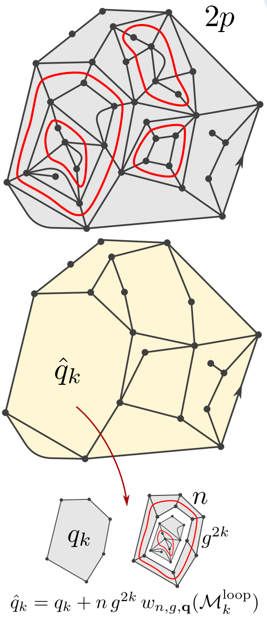

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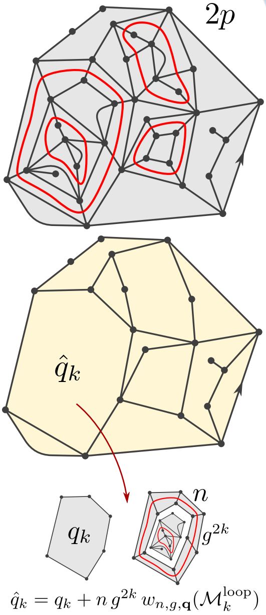
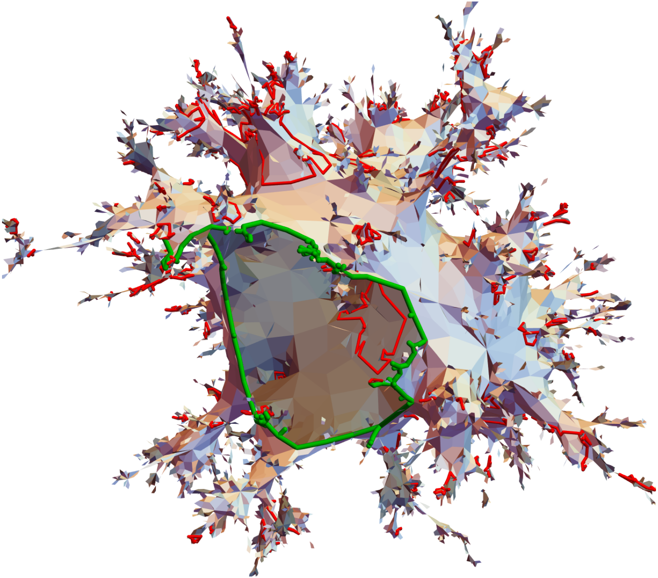
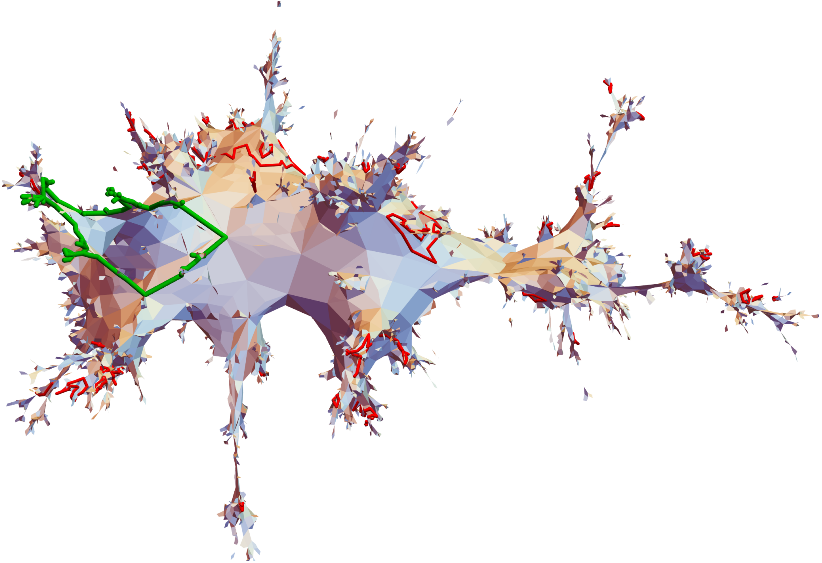
  - $\hat{\mathbf{q}}$ is admissible iff $(n, g, \mathbf{q})$ is.
  - $\hat{\mathbf{q}}$ is non-generic critical with index $\alpha = \frac{3}{2} - \frac{1}{\pi} \arccos \frac{n}{2} \in (1, 3/2)$ iff $(n, g, \mathbf{q})$ is dense critical.
  - $\hat{\mathbf{q}}$ is non-generic critical with index $\alpha = \frac{3}{2} + \frac{1}{\pi} \arccos \frac{n}{2} \in (3/2, 2)$ iff $(n, g, \mathbf{q})$ is dilute critical.



Simulation: dilute quadrangulation ($q_2 > 0, q_1 = q_3 = \ldots = 0$), $p = 50$, $n = 0.6$



Simulation: dilute quadrangulation ($q_2 > 0, q_1 = q_3 = \ldots = 0$), $p = 40$, $n = 0.3$



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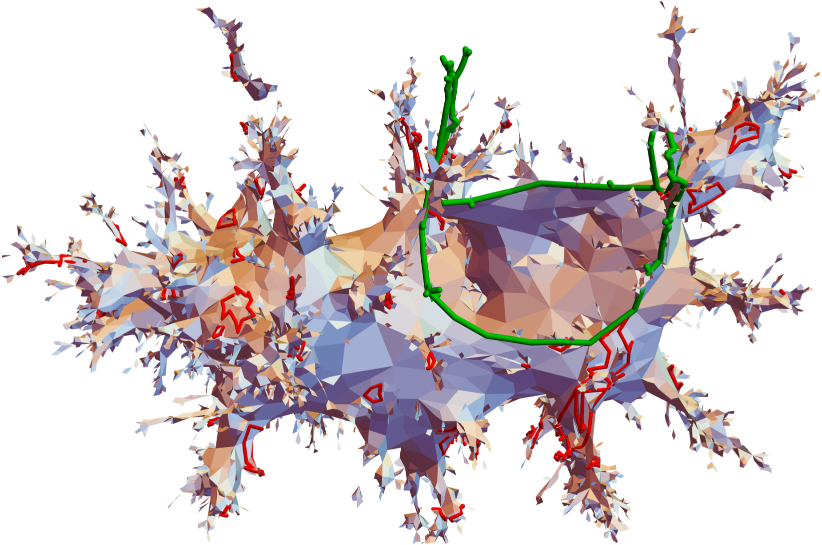
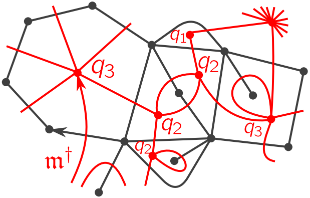


Look at dual graph distances

- The dual map of a non-generic critical Boltzmann planar map supports vertices of high degree.



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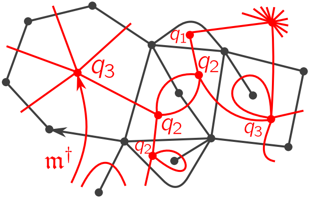

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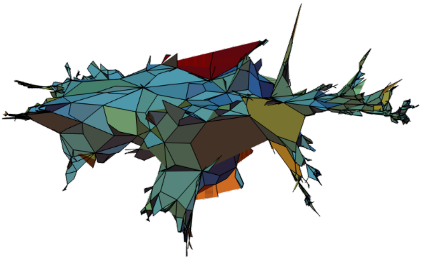

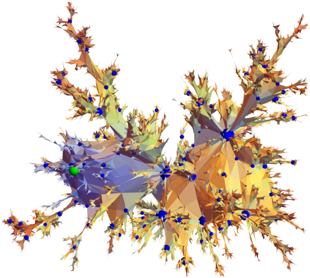

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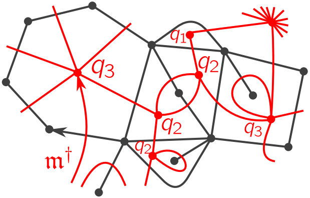

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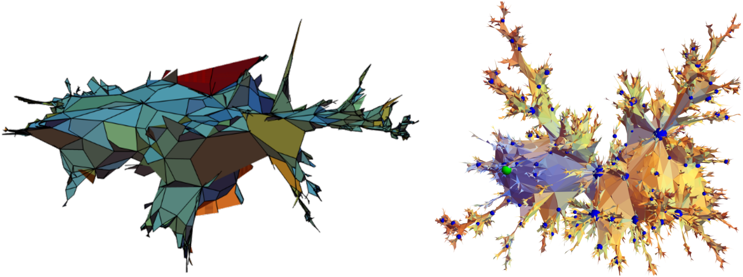
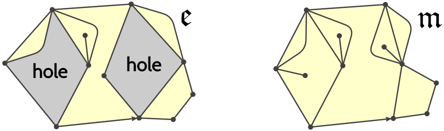


- No tree bijections available: use peeling exploration!

Peeling exploration [Watabiki, Angel, Curien, Le Gall, TB, . . .]

- A planar map with holes $\mathfrak{e}$ is a planar map with a set of distinguished, simple, disjoint faces (the "holes").



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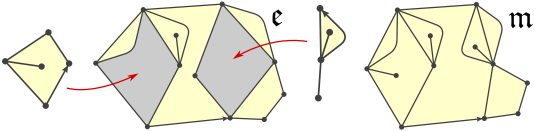

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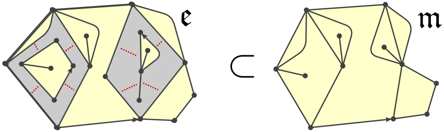

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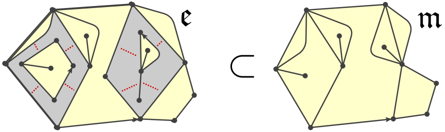

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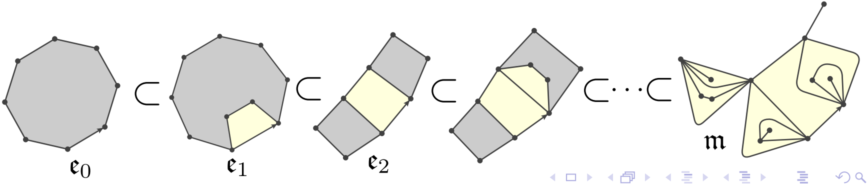

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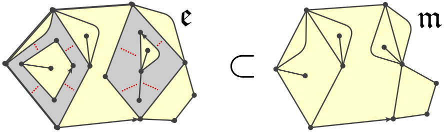

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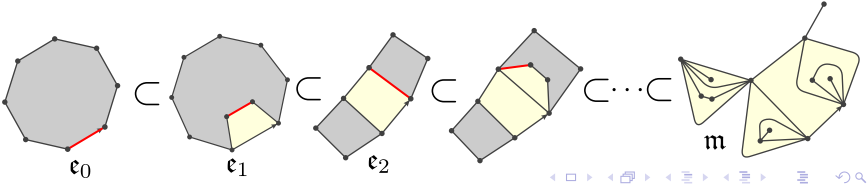

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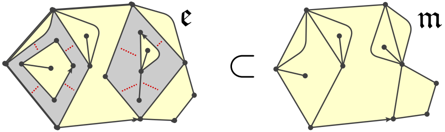

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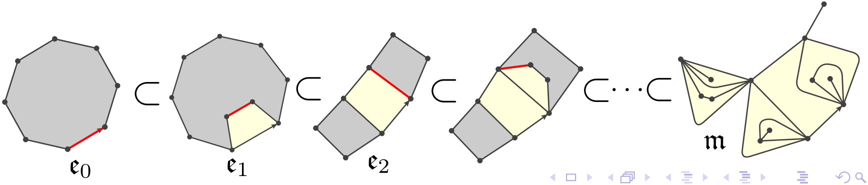
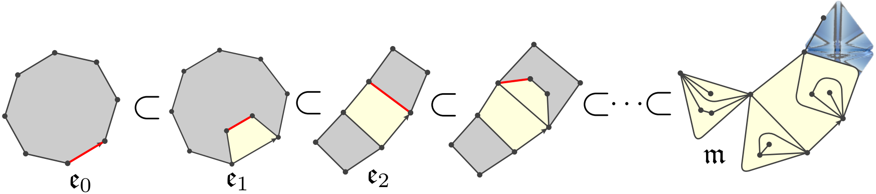
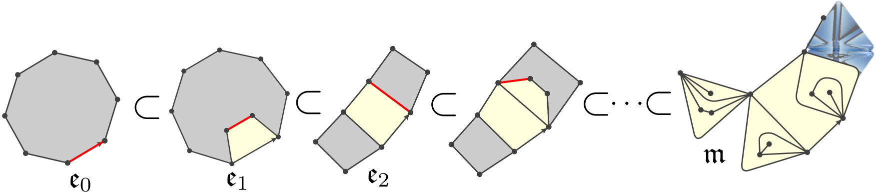
  - $\mathfrak{e}_{i+1}$ is minimal submap, $\mathfrak{e}_i \subset \mathfrak{e}_{i+1} \subset \mathfrak{m}$, such that $\mathcal{A}(\mathfrak{e}_i)$ not incident to hole: reveals new face of $\mathfrak{m}$ or glues pair of edges on the hole.





- For a $\mathbf{q}$-Boltzmann planar map $\mathfrak{m}^{(p)}$ and $\mathfrak{e}$ fixed, conditionally on $\mathfrak{e} \subset \mathfrak{m}$ the maps filling in the holes of $\mathfrak{e}$ are distributed as independent Boltzmann planar maps.



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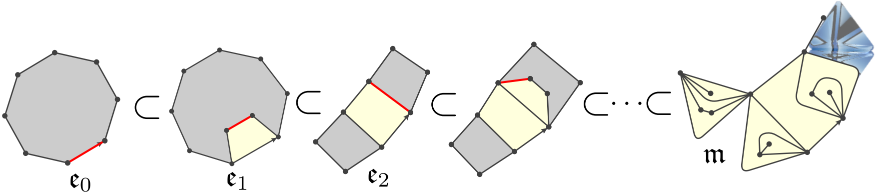

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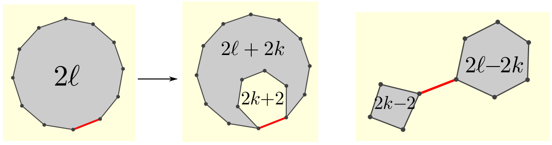
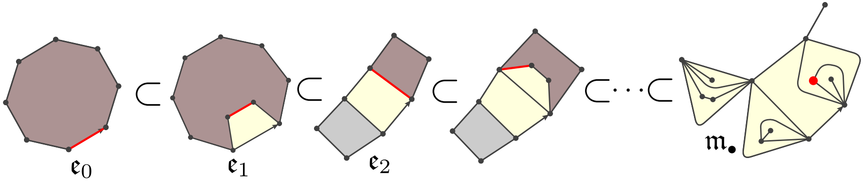
- Hence $(\mathfrak{e}_i)$ is a Markov process with transition probabilities $(W^{(p)} = w_{\mathbf{q}}(\mathcal{M}^{(p)}))$:



Transition probability: $\dfrac{q_{k+1} W^{(l+k)}}{W^{(l)}}$ $\dfrac{W^{(k-1)} W^{(l-k)}}{W^{(l)}}$

Targeted peeling exploration

- If the map $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(p)}$ has a marked vertex, one may track the hole containing the vertex



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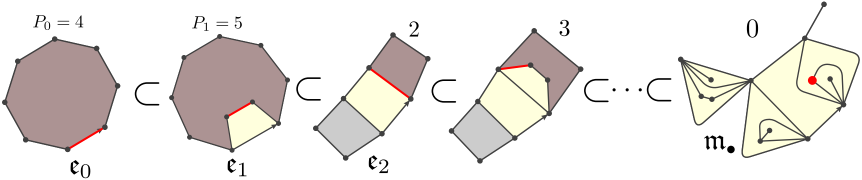
- If the map $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(p)}$ has a marked vertex, one may track the hole containing the vertex and its half-degree (P_i), the perimeter process.



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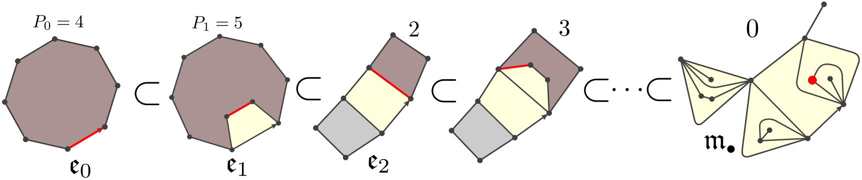
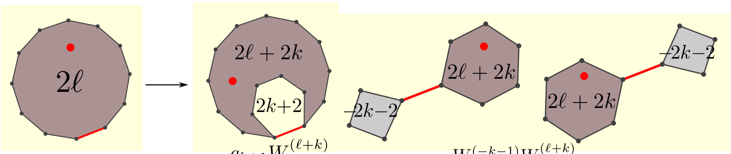


- For a pointed $\mathbf{q}$-Boltzmann planar map $\mathfrak{m}_\bullet^{(p)}$



$$\mathbb{P}(P_{i+1}=\ell+k \mid P_i=\ell) = \underbrace{\frac{q_{k+1} W_\bullet^{(\ell+k)}}{W_\bullet^{(\ell)}}}_{k\geq 0} \qquad \underbrace{2\,\frac{W^{(-k-1)} W_\bullet^{(\ell+k)}}{W_\bullet^{(\ell)}}}_{k<0}$$

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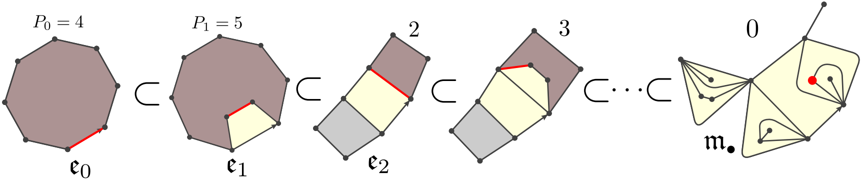

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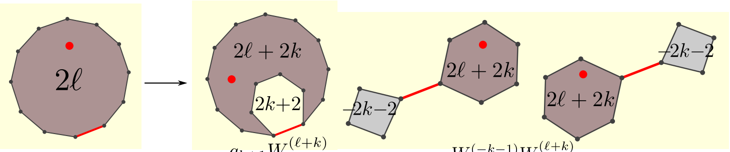

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- As $\ell \to \infty$ it takes the law of a random walk on $\mathbb{Z}$ with distribution

$$\mathbb{P}(P_{i+1} = \ell + k \mid P_i = \ell) \to \nu(k) := \begin{cases} q_{k+1} \kappa^{-k} & k \geq 0 \\ 2W^{(-k-1)} \kappa^{-k} & k < 0 \end{cases}$$

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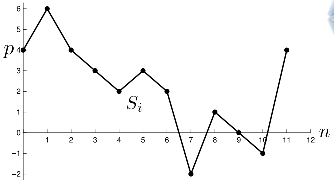

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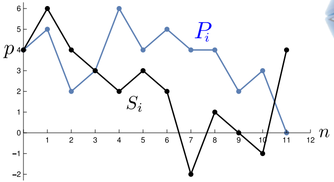

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- How are (S_i) and (P_i) related?

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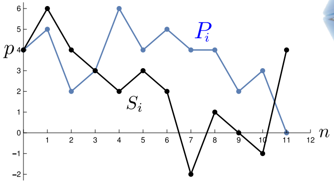

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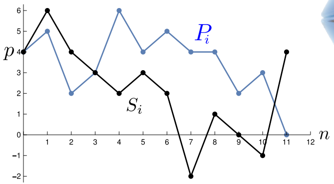

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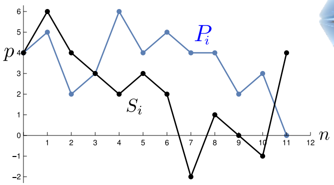

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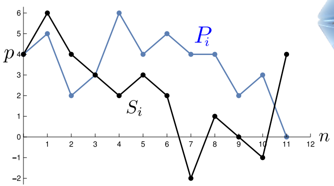

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- $\mathbf{q}$ *non-generic critical of index* $\alpha \in (1,2)$ $\longleftrightarrow$ (S_i) *in the basin of attraction of an* $(\alpha - \frac{1}{2})$*-stable process:* $\nu(\pm k) \sim c_{\pm} k^{-\alpha-1/2}$.

Infinite Boltzmann planar maps

- Benjamini-Schramm-type local limit:
  - Let $\mathbf{q}$ be critical and condition a $\mathbf{q}$-BPM to have n vertices.
  - Then the laws of neighbourhoods of the root converge in distribution as $n \to \infty$ to those of a unique random infinite map:
    [Björnberg, Stefánsson, '14] [Stephenson, '14]

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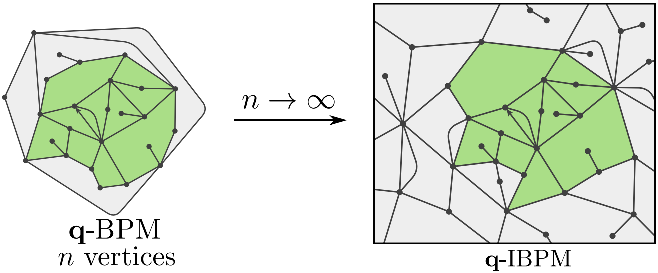

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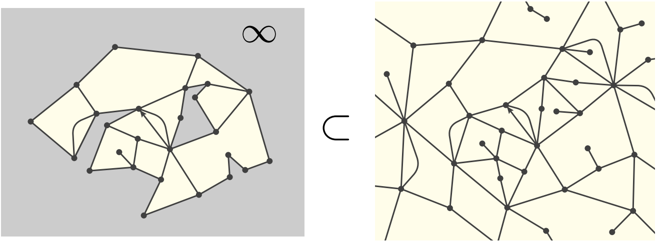

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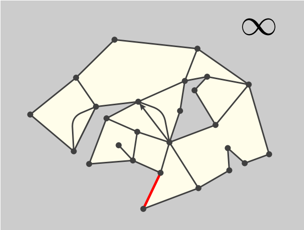
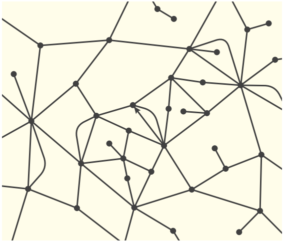


$\subset$



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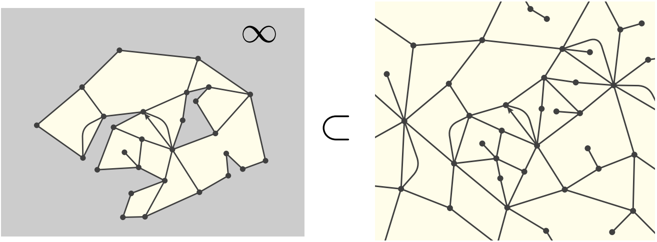

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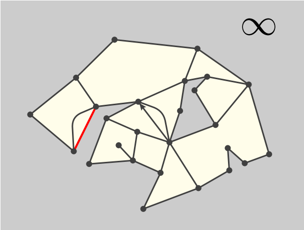
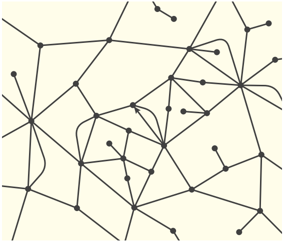


$\subset$



Infinite Boltzmann planar maps

- Benjamini-Schramm-type local limit:
  - Let $\mathbf{q}$ be critical and condition a $\mathbf{q}$-BPM to have n vertices.
  - Then the laws of neighbourhoods of the root converge in distribution as $n \to \infty$ to those of a unique random infinite map:
    [Björnberg, Stefánsson, '14] [Stephenson, '14]
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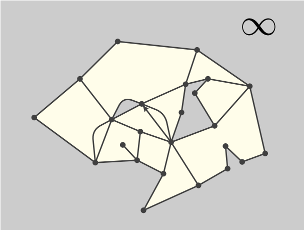

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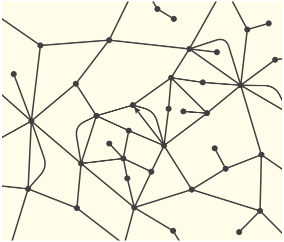

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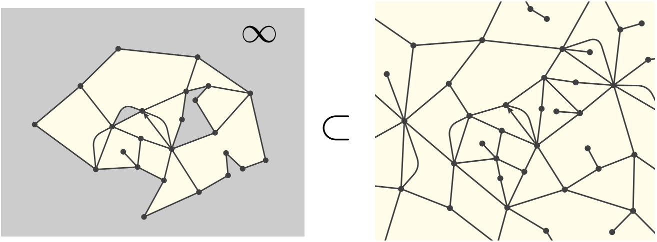
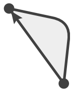
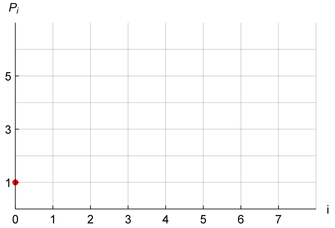


Theorem (TB '15)

For $\mathbf{q}$ critical the perimeter process (P_i) has the law of the random walk (S_i) conditioned to stay positive.

Targeted peeling of infinite Boltzmann maps





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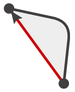

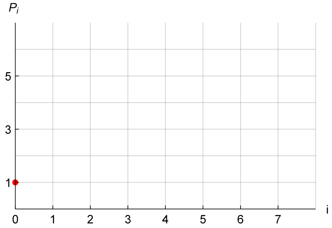

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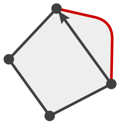

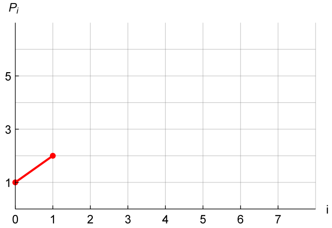

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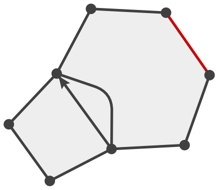

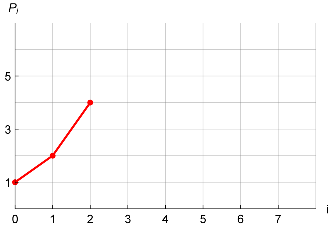

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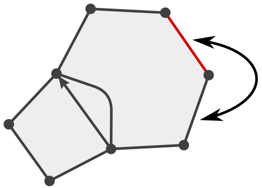

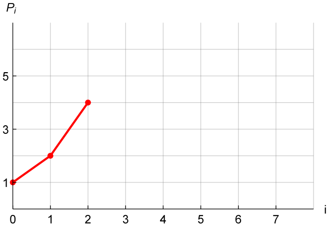

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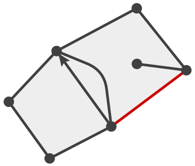

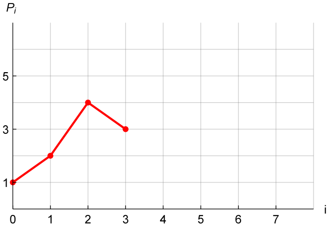

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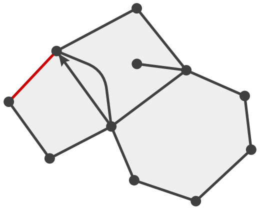

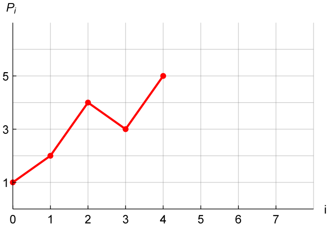

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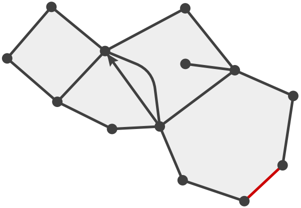

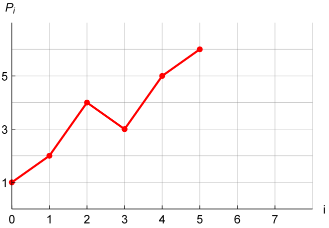

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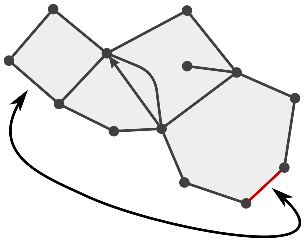

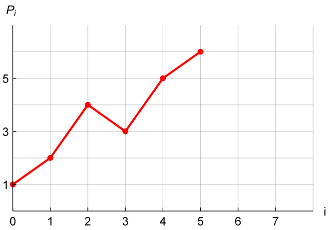

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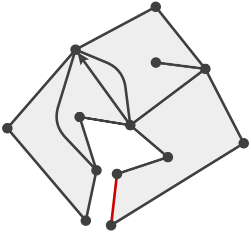

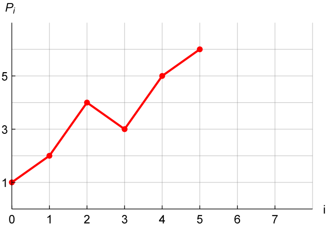

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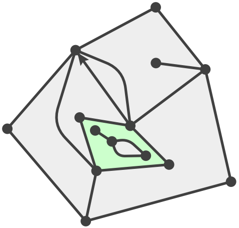

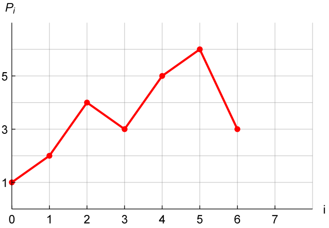

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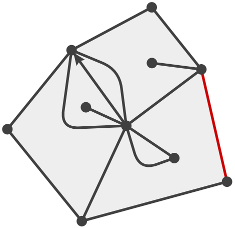

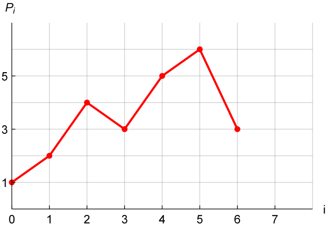

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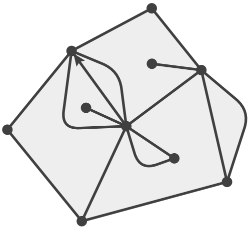

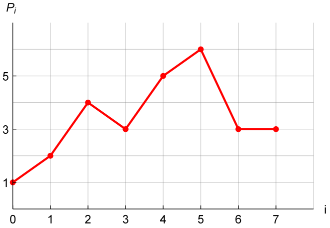

Proposition (TB, Curien, '16)

If $\mathbf{q}$ *is non-generic critical of index* $\alpha \in (1,2)$ *then the perimeter process of* $\mathfrak{m}_\infty$ *satisfies the scaling limit (in the sense of Skorokhod)*

$$\left(\frac{P_{\lfloor \lambda t \rfloor}}{\lambda^{1/(\alpha - 1/2)}} \right)_{t \geq 0} \xrightarrow[\lambda \to \infty]{(\mathrm{d})} c\,(S_t^{\uparrow})_{t \geq 0},$$

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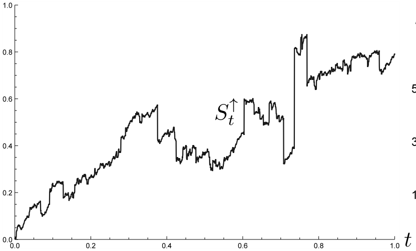

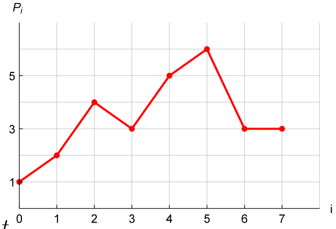

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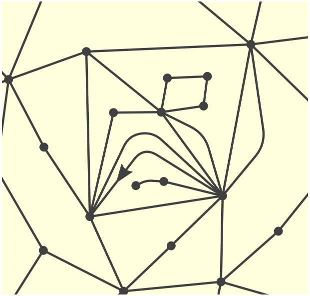

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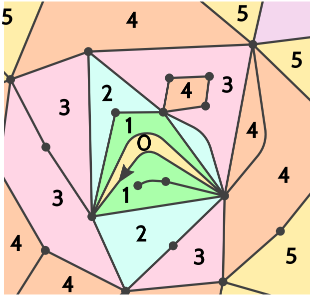

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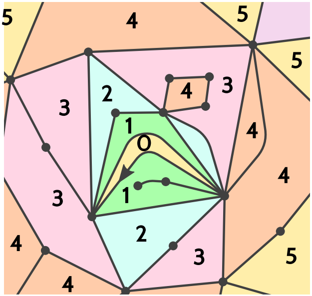

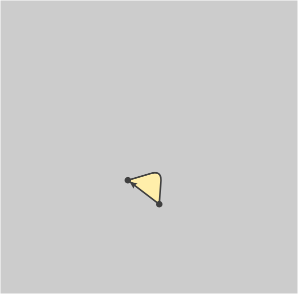

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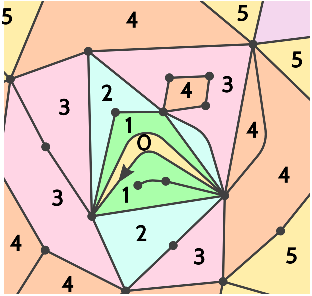

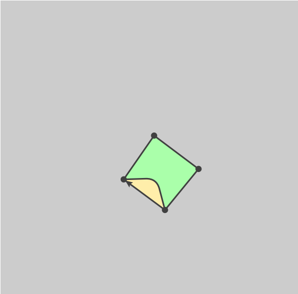

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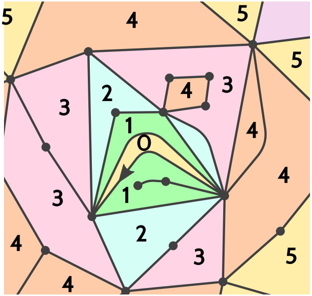

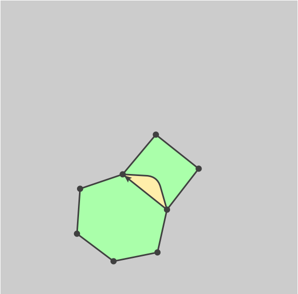

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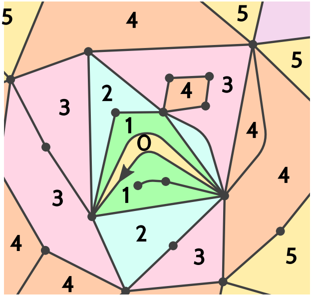

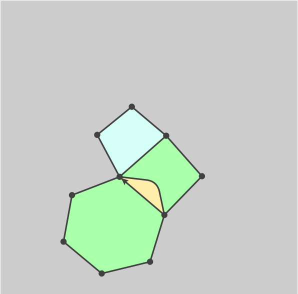

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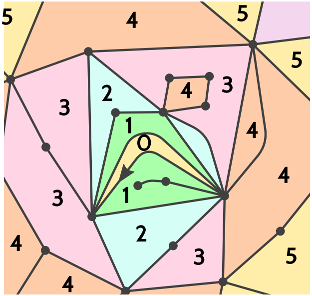

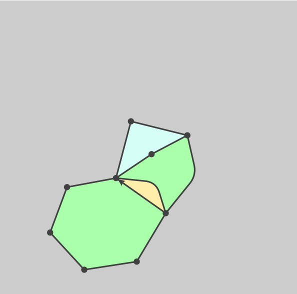

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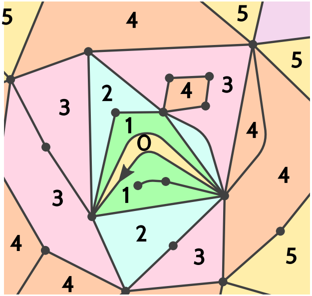

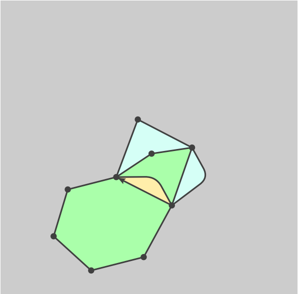

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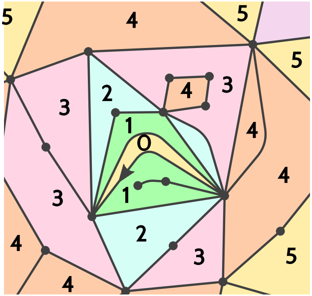

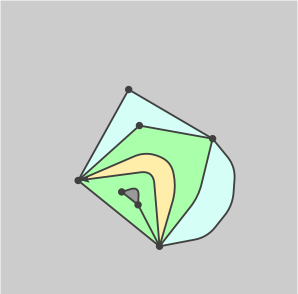

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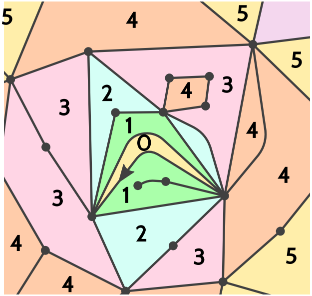

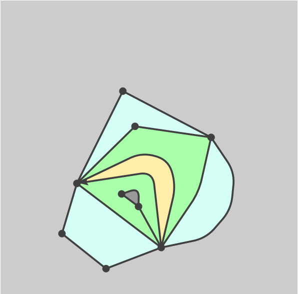

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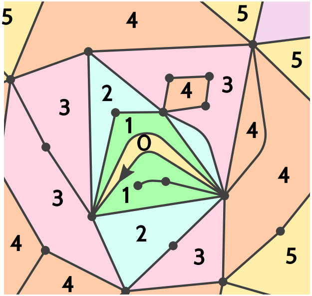

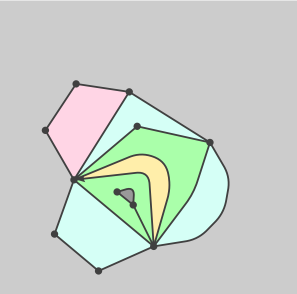

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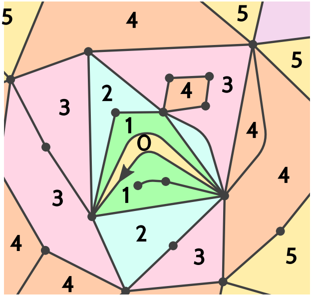

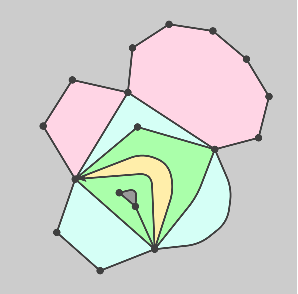

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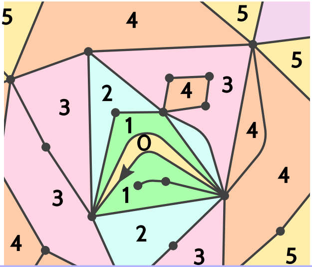

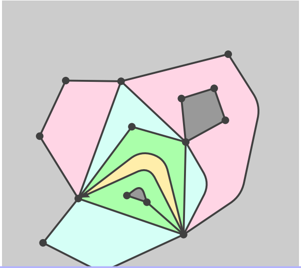

Theorem (TB, Curien, '16; TB, Curien, Marzouk, '17)

If $\mathbf{q}$ *is non-generic of index* $\alpha \in (1,2)$, *then*

$$|Ball_r| \approx \begin{cases} r^{\frac{\alpha}{\alpha-3/2}} & \frac{3}{2} < \alpha < 2 \\ e^{\frac{3\pi}{\sqrt{2}}\sqrt{r}} & \alpha = \frac{3}{2} \\ e^{c\,r} & 1 < \alpha < \frac{3}{2} \end{cases}, \quad |\partial Ball_r| \approx \begin{cases} r^{\frac{1}{\alpha-3/2}} & \frac{3}{2} < \alpha < 2 \\ e^{\pi\sqrt{2}\sqrt{r}} & \alpha = \frac{3}{2} \\ e^{c'\,r} & 1 < \alpha < \frac{3}{2} \end{cases}.$$

For $\alpha > \frac{3}{2}$ *scaling limits for* $\lambda^{\frac{-\alpha}{\alpha-3/2}}|Ball_{\lambda r}|$ *and* $\lambda^{\frac{-1}{\alpha-3/2}}|\partial Ball_{\lambda r}|$ *are known.*

Asymptotic growth [TB, Curien, '16] [TB, Curien, Marzouk, '17]

	$1 < \alpha < \frac{3}{2}$	$\alpha = \frac{3}{2}$	$\frac{3}{2} < \alpha < 2$
P_n	$\approx n^{\frac{1}{\alpha - 1/2}}$	$\approx n$	$\approx n^{\frac{1}{\alpha - 1/2}}$

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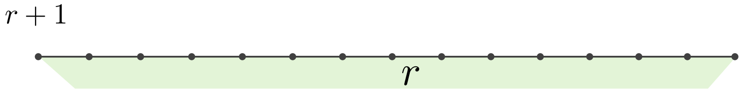

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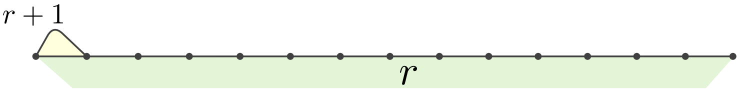

Asymptotic growth [TB, Curien, '16] [TB, Curien, Marzouk, '17]

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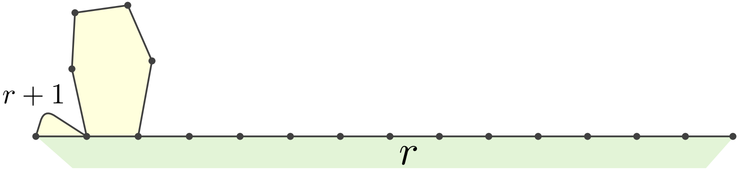

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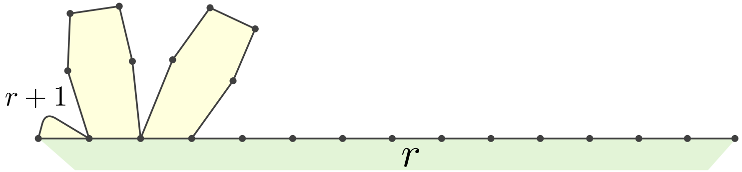

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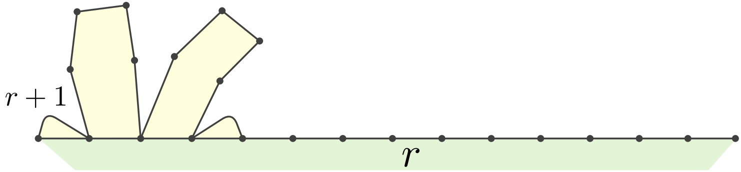

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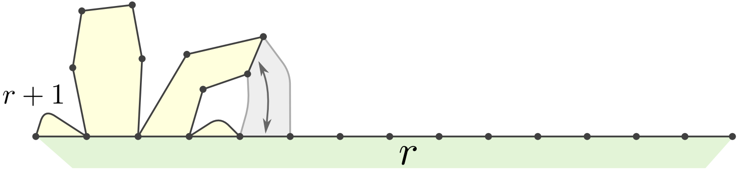

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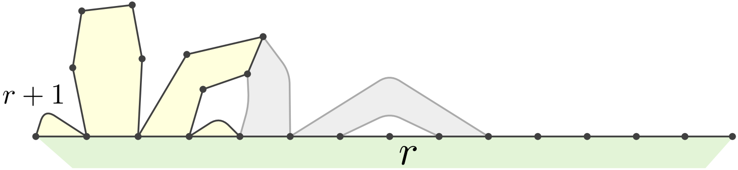

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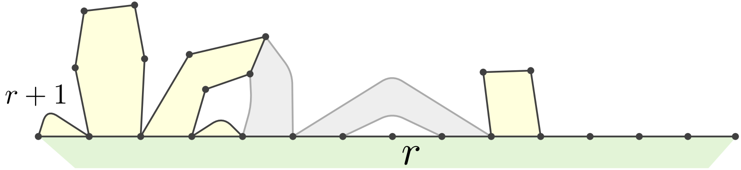

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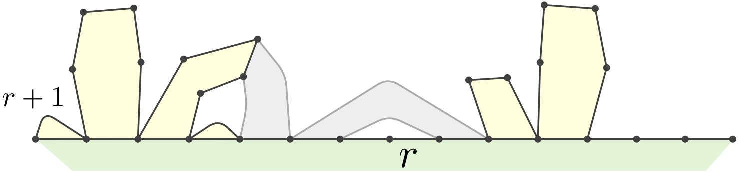

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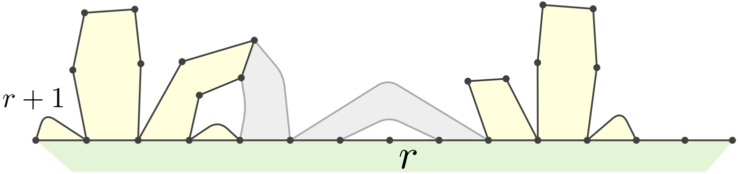

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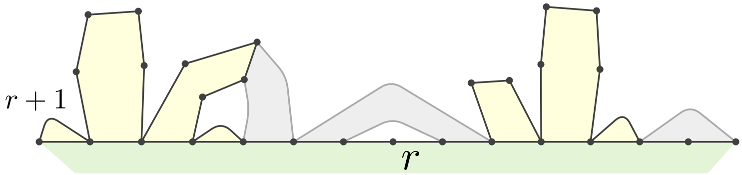

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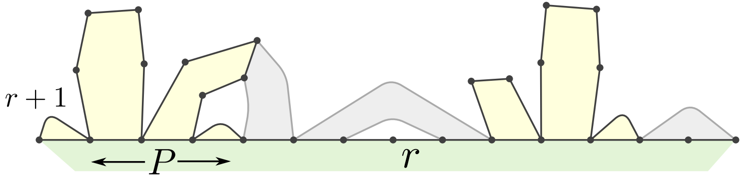
Steps to complete layer of perim. P	$\approx P^{\alpha - 1/2}$	$\approx \frac{P}{\log P}$	$\approx P$



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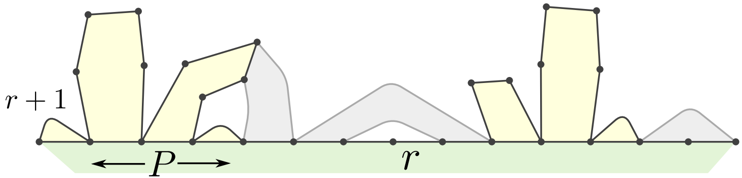
Distance after n steps	$\cdots$	$\sum_{i=0}^{n} \frac{\log P_i}{P_i} \approx (\log n)^2$	$\sum_{i=0}^{n} \frac{1}{P_i} \approx n^{\frac{\alpha - 3/2}{\alpha - 1/2}}$



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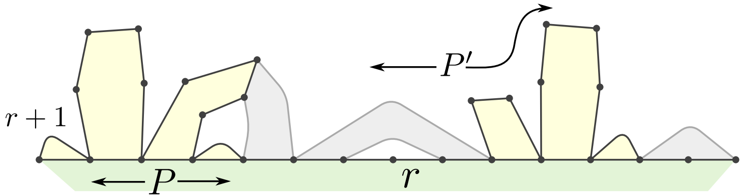
P'/P	$\approx e^{\mathcal{Z}}$ $(\mathbb{E}\mathcal{Z} > 0)$	≈ 1	≈ 1



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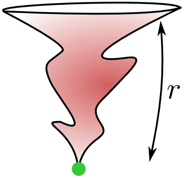
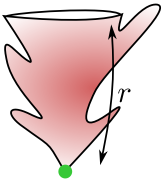
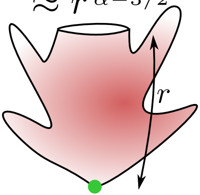
P'/P	$\approx e^{\mathcal{Z}}$ $(\mathbb{E}\mathcal{Z} > 0)$	≈ 1	≈ 1
$\lvert \partial \mathrm{Ball}_r \rvert$	$\approx e^{cr}$	$\approx e^{\pi \sqrt{2} \sqrt{r}}$	$\approx r^{\frac{1}{\alpha - 3/2}}$
			

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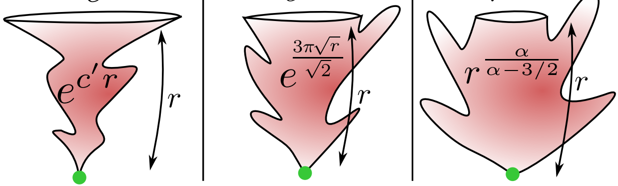
$\lvert \mathrm{Ball}_r \rvert$	$e^{c'r}$	$e^{\frac{3\pi\sqrt{r}}{\sqrt{2}}}$	$r^{\frac{\alpha}{\alpha - 3/2}}$



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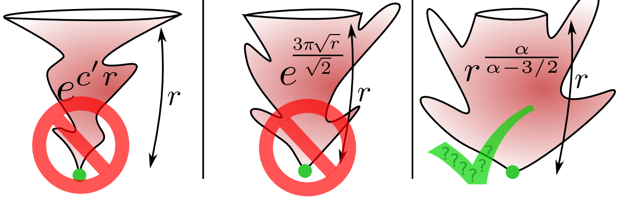
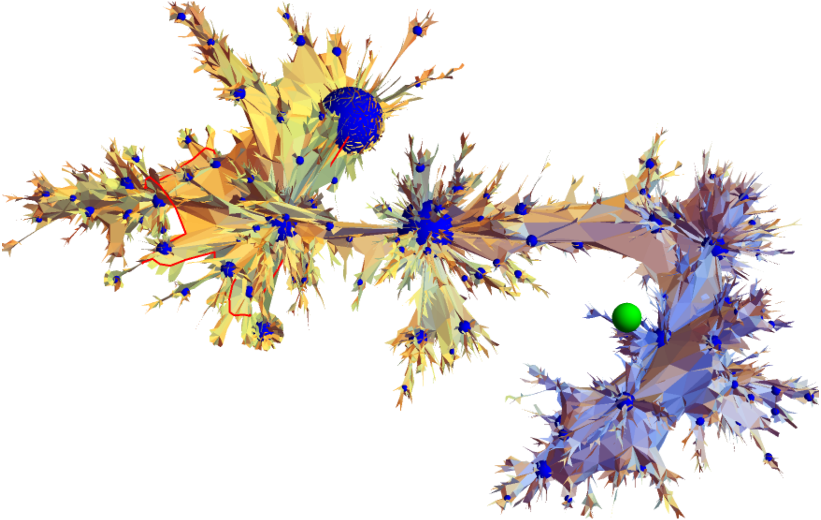
$\lvert\mathrm{Ball}_r\rvert$	$e^{c'r}$	$e^{\frac{3\pi\sqrt{r}}{\sqrt{2}}}$	$r^{\frac{\alpha}{\alpha-3/2}}$
Scaling limit			????



Simulations: dense case

$\alpha = 1.3$



Simulations: dense case

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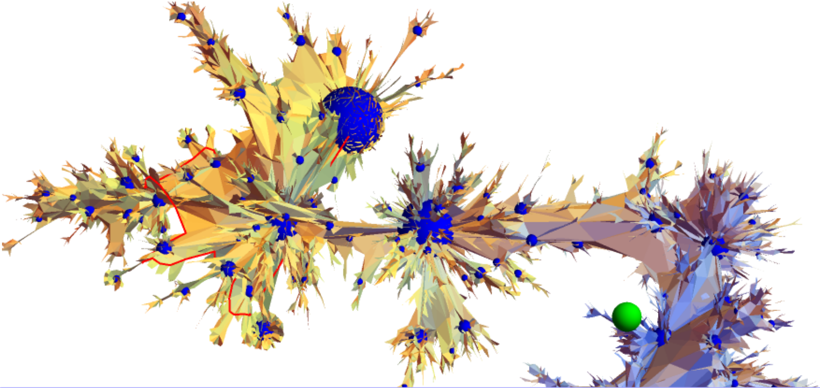
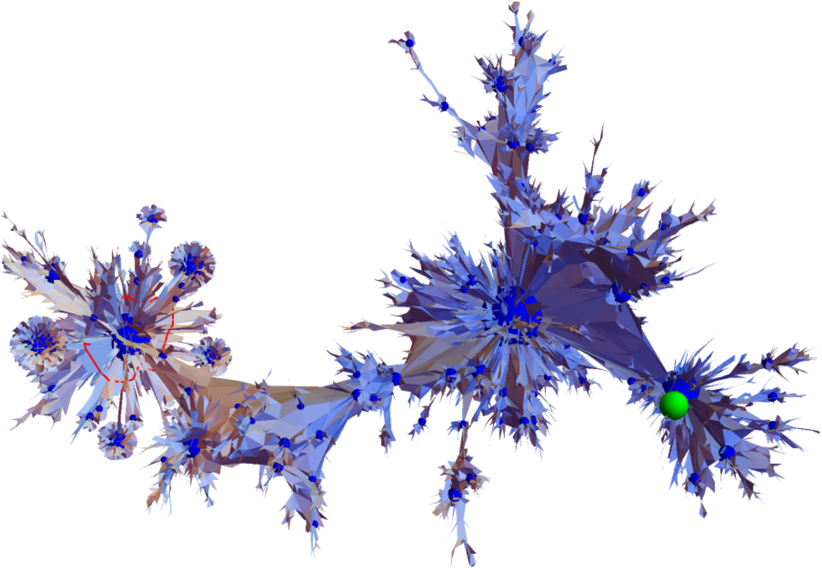


Theorem (TB, Curien, '16)

When $\alpha < 3/2$ the map and its dual both contain infinitely many cut vertices separating root from ∞.

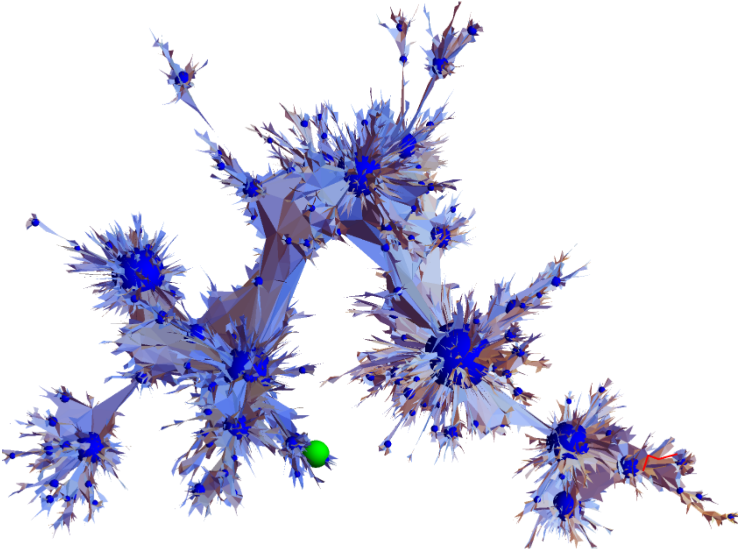
Simulations: dense case

$\alpha = 1.3$



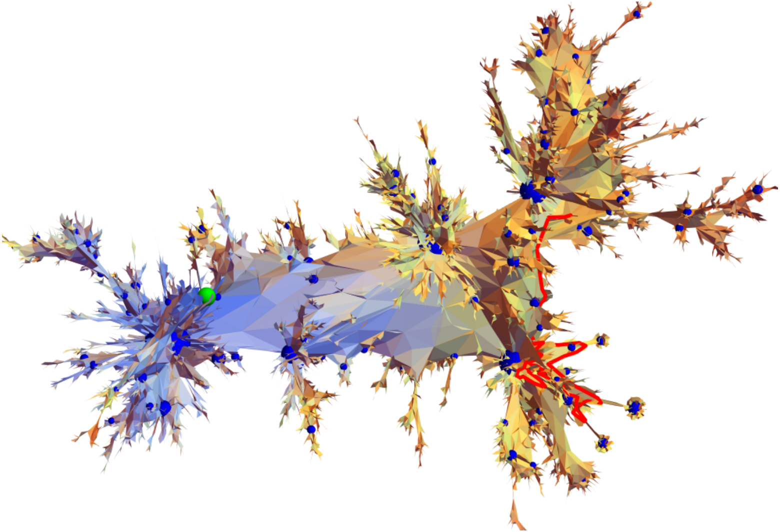
Simulations: dense case

$\alpha = 1.2$



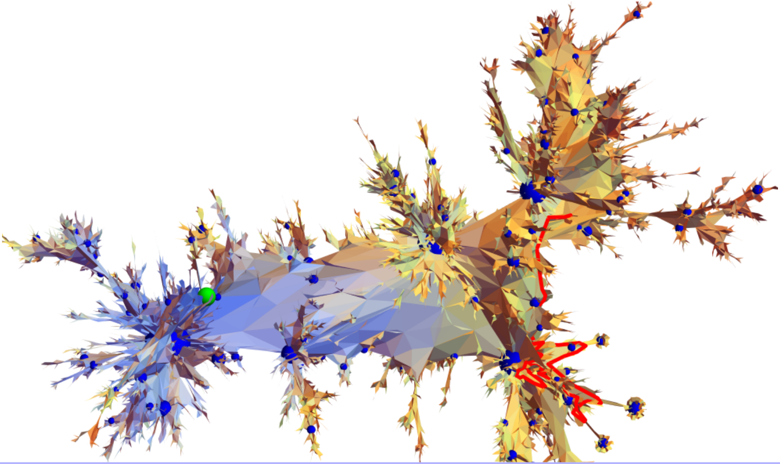
Simulations: Cauchy case

$\alpha = 3/2$



Simulations: Cauchy case

$\alpha = 3/2$

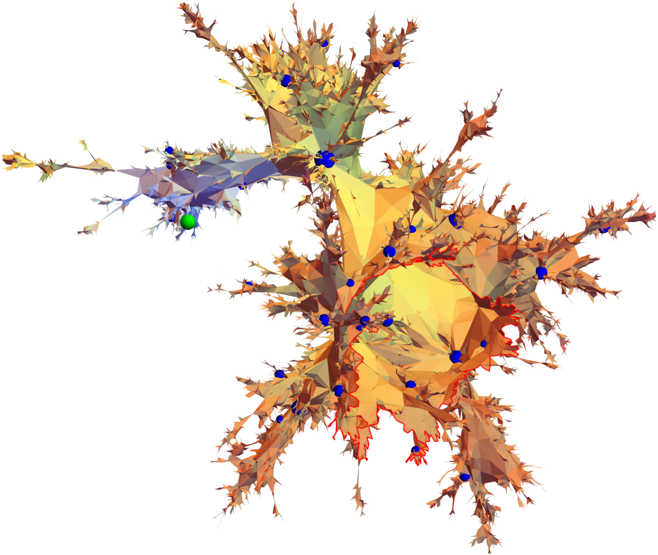


Theorem (TB, Curien, Marzouk '17)

Again infinitely many cut vertices, but their number grows only like $\log\log(\mathit{volume})$.

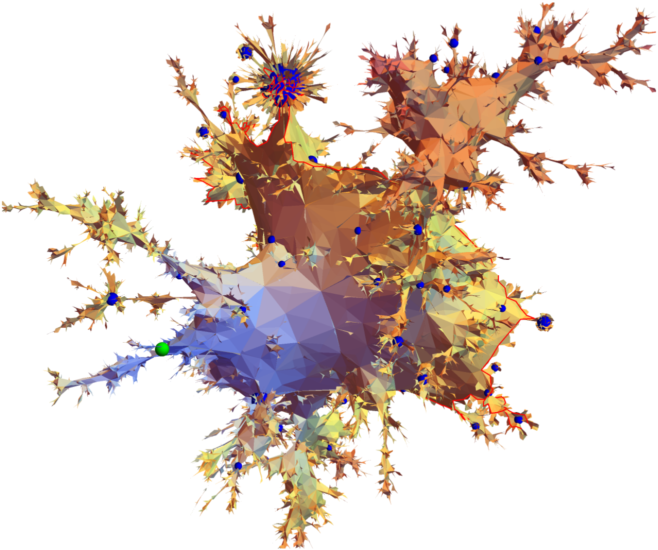
Simulations: dilute case

$\alpha = 1.8$



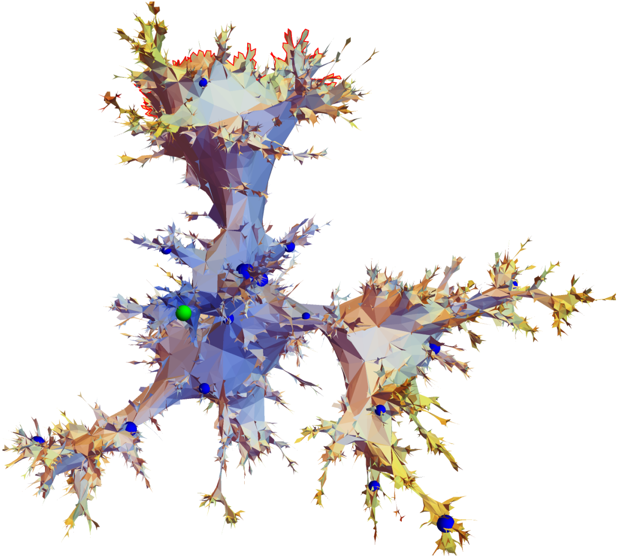
Simulations: dilute case

$\alpha = 1.8$



Simulations: dilute case

$\alpha = 1.85$



Simulations: dilute case

$\alpha = 1.95$