09-07-2021, Random excursions with Jean Bertoin

Random punctured hyperbolic surfaces & the Brownian sphere

Timothy Budd

in collaboration with Nicolas Curien

Radboud University

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/
Random punctured hyperbolic surfaces & the Brownian sphere

Timothy Budd

Martingales in self-similar growth-fragmentations and their connections with random planar maps

Jean Bertoin1 · Timothy Budd2,3 · Nicolas Curien4 · Igor Kortchemski5

in collaboration with Nicolas Curien

Radboud University

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/
Moduli space of punctured spheres

- Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \{ X = \hat{\mathbb{C}} \setminus \{ x_1, \ldots, x_n \} \}$$
Moduli space of punctured spheres

Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \{ X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\} \} / \text{M"obius}.$$
Moduli space of punctured spheres

- Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \left\{X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}\right\} / \text{Möbius}.$$ 

- $\mathcal{M}_{0,n}$ is an orbifold of real dimension $2n - 6$. 

By the uniformization theorem ($n \geq 3$):

$$\mathcal{M}_{0,n} \sim \left\{\text{genus-0 hyperbolic surfaces with } n \text{ cusps}\right\} / \text{isometries}.$$ 

- Locally isometric to hyperbolic plane $\mathbb{H}$, where geodesics are circular arcs or vertical lines.

- Cusps regions are locally isometric to vertical strips with boundaries identified.
Moduli space of punctured spheres

Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \{X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}\} / \text{Möbius}.$$

$\mathcal{M}_{0,n}$ is an orbifold of real dimension $2n - 6$.

By the uniformization theorem ($n \geq 3$):

$$\mathcal{M}_{0,n} \cong \{\text{genus-0 hyperbolic surfaces with } n \text{ cusps}\} / \text{isometries}$$
Moduli space of punctured spheres

Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \{ X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\} \} / \text{Möbius}.$$ 

$\mathcal{M}_{0,n}$ is an orbifold of real dimension $2n - 6$.

By the uniformization theorem ($n \geq 3$):

$$\mathcal{M}_{0,n} \cong \{\text{genus-0 hyperbolic surfaces with } n \text{ cusps}\} / \text{isometries}$$

Locally isometric to hyperbolic plane $\mathbb{H}$.
Moduli space of punctured spheres

▶ Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \left\{ X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\} \right\} / \text{Möbius.}$$

▶ $\mathcal{M}_{0,n}$ is an orbifold of real dimension $2n - 6$.

▶ By the uniformization theorem ($n \geq 3$):

$$\mathcal{M}_{0,n} \cong \{ \text{genus-0 hyperbolic surfaces with } n \text{ cusps} \} / \text{isometries}$$

▶ Locally isometric to hyperbolic plane $\mathbb{H}$, where geodesics are circular arcs or vertical lines.
Moduli space of punctured spheres

Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with $n$ points removed:

$$\mathcal{M}_{0,n} = \left\{ X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\} \right\} / \text{Möbius.}$$

$\mathcal{M}_{0,n}$ is an orbifold of real dimension $2n - 6$.

By the uniformization theorem ($n \geq 3$):

$$\mathcal{M}_{0,n} \cong \{\text{genus-0 hyperbolic surfaces with } n \text{ cusps}\} / \text{isometries}$$

Locally isometric to hyperbolic plane $\mathbb{H}$, where geodesics are circular arcs or vertical lines.

Cusps regions are locally isometric to vertical strips with boundaries identified.

\[ \mathcal{M}_{0,n} \] admits a natural measure \( WP \) arising from its Weil-Petersson symplectic structure.

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.
- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2}\pi^4, \ldots$ are finite. [Wolpert ’83, Penner ’92] An explicit generating function: [Zograf ’95]

$$Z(x) = \sum_{n\geq 3} \frac{2^{3-n}}{(n-2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{Z(x)}}{\pi} J_1(2\pi \sqrt{Z(x)}).$$

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.
- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2}\pi^4, \ldots$ are finite. [Wolpert '83, Penner '92] An explicit generating function: [Zograf '95]

$$Z(x) = \sum_{n \geq 3} \frac{2^{3-n}}{(n-2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{Z(x)}}{\pi} J_1(2\pi \sqrt{Z(x)}).$$

- Normalizing $WP$: the random hyperbolic surface $S_n \in \mathcal{M}_{0,n}$. 

\[ S_{5000} \]

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.
- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2} \pi^4, \ldots$ are finite. [Wolpert ’83, Penner ’92] An explicit generating function: [Zograf ’95]

$$\mathcal{Z}(x) = \sum_{n \geq 3} \frac{2^{3-n}}{(n-2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{\mathcal{Z}(x)}}{\pi} J_1(2\pi \sqrt{\mathcal{Z}(x)}).$$

- Normalizing $WP$: the random hyperbolic surface $S_n \in \mathcal{M}_{0,n}$.
- Unique probability measure invariant under uniform twist along any simple closed geodesic. [Wolpert, ’82]

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.

- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2}\pi^4, \ldots$ are finite. [Wolpert ’83, Penner ’92] An explicit generating function: [Zograf ’95]

$$Z(x) = \sum_{n \geq 3} \frac{2^{3-n}}{(n-2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{Z(x)}}{\pi} J_1(2\pi \sqrt{Z(x)}).$$

- Normalizing $WP$: the random hyperbolic surface $S_n \in \mathcal{M}_{0,n}$.

- Unique probability measure invariant under uniform twist along any simple closed geodesic. [Wolpert, ’82]

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.
- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2} \pi^4, \ldots$ are finite. [Wolpert ’83, Penner ’92] An explicit generating function: [Zograf ’95]

$$Z(x) = \sum_{n \geq 3} \frac{2^3 - n}{(n - 2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{Z(x)}}{\pi} J_1(2\pi \sqrt{Z(x)}).$$

- Normalizing $WP$: the random hyperbolic surface $S_n \in \mathcal{M}_{0,n}$.
- Unique probability measure invariant under uniform twist along any simple closed geodesic. [Wolpert, ’82]

$S_{5000}$

- $\mathcal{M}_{0,n}$ admits a natural measure $WP$ arising from its Weil-Petersson symplectic structure.
- Weil–Petersson volumes $V_{0,n} := WP(\mathcal{M}_{0,n}) = 1, \pi^2, \frac{5}{2}\pi^4, \ldots$ are finite. [Wolpert ’83, Penner ’92] An explicit generating function: [Zograf ’95]

$$Z(x) = \sum_{n \geq 3} \frac{2^{3-n}}{(n-2)!} V_{0,n} x^{n-2}, \quad x = \frac{\sqrt{Z(x)}}{\pi} J_1(2\pi\sqrt{Z(x)}).$$

- Normalizing $WP$: the random hyperbolic surface $S_n \in \mathcal{M}_{0,n}$.
- Unique probability measure invariant under uniform twist along any simple closed geodesic. [Wolpert, ’82]
Main result: scaling limit of metric spaces

- Due to cusps, the metric space \((S_n, d_{hyp})\) is non-compact.
Main result: scaling limit of metric spaces

- Due to cusps, the metric space \((S_n, d_{hyp})\) is non-compact.
- Disjoint length-1 horocycles \(c_1, \ldots, c_n \subset S_n\).

Theorem (TB, Curien, '21+)

We have

\[
\left\{ \{c_1, \ldots, c_n\} \right\}_{n \to \infty} \rightharpoonup_{d_{hyp}} (m_\infty, D^*)
\]

\[
\left\{ (S_n, \text{Area} \frac{2\pi}{n}, d_{hyp}) \right\}_{n \to \infty} \rightharpoonup_{d_{hyp}} (m_\infty, \mu, D^*)
\]

where \(c_{WP} = 2.339\) ...

\(\mu\) is the Brownian sphere with its natural normalized measure.
Main result: scaling limit of metric spaces

- Due to cusps, the metric space \((S_n, d_{hyp})\) is non-compact.
- Disjoint length-1 horocycles \(c_1, \ldots, c_n \subset S_n\).

**Theorem (TB, Curien, ’21+)**

We have

\[
\left( \left\{ c_1, \ldots, c_n \right\}, n^{-\frac{1}{4}} d_{hyp} \right) \xrightarrow{(d)} c_{WP} \left( m_\infty, D^* \right) \quad \text{(Gromov-Hausdorff sense)}
\]

where \(c_{WP} = 2.339 \ldots\) and \((m_\infty, D^*)\) is the Brownian sphere.
Main result: scaling limit of metric spaces

▶ Due to cusps, the metric space \((S_n, d_{hyp})\) is non-compact.
▶ Disjoint length-1 horocycles \(c_1, \ldots, c_n \subset S_n\).
▶ Turn into compact metric space \((S_n^\circ, d_{hyp})\) by removing interiors of \(c_1, \ldots, c_n \subset S_n\).

**Theorem (TB, Curien, ’21+)**

We have

\[
\left( \left\{ c_1, \ldots, c_n \right\}, n^{-\frac{1}{4}} d_{hyp} \right) \xrightarrow{\left( d \right)} \to n \to \infty \quad c_{WP}(m_\infty, D^*) \quad \text{(Gromov-Hausdorff sense)}
\]

\[
\left( S_n^\circ, n^{-\frac{1}{4}} d_{hyp} \right) \xrightarrow{\left( d \right)} \to n \to \infty \quad c_{WP}(m_\infty, D^*) \quad \text{(Gromov-Hausdorff sense)}
\]

where \(c_{WP} = 2.339 \ldots\) and \((m_\infty, D^*)\) is the Brownian sphere.

▶ Implied by \(1^{st}\) convergence: \(\sup_{x \in S_n^\circ} d_{hyp}(x, \{c_1, \ldots, c_n\}) = o(n^{\frac{1}{4}})\).
Main result: scaling limit of metric spaces

- Due to cusps, the metric space \((S_n, d_{hyp})\) is non-compact.
- Disjoint length-1 horocycles \(c_1, \ldots, c_n \subset S_n\).
- Turn into compact metric space \((S_n^\circ, d_{hyp})\) by removing interiors of \(c_1, \ldots, c_n \subset S_n\).

**Theorem (TB, Curien, '21+)**

We have

\[
\left(\{c_1, \ldots, c_n\}, n^{-\frac{1}{4}} d_{hyp}\right) \xrightarrow{\text{(d)}}_{n \to \infty} c_{WP}(m_\infty, D^*) \quad \text{(Gromov-Hausdorff sense)}
\]

\[
\left(S_n^\circ, n^{-\frac{1}{4}} d_{hyp}\right) \xrightarrow{\text{(d)}}_{n \to \infty} c_{WP}(m_\infty, D^*) \quad \text{(Gromov-Hausdorff sense)}
\]

\[
\left(S_n, \frac{\text{Area}}{2\pi n}, n^{-\frac{1}{4}} d_{hyp}\right) \xrightarrow{\text{(d)}}_{n \to \infty} c_{WP}(m_\infty, \mu, D^*) \quad \text{(Gromov-Prokhorov sense)}
\]

where \(c_{WP} = 2.339 \ldots\) and \((m_\infty, D^*)\) is the Brownian sphere with its natural normalized measure \(\mu\).

- Implied by 1\(^{\text{st}}\) convergence: \(\sup_{x \in S_n^\circ} d_{hyp}(x, \{c_1, \ldots, c_n\}) = o(n^{\frac{1}{4}})\).
Setting the scale

Proposition (TB, Curien, ’21+)

If \( c_0, c_1, c_2 \) are horocycles of three uniform cusps in \( S_n \), then

\[
\text{Var} \left[ d_{hyp}(c_1, c_0) - d_{hyp}(c_2, c_0) \right] = \frac{\pi^2}{3} \frac{[\sqrt{n}]^{n-3} J_2(2\pi \sqrt{n})}{[\sqrt{n}]^{n-3} J_1(2\pi \sqrt{n})} \quad n \to \infty \quad \frac{\sqrt{2\pi^5 n}}{3c_0},
\]

where \( x = \frac{\sqrt{n}}{\pi} J_1(2\pi \sqrt{n}) \) and \( c_0 \) is the first Bessel zero \( J_0(c_0) = 0 \).
Setting the scale

Proposition (TB, Curien, ’21+)

If \( c_0, c_1, c_2 \) are horocycles of three uniform cusps in \( S_n \), then

\[
\text{Var}\left[ d_{hyp}(c_1, c_0) - d_{hyp}(c_2, c_0) \right] = \frac{\pi^2}{3} \frac{[x^{n-3}] Z'' n^2 J_2(2\pi \sqrt{Z})}{[x^{n-3}] Z'} \to n \to \infty \frac{\sqrt{2\pi^5 n}}{3c_0},
\]

where \( x = \frac{\sqrt{Z}}{\pi} J_1(2\pi \sqrt{Z}) \) and \( c_0 \) is the first Bessel zero \( J_0(c_0) = 0 \).

▶ Compare \( \text{Var}[D^*(x_1, x_0) - D^*(x_2, x_0)] = \sqrt{\frac{\pi}{8}} \) on Brownian sphere:

\[
c_{\text{WP}} = \frac{2\pi}{\sqrt{3c_0}} = 2.339 \ldots
\]
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map [Cori, Vauquelin] [Schaeffer, '99]

\[
\begin{align*}
\{ & \text{rooted quadrangulations} \\
& \text{with a distinguished vertex} \} \leftrightarrow \{ & \text{rooted plane trees with labels} \\
& \text{in } \mathbb{Z} \text{ that vary by at most 1} \} 
\end{align*}
\]
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map \([\text{Cori, Vauquelin}] [\text{Schaeffer, '99}]
\{\text{rooted quadrangulations with a distinguished vertex}\} \leftrightarrow \{\text{rooted plane trees with labels in } \mathbb{Z} \text{ that vary by at most 1}\}
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map [Cori, Vauquelin] [Schaeffer, '99]

\[
\begin{align*}
\{ & \text{rooted quadrangulations} \\
& \text{with a distinguished vertex} \} \\
\leftrightarrow \\
\{ & \text{rooted plane trees with labels in } \mathbb{Z} \text{ that vary by at most 1} \}
\end{align*}
\]
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map \([\text{Cori, Vauquelin}] \ [\text{Schaeffer, '99}]\)\{rooted quadrangulations with a distinguished vertex\} \leftrightarrow \{rooted plane trees with labels in \(\mathbb{Z}\) that vary by at most 1\}.

The tree labels encode the distances to the distinguished vertex.
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map [Cori, Vauquelin] [Schaeffer, ’99]

\[
\begin{align*}
\{ \text{rooted quadrangulations} \} & \leftrightarrow \{ \text{rooted plane trees with labels in } \mathbb{Z} \text{ that vary by at most 1} \}
\end{align*}
\]
Context: quadrangulations and labeled trees

- A **planar map** is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.
- A **quadrangulation** has faces of degree 4: represents the gluing rules of squares into a topological sphere.
- There exists a 2-to-1 map \cite{Cori, Vauquelin, Schaeffer} \{rooted quadrangulations with a distinguished vertex\} $\leftrightarrow$ \{rooted plane trees with labels in $\mathbb{Z}$ that vary by at most 1\}

The tree labels encode the distances to the distinguished vertex.
A **planar map** is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A **quadrangulation** has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map \([\text{Cori, Vauquelin} [\text{Schaeffer, '99}]
\]

\[
\{ \text{rooted quadrangulations} \quad \text{with a distinguished vertex} \} \leftrightarrow \{ \text{rooted plane trees with labels in } \mathbb{Z} \text{ that vary by at most 1} \}
\]
A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.

A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.

There exists a 2-to-1 map \([Cori, Vauquelin] [Schaeffer, ’99]\)

\[
\begin{align*}
\{ \text{rooted quadrangulations} & \quad \leftrightarrow \quad \{ \text{rooted plane trees with labels} \\
\text{with a distinguished vertex} & \quad \text{in } \mathbb{Z} \text{ that vary by at most 1} \}
\end{align*}
\]
Context: quadrangulations and labeled trees

- A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.
- A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.
- There exists a 2-to-1 map \cite{Cori, Vauquelin} \cite{Schaeffer, '99}:

\[
\left\{ \text{rooted quadrangulations with a distinguished vertex} \right\} \leftrightarrow \left\{ \text{rooted plane trees with labels in } \mathbb{Z} \text{ that vary by at most 1} \right\}
\]

- The tree labels encode the distances to the distinguished vertex.
From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$.
From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$.
The labeled tree is encoded in contour process $C^{(n)}(t)$.
From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$.
- The continuum analogues are the Brownian excursion $e_t$. 

![Diagram of labeled tree and contour process](image)
From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$.
- The continuum analogues are the Brownian excursion $e_t$.
From labeled trees to the Brownian snake

▶ The labeled tree is encoded in contour process $C^{(n)}(t)$ and label process $Z^{(n)}(t)$.
▶ The continuum analogues are the Brownian excursion $e_t$.
From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$ and label process $Z^{(n)}(t)$.
- The continuum analogues are the Brownian excursion $e_t$ and the Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$.

Brownian motion on CRT:
$\text{Var}[Z_s - Z_t] = d_e(s, t)$

CRT
[Aloud, '92]

Credit:
J. Bettinelli
From labeled trees to the Brownian snake

▶ The labeled tree is encoded in contour process $C^{(n)}(t)$ and label process $Z^{(n)}(t)$.

▶ The continuum analogues are the Brownian excursion $e_t$ and the Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$.

Brownian sphere

Brownian motion on CRT:
$\text{Var}[Z_s - Z_t] = d_e(s, t)$

$Z_s = Z_t$ [Aldous, '92]
The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

- More precisely, given Brownian snake \((e_t, Z_t)_{0 \leq t \leq 1}\), define pseudo-distance on \([0, 1]\) via

\[ D^\circ(s, t) = Z_s + Z_t - 2 \max \left\{ \min_{[s,t]} Z, \min_{[t,s]} Z \right\}, \quad s, t \in [0, 1]. \]
The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

- More precisely, given Brownian snake \((e_t, Z_t)_{0 \leq t \leq 1}\), define pseudo-distance on \([0, 1]\) via

\[
D^\circ(s, t) = Z_s + Z_t - 2 \max \left\{ \min_{[s,t]} Z, \min_{[t,s]} Z \right\}, \quad s, t \in [0, 1].
\]

- Writing \(t \sim s\) if identified in CRT, introduce new pseudo-distance

\[
D^*(s, t) = \inf \left\{ D^\circ(s, t_1) + D^\circ(s_1, t_2) + \cdots + D^\circ(s_k, t) : t_i \sim s_i \right\}.
\]
The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

- More precisely, given Brownian snake \((e_t, Z_t)_{0 \leq t \leq 1}\), define pseudo-distance on \([0, 1]\) via

\[
D^\circ(s, t) = Z_s + Z_t - 2 \max\left\{\min_{[s,t]} Z, \min_{[t,s]} Z\right\}, \quad s, t \in [0, 1].
\]

- Writing \(t \sim s\) if identified in CRT, introduce new pseudo-distance

\[
D^*(s, t) = \inf \left\{ D^\circ(s, t_1) + D^\circ(s_1, t_2) + \cdots + D^\circ(s_k, t) : t_i \sim s_i \right\}.
\]

- Brownian sphere is defined as \((m_\infty = [0, 1]/\{D^* = 0\}, D^*)\).
The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

- More precisely, given Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$, define pseudo-distance on $[0, 1]$ via

$$D^\circ(s, t) = Z_s + Z_t - 2 \max \left\{ \min_{[s,t]} Z, \min_{[t,s]} Z \right\}, \quad s, t \in [0, 1].$$

- Writing $t \sim s$ if identified in CRT, introduce new pseudo-distance

$$D^*(s, t) = \inf \left\{ D^\circ(s, t_1) + D^\circ(s_1, t_2) + \cdots + D^\circ(s_k, t) : t_i \sim s_i \right\}.$$

- Brownian sphere is defined as $(m_\infty = [0, 1]/\{D^* = 0\}, D^*)$.

- Gromov-Hausdorff convergence proven for many types of maps, including
  - $p$-angulations [Le Gall, '13][Miermont, '13][Addario-Berry, Albenque, '20]
  - Uniform (bipartite) maps [Bettinelli, Jacob, Miermont, '14][Abraham, '16]
  - Simple triangulations, quadrangulations [Addario-Berry, Albenque, '20]
  - Bipartite maps with prescribed degrees [Marzouk, '18], ...
The Brownian sphere \cite{Marckert, Mokkadem, Le Gall, Miermont, ...}

- More precisely, given Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$, define pseudo-distance on $[0, 1]$ via

$$D^\circ(s, t) = Z_s + Z_t - 2\max\left\{\min_{[s, t]} Z, \min_{[t, s]} Z\right\}, \quad s, t \in [0, 1].$$

- Writing $t \sim s$ if identified in CRT, introduce new pseudo-distance

$$D^*(s, t) = \inf \left\{D^\circ(s, t_1) + D^\circ(s_1, t_2) + \cdots + D^\circ(s_k, t) : t_i \sim s_i \right\}.$$

- Brownian sphere is defined as $(m_\infty = [0, 1]/\{D^* = 0\}, D^*)$.

- Gromov-Hausdorff convergence proven for many types of maps, including
  - $p$-angulations \cite{Le Gall, '13}\cite{Miermont, '13}\cite{Addario-Berry, Albenque, '20}
  - Uniform (bipartite) maps \cite{Bettinelli, Jacob, Miermont, '14}\cite{Abraham, '16}
  - Simple triangulations, quadrangulations \cite{Addario-Berry, Albenque, '20}
  - Bipartite maps with prescribed degrees \cite{Marzouk, '18}, ...

- Can also be recovered from Liouville Quantum Gravity at $\gamma = \sqrt{\frac{8}{3}}$.
  \cite{Miller, Sheffield}
The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, …]

More precisely, given Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$, define pseudo-distance on $[0, 1]$ via

$$D^\circ(s, t) = Z_s + Z_t - 2 \max \left\{ \min_{[s,t]} Z, \min_{[t,s]} Z \right\}, \quad s, t \in [0, 1].$$

Writing $t \sim s$ if identified in CRT, introduce new pseudo-distance

$$D^*(s, t) = \inf \left\{ D^\circ(s, t_1) + D^\circ(s_1, t_2) + \cdots + D^\circ(s_k, t) : t_i \sim s_i \right\}.$$

Brownian sphere is defined as $(m_\infty = [0, 1]/\{D^* = 0\}, D^*)$.

Gromov-Hausdorff convergence proven for many types of maps, including

- $p$-angulations [Le Gall, '13][Miermont, '13][Addario-Berry, Albenque, '20]
- Uniform (bipartite) maps [Bettinelli, Jacob, Miermont, '14][Abraham, '16]
- Simple triangulations, quadrangulations [Addario-Berry, Albenque, '20]
- Bipartite maps with prescribed degrees [Marzouk, '18], …

Can also be recovered from Liouville Quantum Gravity at $\gamma = \sqrt{\frac{8}{3}}$.

[Miller, Sheffield]

Novelty of this work: Brownian sphere limit from continuous model!
Where are the trees in a hyperbolic surface?

- Let $S_n \in M''_{0,n}$ with two distinguished cusps $\star, \triangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$. 
Where are the trees in a hyperbolic surface?

Let $S_n \in \mathcal{M}_{0,n}''$ with two distinguished cusps $\star, \blacklozenge$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$. 

Generically a rooted plane binary tree $T_n \in \text{Bin}_n$ with $n - 1$ leaves.
Where are the trees in a hyperbolic surface?

Let \( S_n \in \mathcal{M}_{0,n}'' \) with two distinguished cusps \( \star, \triangle \) and determine cut locus of \( \star \): points with multiple shortest geodesics to \( \star \).
Where are the trees in a hyperbolic surface?

- Let $S_n \in \mathcal{M}_{0,n}''$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$. 
Where are the trees in a hyperbolic surface?

- Let $S_n \in \mathcal{M}_{0,n}''$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$. 

![Diagram of a hyperbolic surface with trees and cut locus]
Where are the trees in a hyperbolic surface?

- Let $S_n \in \mathcal{M}_{0,n}''$ with two distinguished cusps $\bigstar, \blacktriangle$ and determine cut locus of $\bigstar$: points with multiple shortest geodesics to $\bigstar$. 
Where are the trees in a hyperbolic surface?

Let $S_n \in \mathcal{M}_{0,n}''$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$. 

Generically a rooted plane binary tree $T_n \in \text{Bin}_n$ with $n-1$ leaves.
Where are the trees in a hyperbolic surface?

- Let \( S_n \in M_{0,n}'' \) with two distinguished cusps \( \star, ▲ \) and determine cut locus of \( \star \): points with multiple shortest geodesics to \( \star \).
- Generically a rooted plane binary tree \( T_n \in \text{Bin}_n \) with \( n - 1 \) leaves.
Where are the trees in a hyperbolic surface?

- Let \( S_n \in \mathcal{M}_{0,n}'' \) with two distinguished cusps \( \star, \triangle \) and determine cut locus of \( \star \): points with multiple shortest geodesics to \( \star \).
- Generically a rooted plane binary tree \( T_n \in \text{Bin}_n \) with \( n - 1 \) leaves.

\[ \theta + \sigma > \pi \]
Where are the trees in a hyperbolic surface?

- Let $S_n \in M''_{0,n}$ with two distinguished cusps $\star, \triangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$.
- Generically a rooted plane binary tree $T_n \in \text{Bin}_n$ with $n - 1$ leaves.

**Theorem**

There exists an open subset $M^\circ_{0,n} \subset M''_{0,n}$ of full WP-measure, such that

$$M^\circ_{0,n} \xleftarrow{\text{bijection}} \bigsqcup_{T \in \text{Bin}_n} \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi\}.$$
Where are the trees in a hyperbolic surface?

- Let $S_n \in \mathcal{M}_{0,n}^{''}$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$.
- Generically a rooted plane binary tree $T_n \in \text{Bin}_n$ with $n - 1$ leaves.

**Theorem**

There exists an open subset $\mathcal{M}_{0,n}^\circ \subset \mathcal{M}_{0,n}^{''}$ of full $WP$-measure, such that

\[
\begin{align*}
\mathcal{M}_{0,n}^\circ &\xrightarrow{\text{bijection}} \bigsqcup_{T \in \text{Bin}_n} \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi\}. \\
\text{The WP measures is mapped to Lebesgue: } 2^{n-3}d\alpha_1d\beta_1 \cdots d\alpha_{n-3}d\beta_{n-3}.
\end{align*}
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$. 

\[
\text{The Weil-Petersson measure is } WP = \frac{1}{(n-3)!} \left( \sum \text{corners } d\ell_i \wedge d\ell_j \right)^{n-3} = 2^{n-3} d\alpha_1 d\beta_1 \cdots d\alpha_{n-3} d\beta_{n-3}.
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$. 

Angles are related to hyperbolic distances $\ell_i$ via sine law:

\[
\frac{e^{\ell_1}}{\sin(2\pi - \alpha_1 - \beta_1)} = \frac{e^{\ell_3}}{\sin \alpha_1} = \frac{e^{\ell_2}}{\sin \beta_1} \cdots \frac{e^{\ell_n-3}}{\sin \alpha_{n-3} \sin \beta_{n-3}}.
\]

The Weil-Petersson measure is 

\[
WP = \frac{1}{(n-3)!} \left( -2 \sum_{\text{corners}} d\ell_i \wedge d\ell_j \right) = 2^{n-3} d\alpha_1 d\beta_1 \cdots d\alpha_{n-3} d\beta_{n-3}.
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$. 

\[ \text{Weil-Petersson measure is} \quad WP = \frac{1}{(n-3)!} \left( \sum_{\text{corners}} -2 \ell_i \wedge \ell_j \right) \frac{n-3}{n-3} \]

\[ = 2 \prod_{i=1}^{n-3} \frac{d\alpha_i d\beta_i \cdots d\alpha_{n-3} d\beta_{n-3}}{}. \]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$. 

\[
\text{The Weil-Petersson measure is } \text{WP} = \frac{1}{(n-3)!} \left( \sum \text{corners} \, d\ell_i \wedge d\ell_j \right)_{n-3} = 2^{n-3} \, d\alpha_1 \, d\beta_1 \cdots d\alpha_{n-3} \, d\beta_{n-3}.
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly.

Angles are related to hyperbolic distances $\ell_i$ via sine law:

\[
e^{\ell_1} \sin(2\pi - \alpha_1 - \beta_1) = e^{\ell_3} \sin \alpha_1 = e^{\ell_2} \sin \beta_1 = \cdots e^{\alpha_{n-3}} \sin \beta_{n-3}.
\]

The Weil-Petersson measure is

\[
WP = \frac{1}{(n-3)!} \left( -2 \sum \text{corners} \right) d\ell_i \wedge d\ell_j = 2^{n-3} d\alpha_1 d\beta_1 \cdots d\alpha_{n-3} d\beta_{n-3}.
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\leftrightarrow$ angles at vertex.

\[ WP = \frac{1}{(n-3)!} \left( -\sum_{\text{corners}} d_{\ell_i} \wedge d_{\ell_j} \right)^{n-3} = 2^{n-3} d_{\alpha_1} d_{\beta_1} \cdots d_{\alpha_{n-3}} d_{\beta_{n-3}}. \]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\leftrightarrow$ angles at vertex.

$$\alpha_1, \beta_1 \in (0, \pi)$$
$$\alpha_1 + \beta_1 > \pi$$

The Weil-Petersson measure is

$$\text{WP} = \frac{1}{(n-3)!} \left( -2 \sum \text{corners} d\ell_i \wedge d\ell_j \right)^{n-3} = 2^{n-3} \prod \alpha_i \prod \beta_i.$$
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\longleftrightarrow$ angles at vertex.

<table>
<thead>
<tr>
<th>α₁, β₁ ∈ (0, π)</th>
<th>α₁ + β₁ &gt; π</th>
</tr>
</thead>
</table>

The Weil-Petersson measure is $WP = \frac{1}{(n-3)!} \left( -2 \sum \text{corners } dℓ_i \wedge dℓ_j \right)^{n-3} = 2^n - 3 dα_1 dβ_1 \cdots dα_{n-3} dβ_{n-3}$. 
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\longleftrightarrow$ angles at vertex.

\[
\begin{align*}
\alpha_1, \beta_1 &\in (0, \pi) \\
\alpha_1 + \beta_1 &> \pi
\end{align*}
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\leftrightarrow$ angles at vertex.
- Well-defined precisely when sum of opposing angles $> \pi$.

\[
\ell_1 \sin(2\pi - \alpha_1 - \beta_1) = \ell_3 \sin \alpha_1 = \ell_2 \sin \beta_1
\]

The Weil-Petersson measure is $WP = \frac{1}{(n-3)!} \left( -2 \sum \text{corners} \ d\ell_i \wedge d\ell_j \right) = 2^{n-3} d\alpha_1 d\beta_1 \cdots d\alpha_{n-3} d\beta_{n-3}$.
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $S_n$.
- To reglue: need to know position where red arcs meet sides perpendicularly $\leftrightarrow$ angles at vertex.
- Well-defined precisely when sum of opposing angles $> \pi$.

- Angles are related to hyperbolic distances $\ell_i$ via sine law:

\[
\frac{e^{\ell_1}}{\sin(2\pi - \alpha_1 - \beta_1)} = \frac{e^{\ell_3}}{\sin \alpha_1} = \frac{e^{\ell_2}}{\sin \beta_1}
\]
Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of \( S_n \).
- To reglue: need to know position where red arcs meet sides perpendicularly \( \iff \) angles at vertex.
- Well-defined precisely when sum of opposing angles \( > \pi \).

- Angles are related to hyperbolic distances \( \ell_i \) via sine law:

\[
\frac{e^{\ell_1}}{\sin(2\pi - \alpha_1 - \beta_1)} = \frac{e^{\ell_3}}{\sin \alpha_1} = \frac{e^{\ell_2}}{\sin \beta_1}
\]

- The Weil-Petersson measure is \([\text{Penner, '92}]

\[
WP = \frac{1}{(n-3)!} \left( -2 \sum_{\text{corners}} d\ell_i \wedge d\ell_j \right)^{n-3} = 2^{n-3} d\alpha_1 d\beta_1 \cdots d\alpha_{n-3} d\beta_{n-3}.
\]
Main technical part: convergence to Brownian snake

Random surface $S_n \in \mathcal{M}_{0,n}'' \iff$ Sample binary tree $T_n \in \text{Bin}_n$ proportional to $\text{Leb}(A_T)$ and angles sampled $\text{Leb}$-uniformly from

$$A_T = \{ (\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi \}.$$
Main technical part: convergence to Brownian snake

- Random surface $S_n \in \mathcal{M}_{0,n}''$ $\longleftrightarrow$ Sample binary tree $T_n \in \text{Bin}_n$ proportional to $\text{Leb}(A_T)$ and angles sampled $\text{Leb}$-uniformly from

$$A_T = \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi\}.$$

- Label edges by distance to $c_\star$, 

$$d(e, c_\star)$$
Main technical part: convergence to Brownian snake

- Random surface $S_n \in M'_{0,n} \leftrightarrow$ Sample binary tree $T_n \in \text{Bin}_n$ proportional to $\text{Leb}(A_T)$ and angles sampled Leb-uniformly from

\[ A_T = \{ (\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi \}. \]

- Label edges by distance to $c_\star$, but shifted to have label 0 on root.
Main technical part: convergence to Brownian snake

- Random surface \( S_n \in M''_{0,n} \) \( \iff \) Sample binary tree \( T_n \in \text{Bin}_n \) proportional to \( \text{Leb}(\mathcal{A}_T) \) and angles sampled Leb-uniformly from

\[
\mathcal{A}_T = \{ (\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi \}.
\]

- Label edges by distance to \( c_* \), but shifted to have label 0 on root.

- Then label on edge incident to cusp \( i \) is \( d_{\text{hyp}}(c_i, c_*) - d_{\text{hyp}}(c_\Delta, c_*) \).
Main technical part: convergence to Brownian snake

- Random surface $S_n \in \mathcal{M}_{0,n}'' \leftrightarrow$ Sample binary tree $T_n \in \text{Bin}_n$ proportional to Leb($A_T$) and angles sampled Leb-uniformly from
  $$A_T = \{ (\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi \}.$$

- Label edges by distance to $c_\star$, but shifted to have label 0 on root.

- Then label on edge incident to cusp $i$ is $d_{\text{hyp}}(c_i, c_\star) - d_{\text{hyp}}(c_\Delta, c_\star)$.

- Let $C^{(n)}(t)$ be contour process,
Main technical part: convergence to Brownian snake

- Random surface \( S_n \in \mathcal{M}'_{0,n} \) \( \leftrightarrow \) Sample binary tree \( T_n \in \text{Bin}_n \) proportional to \( \text{Leb}(A_T) \) and angles sampled \( \text{Leb}\)-uniformly from

\[
A_T = \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi \}.
\]

- Label edges by distance to \( c_\star \), but shifted to have label 0 on root.

- Then label on edge incident to cusp \( i \) is \( d_{\text{hyp}}(c_i, c_\star) - d_{\text{hyp}}(c_\Delta, c_\star) \).

- Let \( C^{(n)}(t) \) be contour process, \( Z^{(n)}(t) \) label process,
Main technical part: convergence to Brownian snake

- Random surface $S_n \in \mathcal{M}_{0,n}^{\prime\prime} \iff$ Sample binary tree $T_n \in Bin_n$ proportional to $\text{Leb}(\mathcal{A}_T)$ and angles sampled Leb-uniformly from

$$\mathcal{A}_T = \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi\}.$$  

- Label edges by distance to $c_\star$, but shifted to have label 0 on root.

- Then label on edge incident to cusp $i$ is $d_{\text{hyp}}(c_i, c_\star) - d_{\text{hyp}}(c_\triangle, c_\star)$.

- Let $C^{(n)}(t)$ be contour process, $Z^{(n)}(t)$ label process, $R^{(n)}(t)$ leaf-counting process.
Main technical part: convergence to Brownian snake

- Random surface $S_n \in \mathcal{M}_{0,n}''' \iff$ Sample binary tree $T_n \in \text{Bin}_n$ proportional to $\text{Leb}(A_T)$ and angles sampled $\text{Leb}$-uniformly from $A_T = \{(\alpha_i, \beta_i) \in (0, \pi)^{2n-6} : \alpha_i + \beta_i > \pi, \theta + \sigma > \pi\}$.

- Label edges by distance to $c_*$, but shifted to have label 0 on root.

- Then label on edge incident to cusp $i$ is $d_{\text{hyp}}(c_i, c_*) - d_{\text{hyp}}(c_\triangle, c_*)$.

- Let $C^{(n)}(t)$ be contour process, $Z^{(n)}(t)$ label process, $R^{(n)}(t)$ leaf-counting process.

**Proposition**

\[
\left( \frac{C^{(n)}(t)}{n^{1/2}}, \frac{Z^{(n)}(t)}{n^{1/4}}, \frac{R^{(n)}(t)}{n} \right)_{0 \leq t \leq 1} \xrightarrow{(d)\, n \to \infty} (c_1 e_t, c_2 Z_t, t)_{0 \leq t \leq 1}
\]
A Bienaymé-Galton-Watson (BGW) tree?

- Make \#leaves random and critical,

\[
\mathbb{P}_{x_c}(\mathcal{T}) = \frac{x_c^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \text{Leb}(\mathcal{A}_\mathcal{T}), \quad x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632\ldots
\]

and in the end condition on \#leaves = \(n - 2\).
A Bienaymé-Galton-Watson (BGW) tree?

- Make #leaves random and critical,

\[
\mathbb{P}_{x_c}(\mathcal{T}) = \frac{x_c^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \text{Leb}(\mathcal{A}_\mathcal{T}), \quad x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632\ldots
\]

and in the end condition on #leaves = n − 2.

- (\(\mathcal{T}, (\alpha_i, \beta_i)\)) is a “continuous-type BGW tree” (or a peculiar fragmentation process?):

![Diagram of a tree with labels \(\theta\) and connections between nodes.]
A Bienaymé-Galton-Watson (BGW) tree?

- Make $\#\text{leaves}$ random and critical,

$$\mathbb{P}_{x_c}(T) = \frac{x_c^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \text{Leb}(\mathcal{A}_T), \quad x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632\ldots$$

and in the end condition on $\#\text{leaves} = n - 2.$

- $(T, (\alpha_i, \beta_i))$ is a “continuous-type BGW tree” (or a peculiar fragmentation process?):
A Bienaymé-Galton-Watson (BGW) tree?

- Make \#leaves random and critical,

\[ \mathbb{P}_{x_c}(T) = \frac{x_c^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \text{Leb}(\mathcal{A}_T), \quad x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632 \ldots \]

and in the end condition on \#leaves = \( n - 2 \).

- \((T, (\alpha_i, \beta_i))\) is a “continuous-type BGW tree” (or a peculiar fragmentation process?):

A Bienaymé-Galton-Watson (BGW) tree?

- Make \#leaves random and critical,

\[
P_{x_c}(\mathcal{T}) = \frac{x_c \cdot \#\text{leaves}}{\mathcal{Z}(x_c)} \cdot \text{Leb}(\mathcal{A}_\mathcal{T})
\]

\[x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632 \ldots\]

and in the end condition on \#leaves = \(n - 2\).

- \((\mathcal{T}, (\alpha_i, \beta_i))\) is a “continuous-type BGW tree” (or a peculiar fragmentation process?):

- Unfortunately no good invariance principles (yet)! Can we find a single-type BGW tree instead?
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic:
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic:
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic:
Only some of the edges of $T$ intersect their dual geodesic: canonical partition of the ideal triangulation into “blobs”.
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic: canonical partition of the ideal triangulation into “blobs”.

Connectivity tree $\mathcal{T}$ of the blobs
Only some of the edges of $T$ intersect their dual geodesic: canonical partition of the ideal triangulation into “blobs”.

Connectivity tree $\Xi$ of the blobs has law of a critical BGW tree with explicit offspring dist $(p_k)$, except root has offspring dist $(p_k^\cdot)$.

To recover $T$ from $\Xi$: independently attach to each black vertex of degree $k$ a red leaf with probability $r_k$ in uniform corner ($r_1 = 1$).

Insert independent random blobs of appropriate degree (with or without leaf) sampled according to Leb.
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic: canonical partition of the ideal triangulation into “blobs”.

Connectivity tree $\mathcal{S}$ of the blobs has law of a critical BGW tree with explicit offspring dist $(p_k)$, except root has offspring dist $(p_1^0)$.

To recover $\mathcal{T}$ from $\mathcal{S}$: independently attach to each black vertex of degree $k$ a red leaf with probability $r_k$ in uniform corner $(r_1 = 1)$.
Only some of the edges of $\mathcal{T}$ intersect their dual geodesic: canonical partition of the ideal triangulation into “blobs”.

Connectivity tree $\mathcal{T}$ of the blobs has law of a critical BGW tree with explicit offspring dist $(p_k)$, except root has offspring dist $(p_k^\bullet)$.

To recover $\mathcal{T}$ from $\mathcal{Z}$: independently attach to each black vertex of degree $k$ a red leaf with probability $r_k$ in uniform corner ($r_1 = 1$).

Insert independent random blobs of appropriate degree (with or without leaf) sampled according to Leb.
Adding the labels

Transfer the (distance) labels to the black tree.
Adding the labels

- Transfer the (distance) labels to the black tree.
- Conditionally on $\mathcal{T}$, the increments $(\Delta_1^{(k)}, \ldots, \Delta_k^{(k)})$ at a vertex of degree $k + 1$ are independent of those at other vertices and

\[
\mathbb{E}[\Delta_i^{(k)}] = 0, \quad \mathbb{E}[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \ldots, k.
\]
Adding the labels

▶ Transfer the (distance) labels to the black tree.

▶ Conditionally on $\mathcal{T}$, the increments $(\Delta_1^{(k)}, \ldots, \Delta_k^{(k)})$ at a vertex of degree $k + 1$ are independent of those at other vertices and

$$E[\Delta_i^{(k)}] = 0, \quad E[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \ldots, k.$$
Adding the labels

- Transfer the (distance) labels to the black tree.
- Conditionally on $\mathcal{T}$, the increments $(\Delta_1^{(k)}, \ldots, \Delta_k^{(k)})$ at a vertex of degree $k + 1$ are independent of those at other vertices and

$$
\mathbb{E}[\Delta_i^{(k)}] = 0, \quad \mathbb{E}[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \ldots, k.
$$
Adding the labels

Transfer the (distance) labels to the black tree.

Conditionally on $\mathcal{T}$, the increments $(\Delta_1^{(k)}, \ldots, \Delta_k^{(k)})$ at a vertex of degree $k + 1$ are independent of those at other vertices and

$$E[\Delta_i^{(k)}] = 0, \quad E[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \ldots, k.$$

[Marckert, Miermont, ’07]: Conditioned on $n_\bullet$ the rescaled contour and label process of $\mathcal{T}$ converges to Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$ as $n_\bullet \to \infty$. 

[Marckert, Miermont, ’07]: Conditioned on $n_\bullet$ the rescaled contour and label process of $\mathcal{T}$ converges to Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$ as $n_\bullet \to \infty$. 

Mirror symmetry!
Proof of technical result

$\mathbb{P}(\cdot \mid n_\bullet = n)$

Stretch to convergence on $\mathcal{T}$, still conditioning on $n_\bullet = n$,

$$\left( \frac{\tilde{C}^{(n)}(t)}{n^{\frac{1}{2}}} , \frac{\tilde{Z}^{(n)}(t)}{n^{\frac{1}{4}}} , \frac{\tilde{R}^{(n)}(t)}{n} \right)_{0 \leq t \leq 1} \xrightarrow{(d)} (\tilde{c}_1 e_t, \tilde{c}_2 Z_t, \tilde{c}_3 t)_{0 \leq t \leq 1}.$$
Proof of technical result

- Stretch to convergence on $\mathcal{T}$, still conditioning on $n_\bullet = n$,
  $$\left(\frac{\tilde{C}^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{\tilde{Z}^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{\tilde{R}^{(n)}(t)}{n}\right)_{0 \leq t \leq 1} \overset{(d)}{\underset{n \to \infty}{\longrightarrow}} (\tilde{c}_1 e_t, \tilde{c}_2 Z_t, \tilde{c}_3 t)_{0 \leq t \leq 1}.$$

- Change conditioning to fixed number $n_\circ = n$ of leaves,
  $$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{R^{(n)}(t)}{n}\right)_{0 \leq t \leq 1} \overset{(d)}{\underset{n \to \infty}{\longrightarrow}} (c_1 e_t, c_2 Z_t, t)_{0 \leq t \leq 1}.$$
Bound on distances between arbitrary horocycles

Distances between arbitrary horocycles satisfy deterministic bound

\[ d_{\text{hyp}}(c_i, c_j) \leq d_{\text{hyp}}(c_i, c^*) + d_{\text{hyp}}(c_j, c^*) - 2 \min_{k, \ell} k + 2 \log n + 10 \cdot o\left(n^{1/4}\right). \]
Bound on distances between arbitrary horocycles

Distances between arbitrary horocycles satisfy deterministic bound
\[ d_{\text{hyp}}(c_i, c_j) \leq d_{\text{hyp}}(c_i, c^*) + d_{\text{hyp}}(c_j, c^*) - 2 \min_{k, \ell} k + 2 \log n + 10 \cdot o\left(\frac{1}{n^{1/4}}\right). \]
Bound on distances between arbitrary horocycles

Distances between arbitrary horocycles satisfy deterministic bound

\[ d_{\text{hyp}}(c_i, c_j) \leq d_{\text{hyp}}(c_i, c^*) + d_{\text{hyp}}(c_j, c^*) - 2 \min_k \ell_k + 2 \log n + 10 o(n^{1/4}). \]
Bound on distances between arbitrary horocycles

\[ d(c_2, c_3) \leq \]

\[ d(c_1, c_*), \quad \ell_1, \quad \ell_2, \quad \ell_3, \quad d(c_2, c_*), \quad d(c_3, c_*) \]

\[ 1 \quad 2 \quad 3 \]

\[ \star \text{ at } \infty \]
Bound on distances between arbitrary horocycles

\[ d(c_2, c_3) \leq 3 \]

Distances between arbitrary horocycles satisfy deterministic bound

\[ d_{\text{hyp}}(c_i, c_j) \leq d_{\text{hyp}}(c_i, c^*) + d_{\text{hyp}}(c_j, c^*) - 2 \min_{k \neq \ell} k + 2 \log n + 10. \]
Bound on distances between arbitrary horocycles

\[ d(c_2, c_3) \leq 3 \]

Distances between arbitrary horocycles satisfy deterministic bound

\[
d_{\text{hyp}}(c_i, c) \leq d_{\text{hyp}}(c_i, c_*) + d_{\text{hyp}}(c_j, c_*) - 2 \min_k \ell_k + 2 \log n + 10. \\
\text{o}(n^{\frac{1}{4}})
\]
Convergence to the Brownian sphere

\[ \left( \frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}} \right)_{0 \leq t \leq 1} \xrightarrow{n \to \infty} (c_1 \epsilon_t, c_2 Z_t)_{0 \leq t \leq 1} \]

\[ d^{(n)}_{hyp}(s, t) \leq Z^{(n)}(s) + Z^{(n)}(t) - 2 \max \left\{ \min \left[ Z^{(n)} \right]_{[s, t]}, \min \left[ Z^{(n)} \right]_{[t, s]} \right\} + o(n^{\frac{1}{4}}) \]

Invariance under rerooting

\[ \downarrow \quad \text{[Le Gall, '13]'s rerooting trick} \]

\[ \left( \frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{d^{(n)}_{hyp}(s, t)}{n^{\frac{1}{4}}} \right)_{0 \leq t \leq 1} \xrightarrow{n \to \infty} (c_1 \epsilon_t, c_2 Z_t, c_{WP} D^*_{s, t})_{0 \leq t \leq 1}. \]
Convergence to the Brownian sphere

\[ \left( \frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}} \right) \xrightarrow{n \to \infty} (c_1 e_t, c_2 Z_t)_{0 \leq t \leq 1} \]

\[ + \]

\[ d^{(n)}_{hyp}(s, t) \leq Z^{(n)}(s) + Z^{(n)}(t) - 2 \max \left\{ \min_{[s,t]} Z^{(n)}, \min_{[t,s]} Z^{(n)} \right\} + o(n^{\frac{1}{4}}) \]

\[ + \]

Invariance under rerooting

\[ \downarrow \]

\[ [\text{Le Gall, '13}']s \text{ rerooting trick} \]

\[ \left( \frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{d^{(n)}_{hyp}(s, t)}{n^{\frac{1}{4}}} \right) \xrightarrow{n \to \infty} (c_1 e_t, c_2 Z_t, c_{WP} D_{s,t})_{0 \leq t \leq 1}. \]

\[ \downarrow \]

\[ \left( \{c_1, \ldots, c_n\}, n^{-\frac{1}{4}} d_{hyp} \right) \xrightarrow{n \to \infty} c_{WP} (m_{\infty}, D^*) \quad (\text{Gromov-Hausdorff sense}) \]
Setting the scale

- Random surface but now subcritical and extra marked cusp ♦, \( P_x^x(\mathcal{T}) = \frac{x^{\#\text{leaves}}}{Z'(x)} \text{Leb}(\mathcal{A}_T). \)
Setting the scale

- Random surface but now subcritical and extra marked cusp $\circ$,
  $P^o_x(\mathcal{T}) = \frac{x \# \text{leaves}}{Z'(x)} \text{Leb}(\mathcal{A}_\mathcal{T})$.

- The distance difference satisfies

  \[ \delta = d_{\text{hyp}}(c_\circ, c_\star) - d_{\text{hyp}}(c_\triangledown, c_\star) = \sum_{i=1}^{k} \log \frac{\sin \alpha_i}{\sin(2\pi - \alpha_i - \beta_i)} \]
Setting the scale

- Random surface but now subcritical and extra marked cusp \( \circ \),
  \( \mathbb{P}_x^\circ(\mathcal{T}) = \frac{x}{\mathcal{Z}'(x)} \text{Leb}(\mathcal{A}_\mathcal{T}) \).
- The distance difference satisfies
  \[ \delta = d_{\text{hyp}}(c_\circ, c_*^\circ) - d_{\text{hyp}}(c_\uparrow, c_*) = \sum_{i=1}^{k} \log \frac{\sin \alpha_i}{\sin(2\pi - \alpha_i - \beta_i)} = \int_0^\tau \cot \theta_t \, dt. \]
- \((\theta_t)\) has law of Markov process with drift 1 and \(\downarrow\)-jumps \(\theta \rightarrow \theta'\) at rate \(2F(\theta - \theta') \frac{F_\bullet(\theta')}{F_\bullet(\theta)}\).
  \[ F(\theta) = \frac{\sqrt{Z}}{\theta} J_1(2\theta \sqrt{Z}), F_\bullet(\theta) = J_0(2\theta \sqrt{Z}). \]
Setting the scale

- Random surface but now subcritical and extra marked cusp \( \circ \),
  \[
  \mathbb{P}_x(T) = \frac{x}{Z'(x)} \text{ Leb}(A_T).
  \]
- The distance difference satisfies
  \[
  \delta = d_{hyp}(c_\circ, c_\star) - d_{hyp}(c_\blacktriangle, c_\star) = \sum_{i=1}^{k} \log \frac{\sin \alpha_i}{\sin(2\pi - \alpha_i - \beta_i)} = \int_0^\tau \cot \theta_t \, dt.
  \]
- \((\theta_t)\) has law of Markov process with drift 1 and \(\downarrow\)-jumps \(\theta \to \theta'\) at rate \(2F(\theta - \theta') \frac{F'(\theta')}{F'(\theta)}\). \(F(\theta) = \frac{\sqrt{Z}}{\theta} J_1(2\theta \sqrt{Z}), F'(\theta) = J_0(2\theta \sqrt{Z}).\)

\[
\mathbb{E}_x[\delta^2] = -\frac{2}{F'(\pi)} \int_0^\pi d\alpha \int_0^\alpha d\beta \cot \alpha \cot \beta F'(\pi - \alpha) F'(\alpha - \beta) F'\beta
= \frac{\pi^2}{3} Z'(x) J_2(2\pi \sqrt{Z(\xi))}).
\]
Happy birthday, Jean!