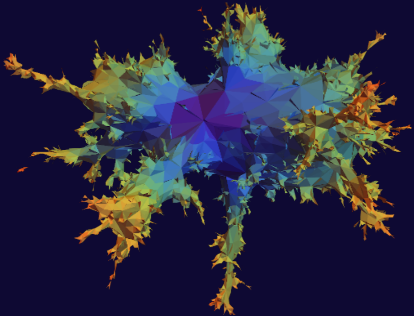
09-07-2021, Random excursions with Jean Bertoin

Random punctured hyperbolic surfaces & the Brownian sphere

Timothy Budd



in collaboration with Nicolas Curien

Radboud University

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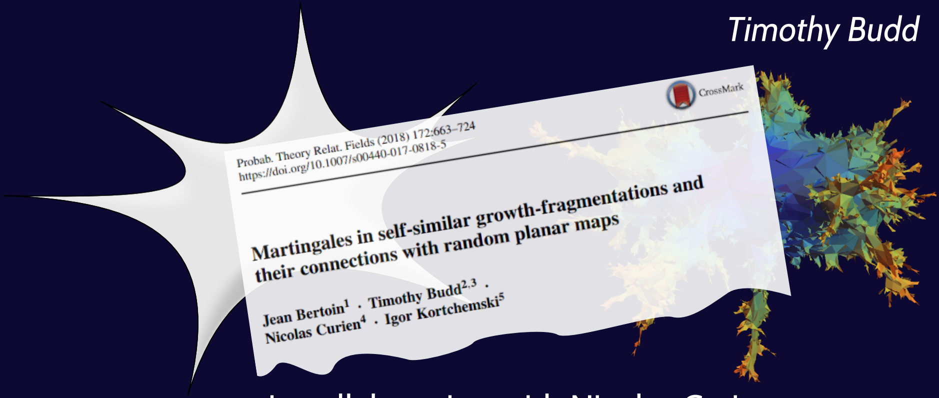

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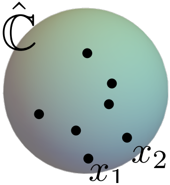
Moduli space of punctured spheres

- Consider the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\hat{\mathbb{C}}$ with n points removed:

$$\mathcal{M}_{0,n} = \{X = \hat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}\}$$



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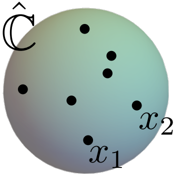

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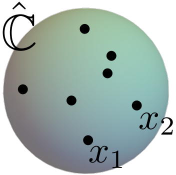

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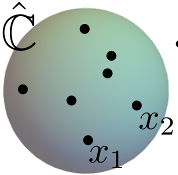

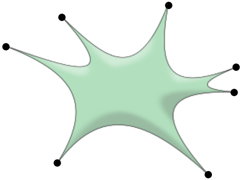

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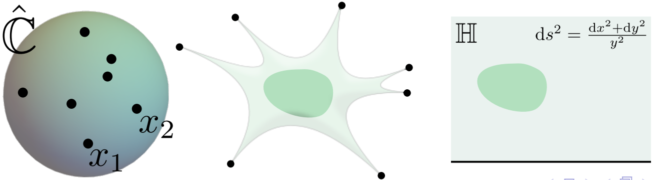

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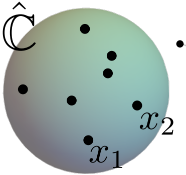

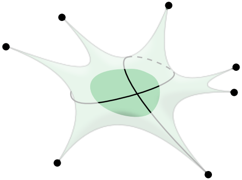

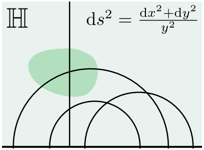

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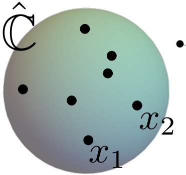
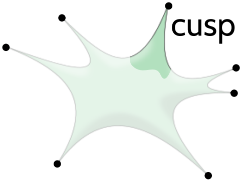
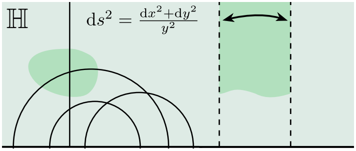
- Cusps regions are locally isometric to vertical strips with boundaries identified.







Random hyperbolic surface? [Guth, Parlier, Young, '11][Mirzakhani, '13]

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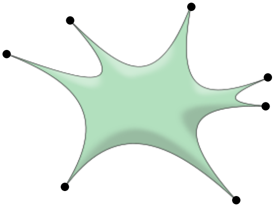

$\mathcal{S}_{5000}$

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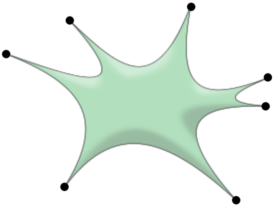

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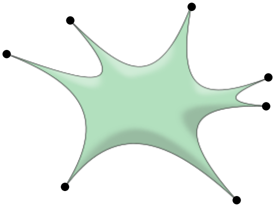

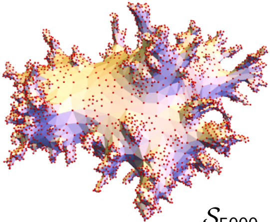

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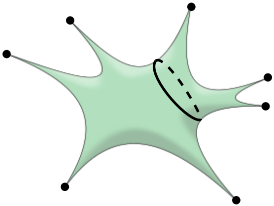

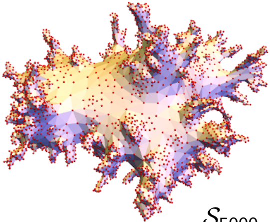

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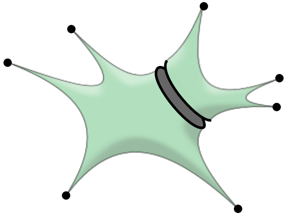

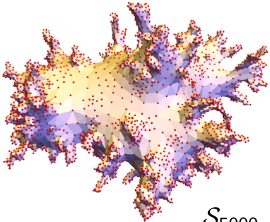

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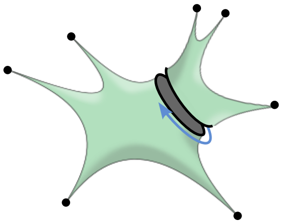

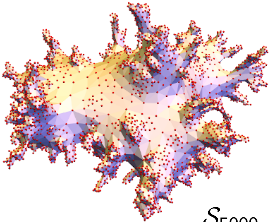

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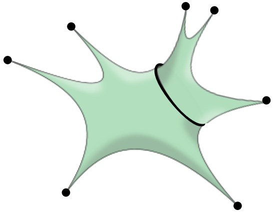

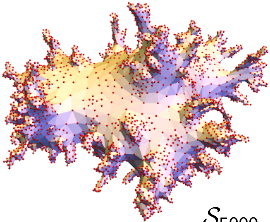

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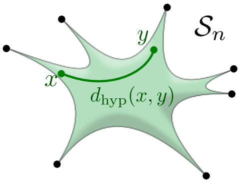
Main result: scaling limit of metric spaces

- Due to cusps, the metric space $(\mathcal{S}_n, d_{\mathrm{hyp}})$ is non-compact.



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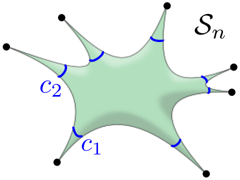

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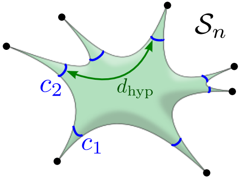

Theorem (TB, Curien, '21+)

We have

$$\left(\{c_1, \ldots, c_n\}, n^{-\frac{1}{4}} d_{hyp}\right) \xrightarrow[n\to\infty]{(d)} \mathrm{c}_{WP}(\mathrm{m}_\infty, D^*) \quad \textit{(Gromov-Hausdorff sense)}$$

where $\mathrm{c}_{WP} = 2.339\ldots$ *and* (m_∞, D^*) *is the Brownian sphere*

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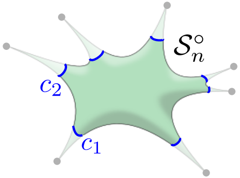
- Turn into compact metric space $(\mathcal{S}_n^\circ, d_{\text{hyp}})$ by removing interiors of $c_1, \ldots, c_n \subset \mathcal{S}_n$.



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- Implied by 1st convergence: $\sup_{x \in \mathcal{S}_n^\circ} d_{\text{hyp}}(x, \{c_1, \ldots, c_n\}) = o(n^{\frac{1}{4}})$.

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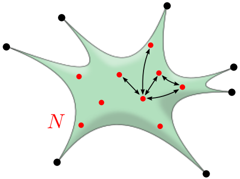

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where $\mathsf{c}_{WP} = 2.339\ldots$ *and* (m_∞, D^*) *is the Brownian sphere with its natural normalized measure* μ*.*

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Setting the scale

Proposition (TB, Curien, '21+)

If c_0, c_1, c_2 are horocycles of three uniform cusps in $\mathcal{S}_n$, then

$$\mathrm{Var}\left[d_{hyp}(c_1, c_0) - d_{hyp}(c_2, c_0)\right] = \frac{\pi^2}{3} \frac{[x^{n-3}]\mathcal{Z}'^2 J_2(2\pi\sqrt{\mathcal{Z}})}{[x^{n-3}]\mathcal{Z}'} \overset{n\to\infty}{\sim} \frac{\sqrt{2\pi^5\, n}}{3\mathrm{c}_0},$$

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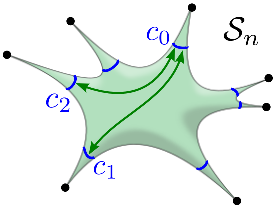

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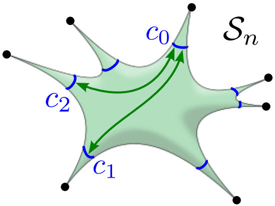
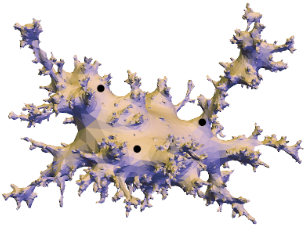
- Compare $\mathrm{Var}[D^*(x_1, x_0) - D^*(x_2, x_0)] = \sqrt{\frac{\pi}{8}}$ on Brownian sphere:

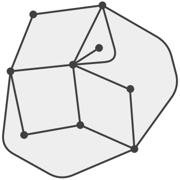
$$\mathrm{c}_{WP} = \frac{2\pi}{\sqrt{3\mathrm{c}_0}} = 2.339\ldots$$





Context: quadrangulations and labeled trees

- A planar map is a planar graph that is properly embedded in the sphere modulo orientation-preserving homeomorphisms.



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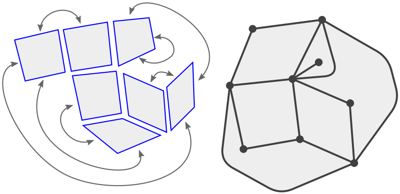
- A quadrangulation has faces of degree 4: represents the gluing rules of squares into a topological sphere.



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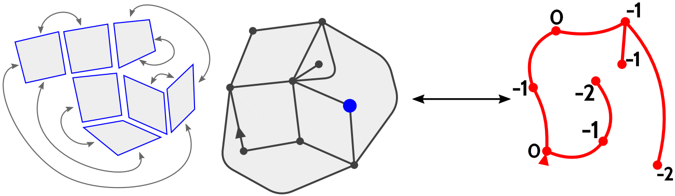

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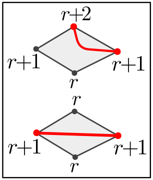

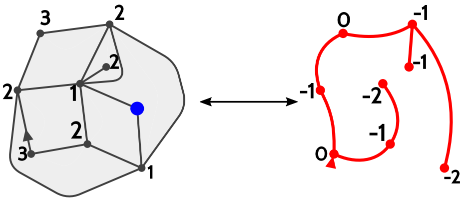

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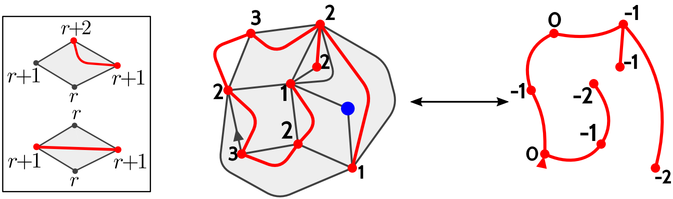

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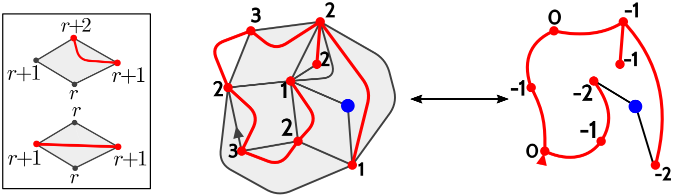

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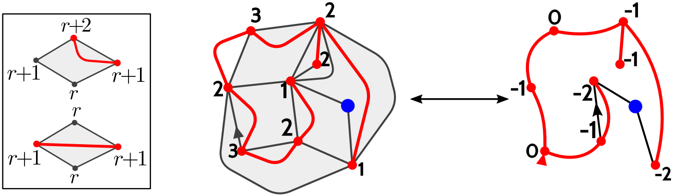

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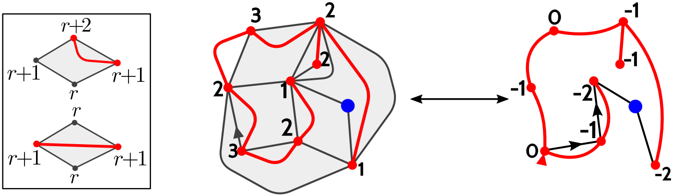

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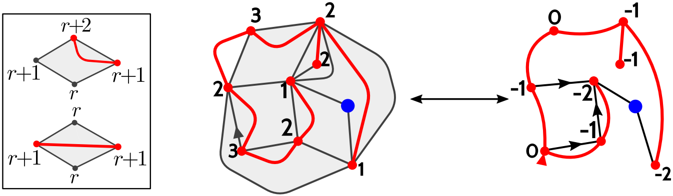

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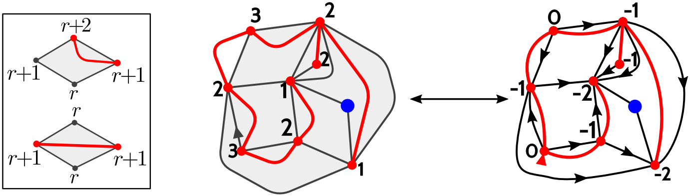

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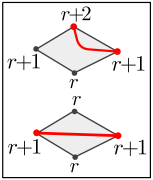

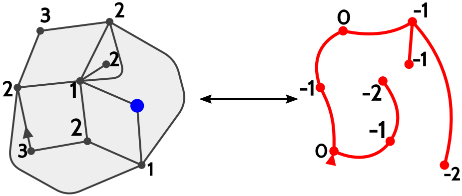

From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$

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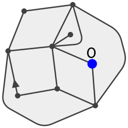

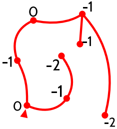

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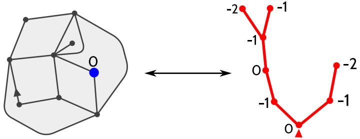

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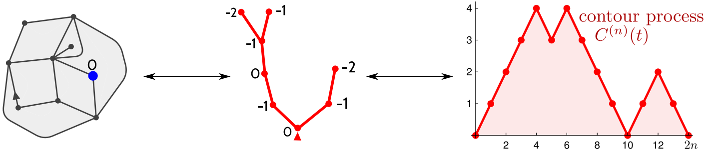

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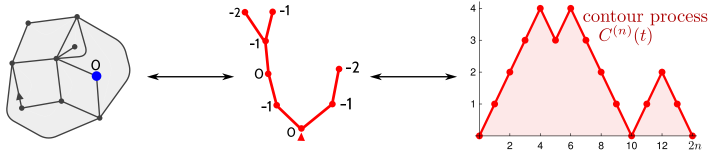

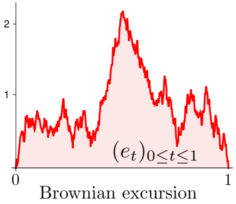

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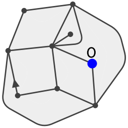
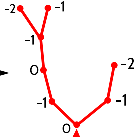
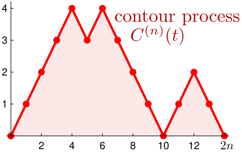
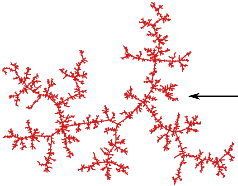
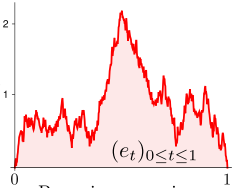








CRT
[Aldous, '92]



Brownian excursion

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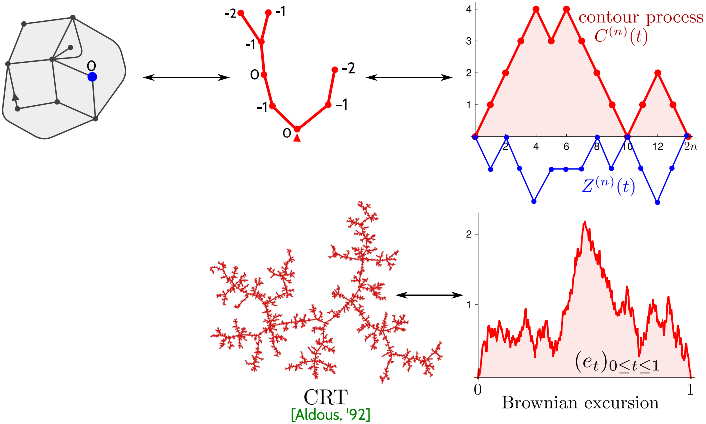

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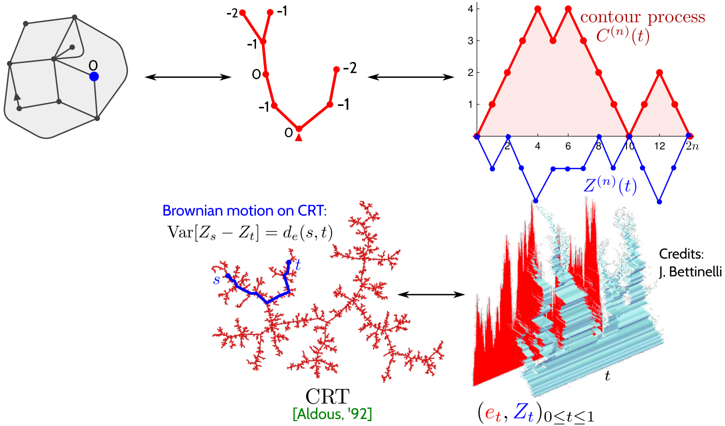

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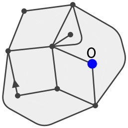
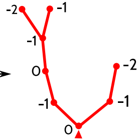
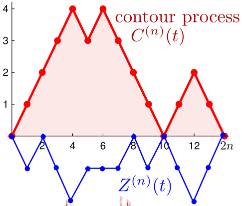
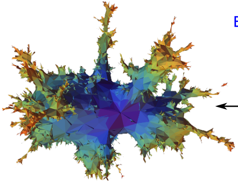
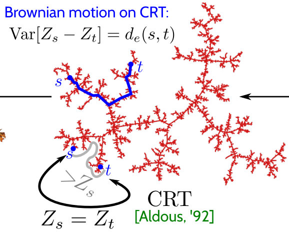


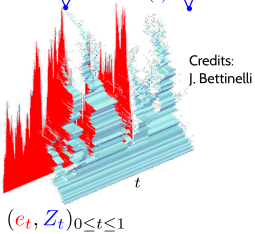






Brownian sphere





The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

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$$D^\circ(s,t) = Z_s + Z_t - 2\max\left\{\min_{[s,t]} Z, \min_{[t,s]} Z\right\}, \quad s,t \in [0,1].$$

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- Novelty of this work: Brownian sphere limit from continuous model!

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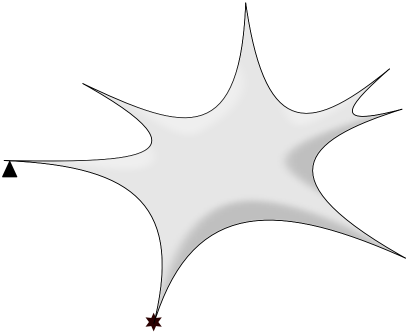
- Let $\mathcal{S}_n \in \mathcal{M}''_{0,n}$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus of $\star$: points with multiple shortest geodesics to $\star$.



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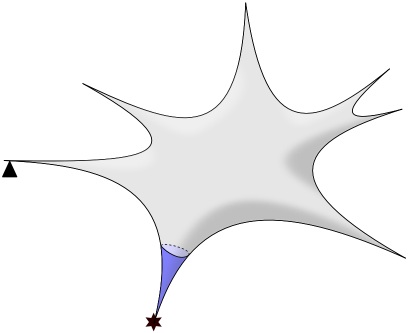

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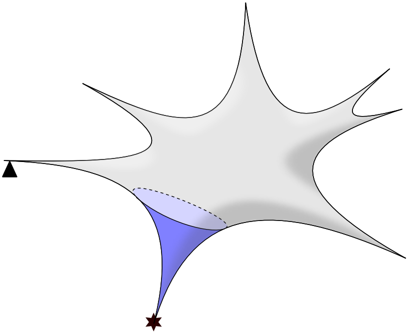

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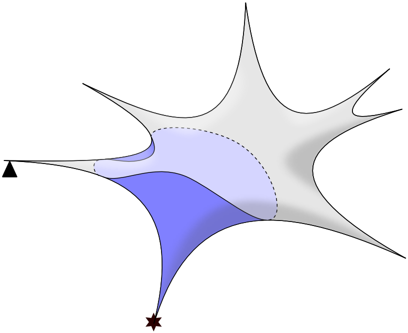

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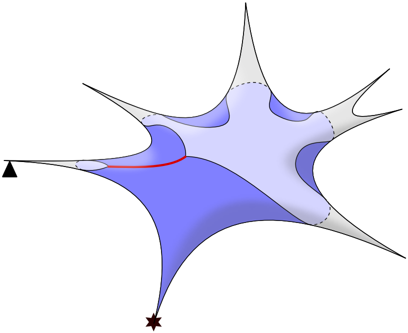

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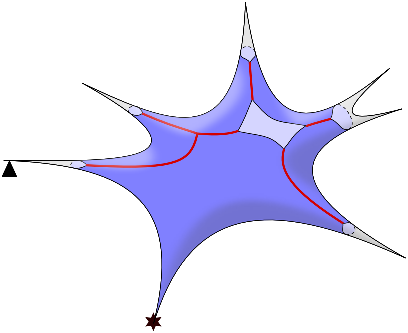

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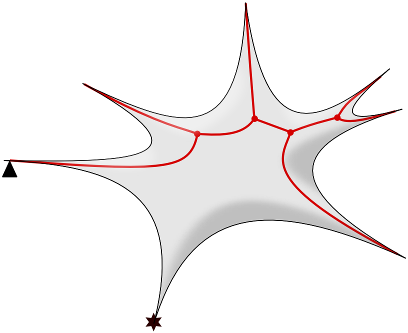

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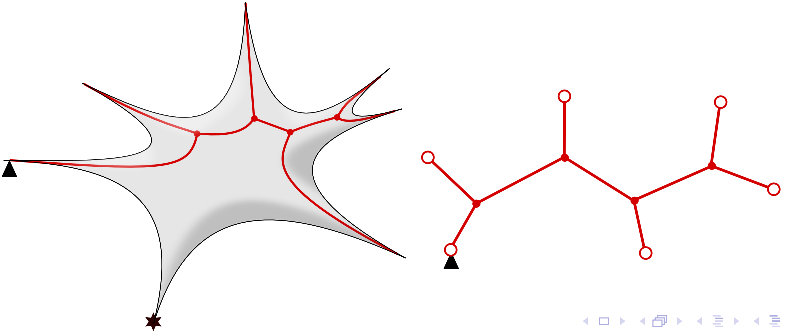

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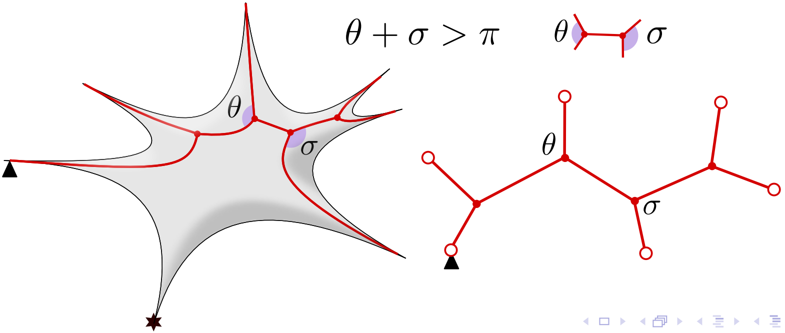

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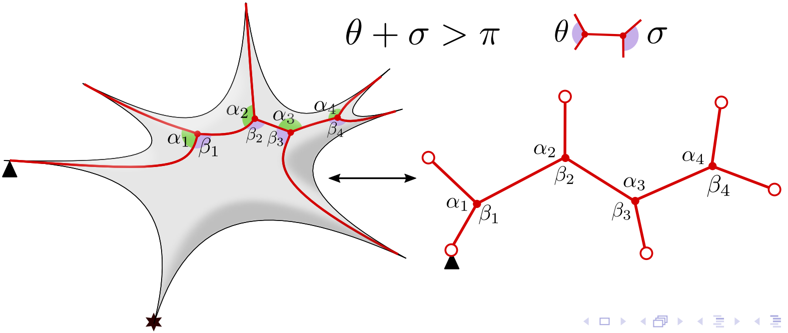

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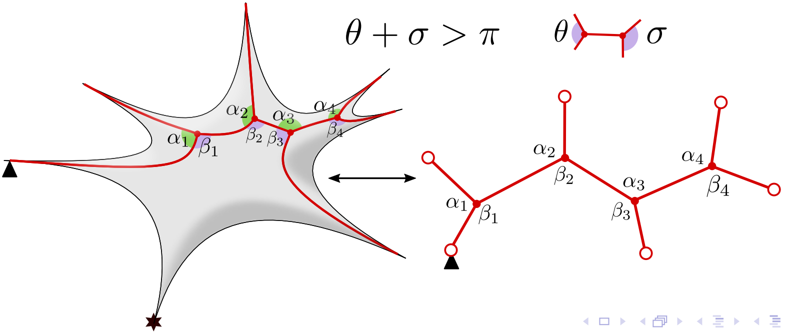
The WP measures is mapped to Lebesgue: $2^{n-3}\mathrm{d}\alpha_1\mathrm{d}\beta_1\cdots\mathrm{d}\alpha_{n-3}\mathrm{d}\beta_{n-3}$.

$\theta+\sigma>\pi$



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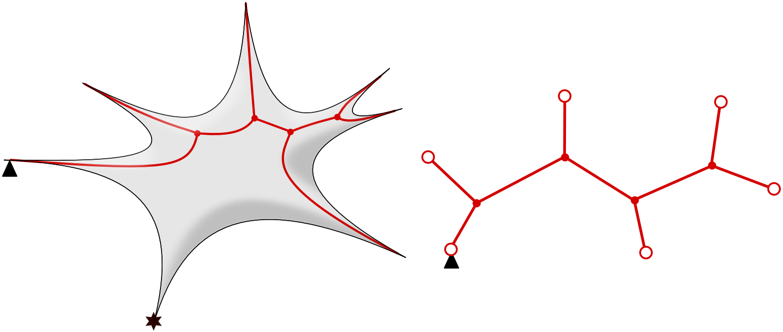

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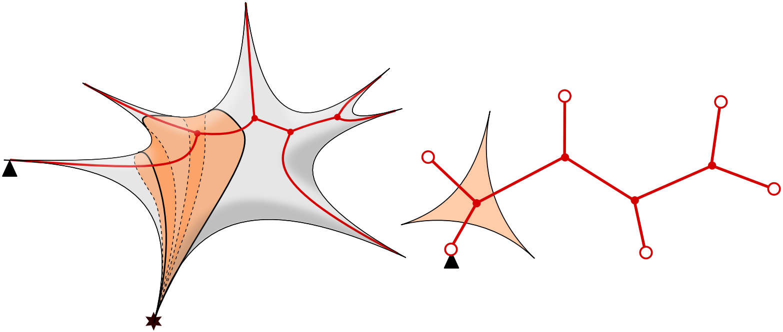

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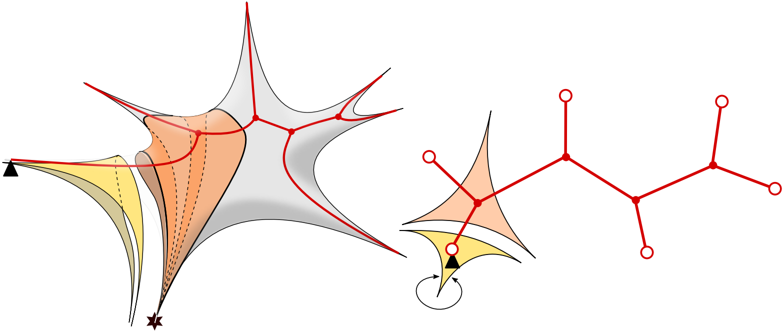

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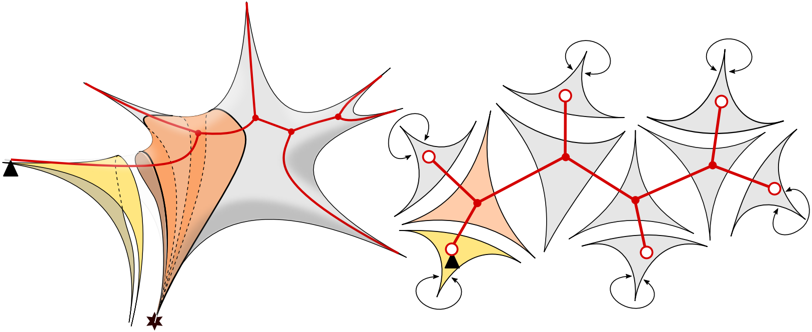

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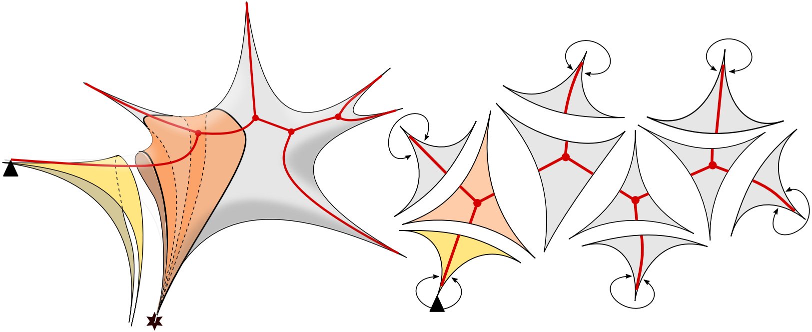

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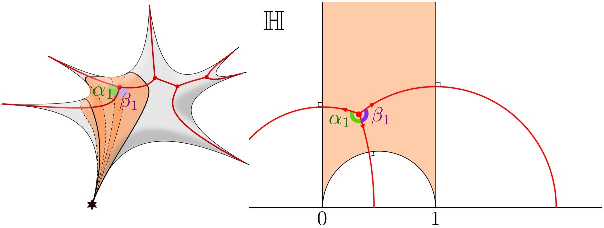

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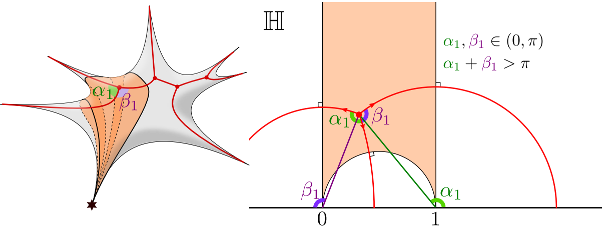

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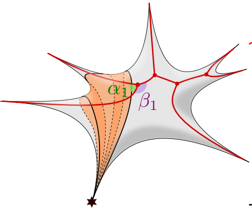

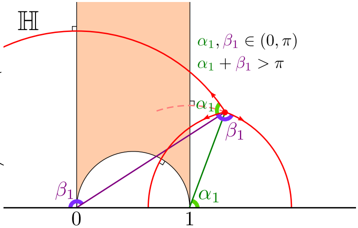

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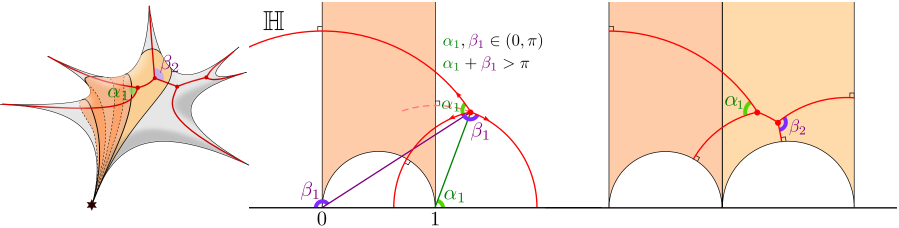

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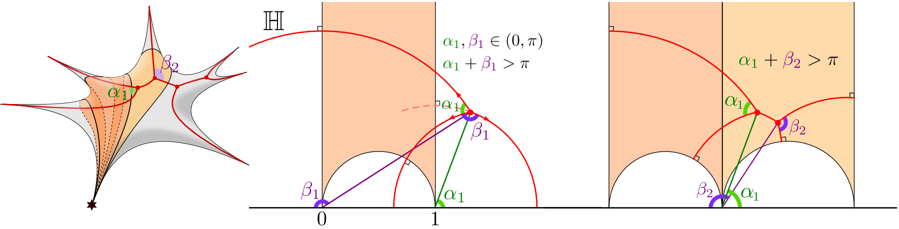

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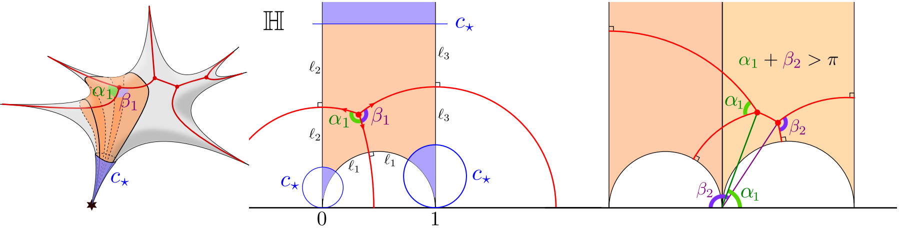

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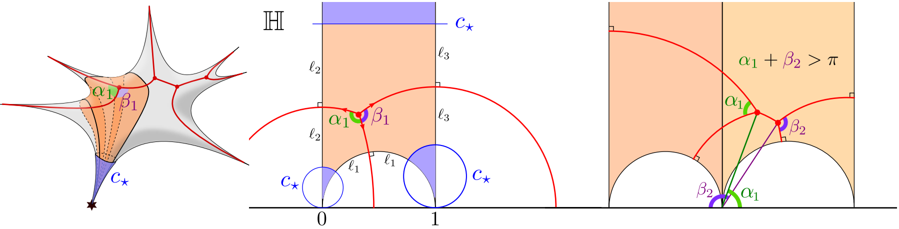

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- The Weil-Petersson measure is [Penner, '92]

$$WP = \frac{1}{(n-3)!}\Big(-2\sum_{\text{corners}} \mathrm{d}\ell_i \wedge \mathrm{d}\ell_j\Big)^{n-3} = 2^{n-3}\mathrm{d}\alpha_1\mathrm{d}\beta_1\cdots\mathrm{d}\alpha_{n-3}\mathrm{d}\beta_{n-3}.$$

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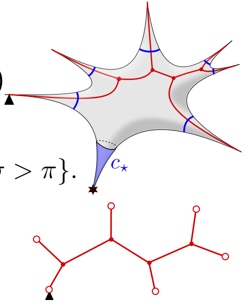

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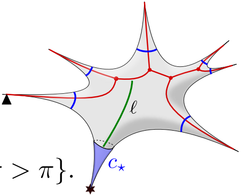

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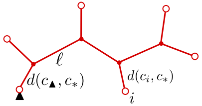

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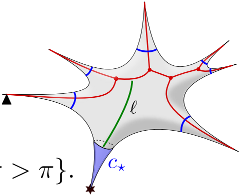

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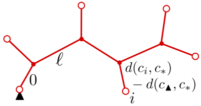

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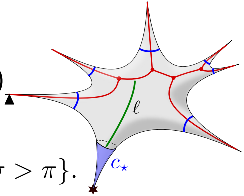

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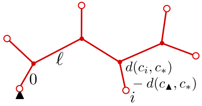

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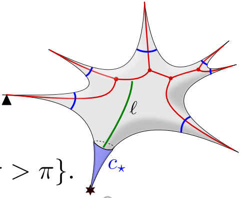

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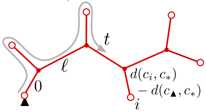

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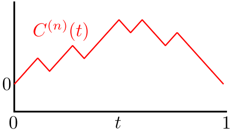

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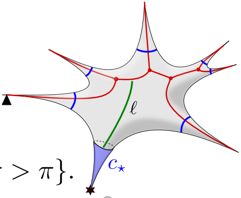

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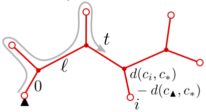

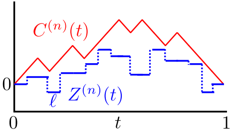

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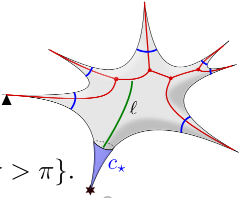

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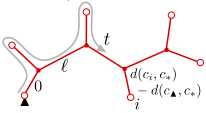

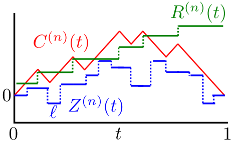

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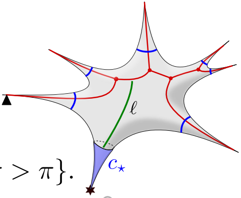

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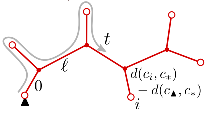

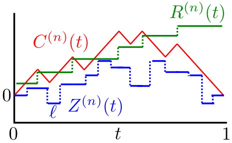

Proposition

$$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{R^{(n)}(t)}{n} \right)_{0 \le t \le 1} \xrightarrow[n\to\infty]{(d)} (\mathrm{c}_1 \mathrm{e}_t, \mathrm{c}_2 Z_t, t)_{0\le t \le 1}$$

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- Make #leaves random and critical,

$$\mathbb{P}_{x_c}(\mathcal{T}) = \frac{{x_c}^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \mathrm{Leb}(\mathcal{A}_{\mathcal{T}}), \qquad x_c = \frac{\mathrm{c}_0 J_1(\mathrm{c}_0)}{2\pi^2} = 0.0632\ldots$$

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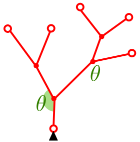

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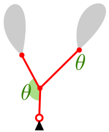

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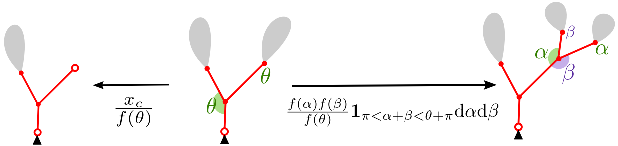


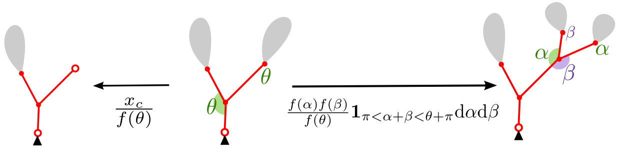
A Bienaymé-Galton-Watson (BGW) tree?

- Make #leaves random and critical,

$$\mathbb{P}_{x_c}(\mathcal{T}) = \frac{{x_c}^{\#\text{leaves}}}{\mathcal{Z}(x_c)} \mathrm{Leb}(\mathcal{A}_{\mathcal{T}}), \qquad x_c = \frac{c_0 J_1(c_0)}{2\pi^2} = 0.0632\ldots$$

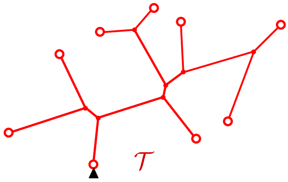
  and in the end condition on #leaves $= n-2$.

- $(\mathcal{T}, (\alpha_i, \beta_i))$ is a "continuous-type BGW tree" (or a peculiar fragmentation process?):



- Unfortunately no good invariance principles (yet)! Can we find a single-type BGW tree instead?

- Only some of the edges of $\mathcal{T}$ intersect their dual geodesic:



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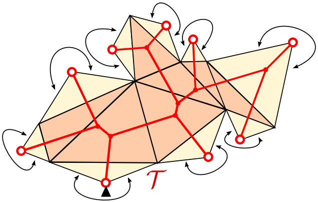

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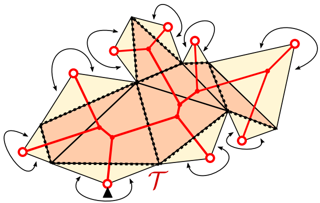
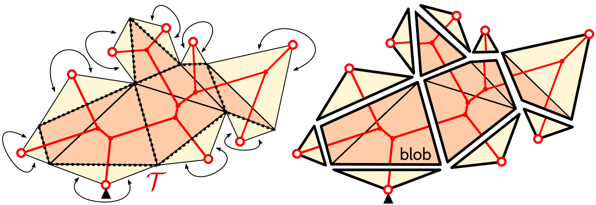


- Only some of the edges of $\mathcal{T}$ intersect their dual geodesic: canonical partition of the ideal triangulation into "blobs".



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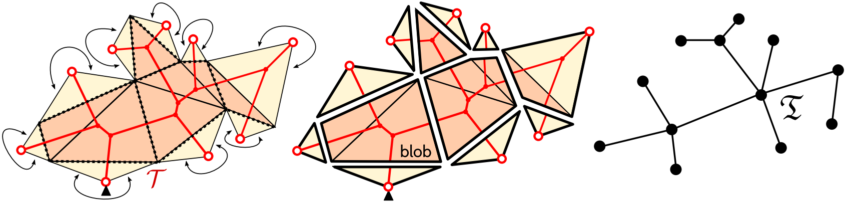
- Connectivity tree $\mathfrak{T}$ of the blobs



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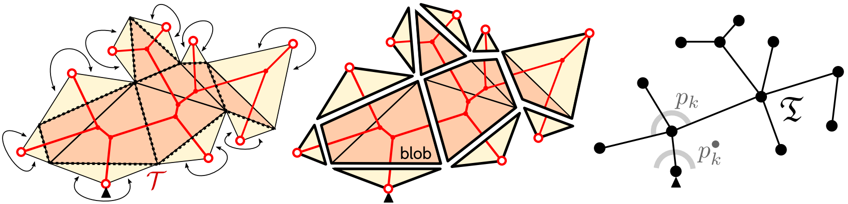

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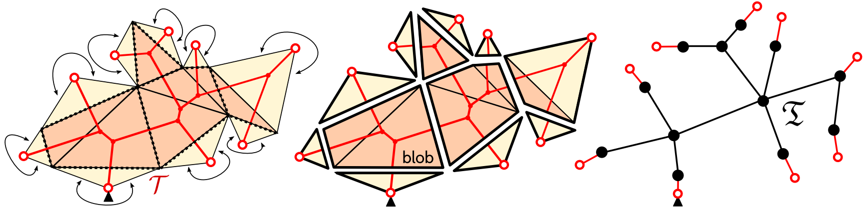

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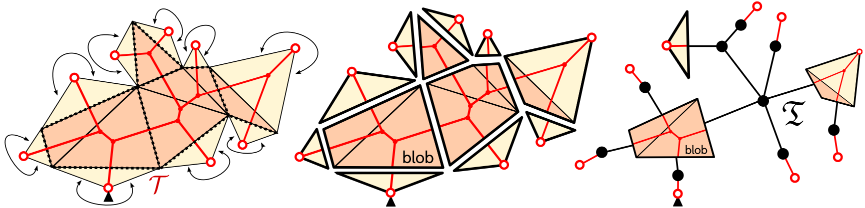

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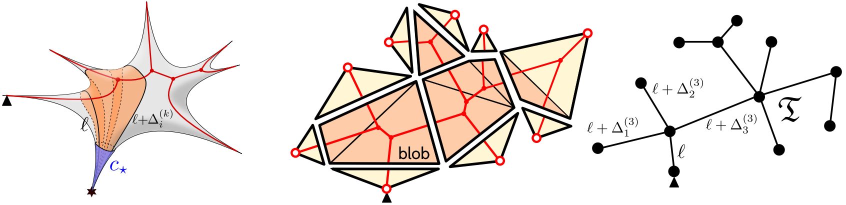
- Insert independent random blobs of appropriate degree (with or without leaf) sampled according to Leb.

Adding the labels



- Transfer the (distance) labels to the black tree.

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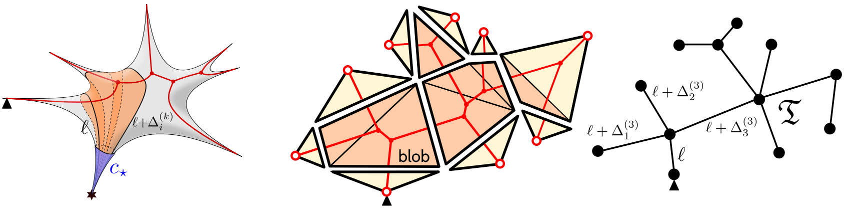

- Transfer the (distance) labels to the black tree.
- Conditionally on $\mathfrak{T}$, the increments $(\Delta_1^{(k)}, \dots, \Delta_k^{(k)})$ at a vertex of degree $k+1$ are independent of those at other vertices and

$$\mathbb{E}[\Delta_i^{(k)}] = 0, \quad \mathbb{E}[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \dots, k.$$

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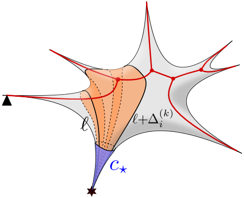

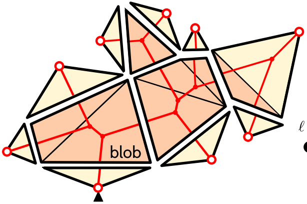

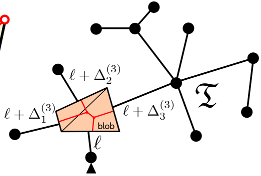

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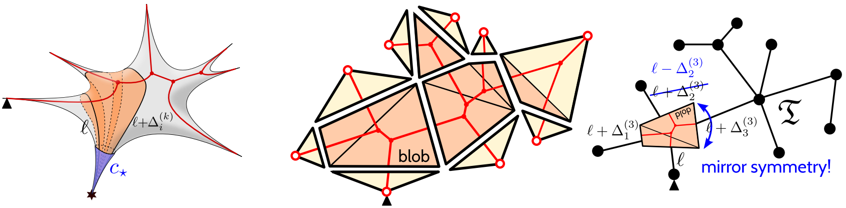

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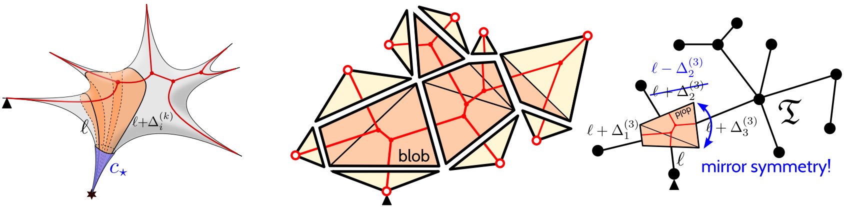

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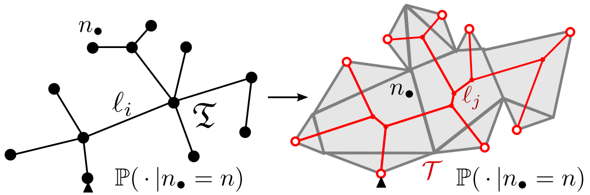
- [Marckert, Miermont, '07]: Conditioned on $n_\bullet$ the rescaled contour and label process of $\mathfrak{T}$ converges to Brownian snake $(e_t, Z_t)_{0\le t\le 1}$ as $n_\bullet \to \infty$.

Proof of technical result



- Stretch to convergence on $\mathcal{T}$, still conditioning on $n_\bullet = n$,

$$\left(\frac{\tilde{C}^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{\tilde{Z}^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{\tilde{R}^{(n)}(t)}{n}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\tilde{c}_1 e_t, \tilde{c}_2 Z_t, \tilde{c}_3 t)_{0\leq t\leq 1}.$$

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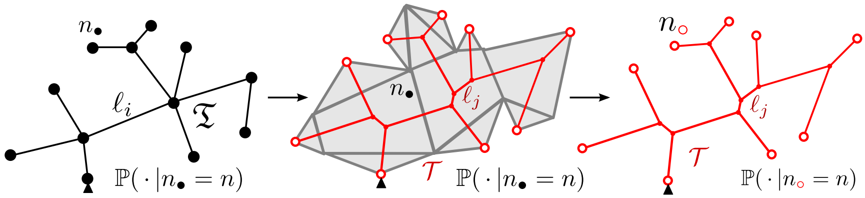

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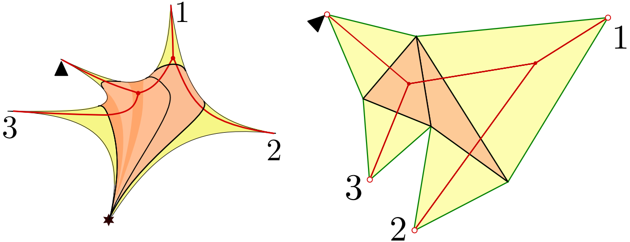
- Change conditioning to fixed number $n_\circ = n$ of leaves,

$$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{R^{(n)}(t)}{n} \right)_{0 \le t \le 1} \xrightarrow[n \to \infty]{(d)} (c_1 e_t, c_2 Z_t, t)_{0 \le t \le 1}.$$

Bound on distances between arbitrary horocycles



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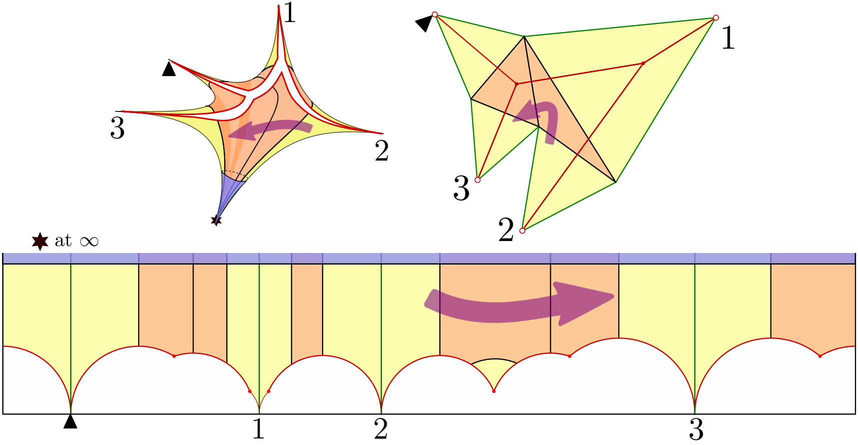

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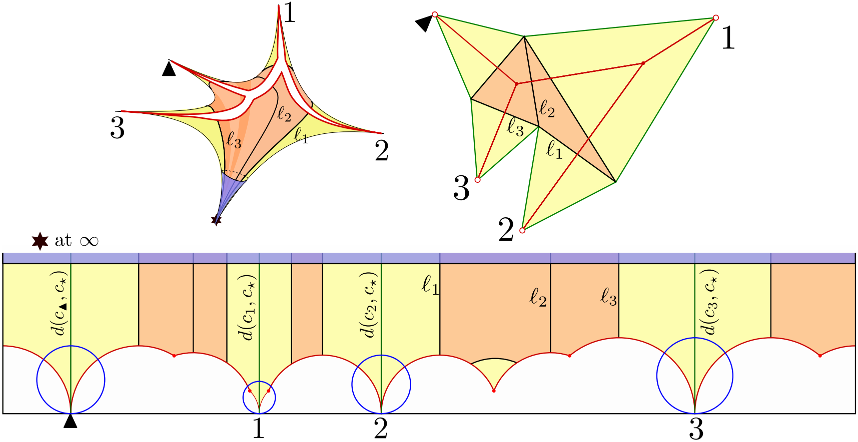
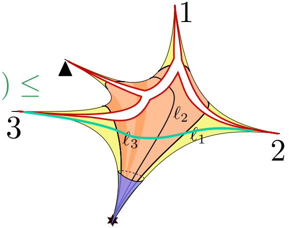
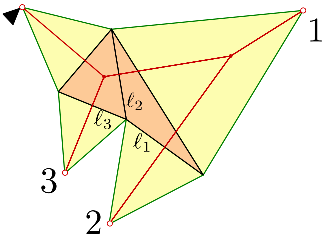
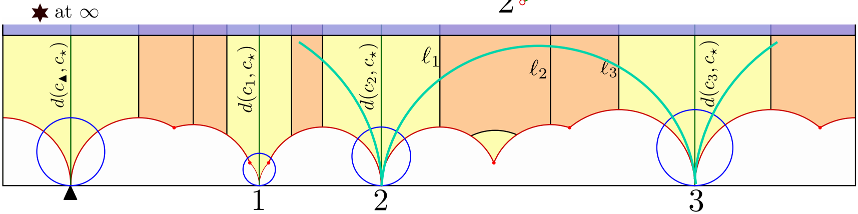


Bound on distances between arbitrary horocycles

$d(c_2, c_3) \leq$







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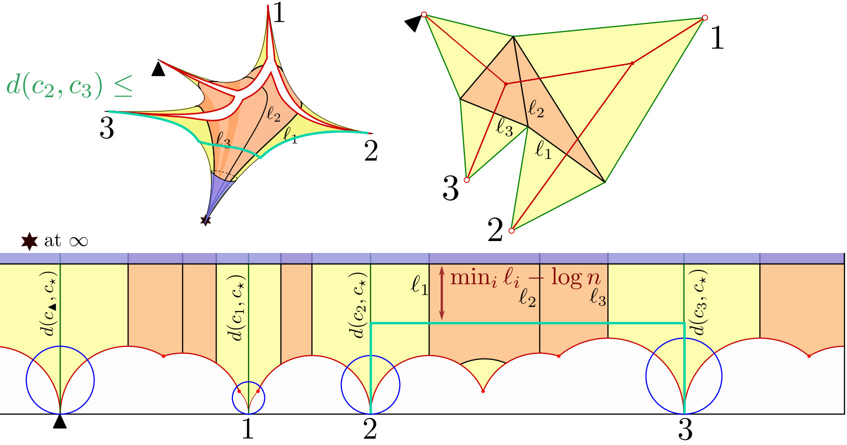

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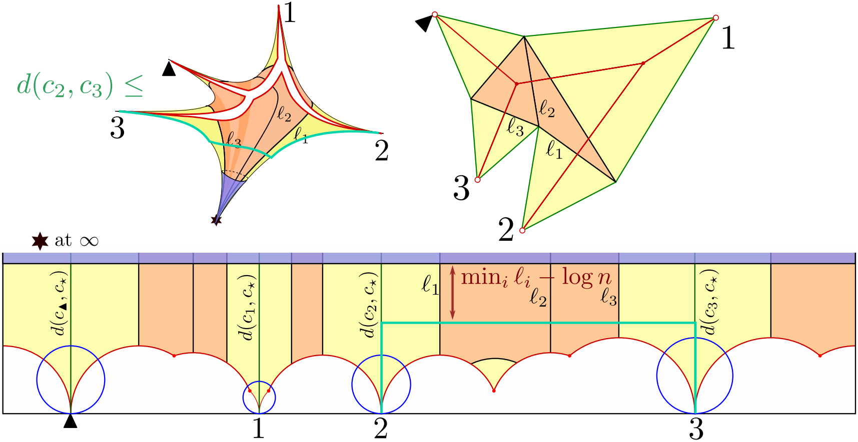

- Distances between arbitrary horocycles satisfy deterministic bound

$$d_{\mathsf{hyp}}(c_i, c_j) \leq d_{\mathsf{hyp}}(c_i, c_*) + d_{\mathsf{hyp}}(c_j, c_*) - 2 \min_k \ell_k + \underbrace{2 \log n + 10}_{o(n^{\frac{1}{4}})}.$$

Convergence to the Brownian sphere

[Le Gall, '13] [Miermont, '13] [Addario-Berry, Albenque, '13] [Bettinelli, Jacob, Miermont, '14]

$$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} (\mathrm{c}_1 e_t, \mathrm{c}_2 Z_t)_{0\le t\le 1}$$

$$+$$

$$d_{\mathrm{hyp}}^{(n)}(s,t) \le Z^{(n)}(s) + Z^{(n)}(t) - 2\max\left\{\min_{[s,t]} Z^{(n)}, \min_{[t,s]} Z^{(n)}\right\} + o(n^{\frac{1}{4}})$$

$$+$$

Invariance under rerooting

$\Downarrow$ [Le Gall, '13]'s rerooting trick

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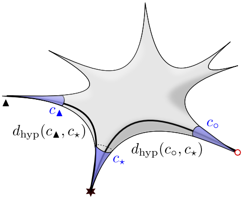
$$\Downarrow$$

$$\left(\{c_1,\dots,c_n\}, n^{-\frac{1}{4}} d_{\mathrm{hyp}}\right) \xrightarrow[n\to\infty]{(d)} \mathrm{c}_{\mathrm{WP}}(\mathrm{m}_\infty, D^*)$$ (Gromov-Hausdorff sense)

Setting the scale

- Random surface but now subcritical and extra marked cusp $\circ$, $\mathbb{P}^{\circ}_{x}(\mathcal{T}) = \frac{x^{\#\text{leaves}}}{\mathcal{Z}'(x)} \,\mathrm{Leb}(\mathcal{A}_{\mathcal{T}})$.



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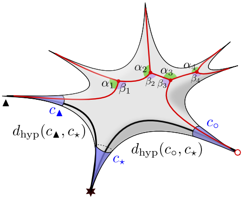
$$\delta = d_{\text{hyp}}(c_{\circ}, c_{*}) - d_{\text{hyp}}(c_{\blacktriangle}, c_{\star}) = \sum_{i=1}^{k} \log \frac{\sin \alpha_i}{\sin(2\pi - \alpha_i - \beta_i)}$$



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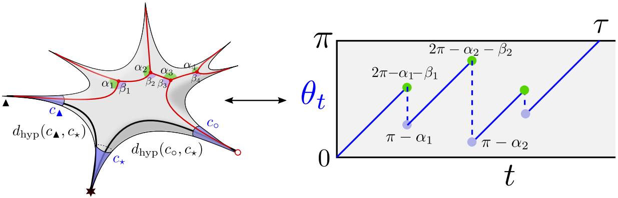
- (θ_t) has law of Markov process with drift 1 and $\downarrow$-jumps $\theta \to \theta'$ at rate $2F(\theta - \theta')\frac{F_\bullet(\theta')}{F_\bullet(\theta)}$. $\quad F(\theta) = \frac{\sqrt{\mathcal{Z}}}{\theta} J_1(2\theta\sqrt{\mathcal{Z}}), F_\bullet(\theta) = J_0(2\theta\sqrt{\mathcal{Z}})$.



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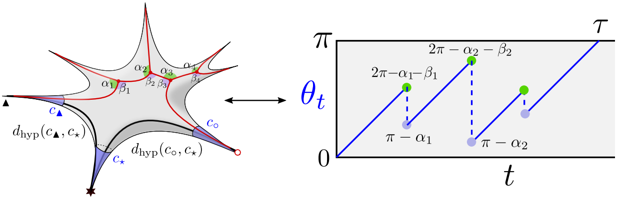

$$\mathbb{E}^\circ_x[\delta^2] = -\frac{2}{F_\bullet(\pi)} \int_0^\pi \mathrm{d}\alpha \int_0^\alpha \mathrm{d}\beta \cot \alpha \cot \beta F_\bullet(\pi - \alpha) F_\bullet(\alpha - \beta) F_\bullet(\beta)$$
$$= \frac{\pi^2}{3} \mathcal{Z}'(x) J_2(2\pi\sqrt{\mathcal{Z}(\S)}).$$

Happy birthday, Jean!