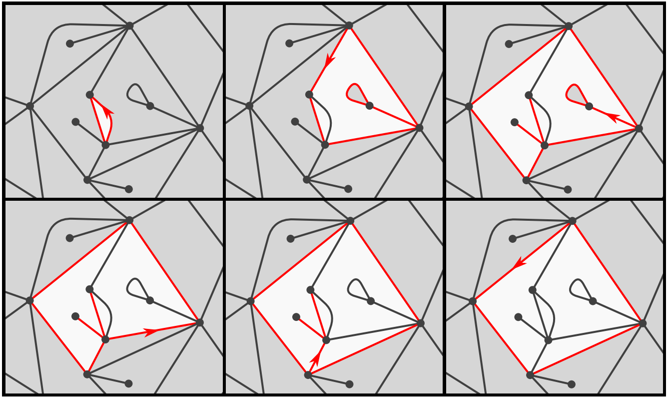
20th Itzykson Conference, IPhT, Saclay, 12-06-2015

Peeling of infinite Boltzmann planar maps

Timothy Budd



Based on arXiv:1506.01590 and work in progress.

Niels Bohr Institute, University of Copenhagen

budd@nbi.dk, http://www.nbi.dk/~budd/

Motivation 1: Wireless sensor networks

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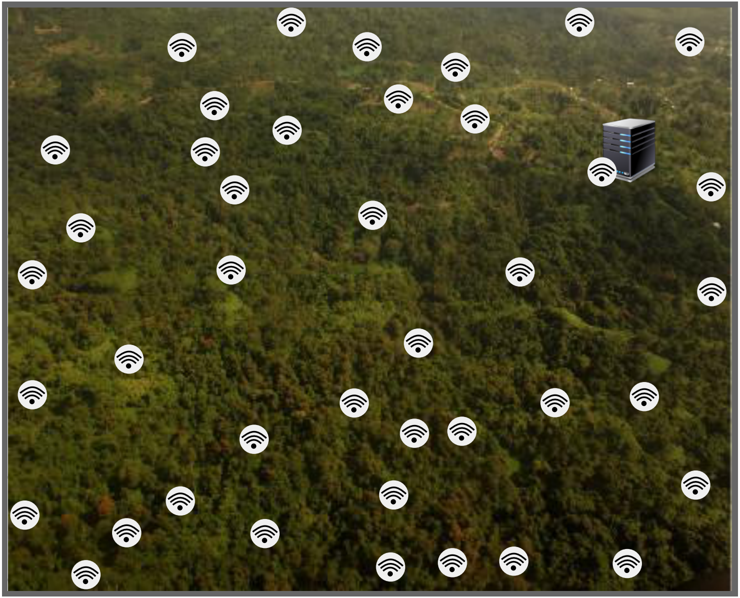

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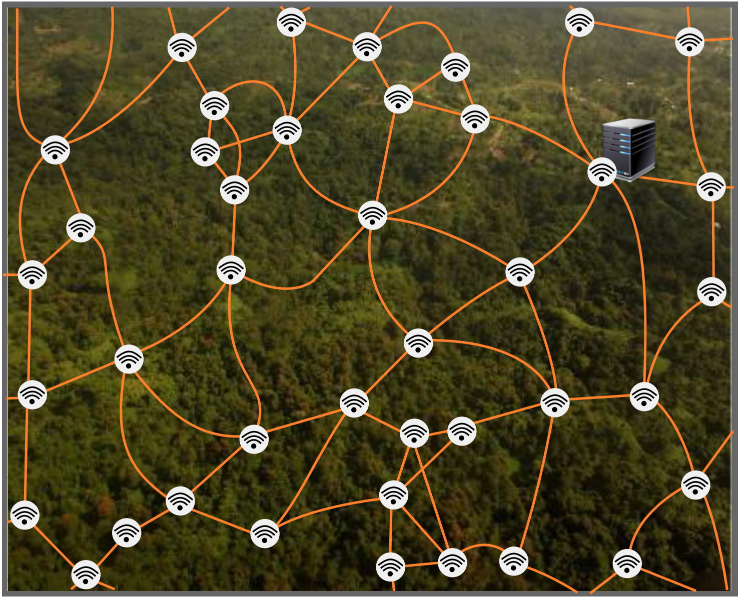

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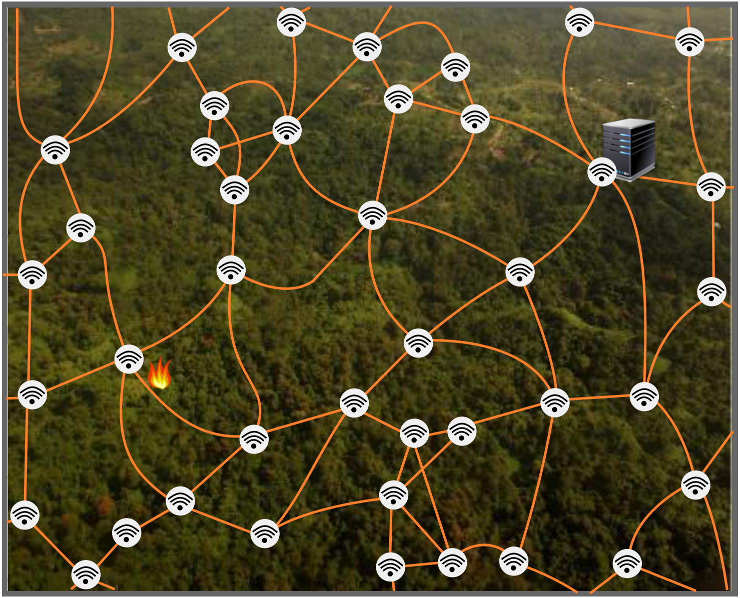

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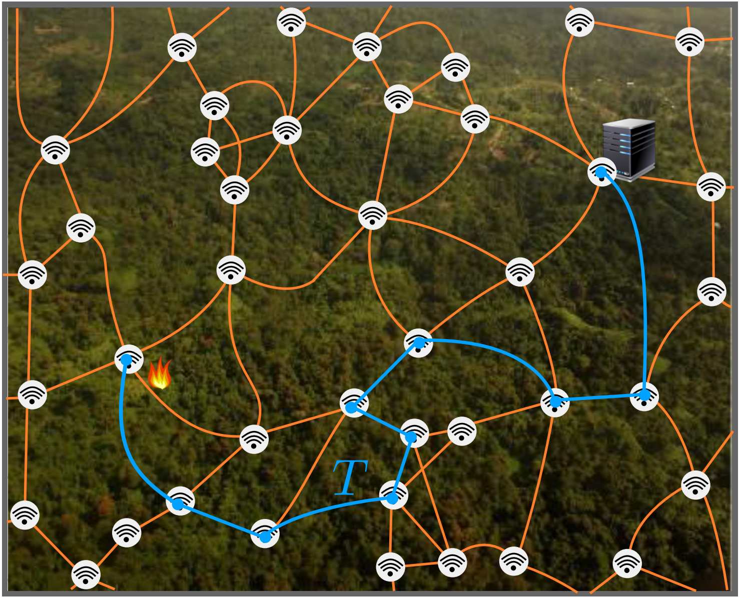

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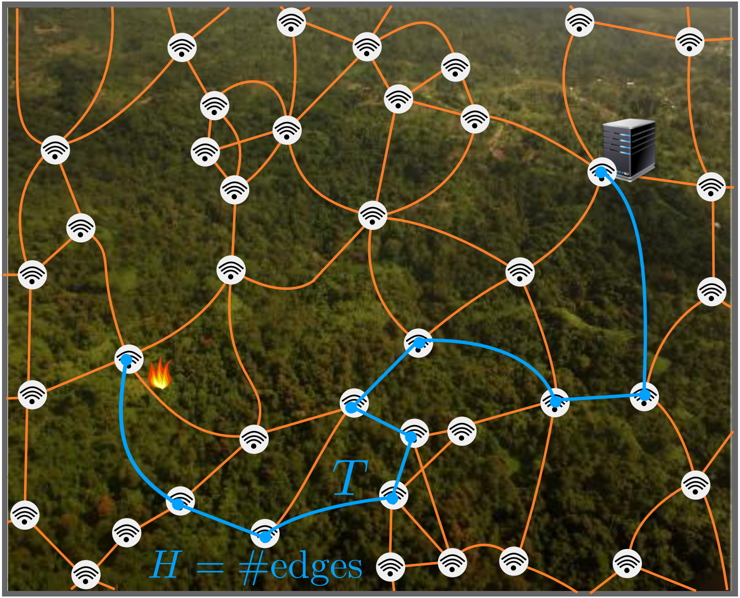

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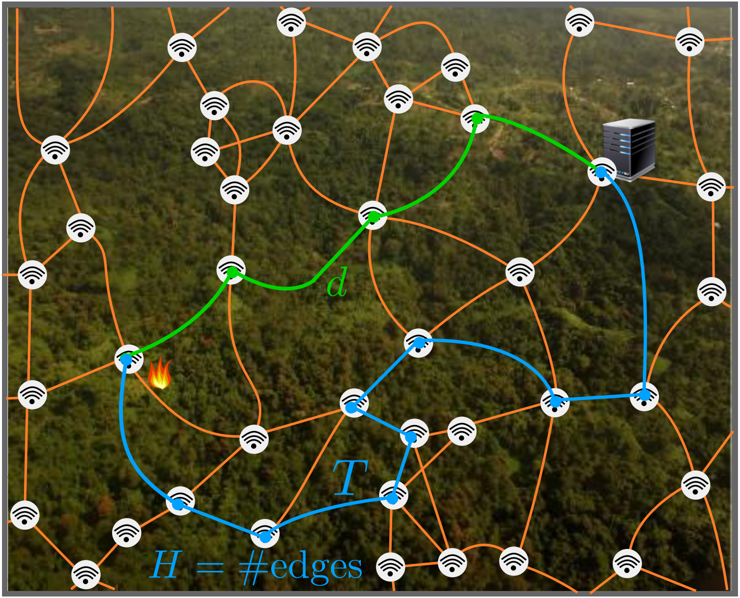

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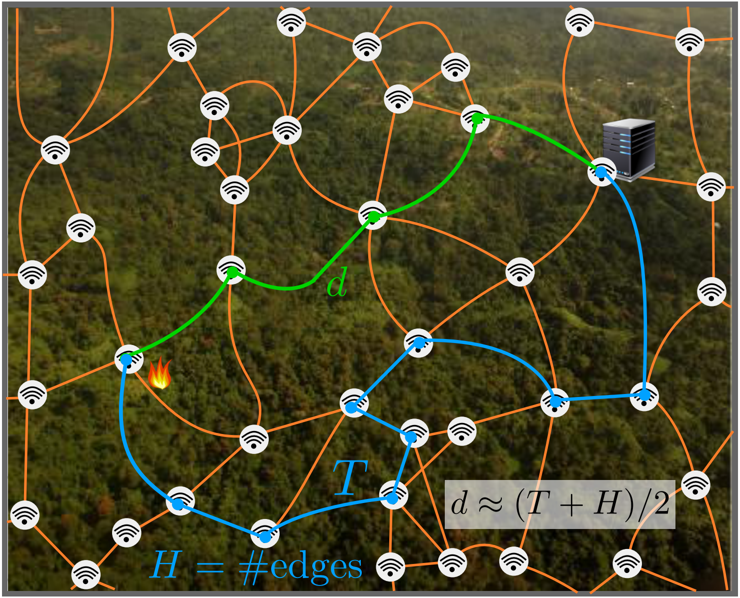

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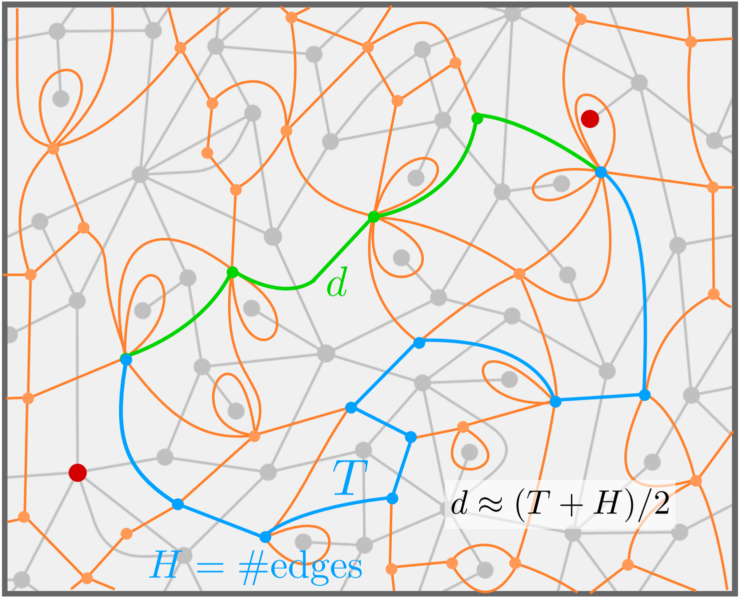

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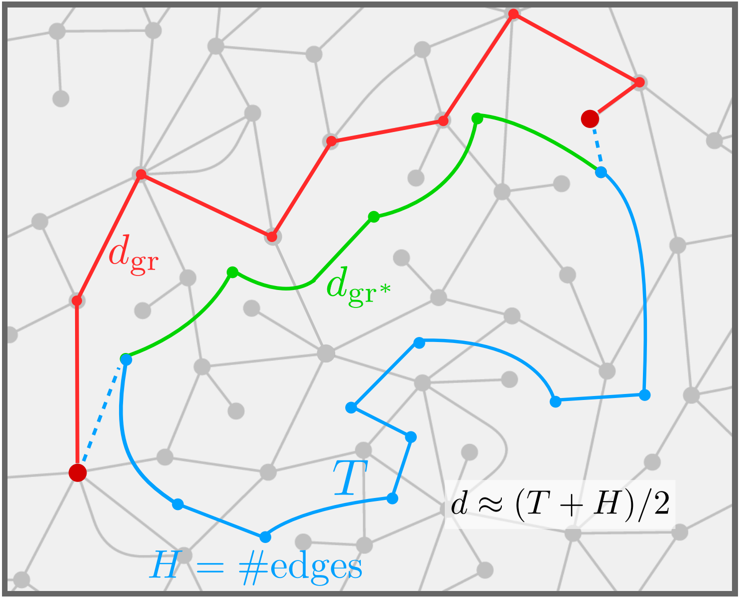
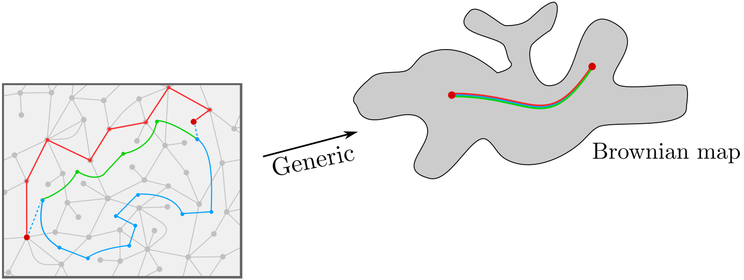


Motivation 2: Geometry of non-generic scaling limits



[Le Gall, Miermont, '11] [Borot, Bouttier, Guitter, '12]

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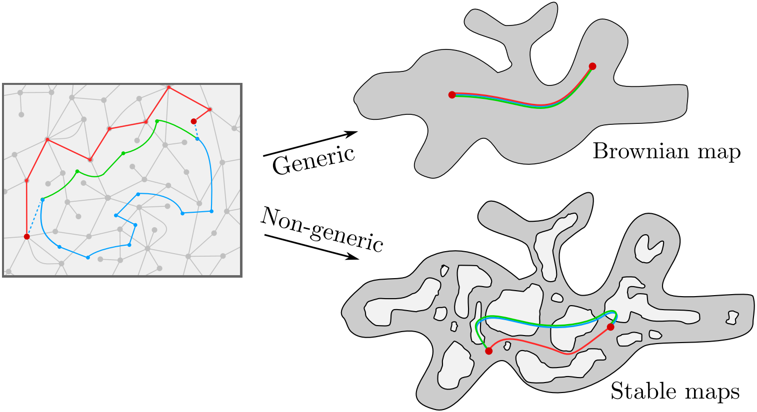

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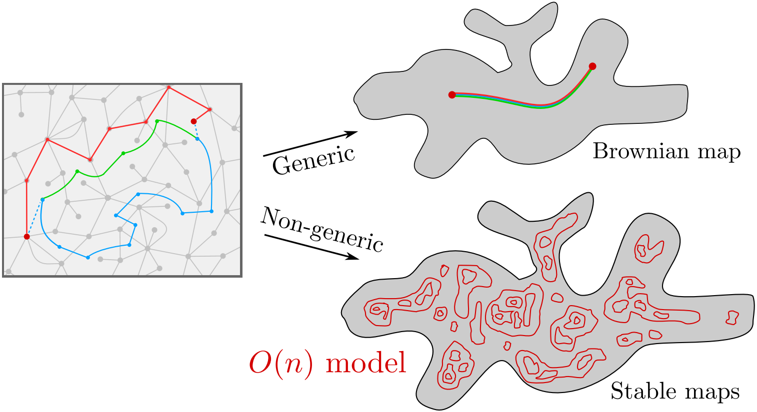

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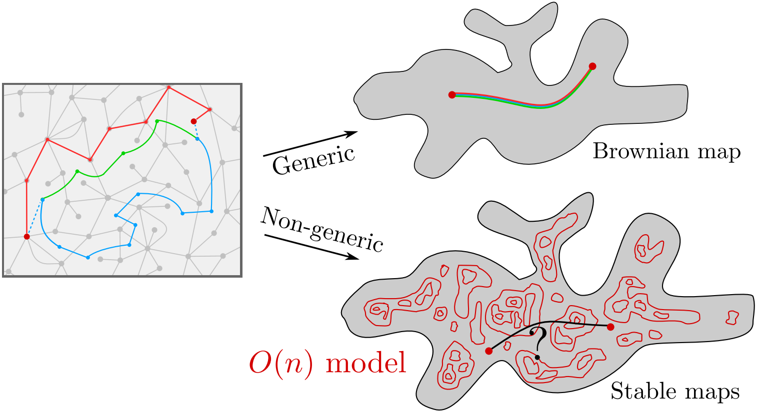

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Outline

- Introduction
  - $\mathbf{q}$-Boltzmann planar maps
  - Lazy peeling process
- Perimeter and volume processes
  - Description in terms of biased random walks
  - Infinite $\mathbf{q}$-Boltzmann planar maps
  - Scaling limit
- Scaling constants from peeling:
  - First-passage time
  - Hop count
  - Dual graph distance
- Miermont's scaling constant for the graph distance
- Example: uniform infinite planar map.
- Outlook

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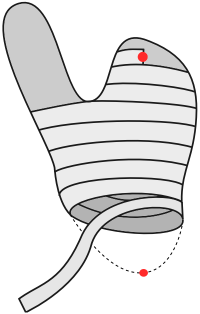

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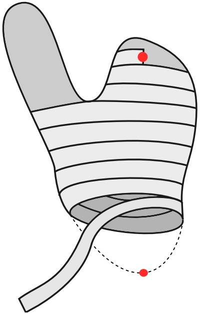

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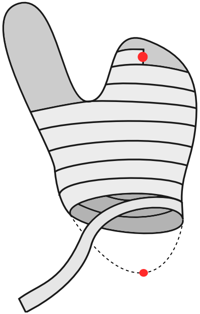

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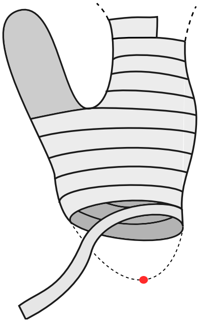

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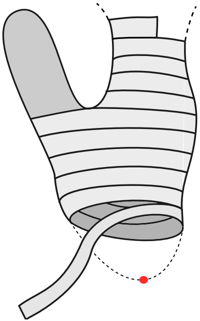

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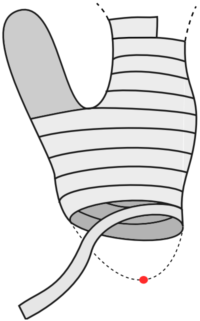

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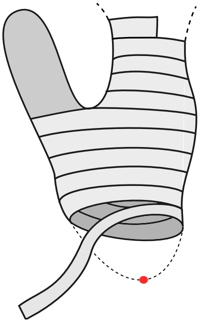
  This talk: extend their results to **q**-IBPM.



Boltzmann planar maps

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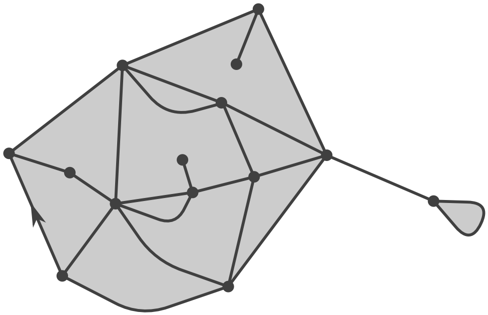

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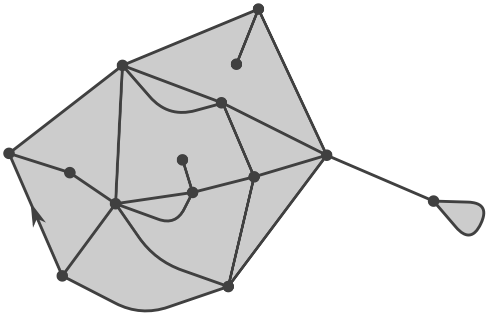

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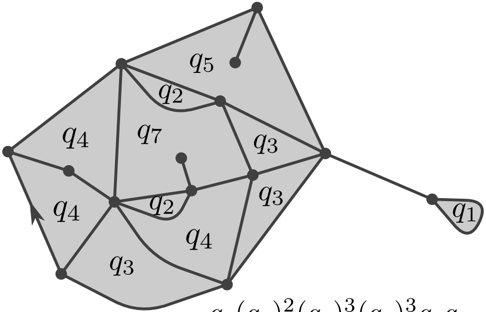
over rooted planar maps m with root face degree l



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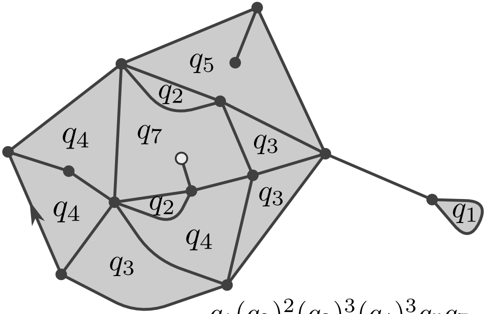

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- If $q_k = 0$ for all odd k, then the $\mathbf{q}$-BPM is *bipartite* and $r = 1$. Otherwise $\mathbf{q}$ *non-bipartite* and $|r| < 1$.

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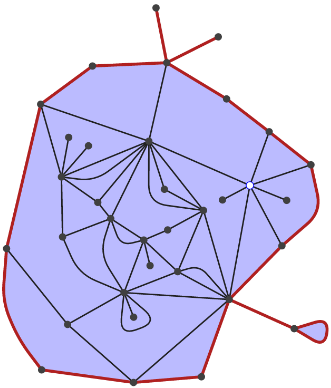

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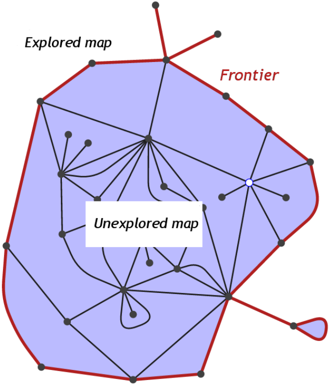

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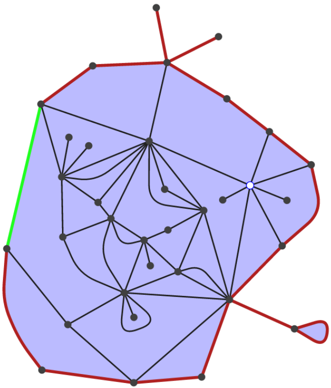

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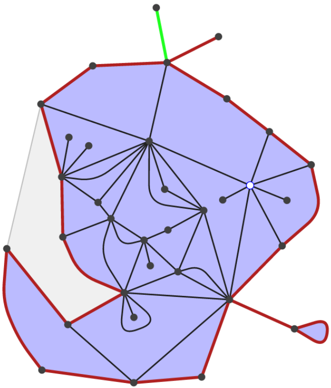

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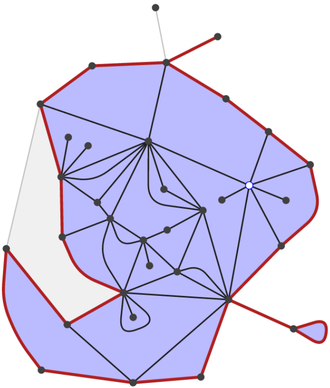

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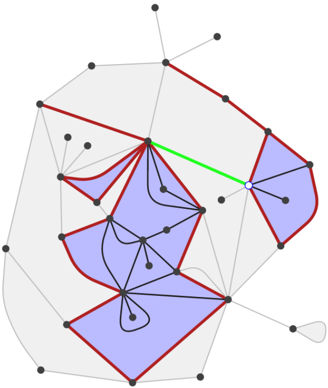

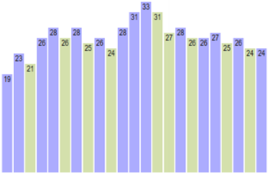

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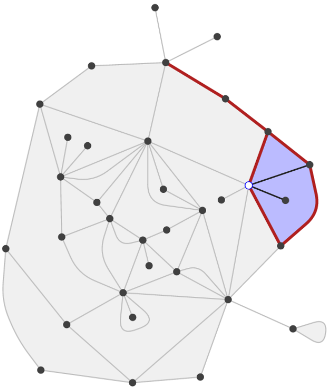

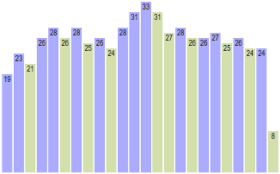

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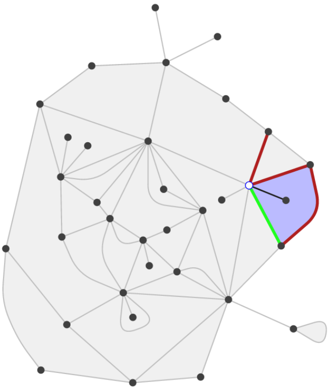

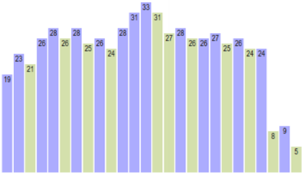

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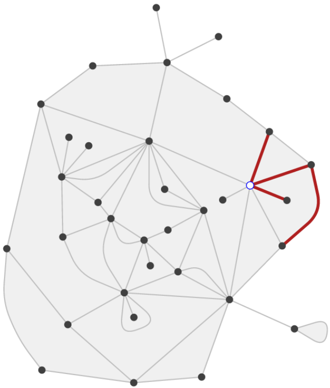

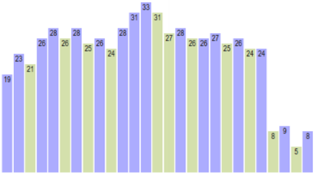

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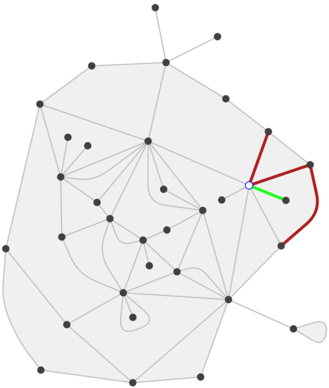

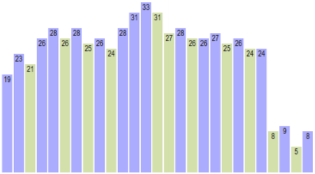

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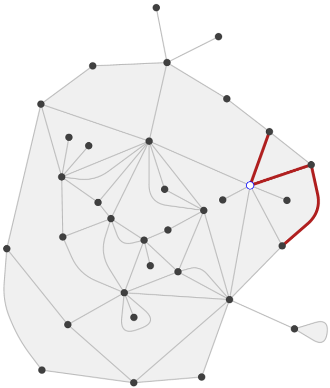

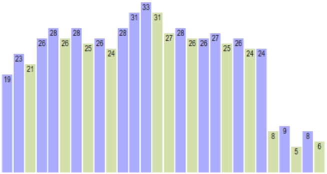

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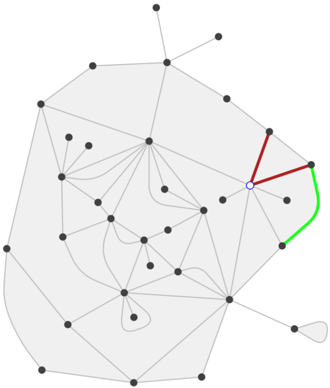

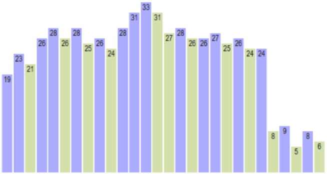

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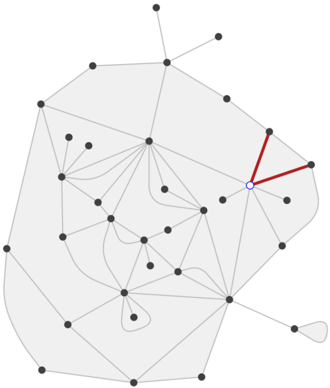

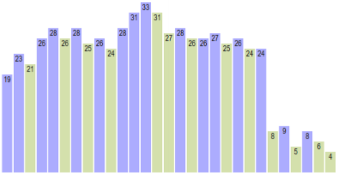

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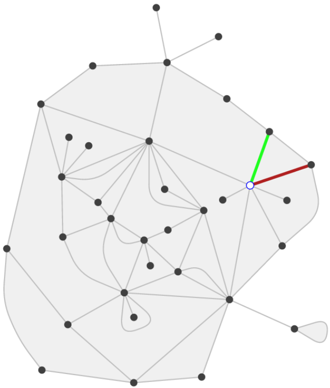

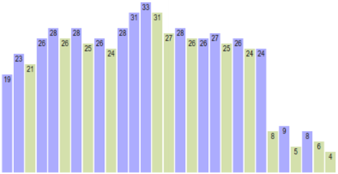

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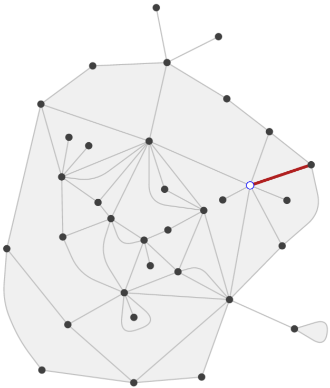

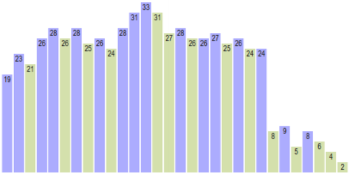

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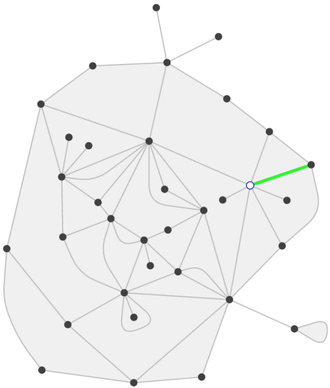

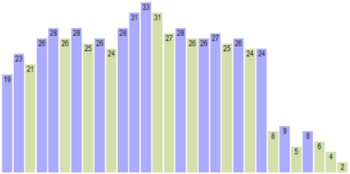

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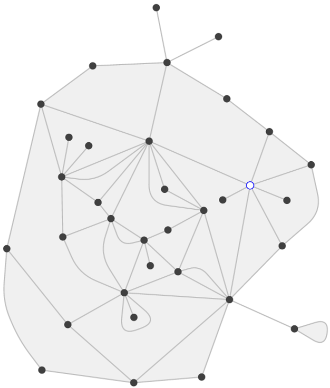

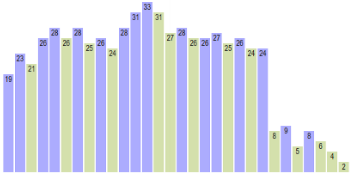

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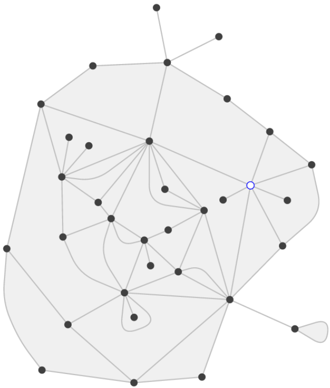

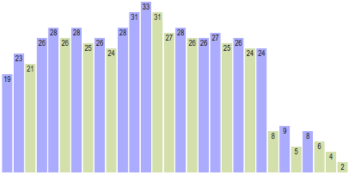

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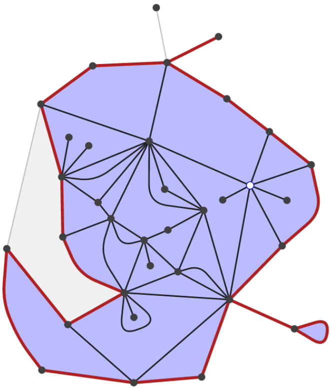

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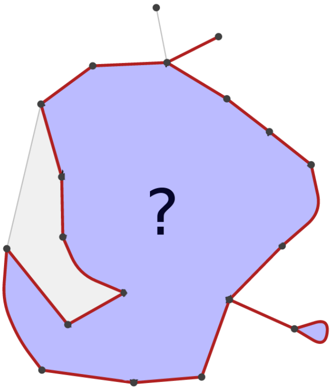

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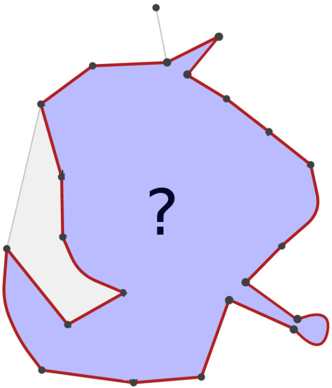

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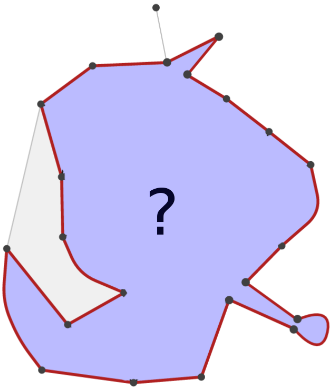
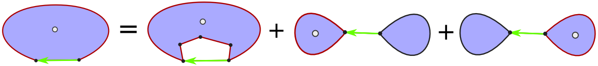


The perimeter process

- Loop equations: $W_\bullet^{(l)} = \sum_{k=0}^{\infty} q_k W_\bullet^{(l+k-2)} + 2\sum_{p=0}^{l-2} W^{(p)} W_\bullet^{(l-p-2)}$



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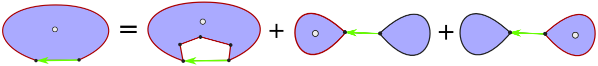

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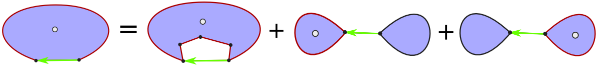

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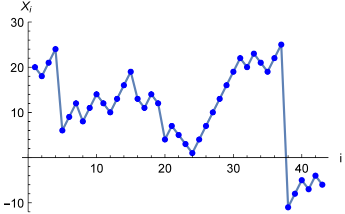

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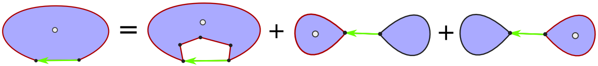

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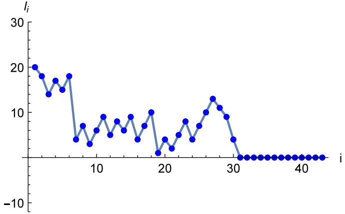

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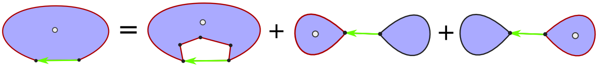

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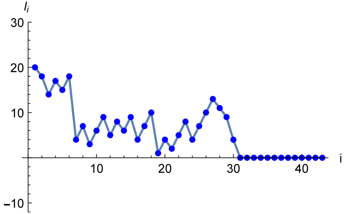

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Properties of critical ν

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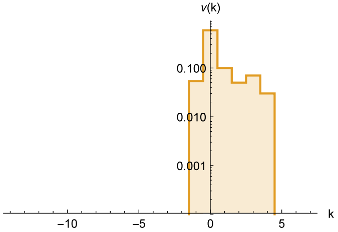

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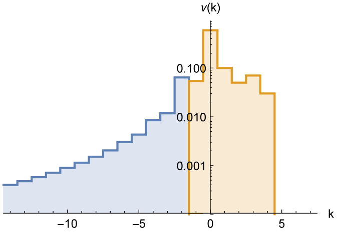

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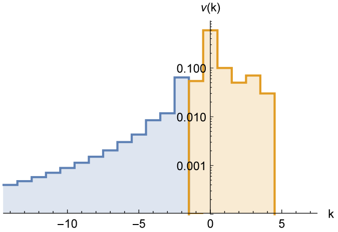

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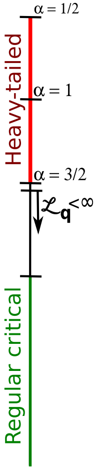

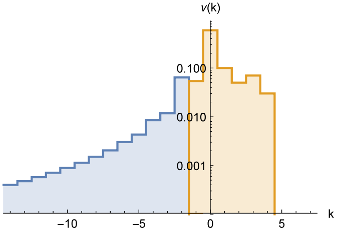

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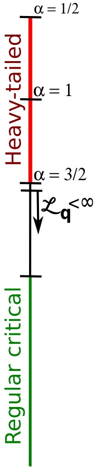

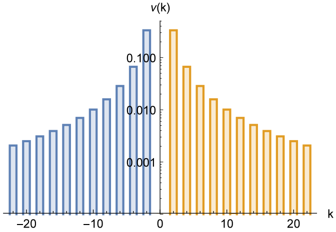

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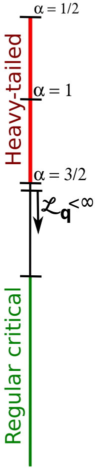
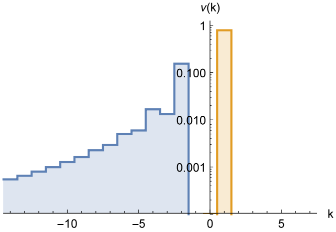
$$\nu(-k) \sim \frac{3\mathcal{L}_{\mathbf{q}}\sqrt{1+r}}{4\sqrt{\pi}} k^{-5/2}$$





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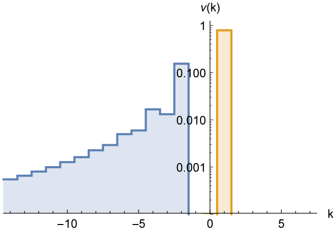
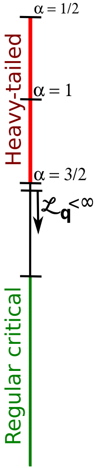
  - *Regular critical*: $\sum_{k=1}^{\infty} \nu(k) C^k < \infty$ for some $C > 1$.





Scaling limit for regular critical $\mathbf{q}$

- Tails and no drift imply (weak) convergence to 3/2-stable process with negative jumps

$$\left(\frac{X_{\lfloor nt \rfloor}}{\left(\sqrt{1 + r\mathcal{L}_{\mathbf{q}} n}\right)^{\frac{2}{3}}} \right)_{t \geq 0} \xrightarrow[n\to\infty]{\text{(d)}} S_{3/2}(t)$$

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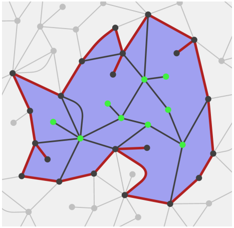

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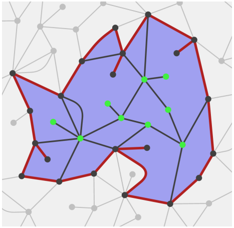

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  - Checking the details of the proof of Curien and Le Gall:

Theorem (TB '15 based on Curien, Le Gall, '14)

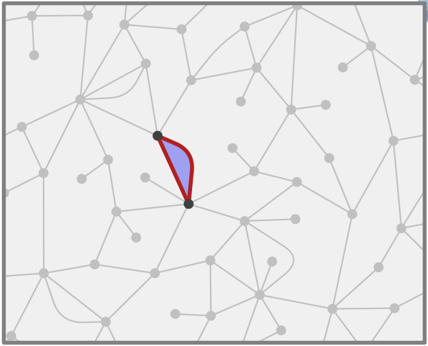
The perimeter $(l_i)_{i\geq 0}$ and volume $(V_i)_{i\geq 0}$ of a peeling of a regular critical $\mathbf{q}$*-IBPM converge jointly in distribution in the sense of Skorokhod to*

$$\left(\frac{l_{\lfloor nt \rfloor}}{\mathbf{p}^{\ell}_{\mathbf{q}} n^{2/3}}, \frac{V_{\lfloor nt \rfloor}}{\mathbf{v}^{\ell}_{\mathbf{q}} n^{4/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{\text{(d)}} (S^{+}_{3/2}(t), Z(t))_{t\geq 0}$$

$$\mathbf{p}^{\ell}_{\mathbf{q}} = (\sqrt{1+r}\mathcal{L}_{\mathbf{q}})^{2/3}$$

$$\mathbf{v}^{\ell}_{\mathbf{q}} = \frac{8}{3c_{+}^{2}}\left(\frac{\mathcal{L}_{\mathbf{q}}}{1+r}\right)^{1/3}$$

First-passage time and hop count



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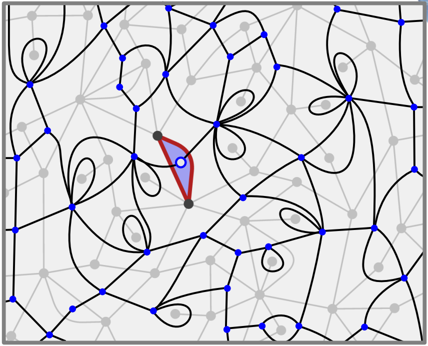

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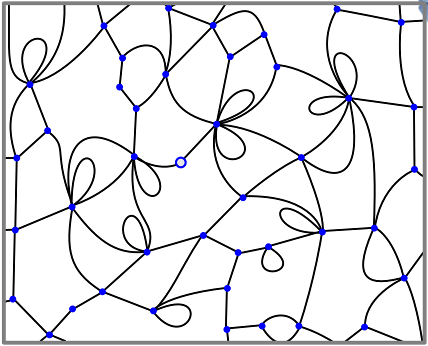

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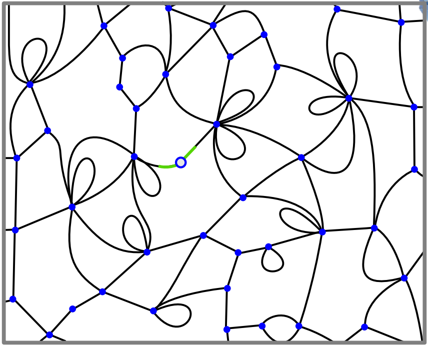

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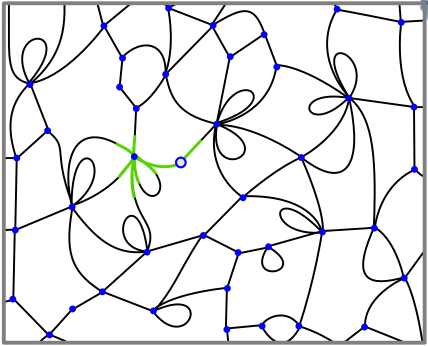

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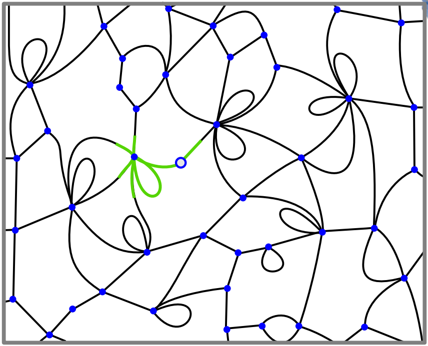

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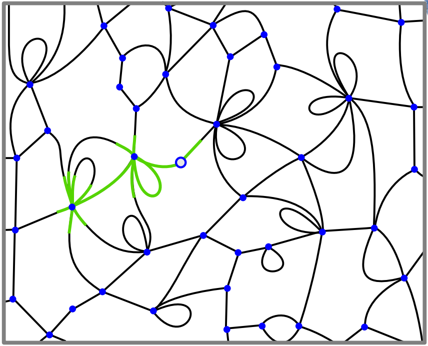

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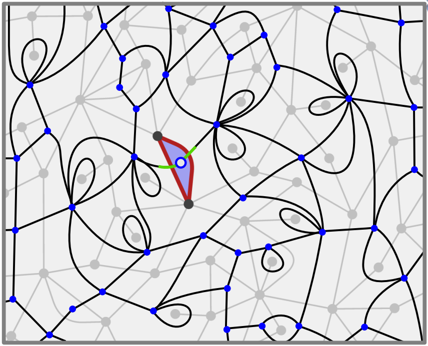

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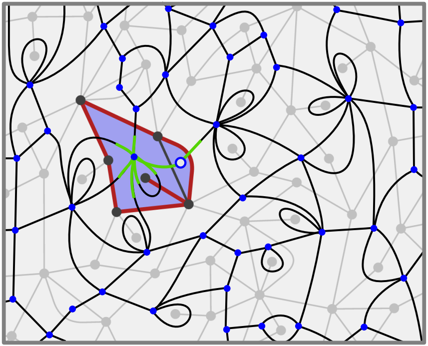

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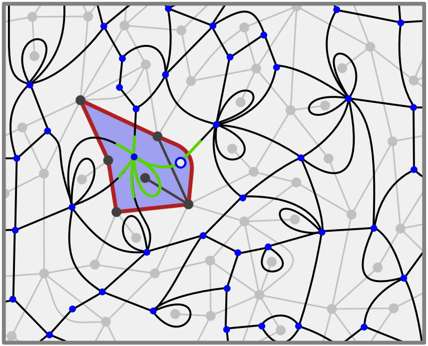

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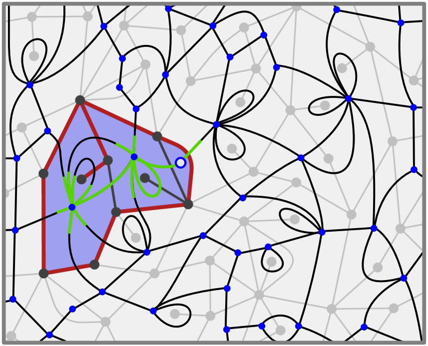

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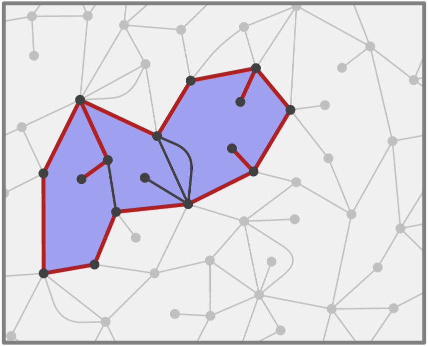

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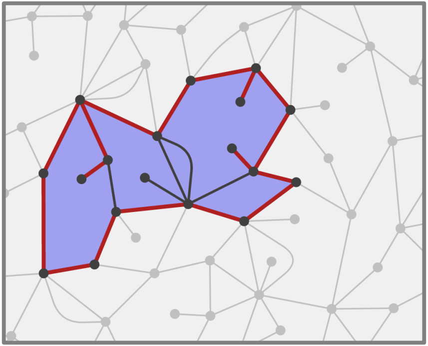

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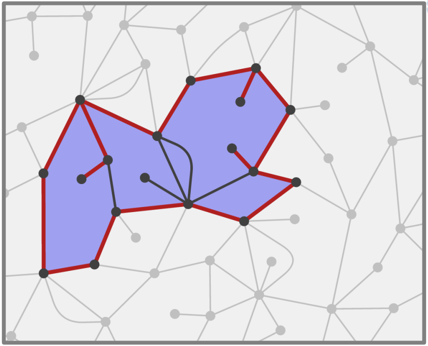

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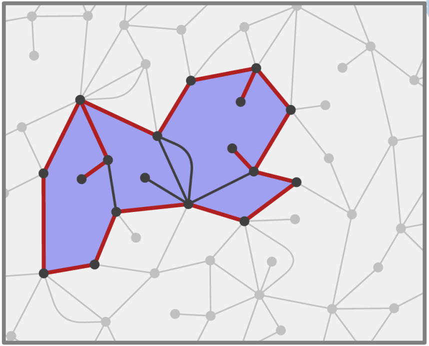

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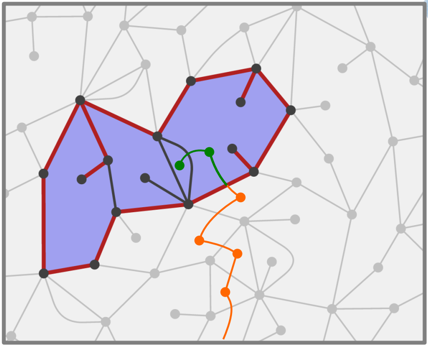

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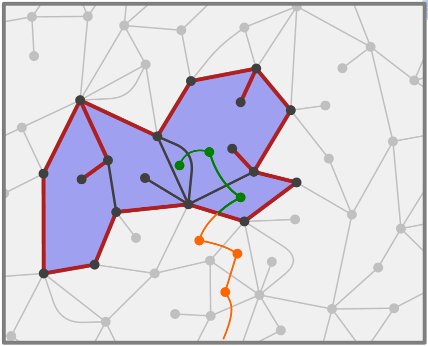

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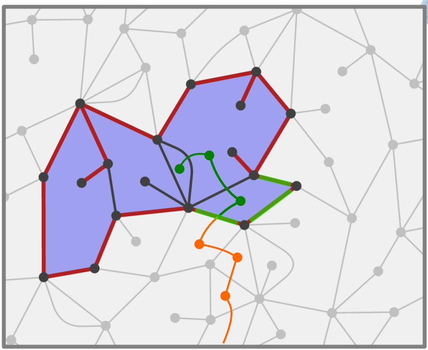

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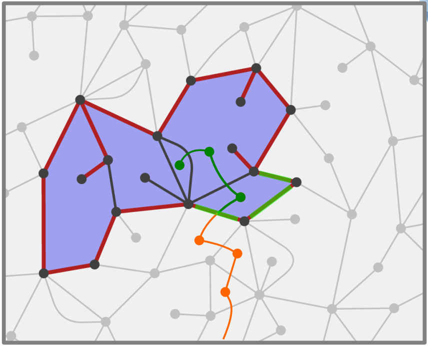

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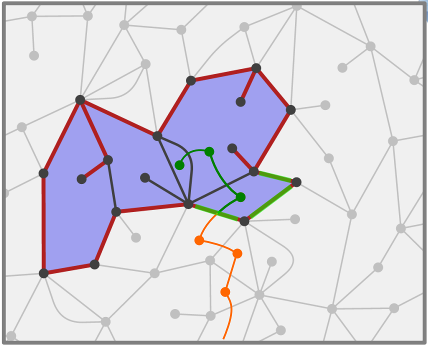

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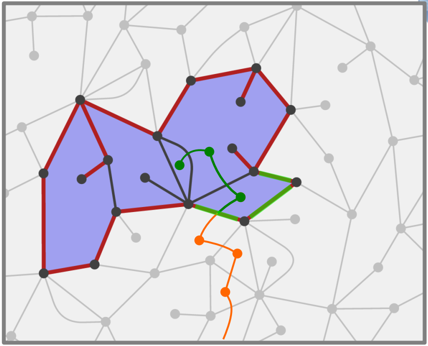
$$\mathbb{E}(H_{i+1} - H_i | l_i) = \sum_{k=0}^{\infty} \frac{k+1}{k+l_i} \frac{h_r^{(1)}(k+l_i)}{h_r^{(1)}(l_i)} \nu(k)$$

First-passage time and hop count



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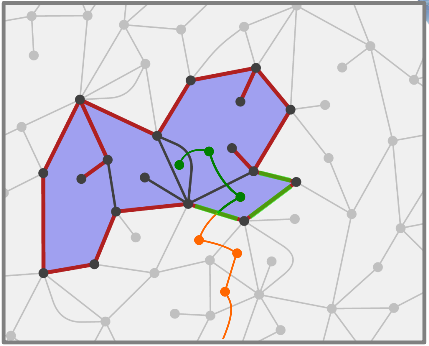
$$\mathbb{E}(H_{i+1}-H_i|l_i) = \sum_{k=0}^{\infty} \frac{k+1}{k+l_i} \frac{h_r^{(1)}(k+l_i)}{h_r^{(1)}(l_i)} \nu(k) = \sum_{k=0}^{\infty} (k+1)\nu(k)\ \mathbb{E}(T_{i+1}-T_i|l_i) + \mathcal{O}(l_i^{-1})$$

First-passage time and hop count

- Assign random exp(1)-lengths to dual edges.
- Associated peeling: choose peel edge uniformly in frontier.
- Let $(T_i)_{i\geq 0}$ be time at which the i'th peeling step occurs.
- Knowing $(l_i)_{i\geq 0}$: $T_i = \sum_{j=1}^{i} \frac{\mathfrak{e}_j}{l_{j-1}}$, where $\mathfrak{e}_j$ are independent exp(1) random variables.



- Let the *hop count* H_i be # of edges explored of a shortest-time path to some faraway vertex after i steps. Knowing $(l_i)_{i\geq 0}$:

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Dual graph distance

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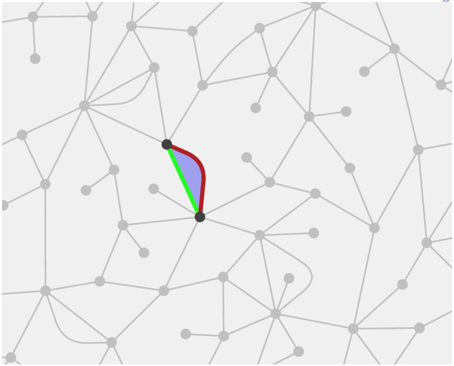

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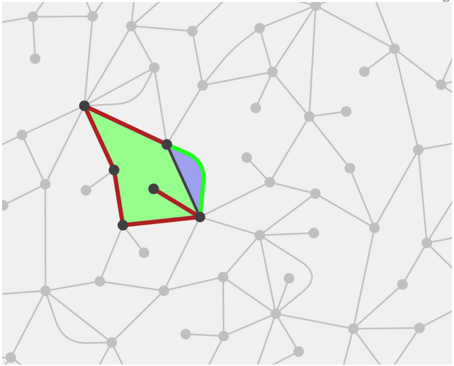

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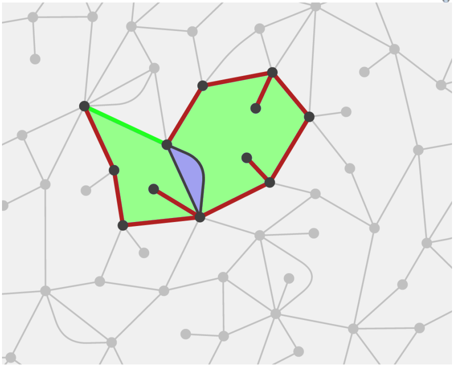

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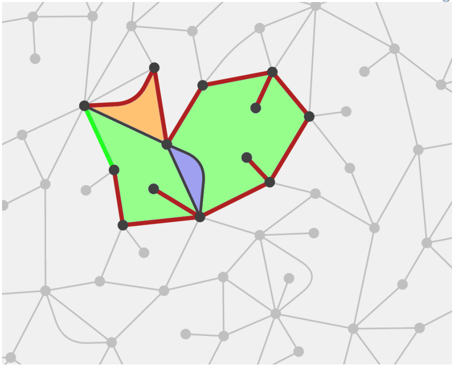

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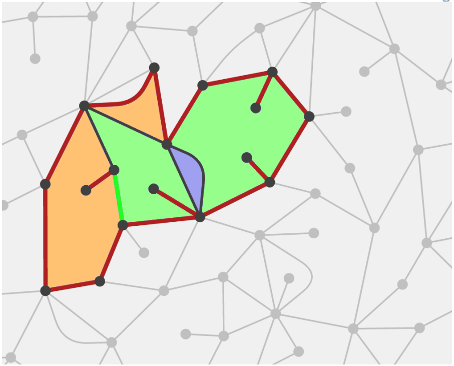

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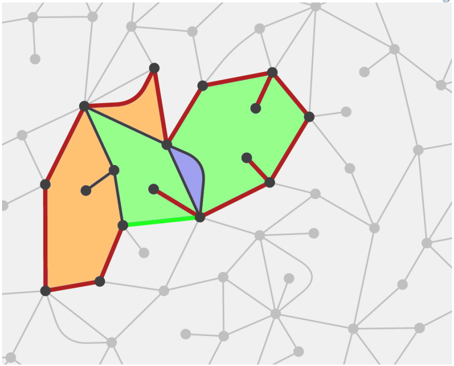

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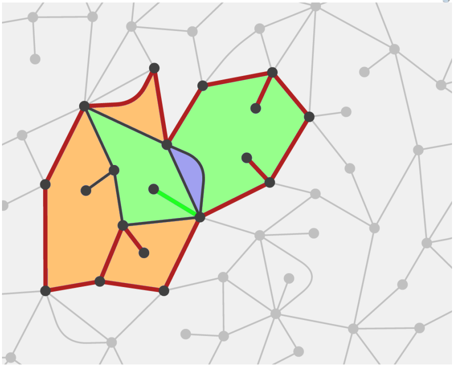

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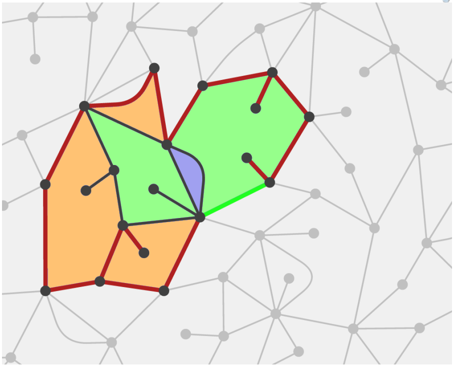

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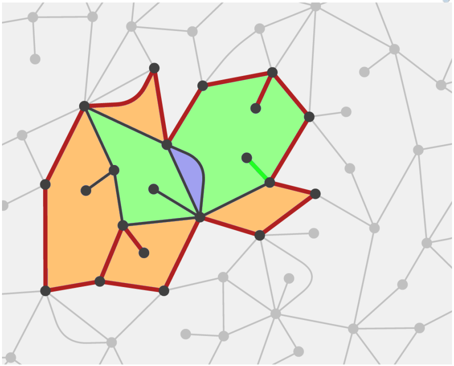

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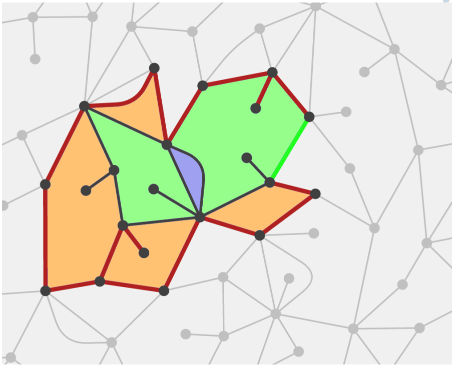

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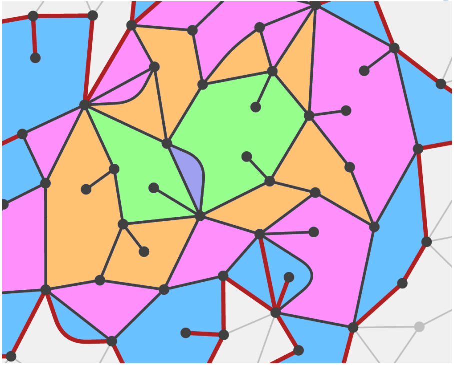

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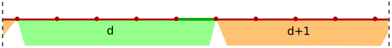

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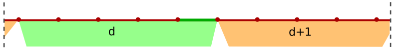

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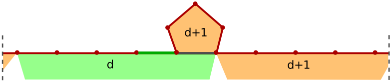

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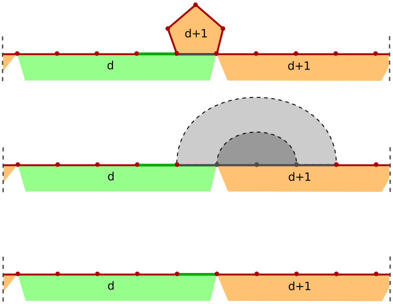

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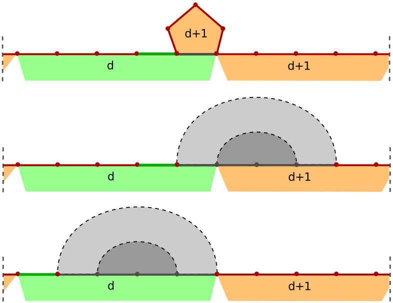

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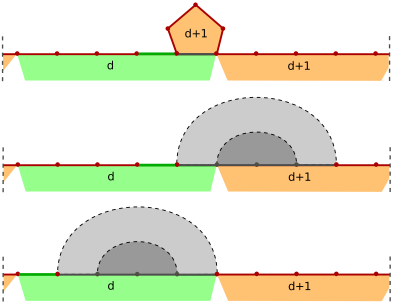

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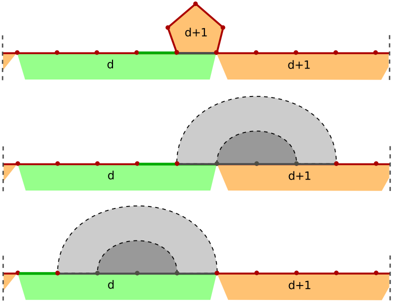

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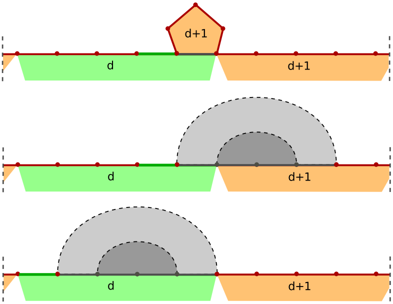

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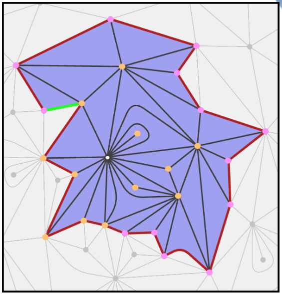
- Using $\mathbb{E}(T_{i+1} - T_i | l_i) = 1/l_i$, and assuming asymptotically linear scaling, this suggests the asymptotic relation:

$d_{\mathrm{gr}^*} \approx \frac{1}{2}(T + H)$ for any regular critical **q**-IBPM

Graph distance

- Can adapt peeling process to *graph distance*: take peel edge to be frontier edge closest to root vertex.



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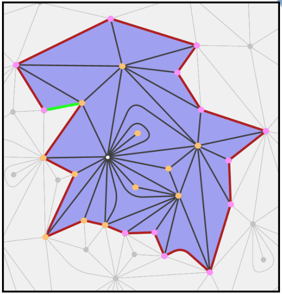

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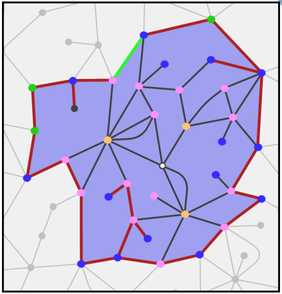

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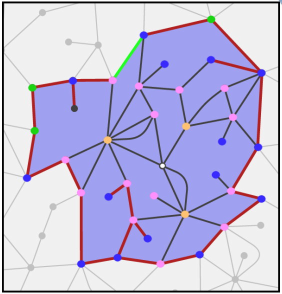

Theorem (Miermont, '06)

If $\mathbf{q}$ is regular critical and m_n is a $\mathbf{q}$-BPM conditioned to have n vertices and v_1, v_2 are random vertices, then there exists a $\mathcal{C}_\mathbf{q} > 0$ and a $\mathbf{q}$-independent random variable d_∞ s.t.

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Conjecture

Let v be a random vertex at distance d_{gr} from the root in a regular $\mathbf{q}$-IBPM, then we have the following limits in probability as $d_{\mathrm{gr}} \to \infty$ for its first-passage time T, hop count H, and dual graph distance d_{gr^*}:

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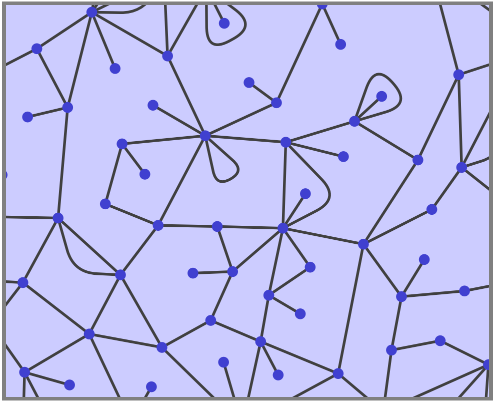
- Seems to be settled for the UIPT. [Curien, Le Gall, to appear]

Example: Uniform infinite planar map (bivariate)

- Local limit of uniform random planar maps with fixed # vertices and faces.



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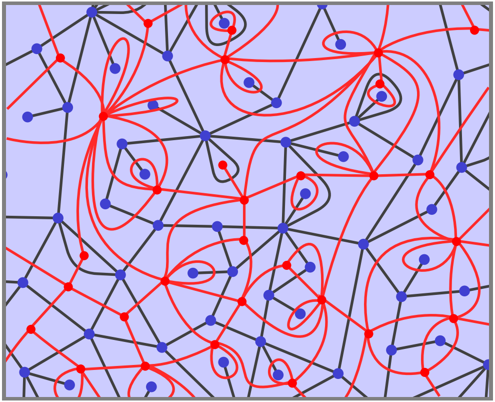

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- Notice UIPM is $\sigma = \frac{5}{6}$, $\mathcal{H}_q = 3$, and duality: $\frac{\mathcal{H}_{\mathbf{q}}-1}{2} \leftrightarrow \frac{2}{\mathcal{H}_{\mathbf{q}}-1}$.

More examples

	r	c_+	$\mathcal{L}_{\mathbf{q}}$	$\mathcal{C}^4_{\mathbf{q}}$
Triangulations	$2\sqrt{3}-3$	$\sqrt{6+4\sqrt{3}}$	$\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)$	$1/3$
Quadrangulations	1	$\sqrt{8}$	$4/3$	$8/9$
Pentangulations	$0.70878\ldots$	$2.6098\ldots$	$2.1704\ldots$	$0.7683\ldots$
$2p$-angulations	1	$\sqrt{\frac{4p}{p-1}}$	$\frac{4}{3}(p-1)$	$\frac{4}{9}p$
Uniform planar maps	$3/5$	$5/\sqrt{3}$	5	$16/9$
Uniform planar maps (biv.)	$\frac{\mathcal{H}^2-3}{\mathcal{H}^2+1}$	$\frac{(\mathcal{H}-1)^{3/2}\sqrt{\mathcal{H}+3}}{2(\mathcal{H}^2+3)}$	$\frac{1}{2}(\mathcal{H}^2+1)$	$\frac{(\mathcal{H}+1)^3}{6(\mathcal{H}+1)}$
	$\frac{\text{vertices}}{\text{faces}}$	$H/T=\mathcal{H}_{\mathbf{q}}$	T/d_{gr}	$d_{\mathrm{gr}^*}/d_{\mathrm{gr}}$
Triangulations	$1/2$	$1+\frac{1}{\sqrt{3}}$	$2\sqrt{3}$	$1+2\sqrt{3}$
Quadrangulations	1	2	$3/2$	$9/4$
Pentangulations	$3/2$	$2.3608\ldots$	$1.0785\ldots$	$1.8123\ldots$
$2p$-angulations	$p-1$	$\frac{2p-1}{p\binom{2p}{p}}2^{2p-1}$	$\frac{3}{2(p-1)}$	$\frac{3}{4}\left(\frac{1}{p-1}+\frac{2^{2p-2}}{p\binom{2p-2}{p}}\right)$
Uniform planar maps	1	3	$1/2$	1
Uniform planar maps (biv.)	$\frac{(\mathcal{H}+3)(\mathcal{H}-1)^2}{8\mathcal{H}}$	$\mathcal{H}$	$\frac{4}{\mathcal{H}^2-1}$	$\frac{2}{\mathcal{H}-1}$

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Thanks for your attention!