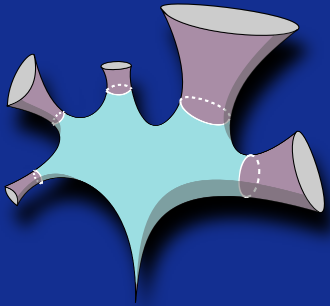
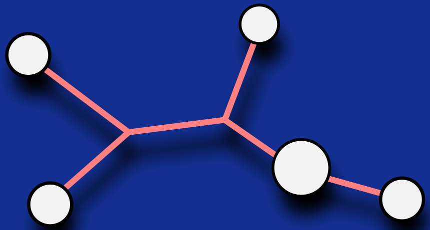
A combinatorial approach to random hyperbolic surfaces

Timothy Budd





w.i.p. with T. Meeusen & B. Zonneveld

(and earlier work with N. Curien)

Radboud University

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http://hef.ru.nl/~tbudd/

Hyperbolic surfaces: a motivation from JT gravity

2D quantum gravity

$$Z = \int [\mathcal{D} g_{ab}] e^{-S[g]}$$

$$\{ g_{ab} \}$$

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[David, Ambjorn, Durhuus, Frohlich, Kazakov, ...]

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[Teitelboim, '83] [Jackiw, '85]

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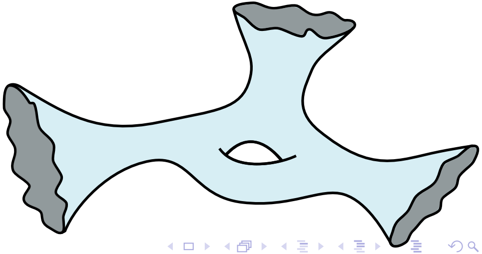

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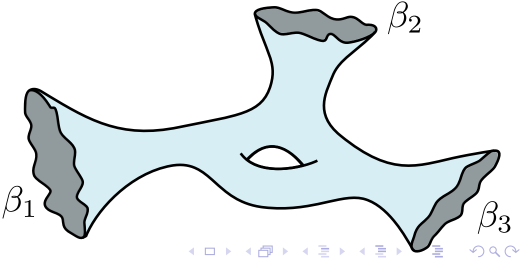

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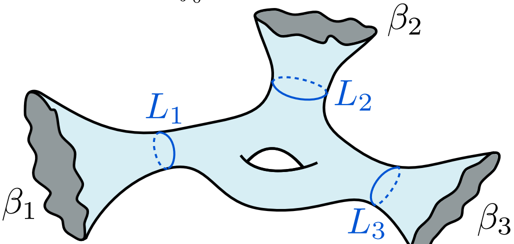

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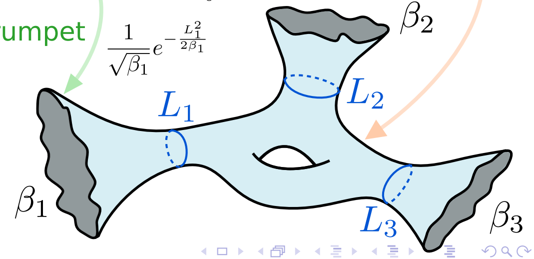
trumpet $\quad \frac{1}{\sqrt{\beta_1}} e^{-\frac{L_1^2}{2\beta_1}}$



The partition function of hyperbolic surfaces: WP volumes

[Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

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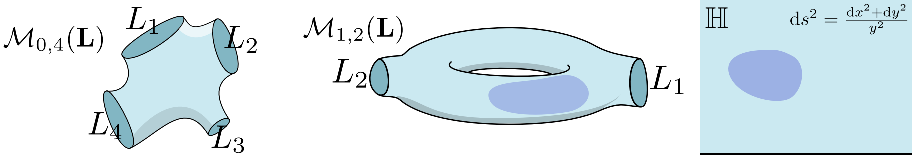

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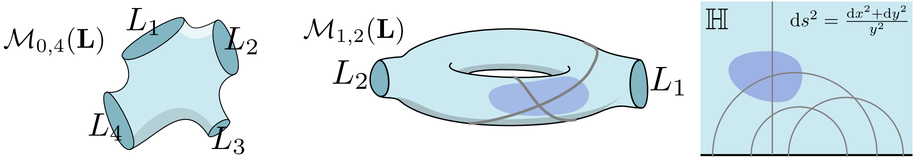

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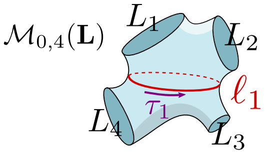

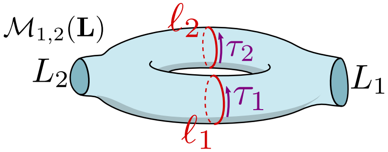

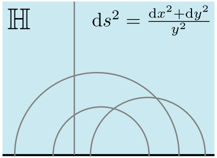

- Carries natural Weil-Petersson volume form μ_{WP}.
  In local Fenchel-Nielsen length & twist coordinates $\ell_1, \tau_1, \ldots, \ell_{3g-3+n}, \tau_{3g-3+n}$ it is

$$\mu_{\mathrm{WP}} = 2^{3-3g-n}\, \mathrm{d}\ell_1 \mathrm{d}\tau_1 \cdots \mathrm{d}\ell_{3g-3+n} \mathrm{d}\tau_{3g-3+n}.$$ [Wolpert, '82]

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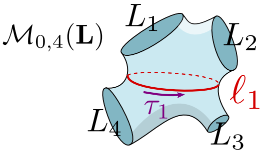

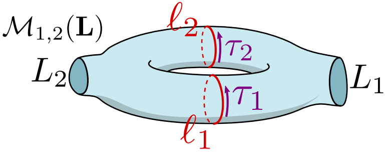

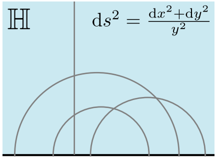

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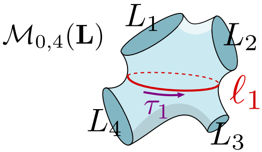

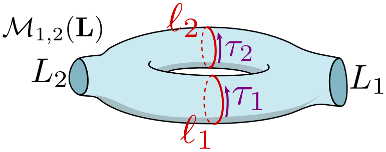

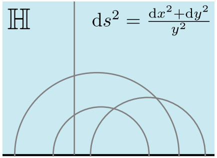

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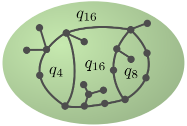
- Examples: $V_{0,3}(\mathbf{L}) = 1, \qquad V_{0,4}(\mathbf{L}) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2) + 2\pi^2,$
  $V_{1,2}(\mathbf{L}) = \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2).$

Bipartite maps on surfaces

- (grand canonical) partition function

$$G_g(q) = \sum_{n\geq 1} \frac{1}{n!} \sum_{d_1=1}^{\infty} q_{d_i} \cdots \sum_{d_n=1}^{\infty} q_{d_n} \# \left\{ \begin{array}{l} \text{genus-}g \text{ maps with} \\ \text{face degrees } 2d_1, \ldots, 2d_n \end{array} \right\}$$



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Hyperbolic surfaces

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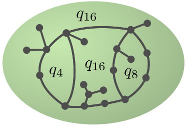

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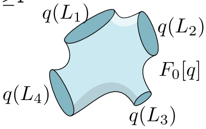

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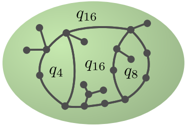

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$e^{\sum_g \lambda^g G_g}$ is a τ-function of KP hierarchy

[Bonzom's talk]

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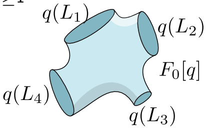

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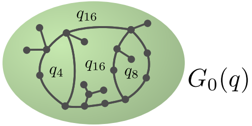

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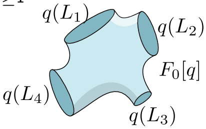

$e^{\sum_g \lambda^g F_g}$ is a τ-function of KdV hierarchy

[Witten, '91][Kontsevich, '92][Kaufmann, Manin, Zagier, '96][Mirzakhani, '07]

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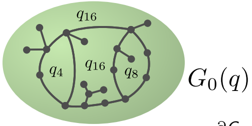
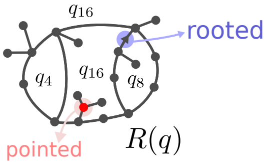


- String equation for $R = \frac{\partial G_0}{\partial q_0 \partial q_1}$:

$$R = 1 + \sum_{k=1}^{\infty} \binom{2k-1}{k} q_{2k} R^k$$



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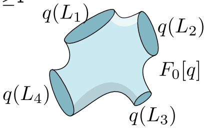

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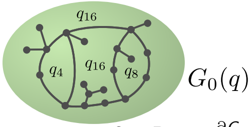

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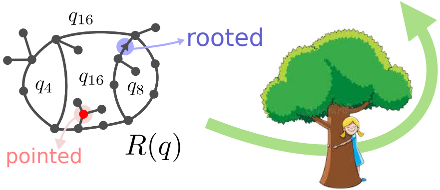

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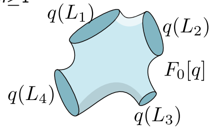

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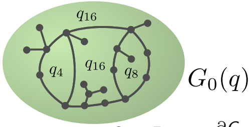

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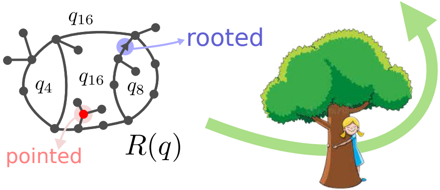

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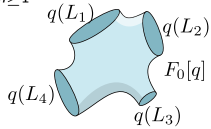

- String equation for $R = \frac{\delta F_0}{\delta q(0)^2}$:

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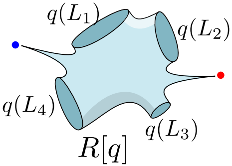
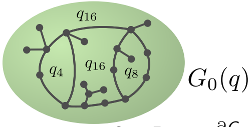


Bipartite maps on surfaces

- (grand canonical) partition function

$$G_g(q) = \sum_{n\geq 1}\frac{1}{n!}\sum_{d_1=1}^{\infty} q_{d_i}\cdots\sum_{d_n=1}^{\infty} q_{d_n}\#\left\{\begin{array}{l}\text{genus-}g\text{ maps with}\\ \text{face degrees }2d_1,\ldots,2d_n\end{array}\right\}$$



- String equation for $R = \frac{\partial G_0}{\partial q_0 \partial q_1}$:

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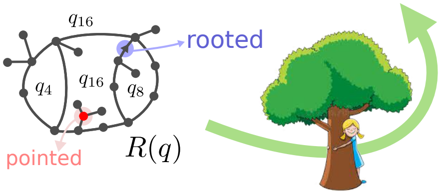

Hyperbolic surfaces

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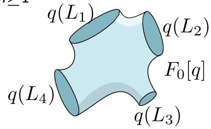

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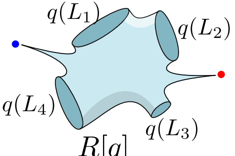

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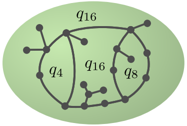

$G_0(q)$

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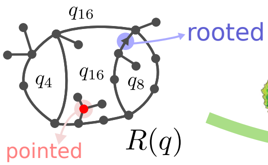

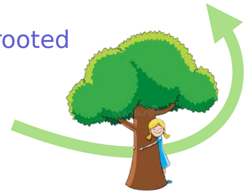

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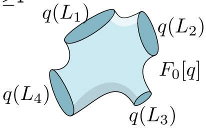

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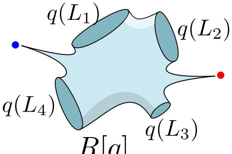
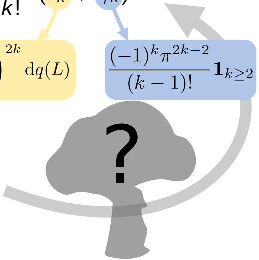
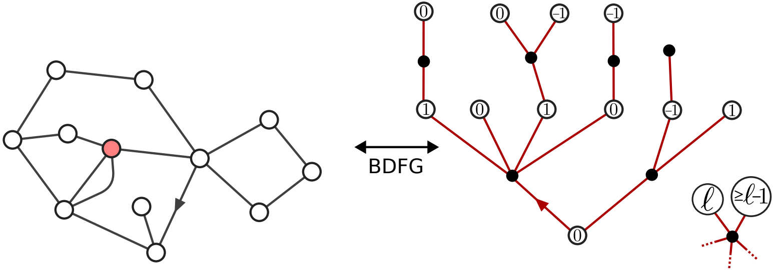




Bouttier–Di Francesco–Guitter bijection [BDFG, '04]

$$\left\{\begin{array}{c}\text{rooted bipartite planar maps}\\ \text{with marked vertex (“origin”)}\end{array}\right\} \overset{\text{2-to-1}}{\longleftrightarrow} \left\{\begin{array}{c}\text{mobiles (bicolored plane trees}\\ \text{with labeled white vertices)}\end{array}\right\}$$



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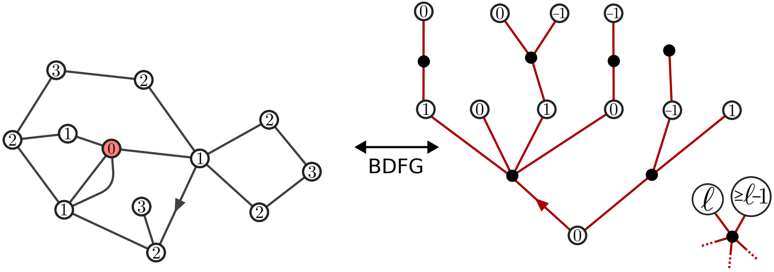

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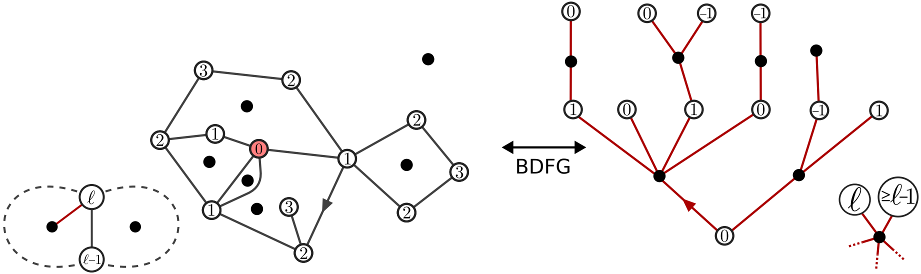

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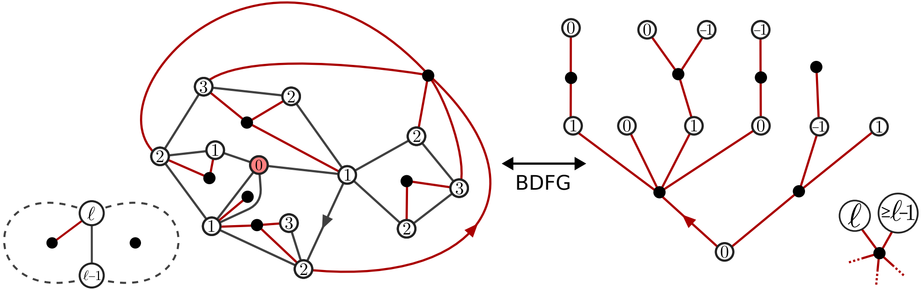

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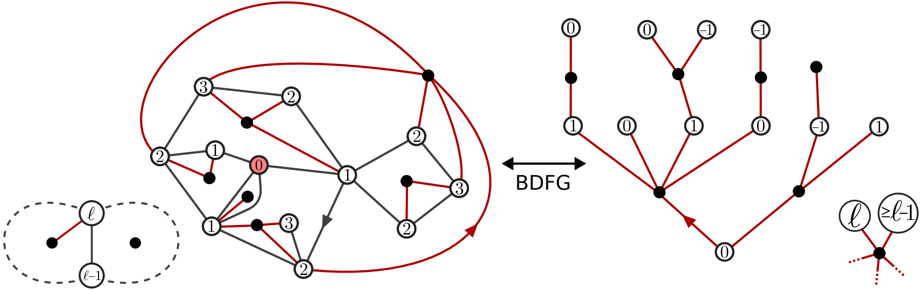

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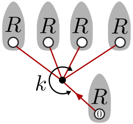
$$R = ⓪ + \sum_{k=1}^{\infty} q_{2k} \sum_{\text{labels}}$$  $$= 1 + \sum_{k=1}^{\infty} q_{2k} \binom{2k-1}{k} R^k,$$

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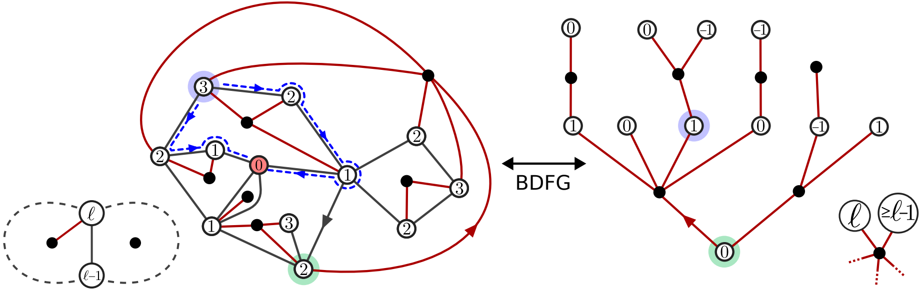

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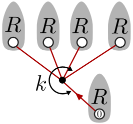

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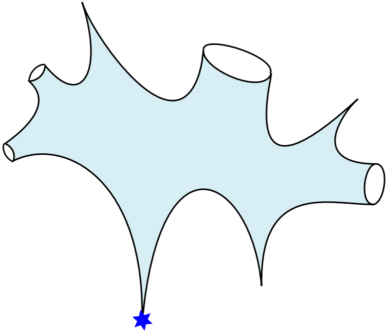
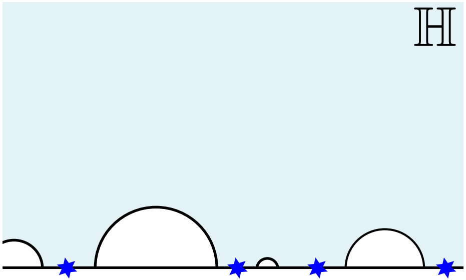
- Vertex with k (left-most) geodesics of length $r > 0$ to origin $\longleftrightarrow$ White vertex of degree k and label $r - r_{\mathrm{root}}$.

Tree in a hyperbolic surface?





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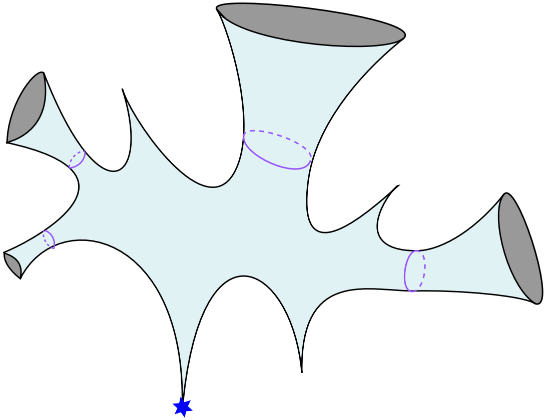

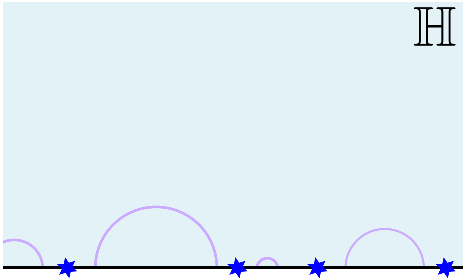

- Extend boundaries with hyperbolic cylinders.

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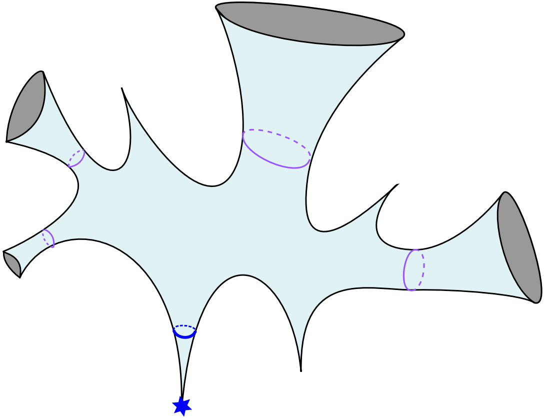

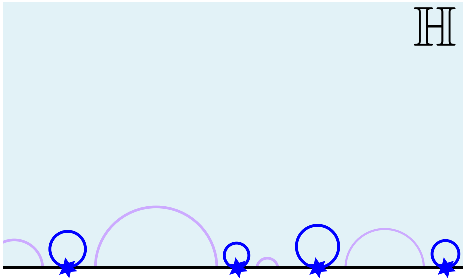

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  [Bowditch, Epstein, '88]

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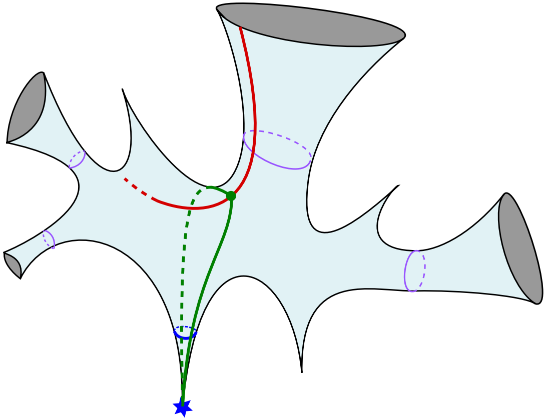

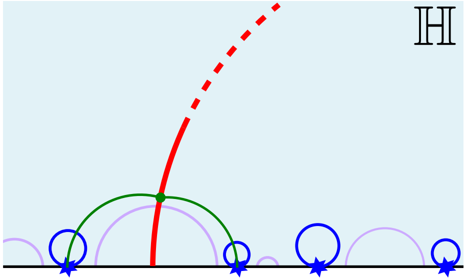

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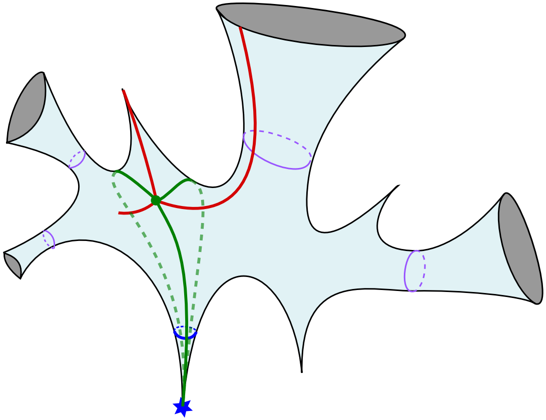

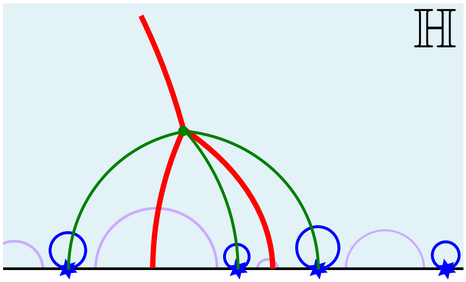

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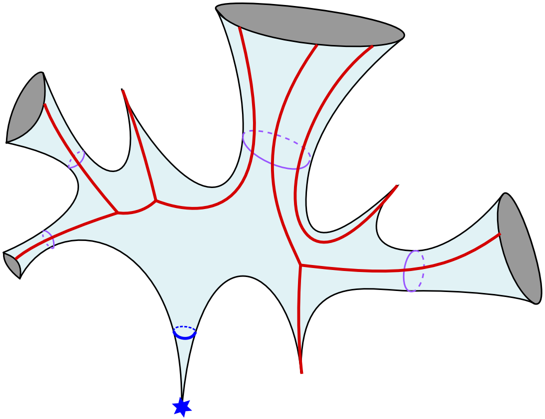

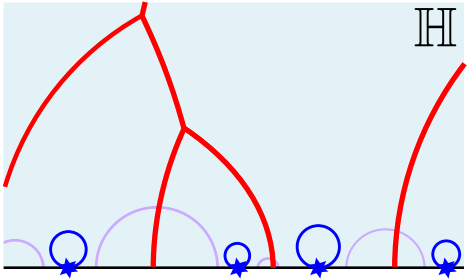

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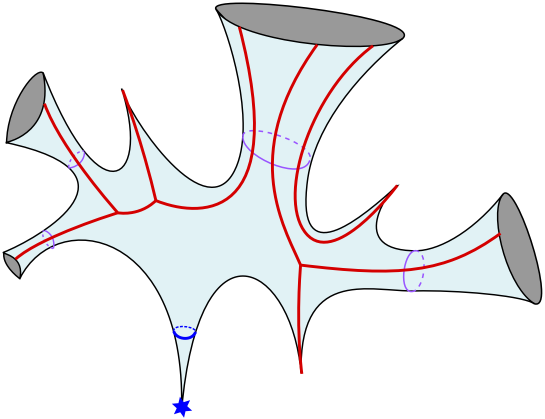

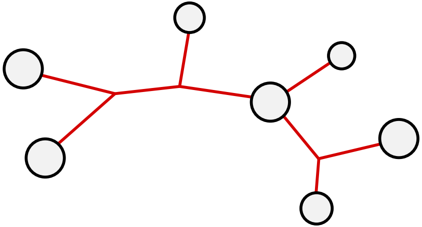

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- Result is a plane tree: boundary/cusp $\longleftrightarrow$ white vertex.

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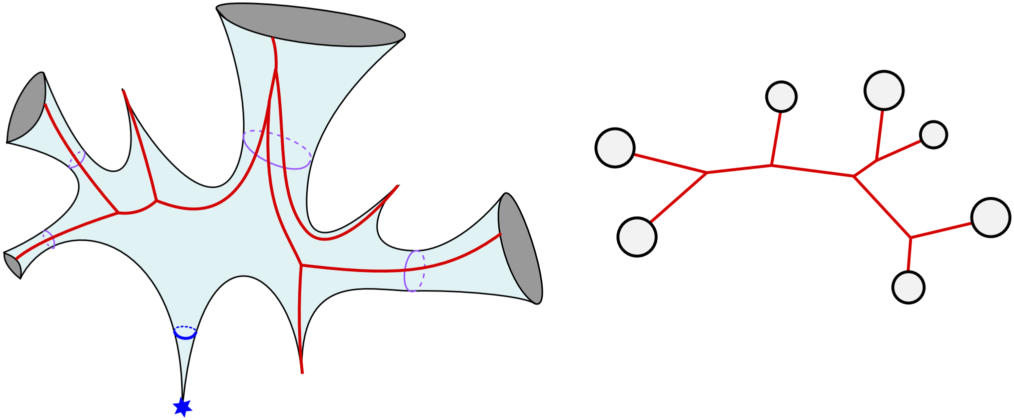

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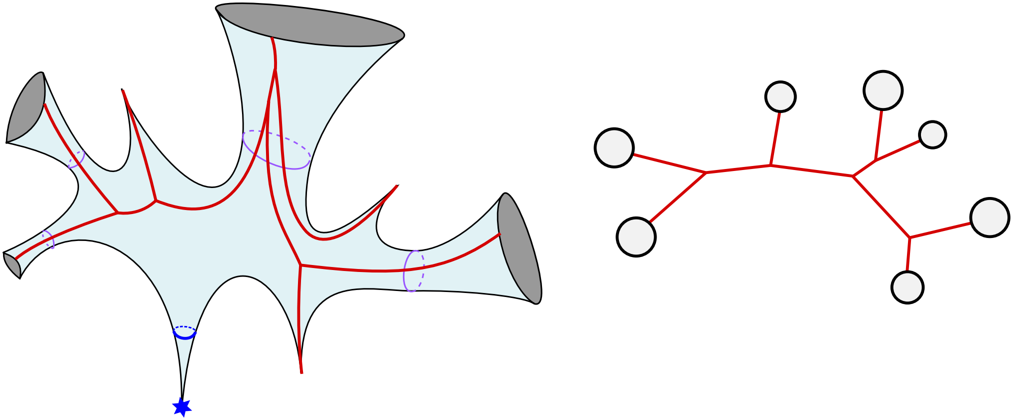

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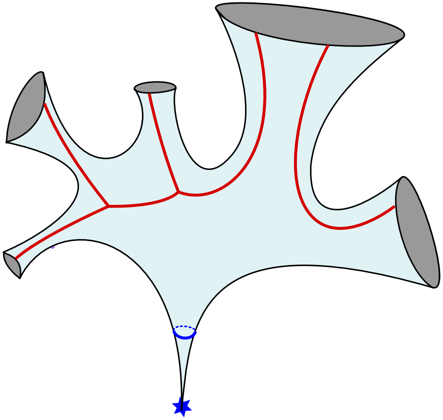
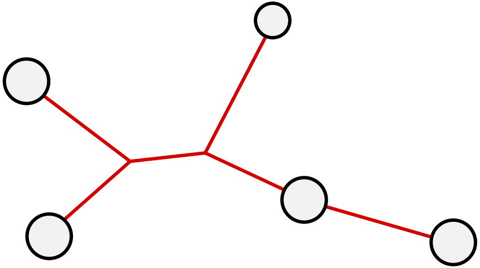
- Note: spine edges can meet in cylinders!
- Can we label the tree to make a bijection?

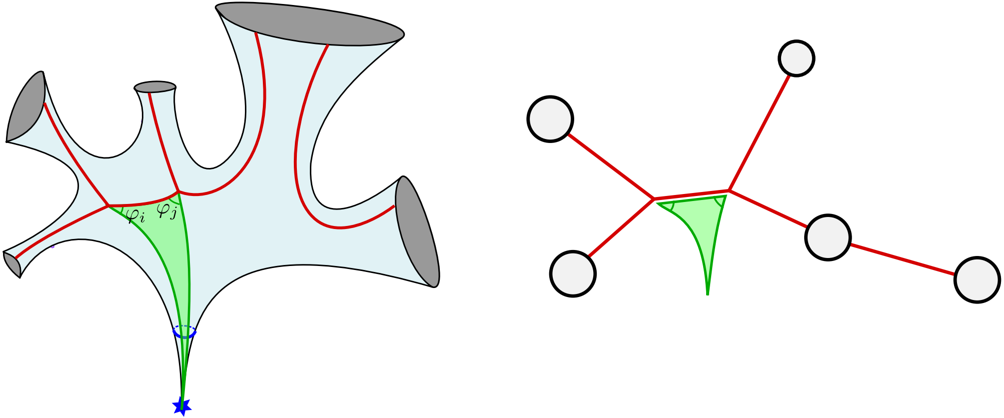
Labels: angles on half edges





- The surface is canonically triangulated by

Labels: angles on half edges



- The surface is canonically triangulated by
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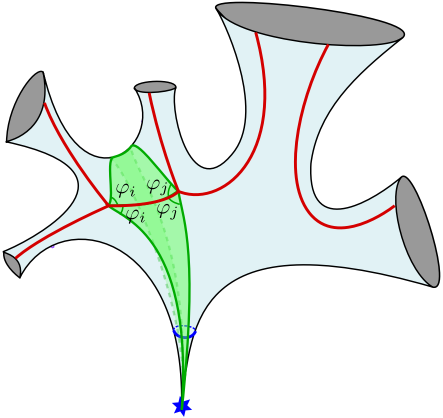
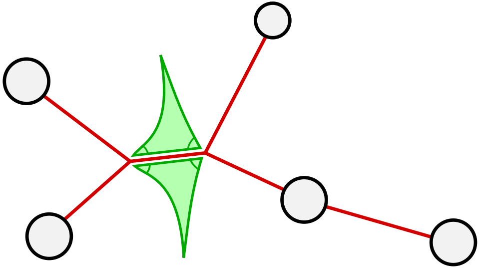
Labels: angles on half edges





- The surface is canonically triangulated by
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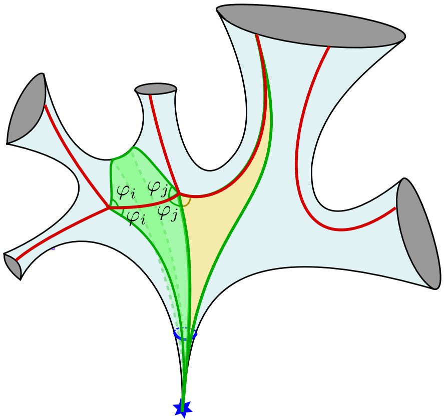

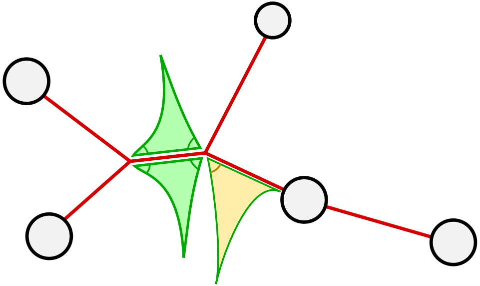

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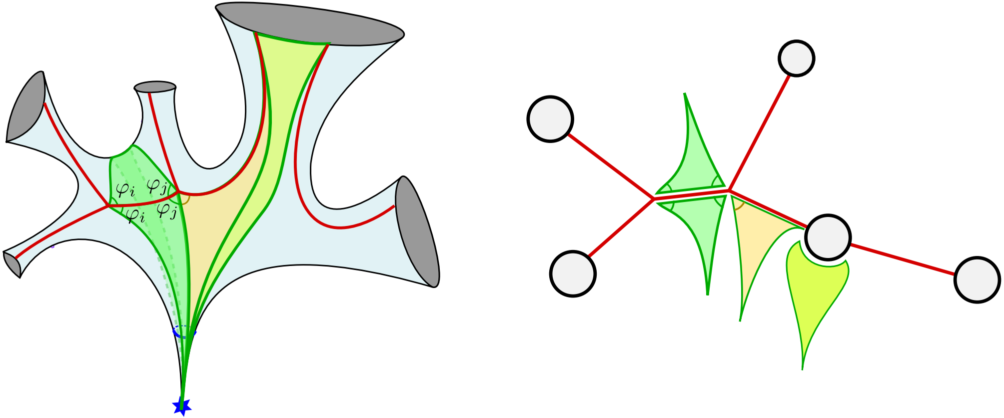
Labels: angles on half edges



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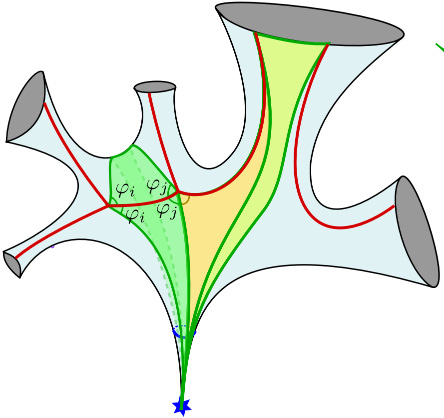

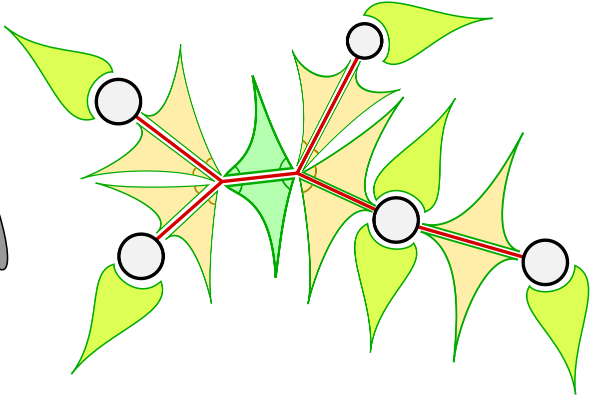

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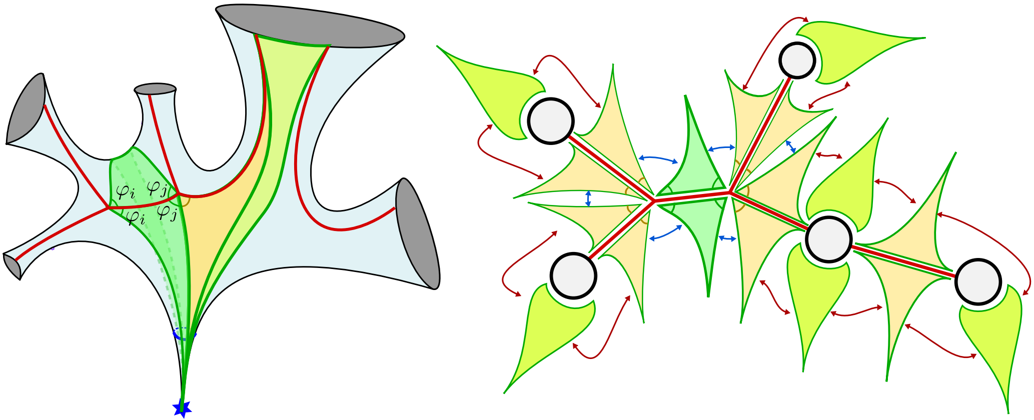

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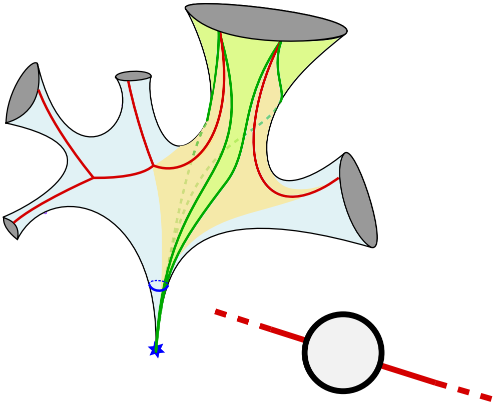
  - for each corner of white vertex: an ideal wedge.
- Gluing of triangles is unique, except for bi-infinite sides: need extra parameters for injectivity.

Labels: geometry around boundary



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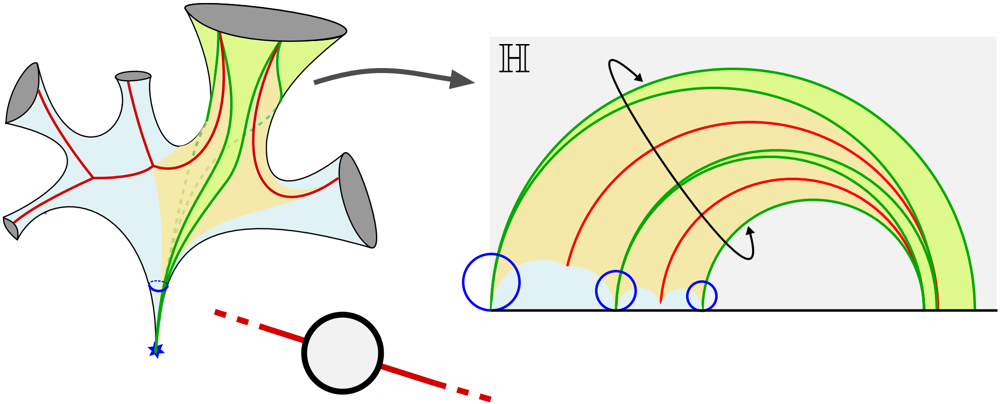
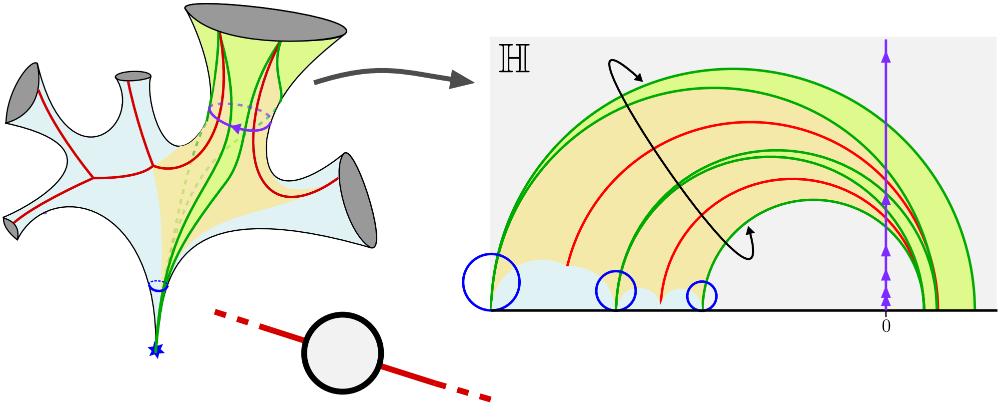


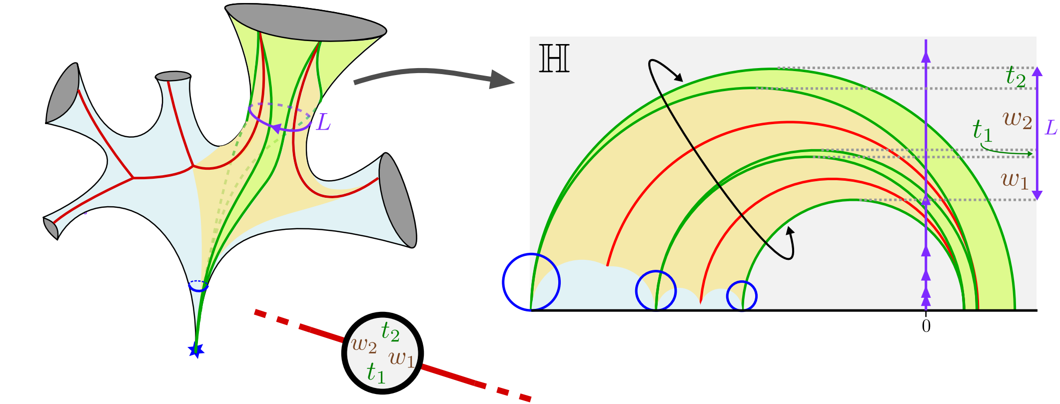
- Consider boundary region in $\mathbb{H}$

Labels: geometry around boundary



- Consider boundary region in $\mathbb{H}$, unique up to scaling if geodesic aligned with imaginary axis.

Labels: geometry around boundary



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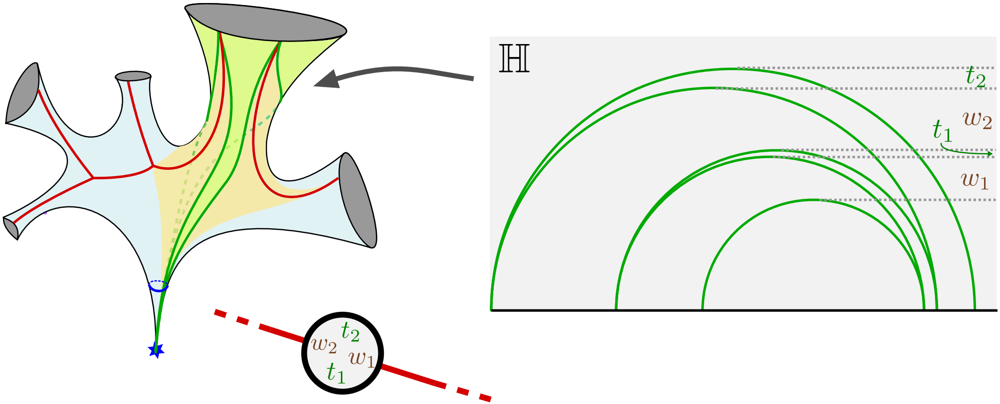

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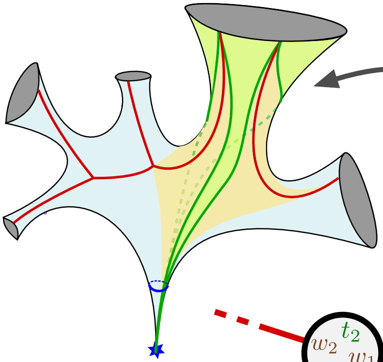

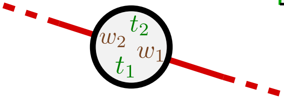

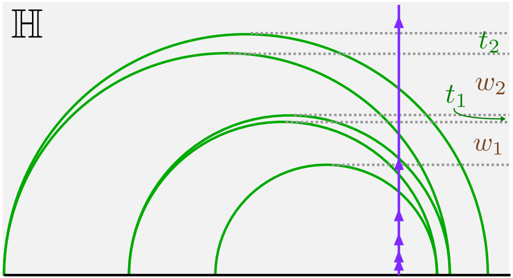

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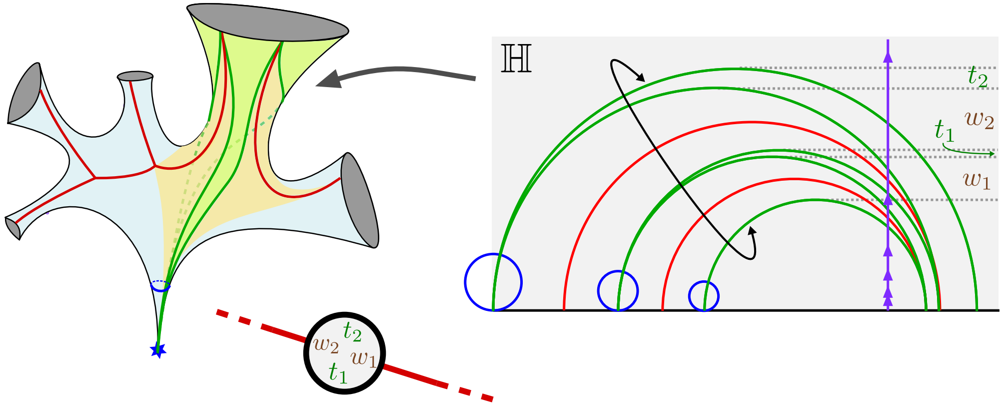

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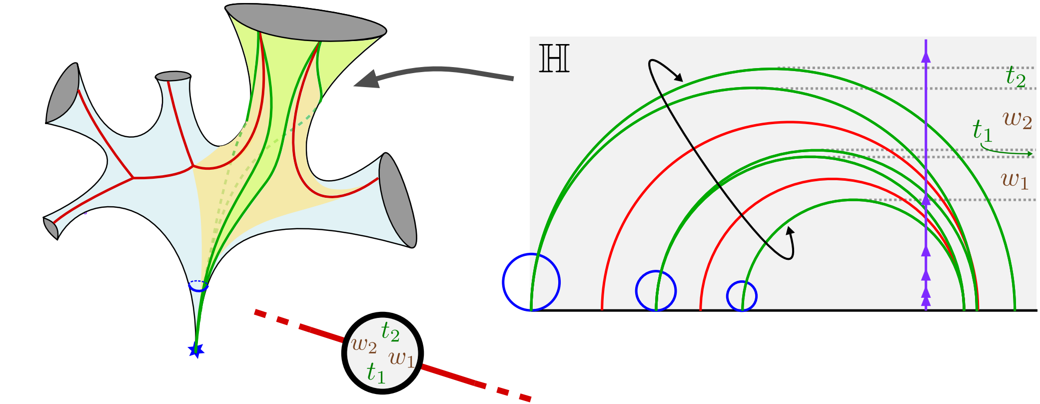

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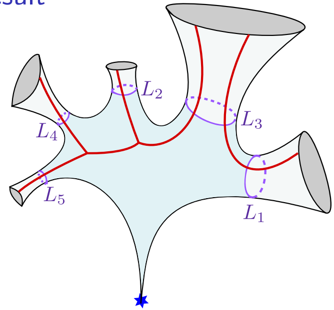
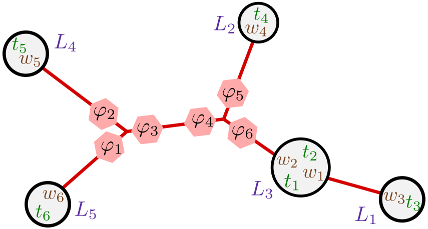
- Label vertex of degree k corresponding to boundary of length L by $\{(w_i, t_i)_{i=1}^k : \sum_{i=1}^k w_i = \sum_{i=1}^k t_i = \frac{L}{2}\}$.

Bijective result





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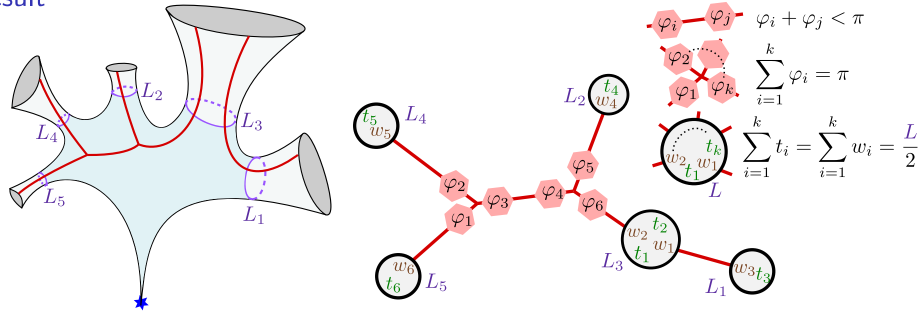

- For tree $\mathfrak{t}$ with n white vertices ($\deg \geq 1$) and red vertices ($\deg \geq 3$),

$$\mathcal{A}_{\mathfrak{t}}(L_1, \ldots, L_n) = \{(\phi_i, t_i, w_i) : \phi_i > 0, t_i \geq 0, w_i > 0, \text{constraints above}\}.$$

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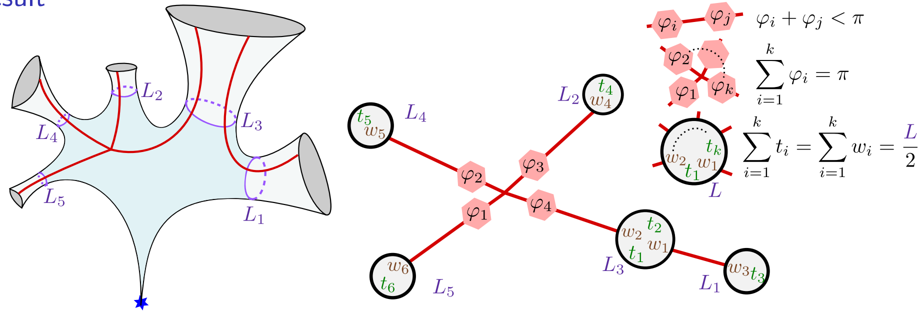

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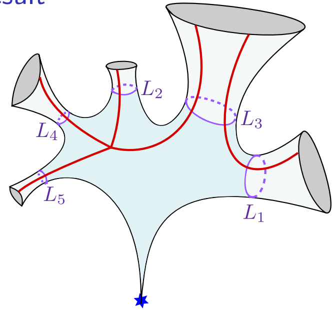

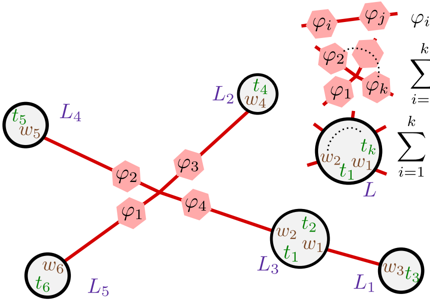

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Theorem (TB, Meeusen, Zonneveld, '23+)

This determines a bijection

$$\Phi_n : \mathcal{M}_{0,n+1}(0, L_1, \ldots, L_n) \longleftrightarrow \bigsqcup_{\mathfrak{t}} \mathcal{A}_{\mathfrak{t}}(L_1, \ldots, L_n).$$

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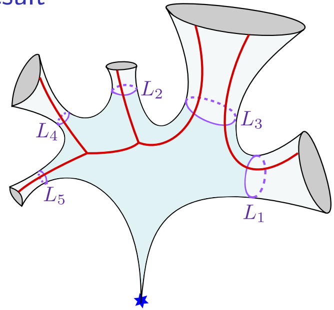

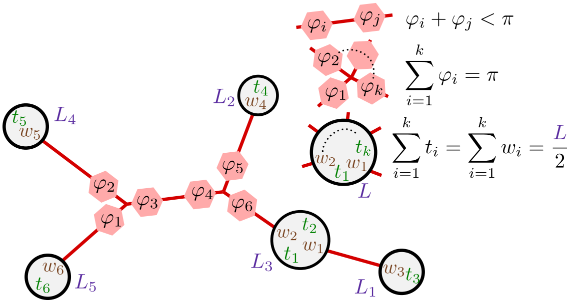

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Weil-Petersson measure

Theorem (TB, Meeusen, Zonneveld, '23+)

The push-forward of the WP volume is simply the Euclidean volume on the polytope $\mathcal{A}_{\mathfrak{t}} \subset \mathbb{R}^{2n-4}$,

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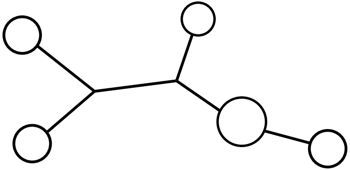

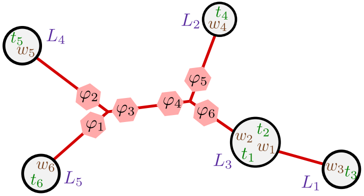

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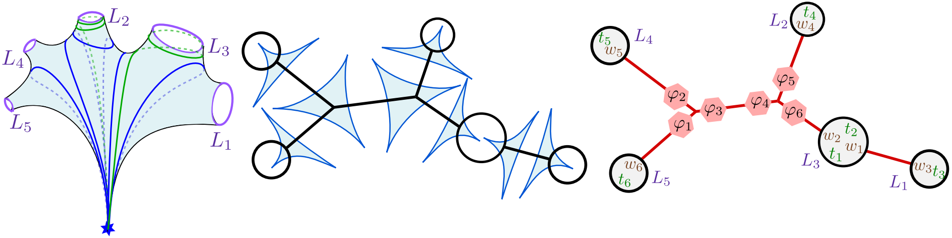

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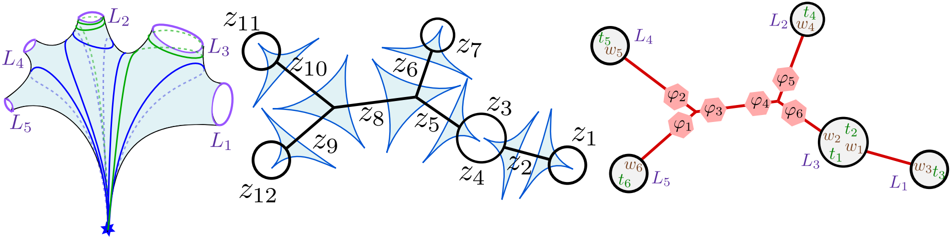

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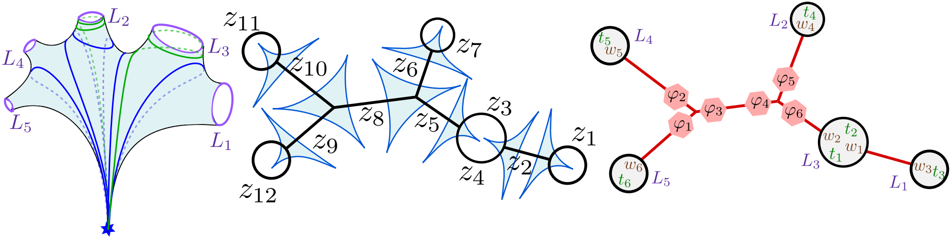

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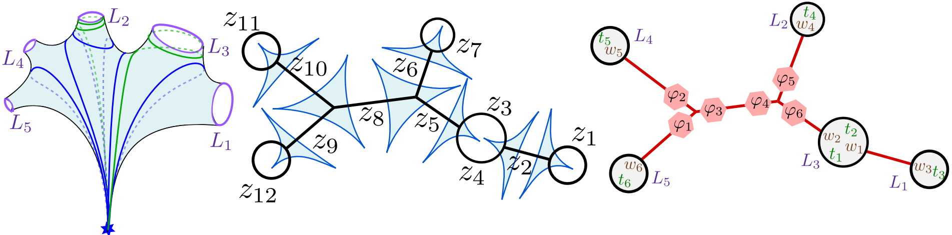

- Consequence: $\mathrm{Vol}(\mathcal{A}_{\mathfrak{t}}) \propto \prod_{\circ} L_{\circ}^{2\deg \circ - 2} \prod_{\bullet} \pi^2 \quad \Longrightarrow \quad V_{0,n} = \sum_{\mathfrak{t}} \mathrm{Vol}(\mathcal{A}_{\mathfrak{t}}) =$ polynomial in π^2, L_i^2.

WP volume generating function

- Why does $R = \sum_{n\geq 1} \frac{1}{n!} \int_0^\infty \mathrm{d}q(L_1)\cdots \mathrm{d}q(L_n)\, V^{\mathrm{WP}}_{0,n+2}(0,0,\mathbf{L})$ satisfy

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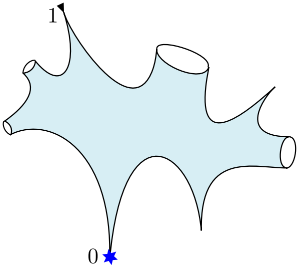

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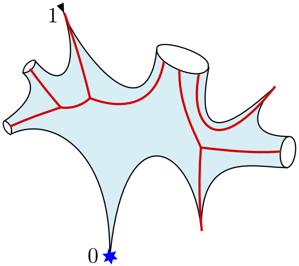

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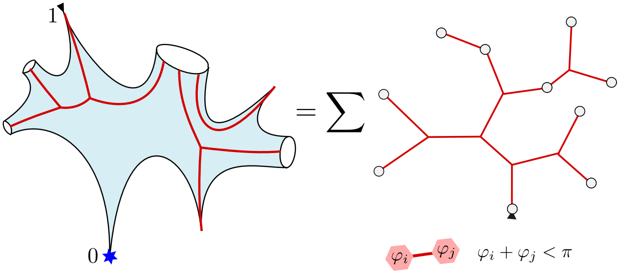

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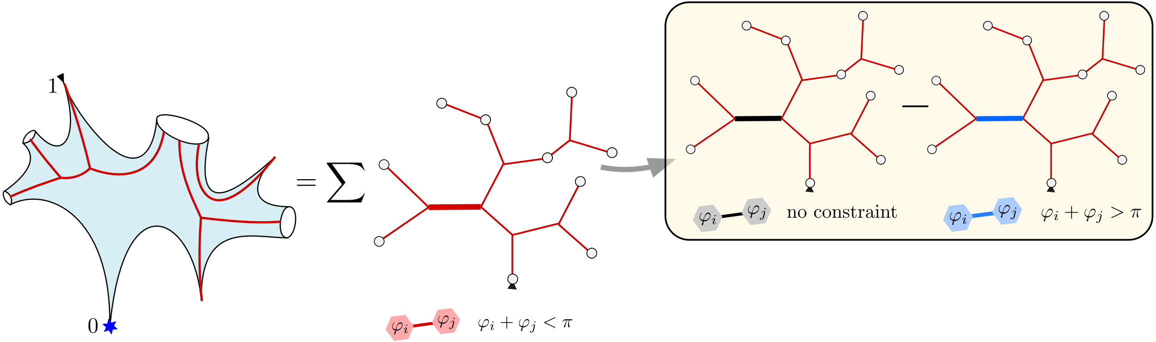

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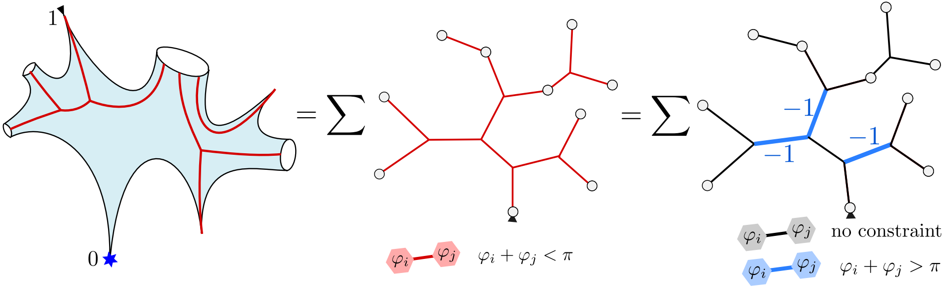

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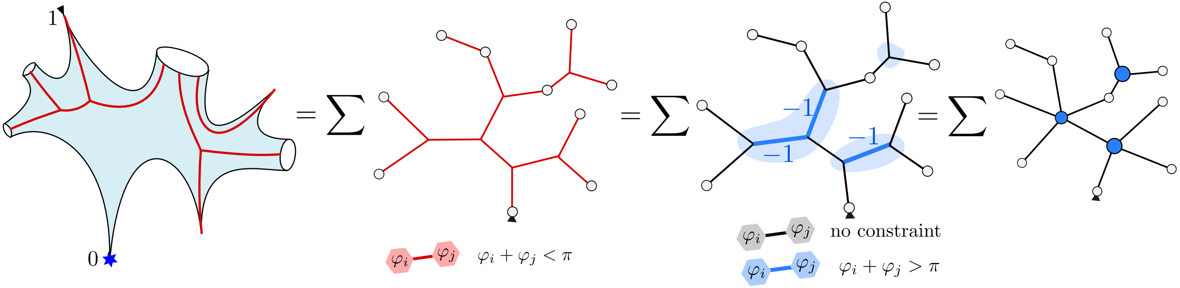

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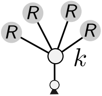
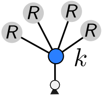
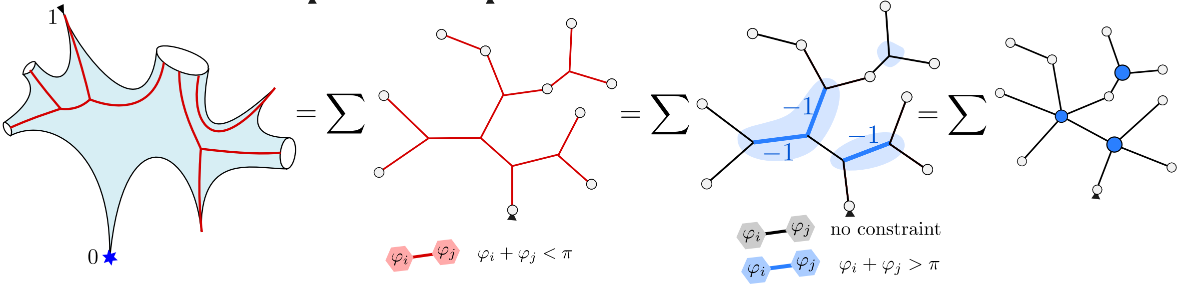
$= \sum_{k=0}^{\infty}$ 

$+ \sum_{k=2}^{\infty}$ 



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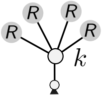
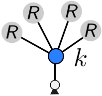
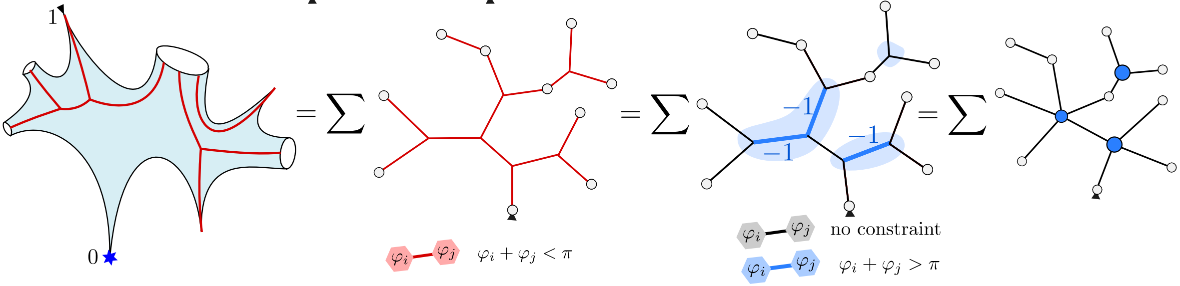
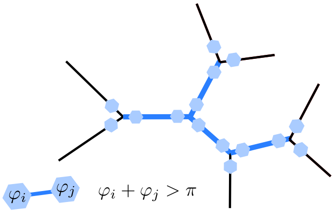
$= \sum_{k=0}^{\infty}$  $+ \sum_{k=2}^{\infty}$ 



WP volume of blue vertices

- The reversed condition $\varphi_i + \varphi_j > \pi$ is simpler, because WP volume is independent of tree structure:



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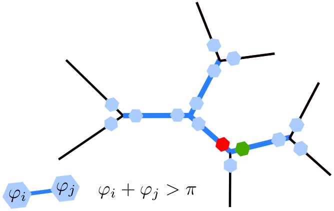

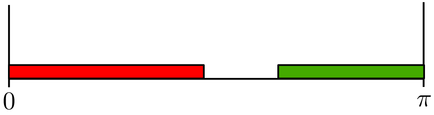

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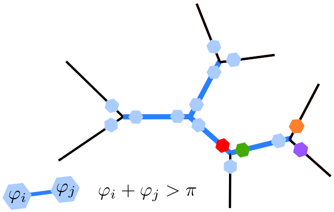

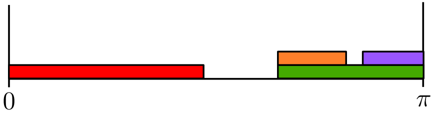

$$(-1)^k \sum_{\text{binary trees}} \int \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{2k-2} \;=$$

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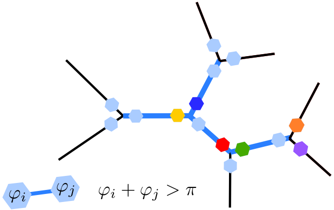

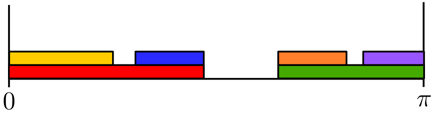

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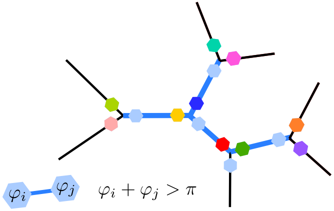

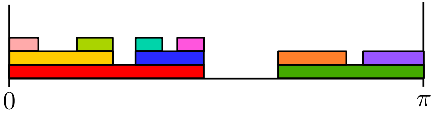

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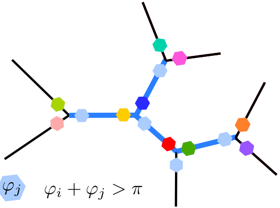

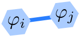

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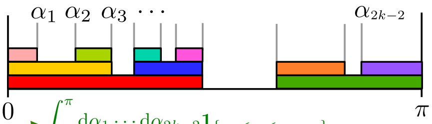

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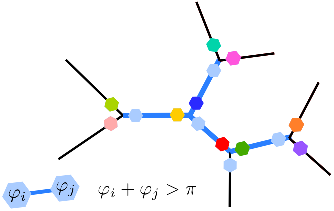

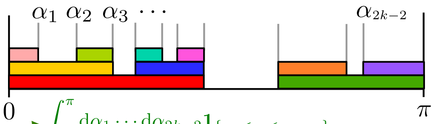

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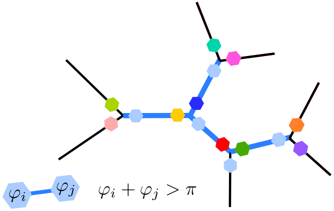

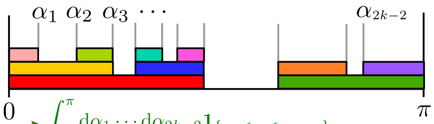

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- Find a full combinatorial explanation for the string equation and disk generating function

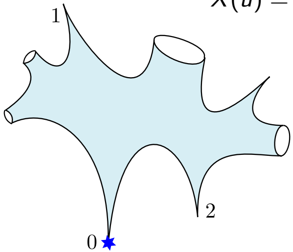
$$W(L) = \sum_{n \geq 1} \frac{1}{n!} \int_0^\infty \left[\prod_{i=1}^n \mathrm{d}q(L_i) \right] V^{\mathrm{WP}}_{0,n+2}(0, L, L_1, L_2, \ldots)$$

$$= \sum_{k=1}^\infty 2^{k-1} \frac{1}{k!(k-1)!} \left(\frac{L}{2}\right)^{2k-2} R^k = \frac{\sqrt{2R}}{L} I_1(L\sqrt{2R}).$$

Not just volumes: geodesic distance control!

- Consider the distance-dependent generating function of triply-cusped surfaces

$$X(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty \left[\prod_{i=1}^{n} \mathrm{d}q(L_i)\right] \int_{\mathcal{M}_{n+3}(0,0,0,\mathbf{L})} e^{2u(d_1-d_2)} \mathrm{d}\mu_{\mathrm{WP}}.$$



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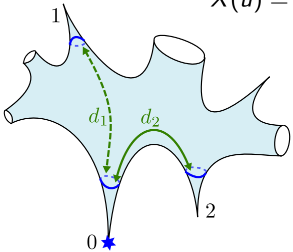

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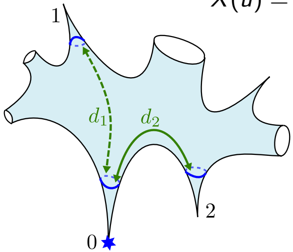

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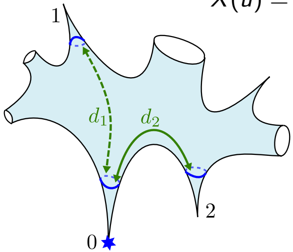

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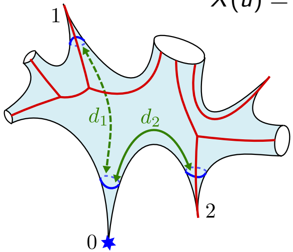

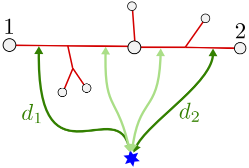

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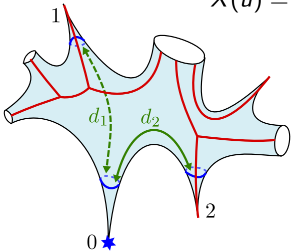

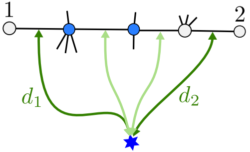

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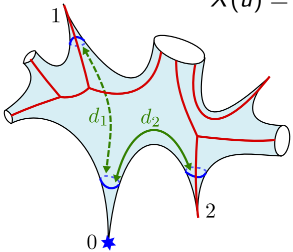

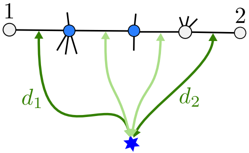

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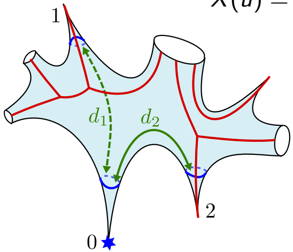

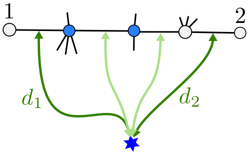

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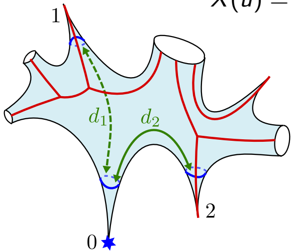

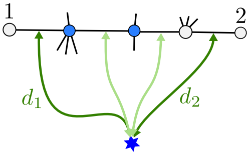

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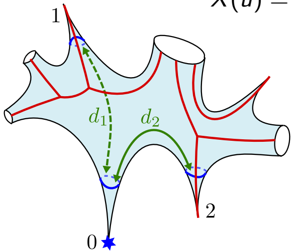

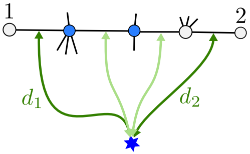

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- Random hyperbolic surface with n boundaries in same universality class as random triangulations of size n? Hausdorff dimension 4?

Control on hyperbolic distances

- In the case of only cusps, $q(L) = x\delta(L)$, this is indeed true:

Theorem (TB, Curien, '22+)

If $S_n \in \mathcal{M}_{0,n}(0)$ is sampled with probability density $\mu_{WP}/V_{0,n}(0)$, then we have the convergence in distribution of the random metric space in the Gromov–Prokhorov topology

$$\left(S_n, \frac{d_{\mathrm{hyp}}}{c\, n^{-1/4}}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \textit{Brownian sphere}, \quad c = 2.339\ldots.$$

- Same limit as uniform planar triangulations/quadrangulations! [Le Gall, '10][Miermont, '10]

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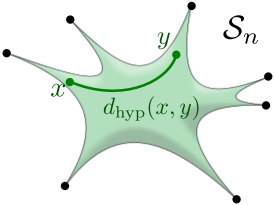

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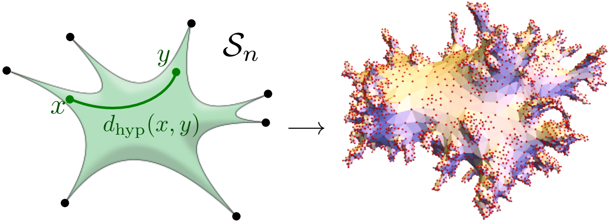

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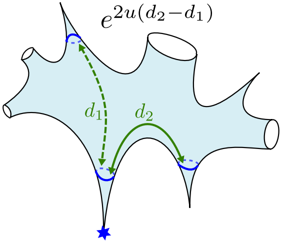
Topological recursion



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Theorem (TB, Zonneveld, '23+)

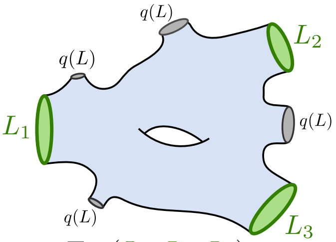
The invariants $\omega_{g,n}(\mathbf{z})$ of the curve $(x(u) = u^2, y(u))$ with initial condition $\omega_{0,2}(z) = (z_1 - z_2)^2$ and topological recursion

$$\omega_{g,n}(\mathbf{z}) = \operatorname*{Res}_{u\to 0} \frac{1}{(z_1^2 - u^2)y(u)} \left[\omega_{g-1,n+1}(u, -u, \mathbf{z}_{\widehat{\{1\}}}) + \sum_{\substack{g_1+g_2=g \\ I \amalg J = \{2,\dots,n\}}} \omega_{g_1,n_1}(u, \mathbf{z}_I)\omega_{g_2,n_2}(-u, \mathbf{z}_J) \right]$$

give the Laplace transforms of "Tight Weil–Petersson volumes" $T_{g,n}(L_1, \dots, L_n)$,

$$\omega_{g,n}(\mathbf{z}) = \int_0^\infty \mathrm{d}L_1 L_1 e^{-z_1 L_1} \cdots \int_0^\infty \mathrm{d}L_n L_n e^{-z_n L_n} T_{g,n}(L_1, \dots, L_n).$$

Tight Weil-Petersson volumes



$T_{1,3}(L_1, L_2, L_3)$

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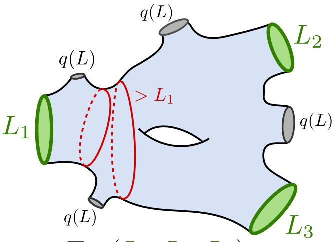
Tight Weil-Petersson volumes

$$T_{g,n}(\mathbf{L}) = \sum_{p=0}^{\infty} \frac{1}{p!} \int \mathrm{d}q(L_{n+1}) \int \mathrm{d}q(L_{n+p}) \int_{\mathcal{M}_{g,n+p}(\mathbf{L},\mathbf{L})} \mathrm{d}\mu_{\mathrm{WP}} \mathbf{1}_{\{\mathrm{tight}\}}$$



$T_{1,3}(L_1, L_2, L_3)$