Probability on trees and planar graphs, Banff, Canada, 15-09-2014

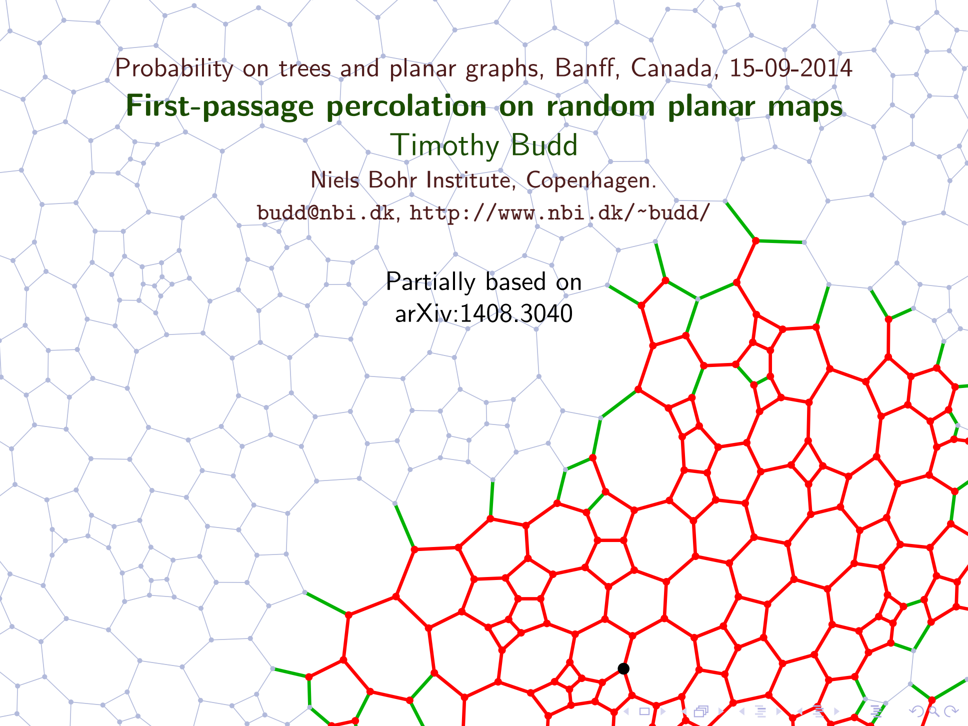
First-passage percolation on random planar maps

Timothy Budd

Niels Bohr Institute, Copenhagen.

budd@nbi.dk, http://www.nbi.dk/~budd/

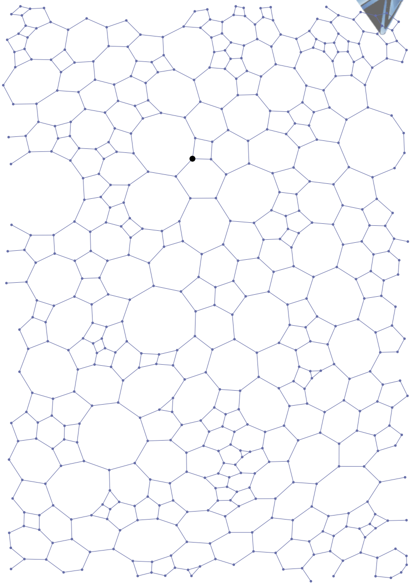
Partially based on arXiv:1408.3040



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

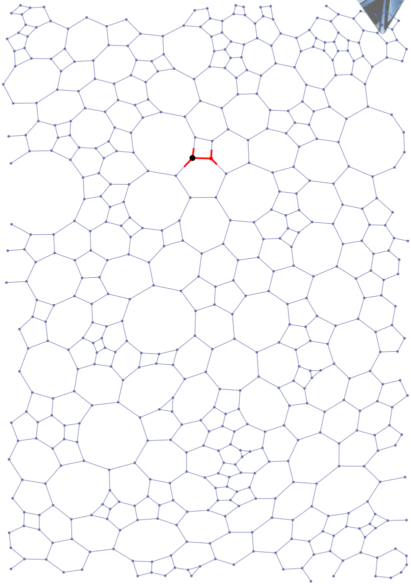
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

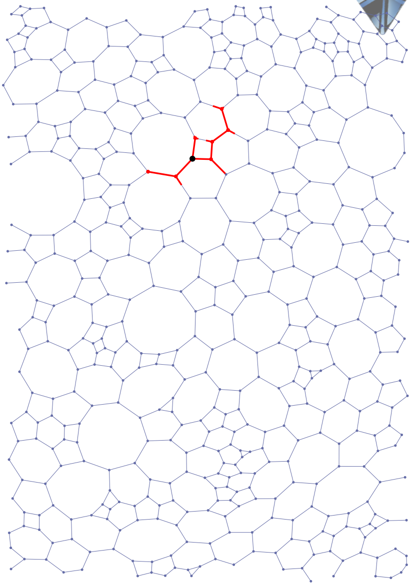
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

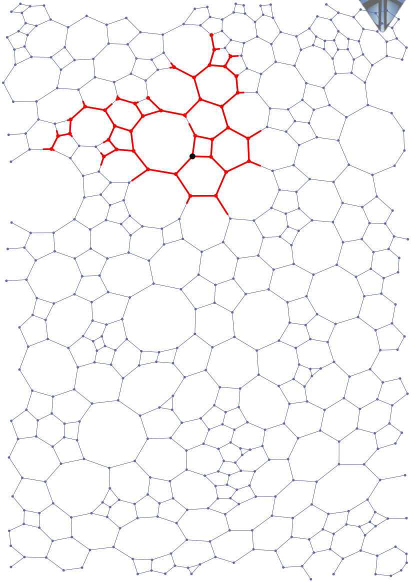
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

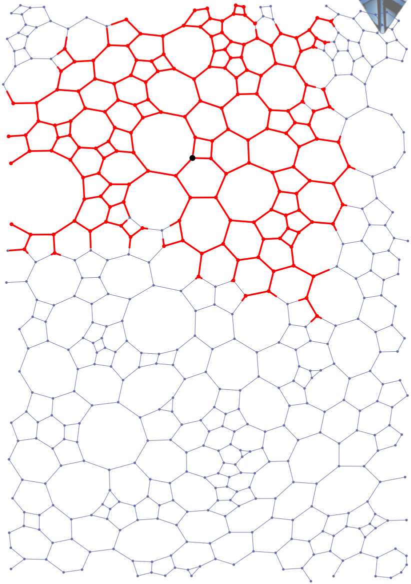
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

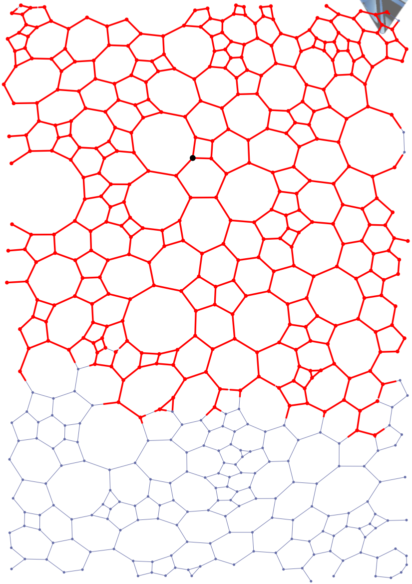
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

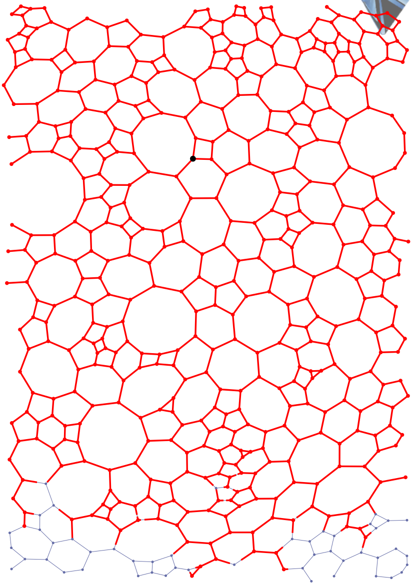
$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$



First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

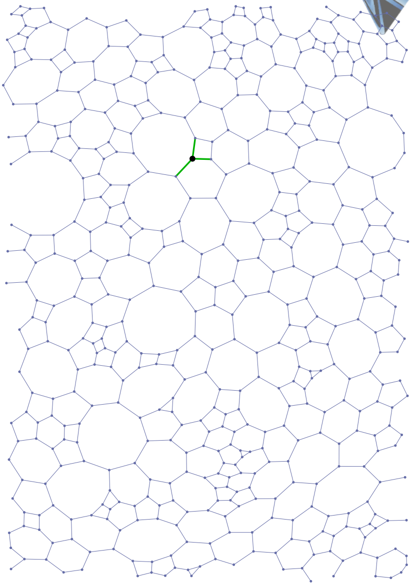


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

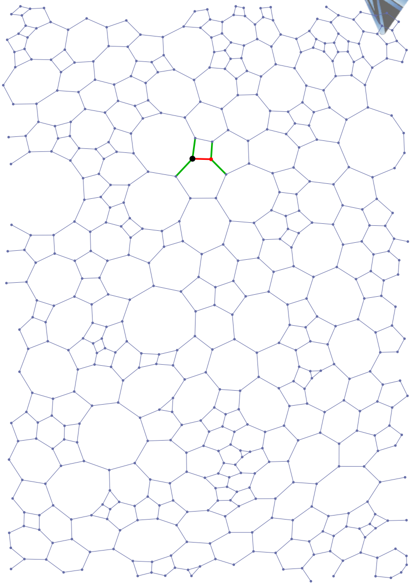


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma: v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

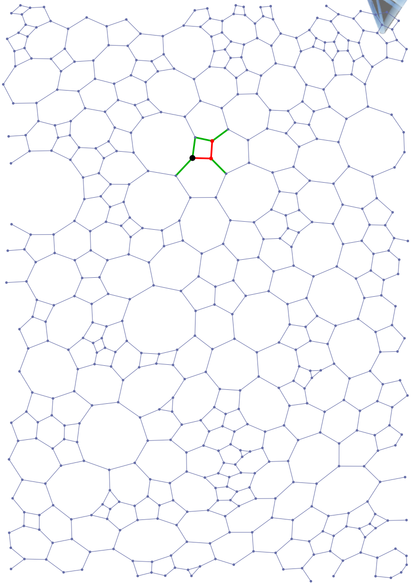


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

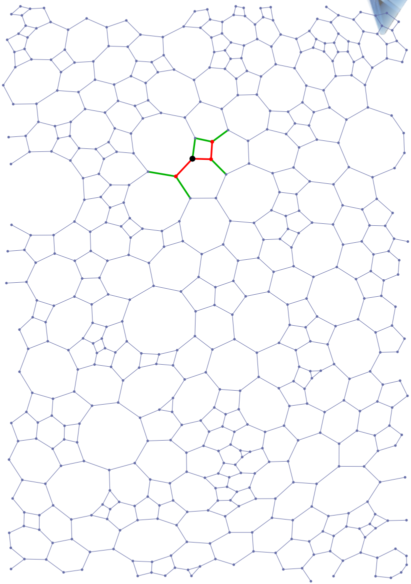


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma: v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

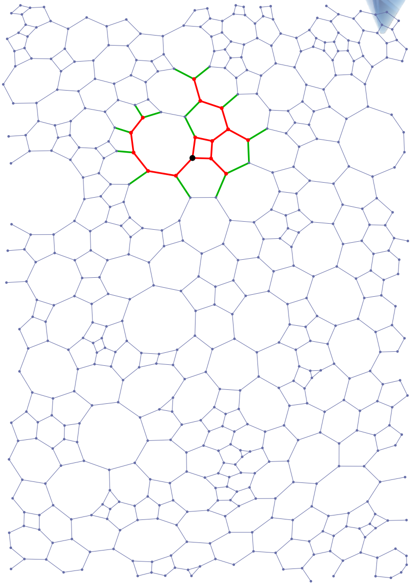


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

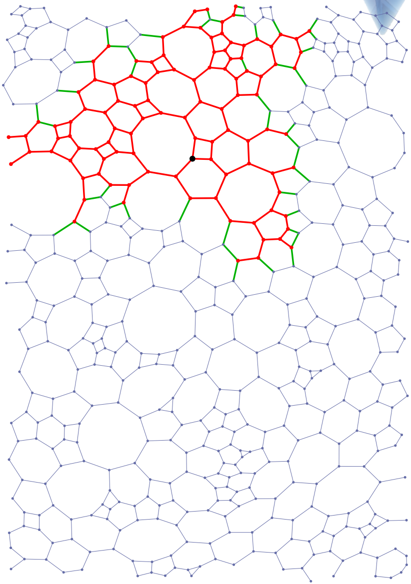


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

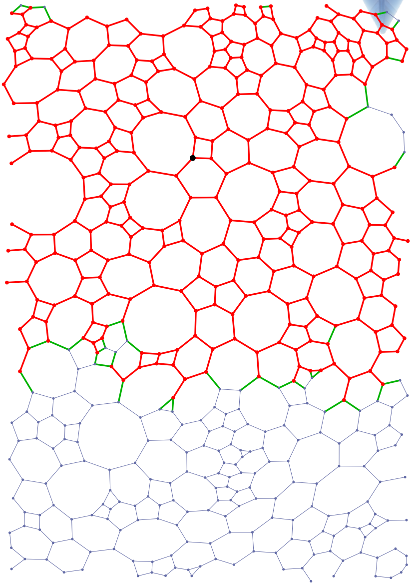


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.

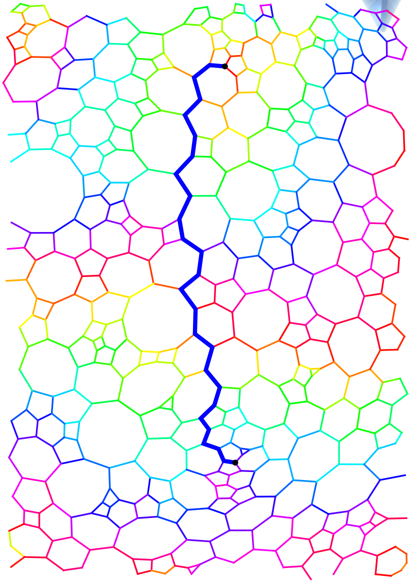


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.
- How does T compare to the graph distance d (asymptotically)?

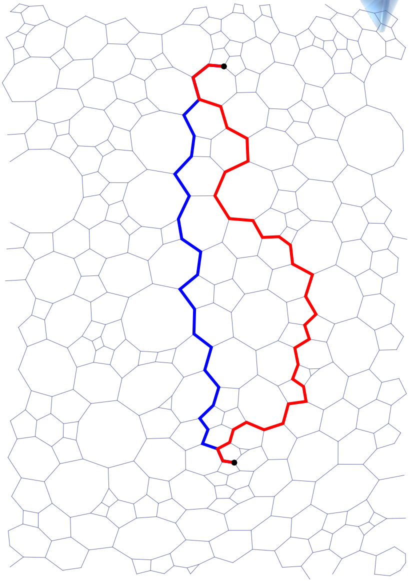


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.
- How does T compare to the graph distance d (asymptotically)?
- *Hop count* h is the number of edges in shortest time path.

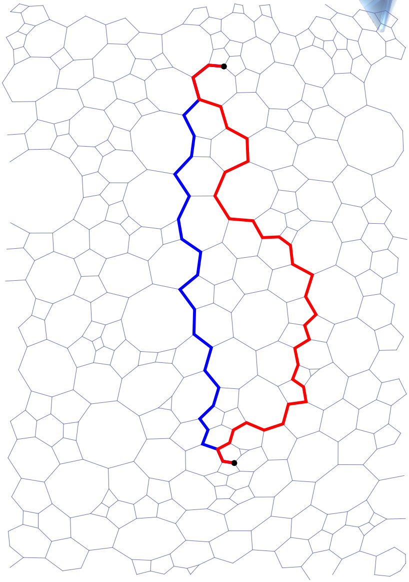


First-passage percolation on a graph

- Random i.i.d. edge weights $w(e)$ with mean 1.
- Passage time $v_1 \to v_2$

$$T = \min_{\gamma : v_1 \to v_2} \sum_{e \in \gamma} w(e)$$

- Exponential distribution: equivalent to an Eden model.
- How does T compare to the graph distance d (asymptotically)?
- *Hop count* h is the number of edges in shortest time path.
- Asymptotically $T < d < h$.



Time constants

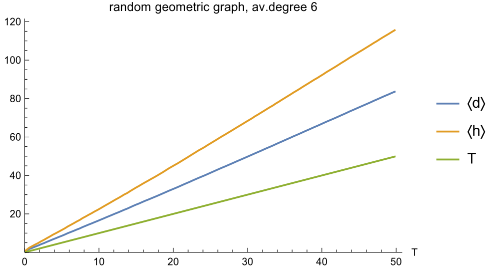
- For many plane graphs ($\mathbb{Z}^2$, Poisson-Voronoi, Random Geometric) one can show that the *time constants* $\lim_{d\to\infty} T/d$ and $\lim_{d\to\infty} h/d$ exist almost surely using Kingman's subadditive ergodic theorem.

Time constants

- For many plane graphs ($\mathbb{Z}^2$, Poisson-Voronoi, Random Geometric) one can show that the *time constants* $\lim_{d\to\infty} T/d$ and $\lim_{d\to\infty} h/d$ exist almost surely using Kingman's subadditive ergodic theorem.
- However, very few exact *time constants* known (ladder graph, . . . ?).

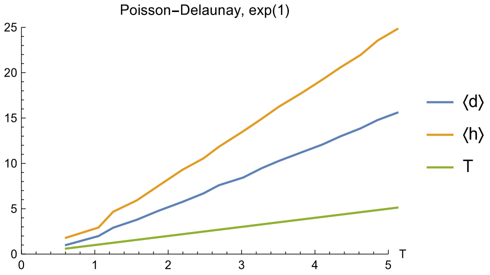
Time constants

- For many plane graphs ($\mathbb{Z}^2$, Poisson-Voronoi, Random Geometric) one can show that the *time constants* $\lim_{d\to\infty} T/d$ and $\lim_{d\to\infty} h/d$ exist almost surely using Kingman's subadditive ergodic theorem.
- However, very few exact *time constants* known (ladder graph, . . . ?).
- Numerically, many models seem to have d right in the middle of the bounds ($T < d < h$). Explanation?



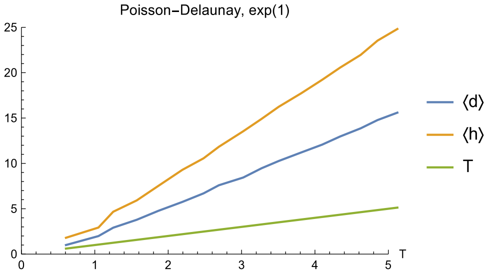
Time constants

- For many plane graphs ($\mathbb{Z}^2$, Poisson-Voronoi, Random Geometric) one can show that the *time constants* $\lim_{d\to\infty} T/d$ and $\lim_{d\to\infty} h/d$ exist almost surely using Kingman's subadditive ergodic theorem.
- However, very few exact *time constants* known (ladder graph,...?).
- Numerically, many models seem to have d right in the middle of the bounds ($T < d < h$). Explanation?



Time constants

- For many plane graphs ($\mathbb{Z}^2$, Poisson-Voronoi, Random Geometric) one can show that the *time constants* $\lim_{d\to\infty} T/d$ and $\lim_{d\to\infty} h/d$ exist almost surely using Kingman's subadditive ergodic theorem.
- However, very few exact *time constants* known (ladder graph,. . . ?).
- Numerically, many models seem to have d right in the middle of the bounds ($T < d < h$). Explanation?
- Let us look at random planar maps: assuming existence of the time constants we can compute them!

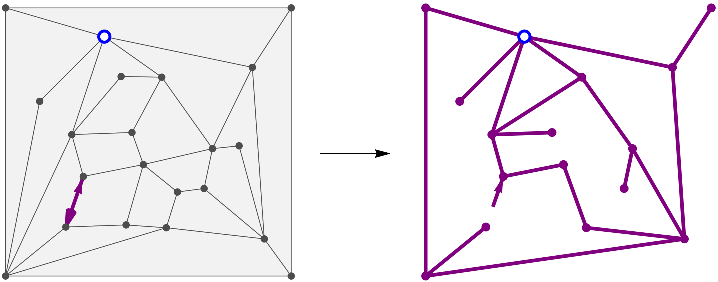


Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.

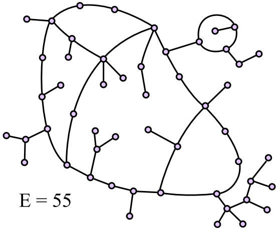
Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]



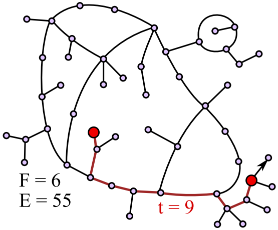
Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]
- Random planar map with E edges converges to Brownian map as $E \to \infty$. [Bettinelli,Jacob,Miermont,'13]



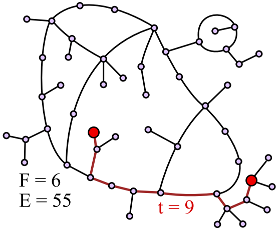
Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]
- Random planar map with E edges converges to Brownian map as $E \to \infty$. [Bettinelli,Jacob,Miermont,'13]
- Also allowed to derive various distance statistics, while controlling both the number E of edges and the number F of faces:
  - Two-point function (rooted) [Ambjorn, TB, '13]



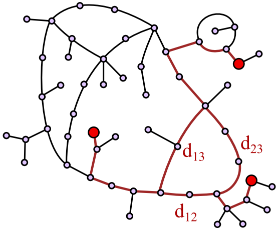
Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]
- Random planar map with E edges converges to Brownian map as $E \to \infty$. [Bettinelli,Jacob,Miermont,'13]
- Also allowed to derive various distance statistics, while controlling both the number E of edges and the number F of faces:
  - Two-point function (rooted) [Ambjorn, TB, '13]
  - Two-point function (unrooted) [Bouttier, Fusy, Guitter, '13]



Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]
- Random planar map with E edges converges to Brownian map as $E \to \infty$. [Bettinelli,Jacob,Miermont,'13]
- Also allowed to derive various distance statistics, while controlling both the number E of edges and the number F of faces:
  - Two-point function (rooted) [Ambjorn, TB, '13]
  - Two-point function (unrooted) [Bouttier, Fusy, Guitter, '13]
  - Three-point function [Fusy, Guitter, '14]

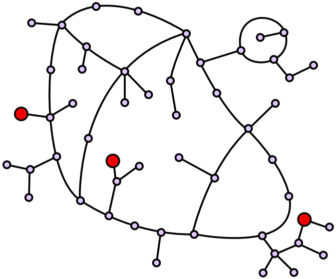


Multi-point functions of general planar maps

- FPP on a random cubic map with exponentially distributed edge weights shows up as a by-product of the study of distances on a random general planar map.
- Bijection between quadrangulations and planar maps preserving distance. [Miermont, '09] [Ambjorn,TB, '13]
- Random planar map with E edges converges to Brownian map as $E \to \infty$. [Bettinelli,Jacob,Miermont,'13]
- Also allowed to derive various distance statistics, while controlling both the number E of edges and the number F of faces:
  - Two-point function (rooted) [Ambjorn, TB, '13]
  - Two-point function (unrooted) [Bouttier, Fusy, Guitter, '13]
  - Three-point function [Fusy, Guitter, '14]
- Consider the limit $E \to \infty$ keeping F fixed. In terms of ...
  - ...quadrangulation: *Generalized Causal Dynamical Triangulations* (GCDT)
  - ...general planar maps: after "removal of trees" cubic maps with edges of random length.

From general maps to weighted cubic maps

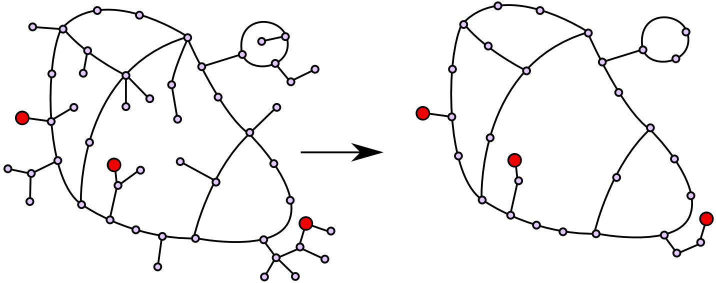
- Take a planar map with marked vertices and weight x per edge.



From general maps to weighted cubic maps

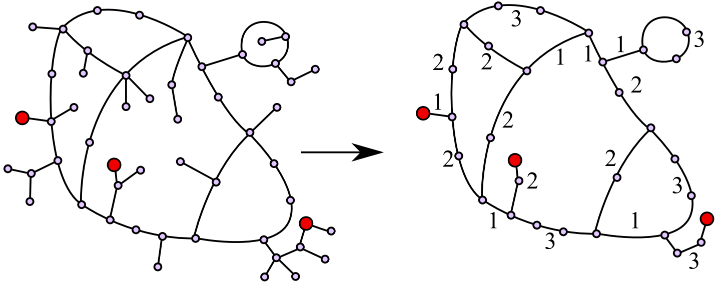
- Take a planar map with marked vertices and weight x per edge.
- Remove dangling edges

.



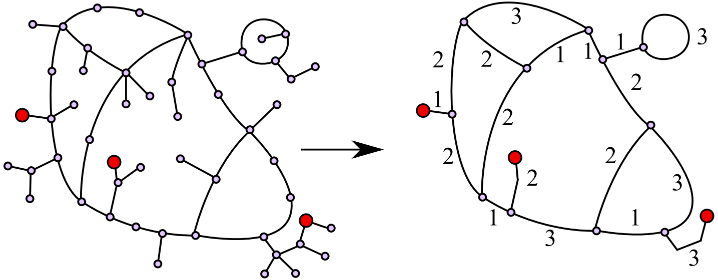
From general maps to weighted cubic maps

- Take a planar map with marked vertices and weight x per edge.
- Remove dangling edges and combine neighboring edges while keeping track of length.



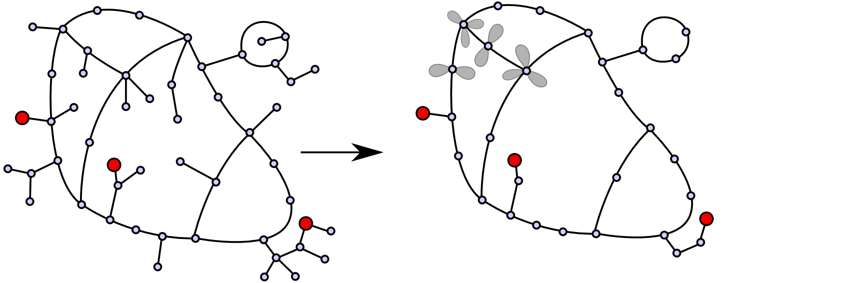
From general maps to weighted cubic maps

- Take a planar map with marked vertices and weight x per edge.
- Remove dangling edges and combine neighboring edges while keeping track of length.



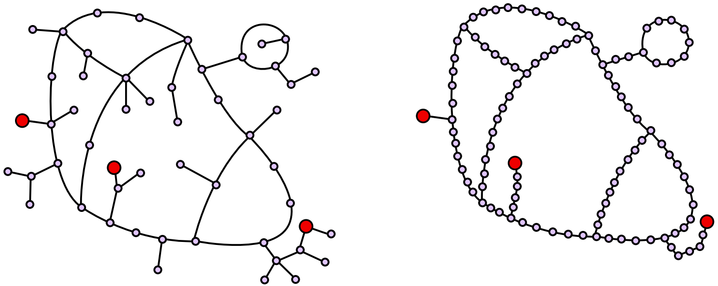
From general maps to weighted cubic maps

- Take a planar map with marked vertices and weight x per edge.
- Remove dangling edges and combine neighboring edges while keeping track of length.
- Each "subedge" comes with weight $x((1-\sqrt{1-4x})/x)^2$, hence length is geometrically distributed.



From general maps to weighted cubic maps

- Take a planar map with marked vertices and weight x per edge.
- Remove dangling edges and combine neighboring edges while keeping track of length.
- Each "subedge" comes with weight $x((1-\sqrt{1-4x})/x)^2$, hence length is geometrically distributed.
- Scaling limit $x \to 1/4$ and edge lengths $\ell(x) := \sqrt{4-16x}$: lengths are exponentially distributed with mean 1. Maximal number of edges preferred: almost surely cubic and univalent marked vertices.

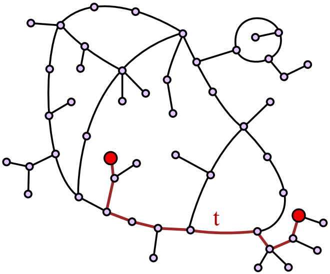


Two-point function

- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.



Two-point function

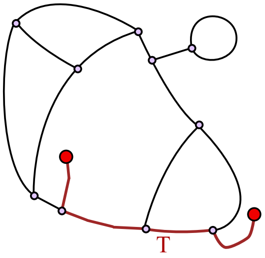
- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

  where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.
- Scaling $z = \ell(x)^3 g$, $t = T/\ell(x)$, $x \to 1/4$:

$$G^{(1,1)}_{g,2}(T) = \partial_T^3 \log(\Sigma \cosh \Sigma T + \alpha \sinh \Sigma T), \quad \Sigma := \sqrt{\tfrac{3}{2}\alpha^2 - \tfrac{1}{8}},$$

  and α is solution to $\alpha^3 - \alpha/4 + g = 0$.



Two-point function

- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

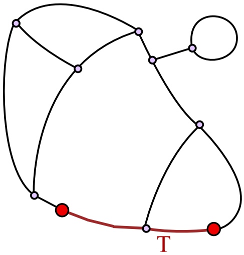
$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

  where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.
- Scaling $z = \ell(x)^3 g$, $t = T/\ell(x)$, $x \to 1/4$:

$$G^{(1,1)}_{g,2}(T) = \partial_T^3 \log(\Sigma \cosh \Sigma T + \alpha \sinh \Sigma T), \quad \Sigma := \sqrt{\tfrac{3}{2}\alpha^2 - \tfrac{1}{8}},$$

  and α is solution to $\alpha^3 - \alpha/4 + g = 0$.
- Bivalent marked vertices:

$$G^{(2,2)}_{g,2}(T) = \tfrac{1}{4}(1+\partial_T)^2 G^{(1,1)}_{g,2}(T)$$



Two-point function

- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\,\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

  where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.
- Scaling $z = \ell(x)^3 g$, $t = T/\ell(x)$, $x \to 1/4$:

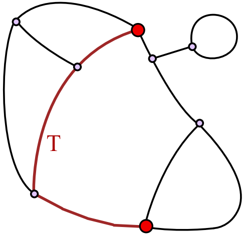
$$G^{(1,1)}_{g,2}(T) = \partial_T^3 \log(\Sigma \cosh \Sigma T + \alpha \sinh \Sigma T), \quad \Sigma := \sqrt{\tfrac{3}{2}\alpha^2 - \tfrac{1}{8}},$$

  and α is solution to $\alpha^3 - \alpha/4 + g = 0$.
- Bivalent marked vertices:

$$G^{(2,2)}_{g,2}(T) = \tfrac{1}{4}(1+\partial_T)^2 G^{(1,1)}_{g,2}(T)$$

- Completely cubic:

$$G^{(3,3)}_{g,2}(T) = \tfrac{1}{9}(2+\partial_T)^2 G^{(2,2)}_{g,2}(T)$$



Two-point function

- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.

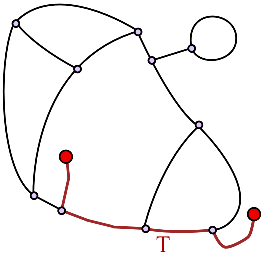
- Scaling $z = \ell(x)^3 g$, $t = T/\ell(x)$, $x \to 1/4$:

$$G^{(1,1)}_{g,2}(T) = \partial_T^3 \log(\Sigma\cosh\Sigma T + \alpha\sinh\Sigma T), \quad \Sigma := \sqrt{\tfrac{3}{2}\alpha^2 - \tfrac{1}{8}},$$

and α is solution to $\alpha^3 - \alpha/4 + g = 0$.

- Scaling limit: $g \to \frac{1}{12\sqrt{3}}$, i.e. $\alpha \to \frac{1}{\sqrt{12}}$ and $\Sigma \to 0$. Renormalizing $\mathcal{T} := \Sigma T$ gives

$$G^{(1,1)}_{g,2}(T)\Sigma^{-3} \to 2\frac{\cosh\mathcal{T}}{\sinh^3\mathcal{T}}.$$



Two-point function

- 2-point function for general planar maps with weight x per edge and z per face and distance t between marked vertices: [Bouttier et al, '13]

$$\mathcal{G}_{z,x}(t) = \log\left[\frac{(1-a\,\sigma^{t+1})^3}{(1-a\,\sigma^{t+2})^3}\frac{(1-a\,\sigma^{t+3})}{(1-a\,\sigma^{t})}\right], \qquad (1)$$

where $a = a(z,x)$ and $\sigma = \sigma(z,x)$ solution of algebraic equations.

- Scaling $z = \ell(x)^3 g$, $t = T/\ell(x)$, $x \to 1/4$:

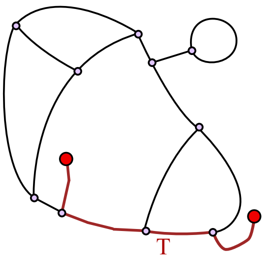
$$G^{(1,1)}_{g,2}(T) = \partial^3_T \log(\Sigma \cosh \Sigma T + \alpha \sinh \Sigma T), \quad \Sigma := \sqrt{\tfrac{3}{2}\alpha^2 - \tfrac{1}{8}},$$

and α is solution to $\alpha^3 - \alpha/4 + g = 0$.

- Scaling limit: $g \to \frac{1}{12\sqrt{3}}$, i.e. $\alpha \to \frac{1}{\sqrt{12}}$ and $\Sigma \to 0$.
Renormalizing $\mathcal{T} := \Sigma T$ gives

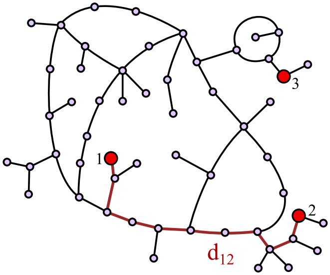
$$G^{(1,1)}_{g,2}(T)\Sigma^{-3} \to 2\frac{\cosh \mathcal{T}}{\sinh^3 \mathcal{T}}.$$

- Agrees with distance-profile of the Brownian map.



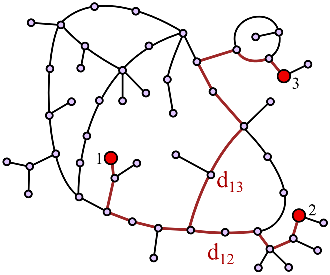
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.



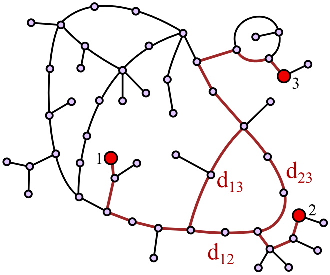
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.



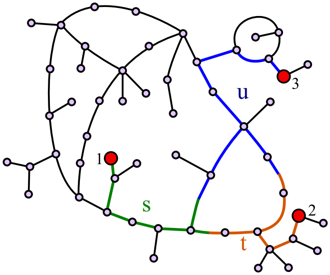
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.



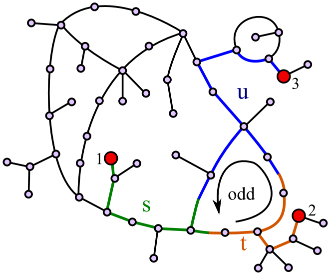
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.



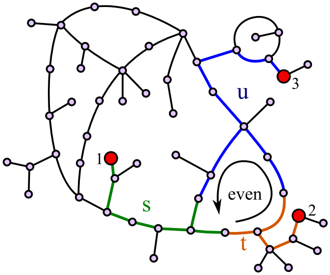
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.



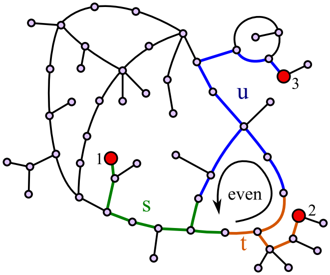
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.



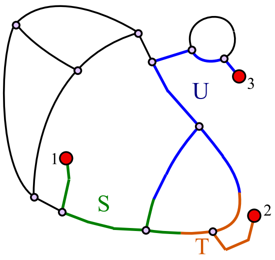
Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\mathrm{even}}_{z,x}(s,t,u)$, $\mathcal{G}^{\mathrm{odd}}_{z,x}(s,t,u)$ known. [Fusy, Guitter, '14]



Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\text{even}}_{z,x}(s,t,u)$, $\mathcal{G}^{\text{odd}}_{z,x}(s,t,u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G^{(1)}_{g,3}(S,T,U) = G^{\text{even}}_{g,3}(S,T,U) + G^{\text{odd}}_{g,3}(S,T,U)$.



Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}_{z,x}^{\mathrm{even}}(s, t, u)$, $\mathcal{G}_{z,x}^{\mathrm{odd}}(s, t, u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G_{g,3}^{(1)}(S, T, U) = G_{g,3}^{\mathrm{even}}(S, T, U) + G_{g,3}^{\mathrm{odd}}(S, T, U)$.
- Scaling again $g \to \frac{1}{12\sqrt{3}}$, $\mathcal{T} := \Sigma T$, etc., gives

$$G_{g,3}^{(1)}(S, T, U)\Sigma^{-2} \to \frac{1}{12}\partial_{\mathcal{S}}\partial_{\mathcal{T}}\partial_{\mathcal{U}} \frac{\sinh^2 \mathcal{S} \sinh^2 \mathcal{T} \sinh^2 \mathcal{U} \sinh^2(\mathcal{S} + \mathcal{T} + \mathcal{U})}{\sinh^2(\mathcal{S} + \mathcal{T}) \sinh^2(\mathcal{T} + \mathcal{U}) \sinh^2(\mathcal{U} + \mathcal{S})},$$

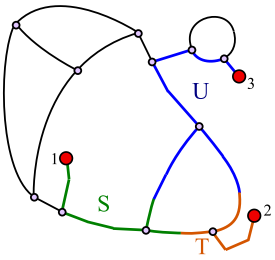
which is the Brownian map 3-point function.

Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\mathrm{even}}_{z,x}(s, t, u)$, $\mathcal{G}^{\mathrm{odd}}_{z,x}(s, t, u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G^{(1)}_{g,3}(S, T, U) = G^{\mathrm{even}}_{g,3}(S, T, U) + G^{\mathrm{odd}}_{g,3}(S, T, U)$.

- Unless cycle vanishes, i.e. completely confluent, even and odd occur with equal probability. Hence

$$G^{conf}_{g,3} = G^{\mathrm{even}}_{g,3} - G^{\mathrm{odd}}_{g,3}$$

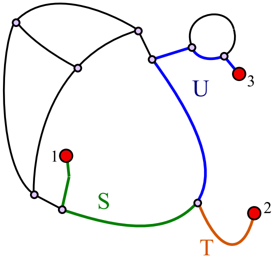


Three-point function

- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\mathrm{even}}_{z,x}(s,t,u)$, $\mathcal{G}^{\mathrm{odd}}_{z,x}(s,t,u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G^{(1)}_{g,3}(S,T,U) = G^{\mathrm{even}}_{g,3}(S,T,U) + G^{\mathrm{odd}}_{g,3}(S,T,U)$.

- Unless cycle vanishes, i.e. completely confluent, even and odd occur with equal probability. Hence

$$G^{conf}_{g,3} = G^{\mathrm{even}}_{g,3} - G^{\mathrm{odd}}_{g,3}$$



Three-point function

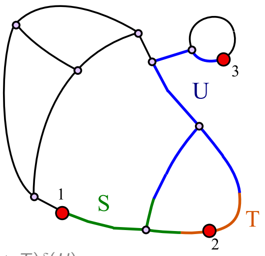
- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\mathrm{even}}_{z,x}(s, t, u)$, $\mathcal{G}^{\mathrm{odd}}_{z,x}(s, t, u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G^{(1)}_{g,3}(S, T, U) = G^{\mathrm{even}}_{g,3}(S, T, U) + G^{\mathrm{odd}}_{g,3}(S, T, U)$.

- Unless cycle vanishes, i.e. completely confluent, even and odd occur with equal probability. Hence

$$G^{conf}_{g,3} = G^{\mathrm{even}}_{g,3} - G^{\mathrm{odd}}_{g,3}$$

- Bivalent marked vertices

$$G^{(2)}_{g,3} = \frac{1}{8}(1 + \partial_S)(1 + \partial_T)(1 + \partial_U)G^{(1)}_{g,3}(S, T, U)$$
$$+ G^{(2)}_{g,2}(T + U)\delta(S) + G^{(2)}_{g,2}(U + S)\delta(T) + G^{(2)}_{g,2}(S + T)\delta(U)$$



Three-point function

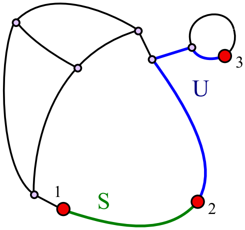
- Three marked vertices with pair-wise distances d_{12}, d_{13} and d_{23}.
- Reparametrize $d_{12} = s + t$, $d_{13} = s + u$, $d_{23} = u + t$.
- Even case: $s, t, u \in \mathbb{Z}$. Odd case: $s, t, u \in \mathbb{Z} + \frac{1}{2}$.
- Explicit expressions $\mathcal{G}^{\mathrm{even}}_{z,x}(s,t,u)$, $\mathcal{G}^{\mathrm{odd}}_{z,x}(s,t,u)$ known. [Fusy, Guitter, '14]
- Scaling as before: $G^{(1)}_{g,3}(S,T,U) = G^{\mathrm{even}}_{g,3}(S,T,U) + G^{\mathrm{odd}}_{g,3}(S,T,U)$.

- Unless cycle vanishes, i.e. completely confluent, even and odd occur with equal probability. Hence

$$G^{conf}_{g,3} = G^{\mathrm{even}}_{g,3} - G^{\mathrm{odd}}_{g,3}$$

- Bivalent marked vertices

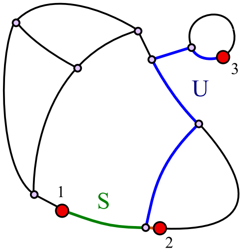
$$G^{(2)}_{g,3} = \tfrac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)G^{(1)}_{g,3}(S,T,U) + G^{(2)}_{g,2}(T+U)\delta(S) + G^{(2)}_{g,2}(U+S)\delta(T) + G^{(2)}_{g,2}(S+T)\delta(U)$$



Expected hop count for random cubic map

- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

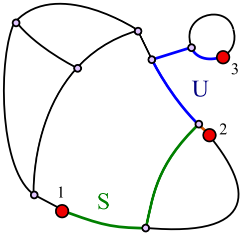
$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \tfrac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\text{even}}_{g,3}(S,T,U) - G^{\text{odd}}_{g,3}(S,T,U)]$$



Expected hop count for random cubic map

- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

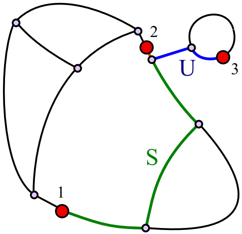
$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \frac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\text{even}}_{g,3}(S,T,U) - G^{\text{odd}}_{g,3}(S,T,U)]$$



Expected hop count for random cubic map

- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \frac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\text{even}}_{g,3}(S,T,U) - G^{\text{odd}}_{g,3}(S,T,U)]$$



Expected hop count for random cubic map

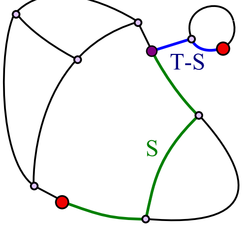
- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \frac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\mathrm{even}}_{g,3}(S,T,U)-G^{\mathrm{odd}}_{g,3}(S,T,U)]$$

- Hence, for fixed number of faces F,

$$\frac{\hat{G}^{(2)}_{F,2}(S,T-S)}{G^{(2)}_{F,2}(T)}$$

is the expected density of vertices (at distance S) along a geodesic of length/passage time T.



Expected hop count for random cubic map

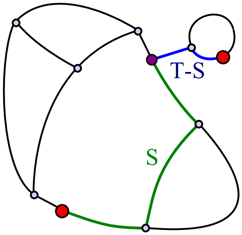
- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \tfrac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\mathrm{even}}_{g,3}(S,T,U) - G^{\mathrm{odd}}_{g,3}(S,T,U)]$$

- Hence, for fixed number of faces F,

$$\frac{\hat{G}^{(2)}_{F,2}(S,T-S)}{G^{(2)}_{F,2}(T)}$$

  is the expected density of vertices (at distance S) along a geodesic of length/passage time T.



- Scaling limit:
  $\hat{G}^{(2)}_{g,2}(S,T-S) \to (1+\frac{1}{\sqrt{3}})G^{(2)}_{g,2}(T)$.

Expected hop count for random cubic map

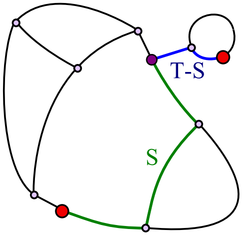
- Consider the limit $T \to 0$ (but $S, T, U > 0$) of the completely confluent three-point function, i.e.

$$\hat{G}^{(2)}_{g,2}(S,U) := \lim_{T\to 0} \tfrac{1}{8}(1+\partial_S)(1+\partial_T)(1+\partial_U)[G^{\mathrm{even}}_{g,3}(S,T,U) - G^{\mathrm{odd}}_{g,3}(S,T,U)]$$

- Hence, for fixed number of faces F,

$$\frac{\hat{G}^{(2)}_{F,2}(S, T-S)}{G^{(2)}_{F,2}(T)}$$

  is the expected density of vertices (at distance S) along a geodesic of length/passage time T.

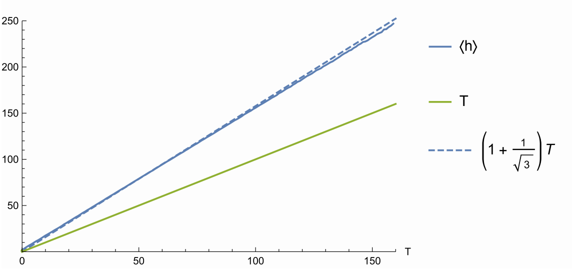


- Scaling limit: $\hat{G}^{(2)}_{g,2}(S, T-S) \to (1 + \frac{1}{\sqrt{3}}) G^{(2)}_{g,2}(T)$.

Asymptotic hop count to passage time ratio is $\langle h \rangle / T \to 1 + \frac{1}{\sqrt{3}}$.

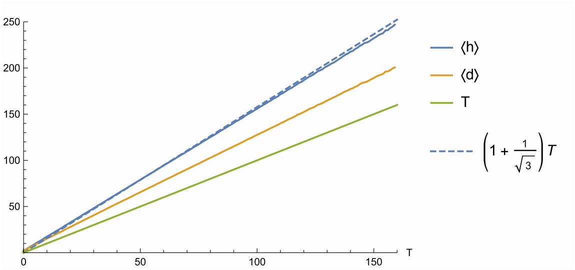
Time constants for FPP on a random cubic graph

- We have seen $\langle h \rangle / T \to 1 + \frac{1}{\sqrt{3}}$. This is confirmed numerically (random triangulation 128k triangles).



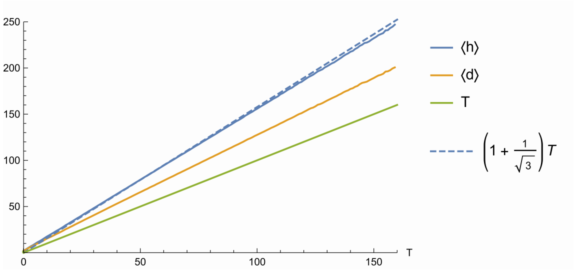
Time constants for FPP on a random cubic graph

- We have seen $\langle h \rangle / T \to 1 + \frac{1}{\sqrt{3}}$. This is confirmed numerically (random triangulation 128k triangles).
- Let us assume that h, T and the graph distance d are asymptotically (almost surely) linearly related.



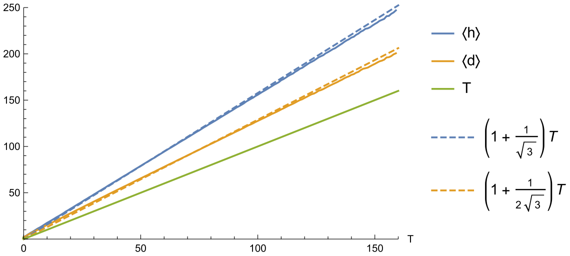
Time constants for FPP on a random cubic graph

- We have seen $\langle h \rangle / T \to 1 + \frac{1}{\sqrt{3}}$. This is confirmed numerically (random triangulation 128k triangles).
- Let us assume that h, T and the graph distance d are asymptotically (almost surely) linearly related.
- Then we can determine the ratio d/T simply by comparing the corresponding two-point functions.



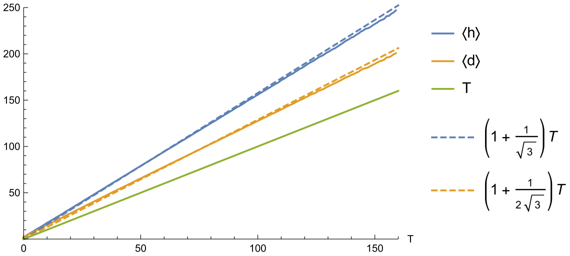
Time constants for FPP on a random cubic graph

- We have seen $\langle h\rangle/T \to 1 + \frac{1}{\sqrt{3}}$. This is confirmed numerically (random triangulation 128k triangles).
- Let us assume that h, T and the graph distance d are asymptotically (almost surely) linearly related.
- Then we can determine the ratio d/T simply by comparing the corresponding two-point functions.
- Deduce from transfer matrix approach in [Kawai et al, '93]: $h/T \to 1 + \frac{1}{2\sqrt{3}}$.



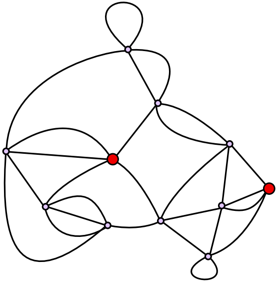
Time constants for FPP on a random cubic graph

- We have seen $\langle h \rangle / T \to 1 + \frac{1}{\sqrt{3}}$. This is confirmed numerically (random triangulation 128k triangles).
- Let us assume that h, T and the graph distance d are asymptotically (almost surely) linearly related.
- Then we can determine the ratio d/T simply by comparing the corresponding two-point functions.
- Deduce from transfer matrix approach in [Kawai et al, '93]: $h/T \to 1 + \frac{1}{2\sqrt{3}}$.
- Simple relation: $d = (h+T)/2$. Is it true more generally?



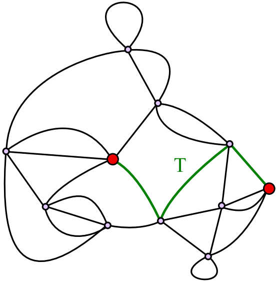
Expected hop count for p-regular maps

- Two vertices a distance T apart.



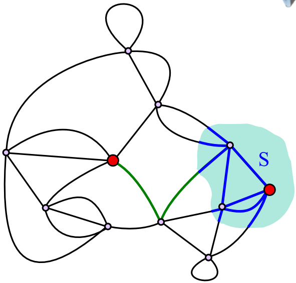
Expected hop count for p-regular maps

- Two vertices a distance T apart.



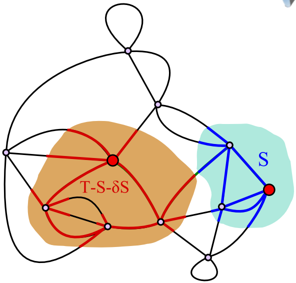
Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.



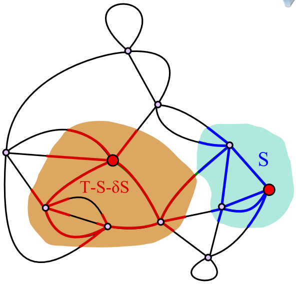
Expected hop count for p-regular maps

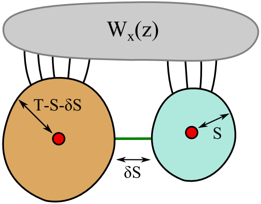
- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.



Expected hop count for p-regular maps

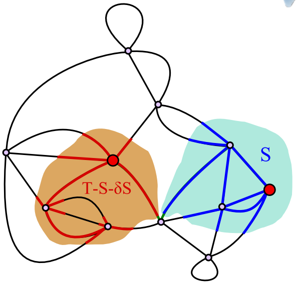
- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.

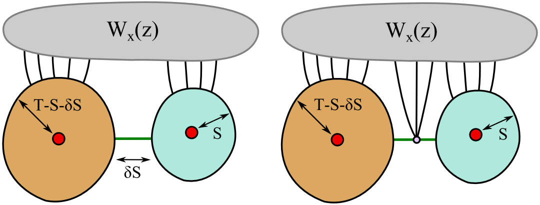




Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.



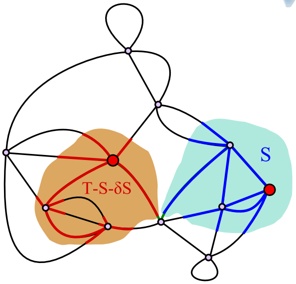


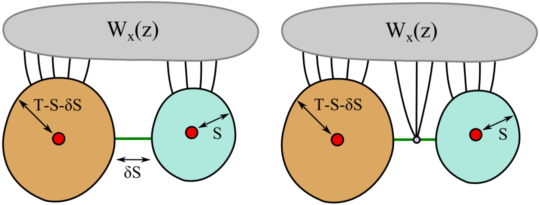
Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}}$$

in terms of the disk function $W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



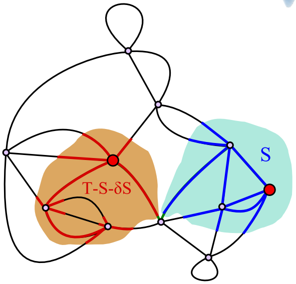


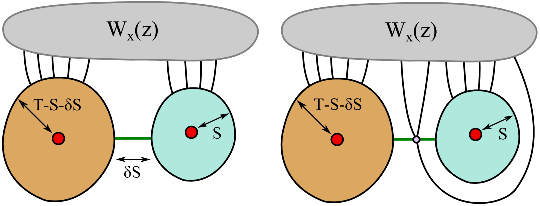
Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T-S-\delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}}$$

in terms of the disk function $W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



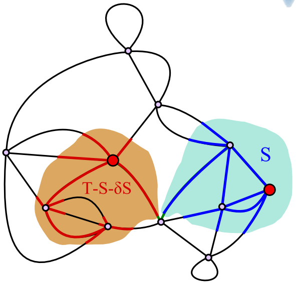


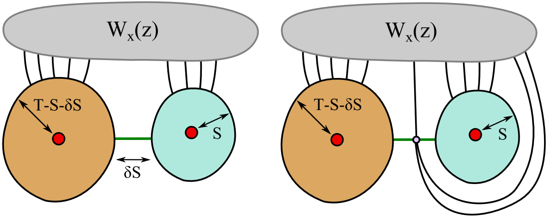
Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}}$$

in terms of the disk function
$W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



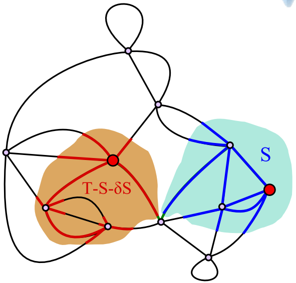


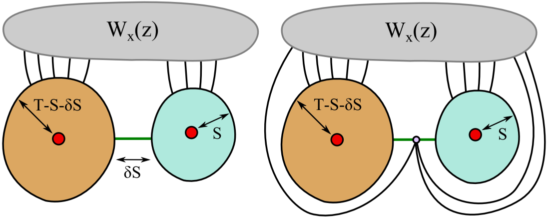
Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}} \underset{N,d\to\infty}{\to} (p-1)x_c z_c^{p-2},$$

in terms of the disk function $W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



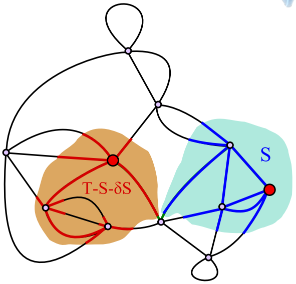


Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T-S-\delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}} \underset{N,d\to\infty}{\to} (p-1)x_c z_c^{p-2},$$

in terms of the disk function
$W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



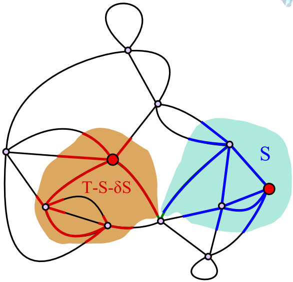
Asymptotic hop count to passage time ratio is $\langle h\rangle / T \to (p-1)x_c z_c^{p-2}$.

Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}} \underset{N,d\to\infty}{\to} (p-1)x_c z_c^{p-2},$$

in terms of the disk function $W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.



Asymptotic hop count to passage time ratio is $\langle h\rangle / T \to (p-1)x_c z_c^{p-2}$.

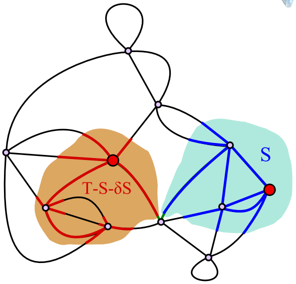
- $p = 3$: $x_c = 1/(2\,3^{3/4})$, $z_c = 3^{3/4}(1 + 1/\sqrt{3})$, $\frac{\langle h\rangle}{T} = 1 + 1/\sqrt{3}$.

Expected hop count for p-regular maps

- Two vertices a distance T apart.
- Determine the balls of radius S and $T - S - \delta S$. They almost touch.
- As $\delta S \to 0$, conditioned on the balls, a vertex occurs with probability $P\,\delta S$ with

$$P = (p-1)\frac{W_{N-1,d+p-2}}{W_{N,d}} \underset{N,d\to\infty}{\to} (p-1)x_c z_c^{p-2},$$

in terms of the disk function $W_x(z) = \sum_{N=0}^{\infty}\sum_{d=1}^{\infty} z^{-d-1}x^N W_{N,d}$.

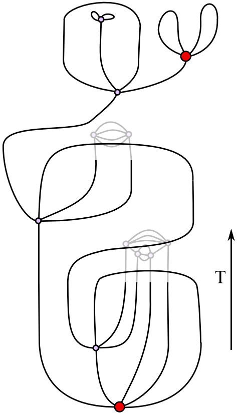


Asymptotic hop count to passage time ratio is $\langle h\rangle/T \to (p-1)x_c z_c^{p-2}$.

- $p = 3$: $x_c = 1/(2\,3^{3/4})$, $z_c = 3^{3/4}(1+1/\sqrt{3})$, $\frac{\langle h\rangle}{T} = 1 + 1/\sqrt{3}$.
- p even: $x_c = \left(\frac{p-2}{p}\right)^{\frac{p}{2}} \frac{4}{p-2}\binom{p}{p/2}^{-1}$, $z_c = \sqrt{\frac{4p}{p-2}}$, $\frac{\langle h\rangle}{T} = 2^{p-2}\binom{p-2}{\frac{p-2}{2}}^{-1}$.

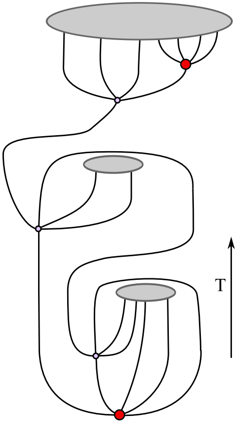
Transfer matrix for FPP

- Take a p-regular weighted planar map with two marked points.



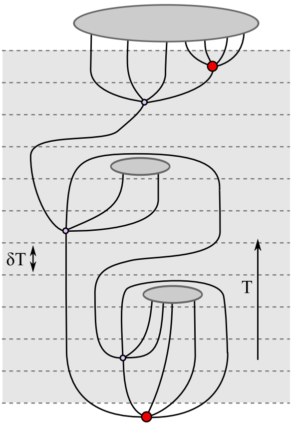
Transfer matrix for FPP

- Take a p-regular weighted planar map with two marked points.
- Identify baby universes.



Transfer matrix for FPP

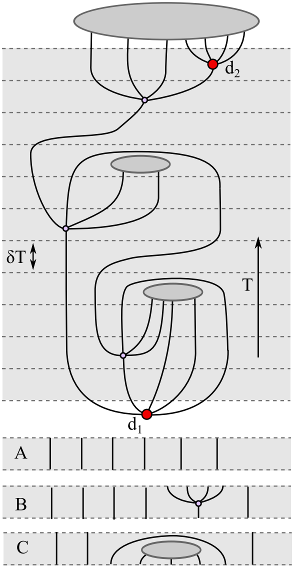
- Take a p-regular weighted planar map with two marked points.
- Identify baby universes.
- Cut into small passage time intervals.



Transfer matrix for FPP

- Take a p-regular weighted planar map with two marked points.
- Identify baby universes.
- Cut into small passage time intervals.
- Write $G_x(z,T) := \sum_{d_1} z^{-d_1-1} G_{x,2}^{(d_1,d_2)}(T)$ and the disk function $W_x(z) := \sum_d z^{-d-1} G_{x,1}^{(d)}$ (weight x per vertex). Then

$$\frac{\partial}{\partial T} G_x(z,T) = \frac{\partial}{\partial z}\Big[\big(\underbrace{z}_{A} - \underbrace{x z^{p-1}}_{B} - 2\underbrace{W_x(z)}_{C}\big) G_x(z,T)\Big]$$

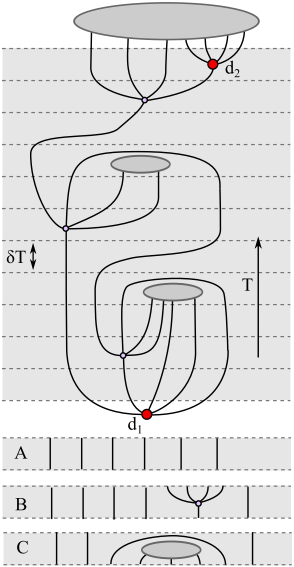


Transfer matrix for FPP

- Take a p-regular weighted planar map with two marked points.
- Identify baby universes.
- Cut into small passage time intervals.
- Write $G_x(z,T) := \sum_{d_1} z^{-d_1-1} G_{x,2}^{(d_1,d_2)}(T)$ and the disk function $W_x(z) := \sum_d z^{-d-1} G_{x,1}^{(d)}$ (weight x per vertex). Then

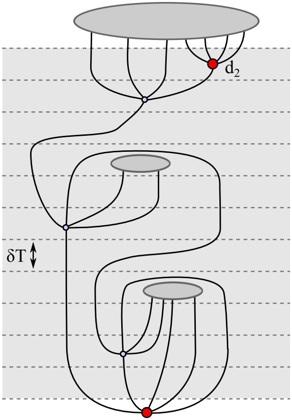
$$\frac{\partial}{\partial T} G_x(z,T) = \frac{\partial}{\partial z}\left[\left(z - xz^{p-1} - 2W_x(z)\right) G_x(z,T)\right]$$

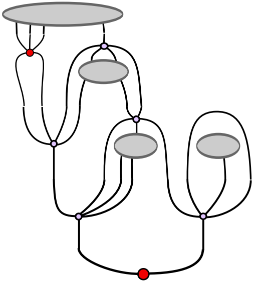
- Precisely this formula was used in [Ambjorn, Watabiki, '95] as an approximation to derive the 2-point function for triangulations. Now we know it is not just an approximation!



Transfer matrix for graph distance [Kawai et al, '93]

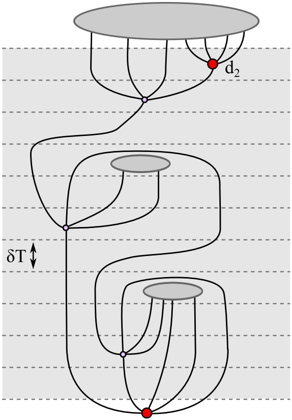
- Now take all edges to have length 1.
  Again we can build a transfer matrix.

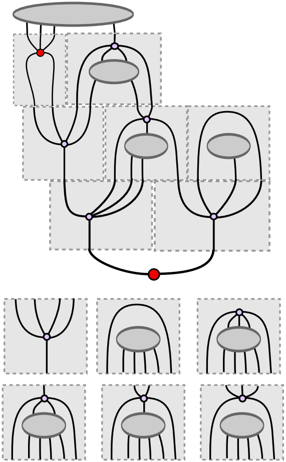




Transfer matrix for graph distance [Kawai et al, '93]

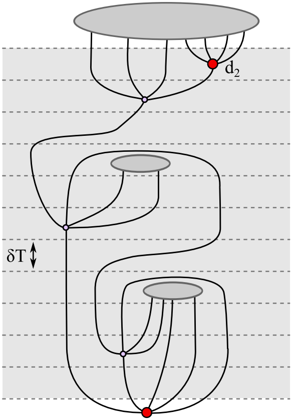
- Now take all edges to have length 1.
  Again we can build a transfer matrix.

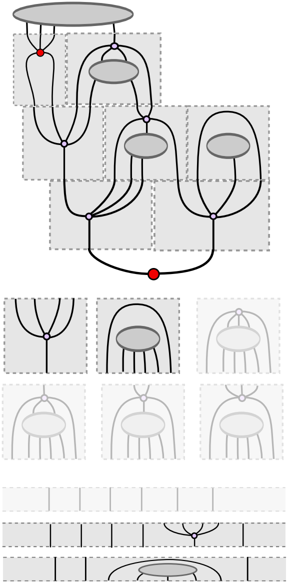




Transfer matrix for graph distance [Kawai et al, '93]

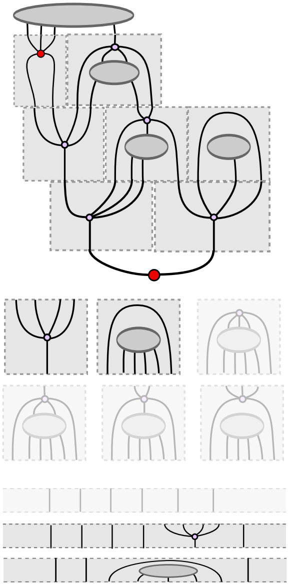
- Now take all edges to have length 1. Again we can build a transfer matrix.





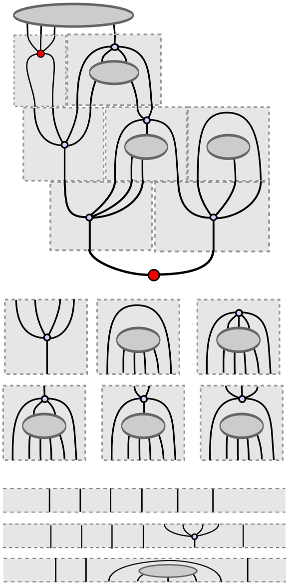
Transfer matrix for graph distance [Kawai et al, '93]

- Now take all edges to have length 1. Again we can build a transfer matrix.
- Two of the building blocks are quite similar...



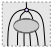
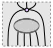
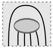
Transfer matrix for graph distance [Kawai et al, '93]

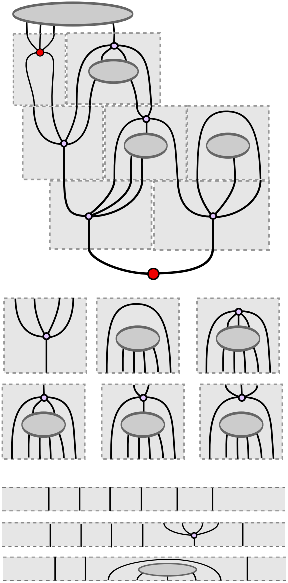
- Now take all edges to have length 1. Again we can build a transfer matrix.
- Two of the building blocks are quite similar…
- …but there are $p-1$ extra ones.



Transfer matrix for graph distance [Kawai et al, '93]

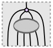
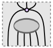
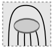
- Now take all edges to have length 1. Again we can build a transfer matrix.
- Two of the building blocks are quite similar…
- …but there are $p - 1$ extra ones.
- Can work out transfer matrix PDE explicitly and compare, but heuristically

 $\sim$  $\sim x_c\, z_c^{p-2}$ 

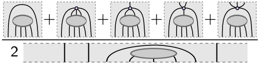


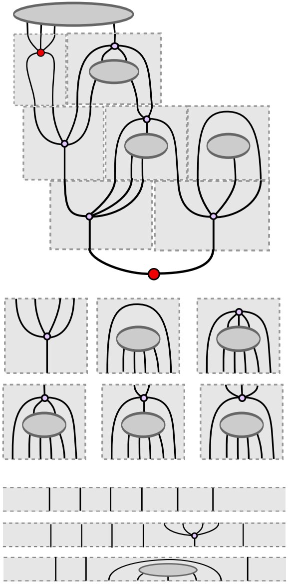
Transfer matrix for graph distance [Kawai et al, '93]

- Now take all edges to have length 1. Again we can build a transfer matrix.
- Two of the building blocks are quite similar...
- ...but there are $p-1$ extra ones.
- Can work out transfer matrix PDE explicitly and compare, but heuristically

 $\sim$  $\sim x_c\, z_c^{p-2}$ 

- and therefore

 $\to \frac{1}{2}(1+(p-1)x_c\, z_c^{p-2})$



Graph distance to passage time ratio is $d/T \to (1+h/T)/2$.

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.
- Open questions
  - Can weighted cubic maps be shown to converge to the Brownian map?

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.
- Open questions
  - Can weighted cubic maps be shown to converge to the Brownian map?
  - Is $d = (h + T)/2$ a coincidence or does it hold for a (much) larger class of random graphs? Does it hold in the case of ...
    - ... weights on the faces, e.g. p-angulations?
    - ... other edge length distributions?
    - ... random geometric graphs (in certain limits)?

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.
- Open questions
  - Can weighted cubic maps be shown to converge to the Brownian map?
  - Is $d = (h + T)/2$ a coincidence or does it hold for a (much) larger class of random graphs? Does it hold in the case of ...
    - ...weights on the faces, e.g. p-angulations?
    - ...other edge length distributions?
    - ...random geometric graphs (in certain limits)?
  - Is there a bijective explanation for $d = (h + T)/2$?

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.
- Open questions
  - Can weighted cubic maps be shown to converge to the Brownian map?
  - Is $d = (h + T)/2$ a coincidence or does it hold for a (much) larger class of random graphs? Does it hold in the case of ...
    - ...weights on the faces, e.g. p-angulations?
    - ...other edge length distributions?
    - ...random geometric graphs (in certain limits)?
  - Is there a bijective explanation for $d = (h + T)/2$?
  - How do the relative fluctuations of d, h, T scale? Kardar-Parisi-Zhang scaling exponents on a random surface?

Conclusion & open questions

- Conclusions
  - Both the two- and three-point functions of weighted cubic maps converge to those of the Brownian map in the scaling limit.
  - We can compute the time constants of FPP on random p-regular planar maps, and in each case they satisfy $d = (h + T)/2$.
- Open questions
  - Can weighted cubic maps be shown to converge to the Brownian map?
  - Is $d = (h + T)/2$ a coincidence or does it hold for a (much) larger class of random graphs? Does it hold in the case of ...
    - ...weights on the faces, e.g. p-angulations?
    - ...other edge length distributions?
    - ...random geometric graphs (in certain limits)?
  - Is there a bijective explanation for $d = (h + T)/2$?
  - How do the relative fluctuations of d, h, T scale? Kardar-Parisi-Zhang scaling exponents on a random surface?

Thanks!