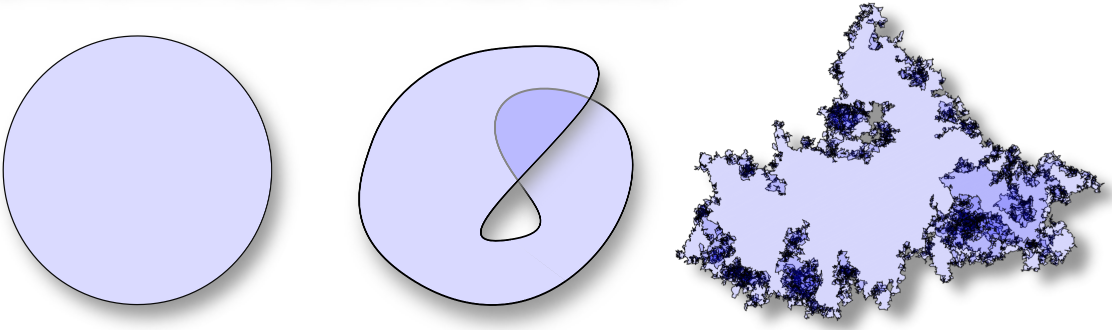
Timothy Budd

Uniform random flat disks



Radboud University

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/

Motivation: Quantum JT gravity

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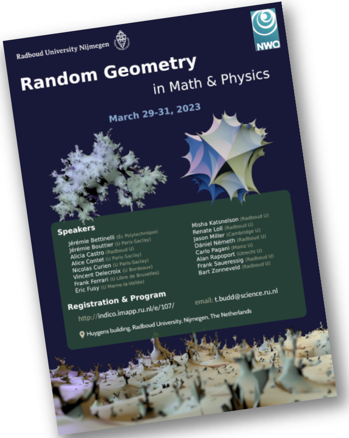

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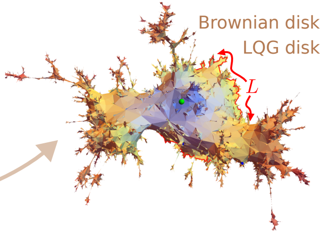

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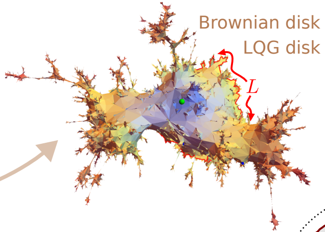

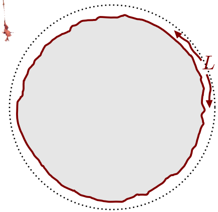

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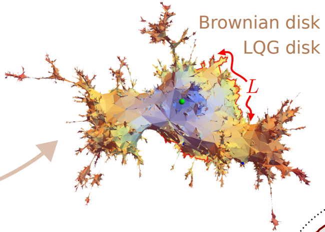

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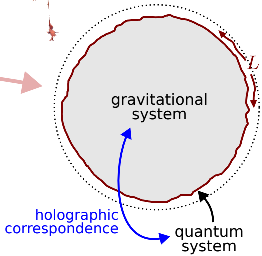

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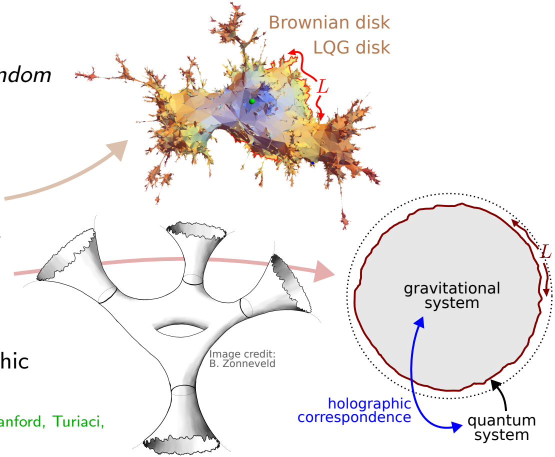

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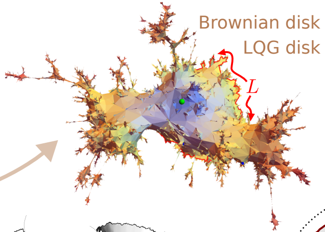

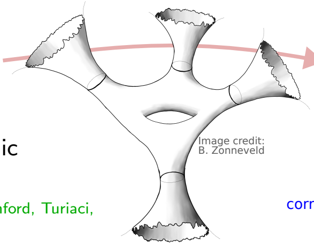

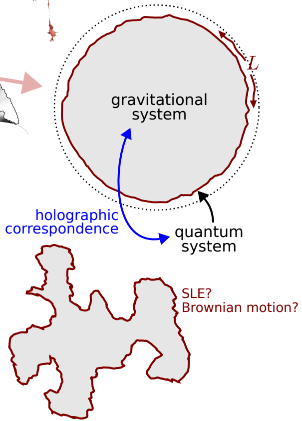

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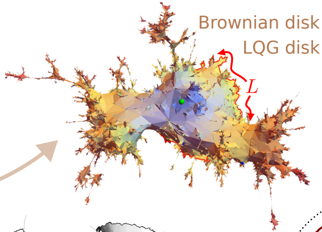

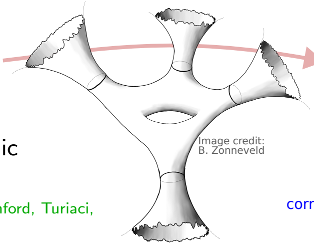

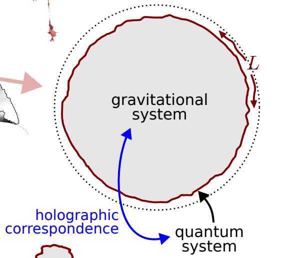

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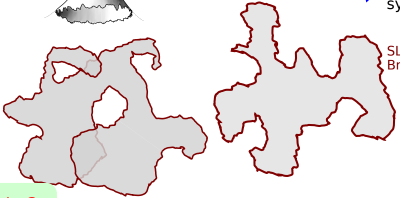
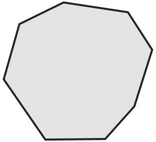
- Ferrari: should allow disks to self-overlap. [Ferrari, '24]



Is there a tractable model of uniform random discrete flat disks?

What is a discrete flat disks?

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]



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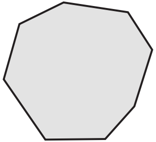

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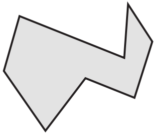

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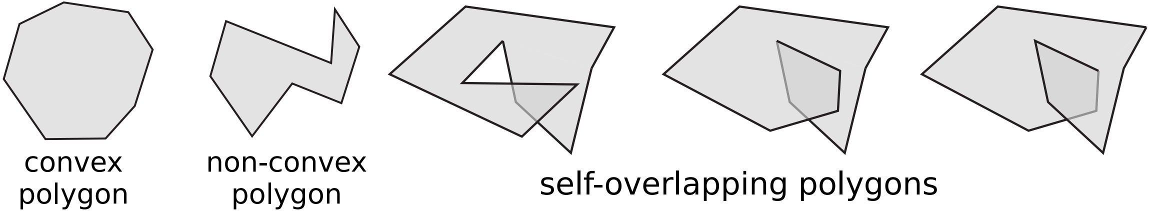

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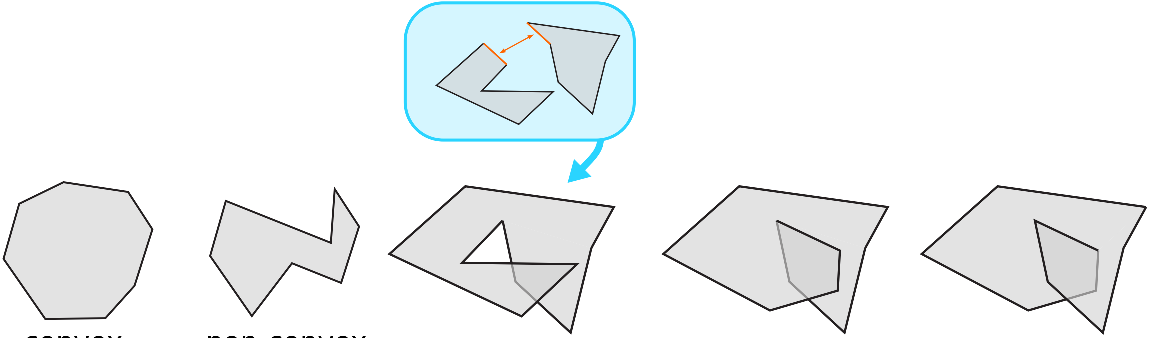

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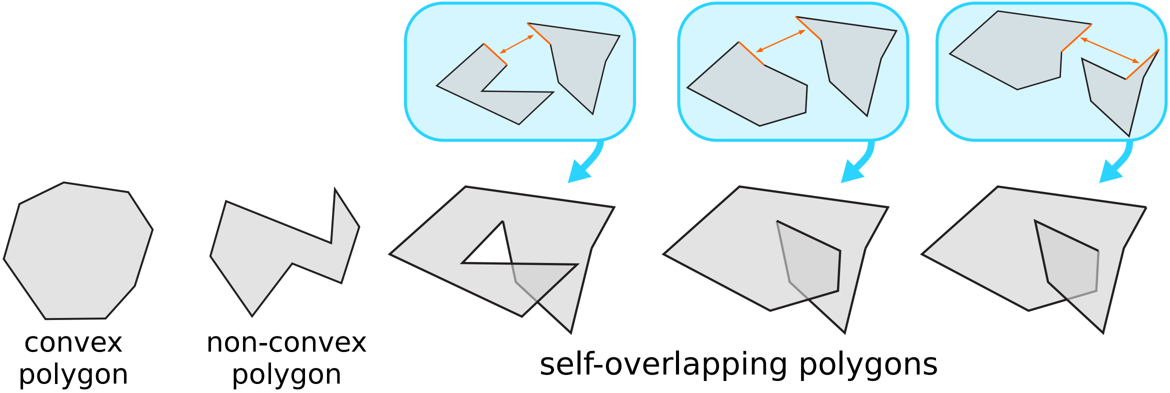

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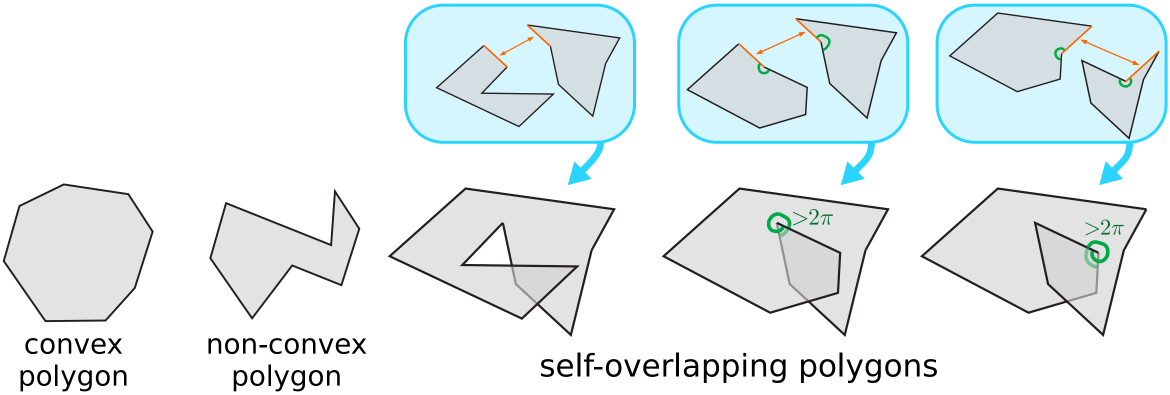

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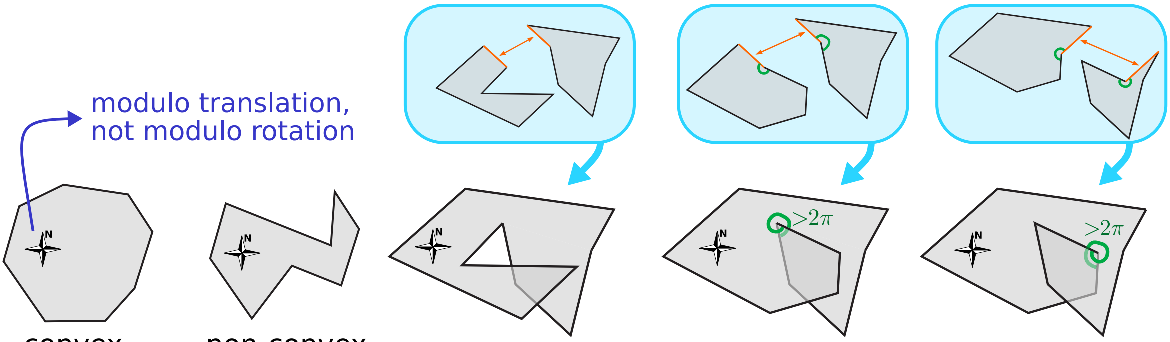

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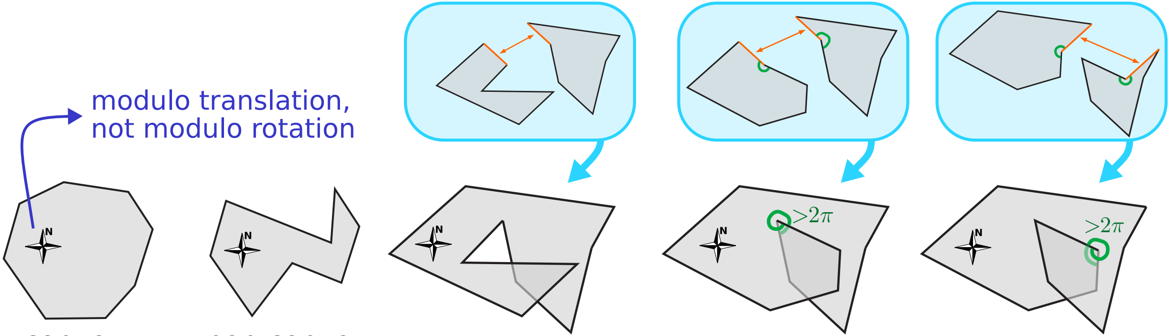

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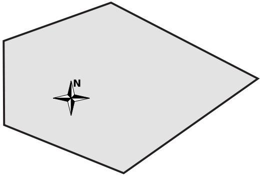
How many discrete flat disks with n sides are there?

First model: fix the sides!



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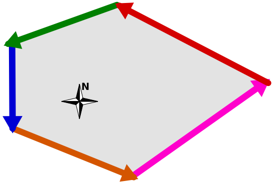

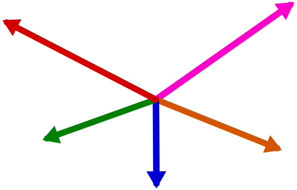

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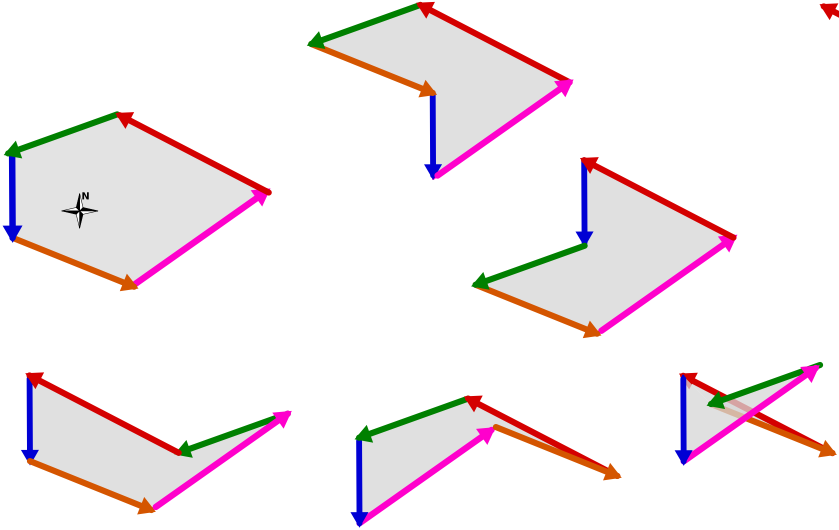

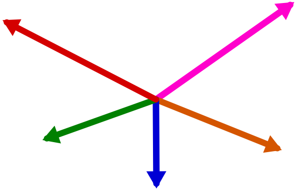

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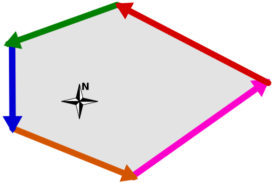

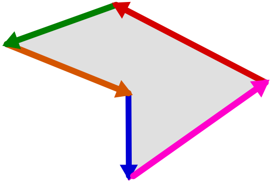

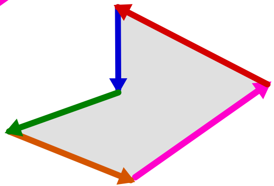

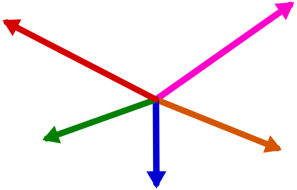

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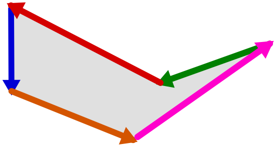

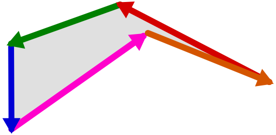

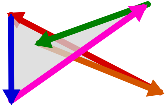

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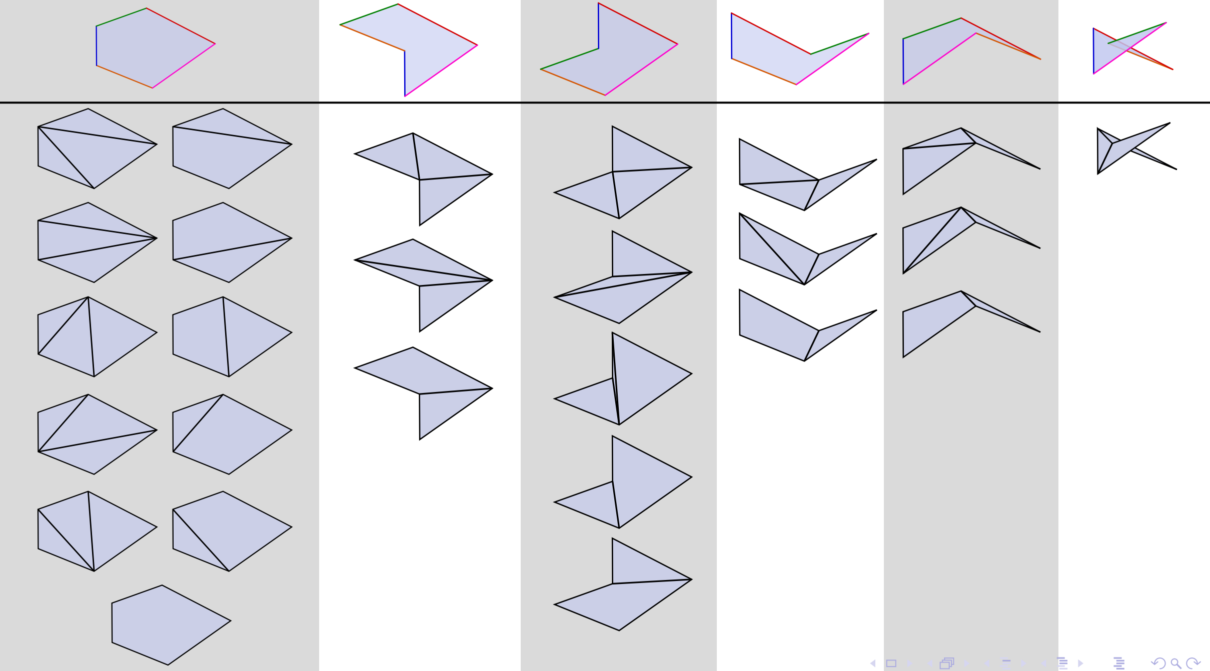
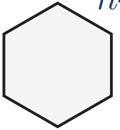
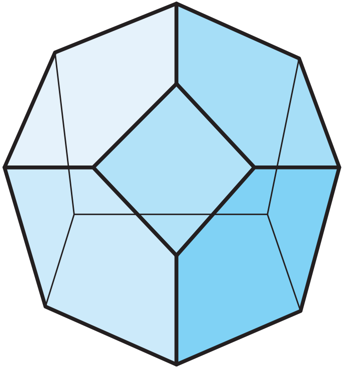


Images adapted from Devadoss, Shah, Shao, Winston

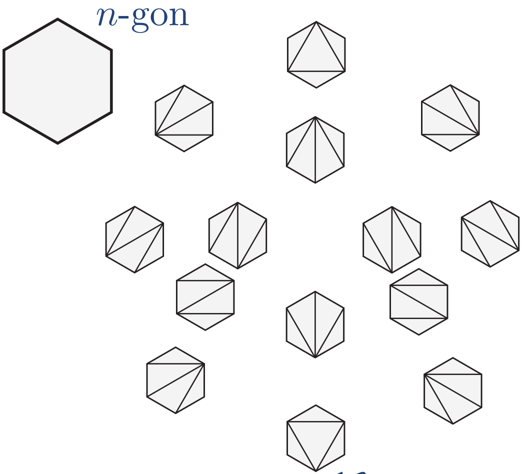


n-gon



Associahedron K_n

[Tamari, '51] [Stasheff, '63]



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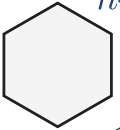

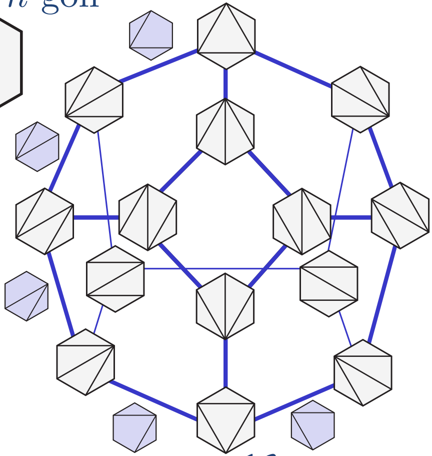

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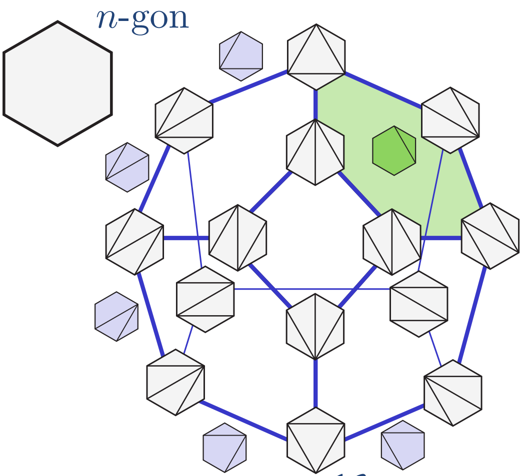

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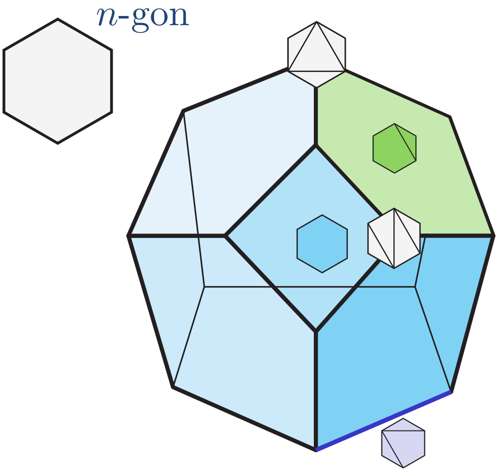

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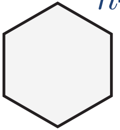

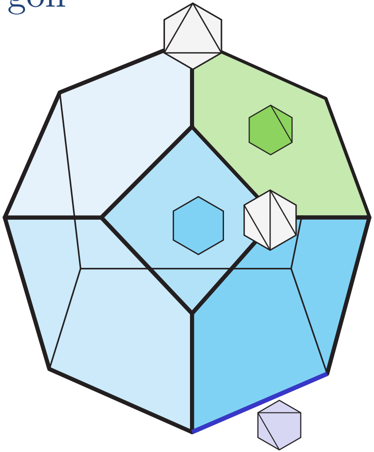

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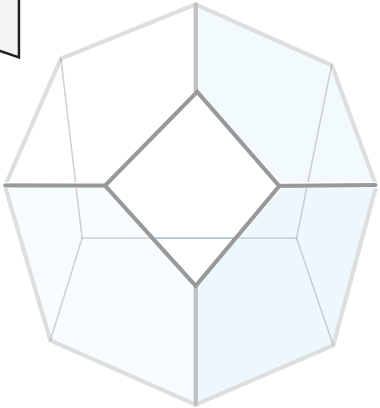

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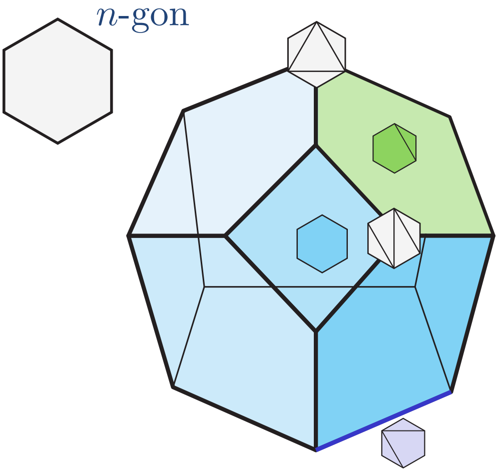

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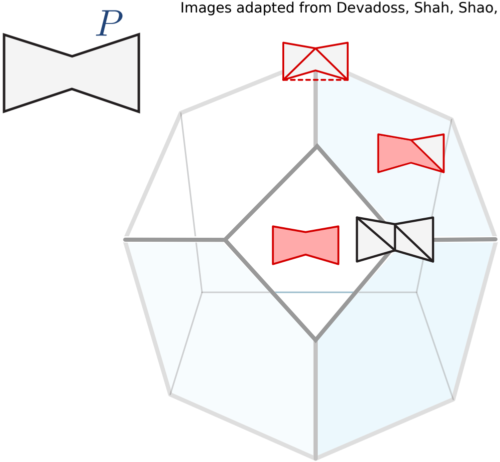

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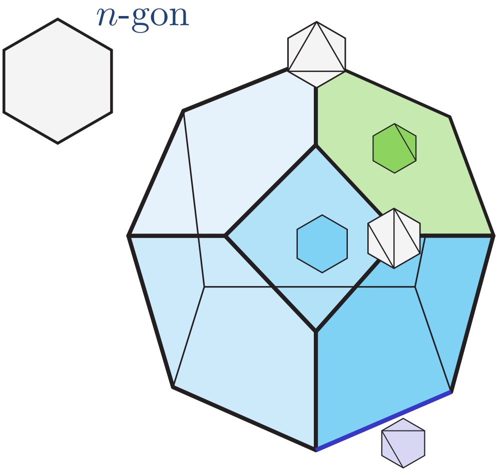

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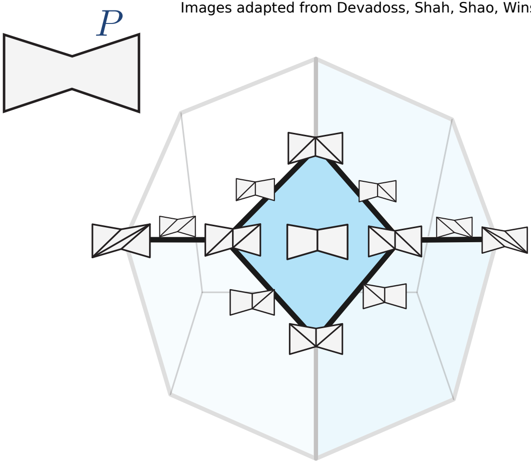

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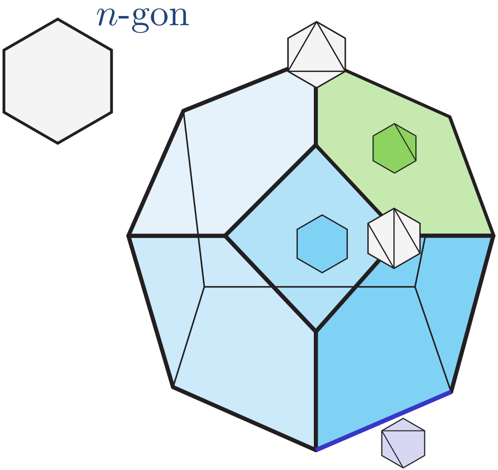

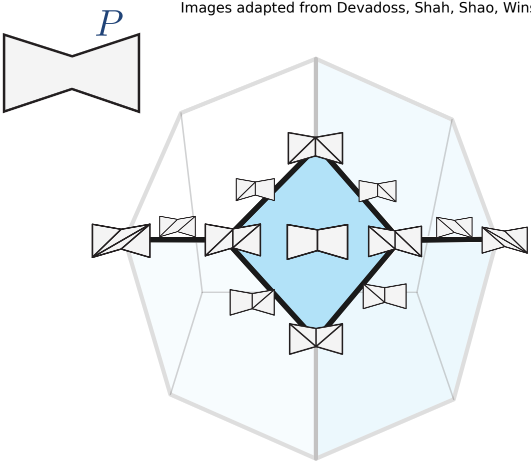

Associahedron $\mathcal{K}_n$

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Convex diagonalizations $\mathcal{K}_P$

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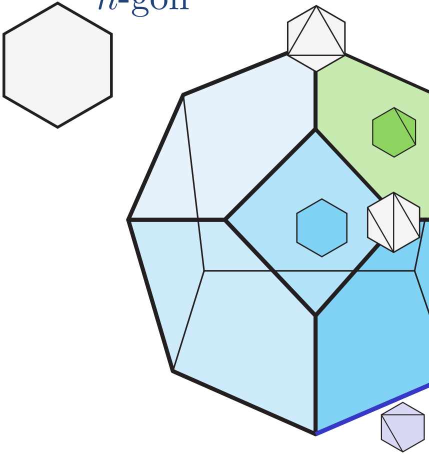
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon P, the complex $\mathcal{K}_P$ is contractible.

Images adapted from Devadoss, Shah, Shao, Winston

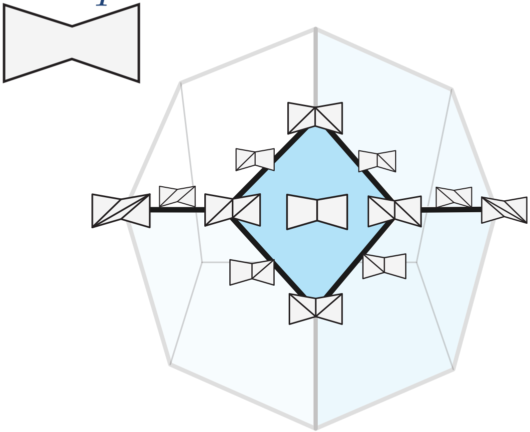
n-gon



Associahedron $\mathcal{K}_n$

[Tamari, '51] [Stasheff, '63]

P

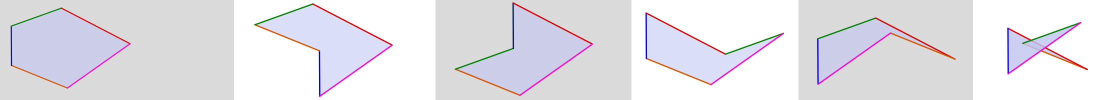


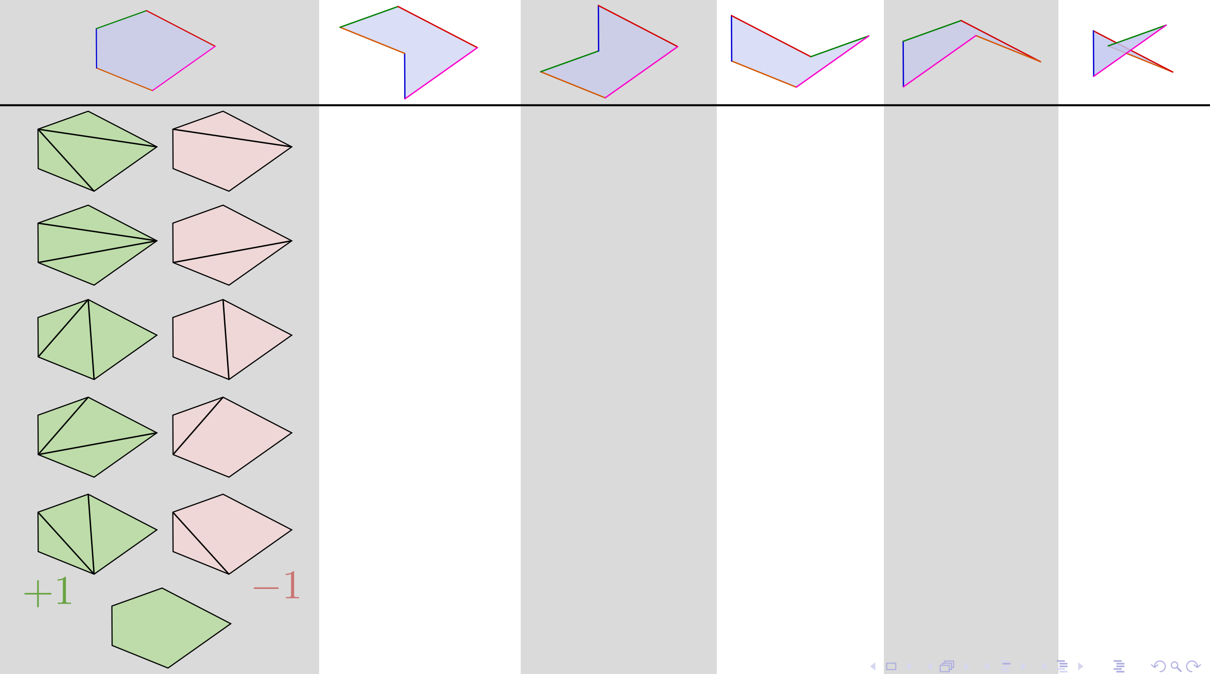
Convex diagonalizations $\mathcal{K}_P$

[Devadoss, Shah, Shao, Winston, '09]

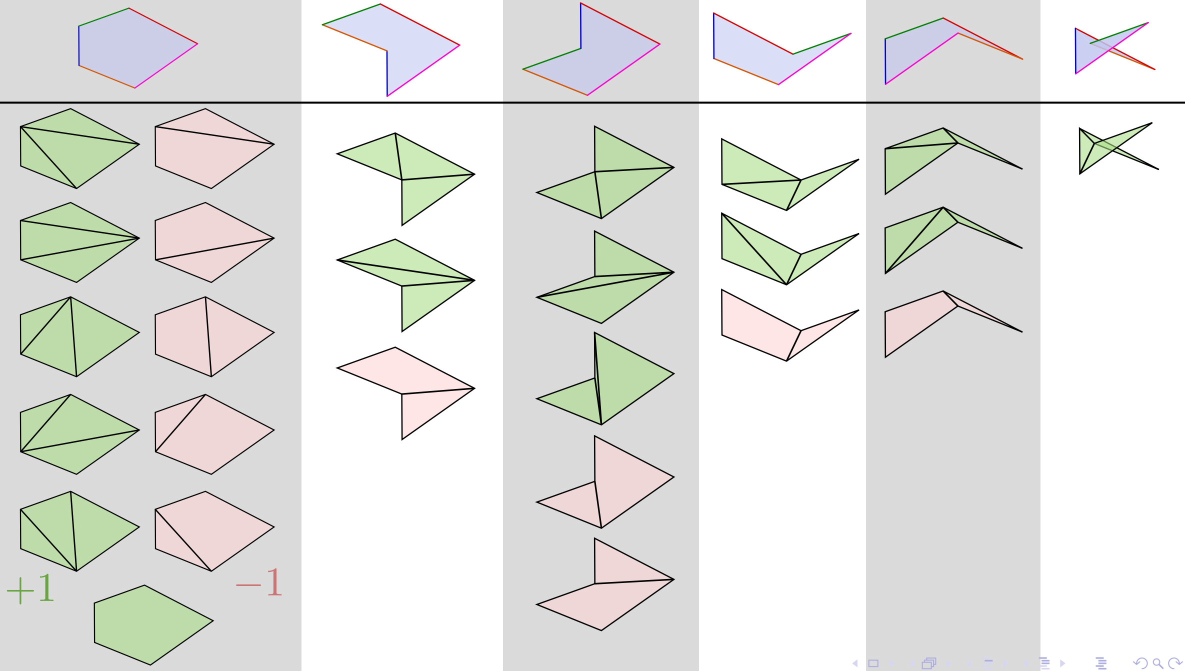
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon P, the complex $\mathcal{K}_P$ is contractible. Hence, Euler characteristic $\chi = \sum_{\text{cells } \sigma} (-1)^{\dim \sigma} = 1.$

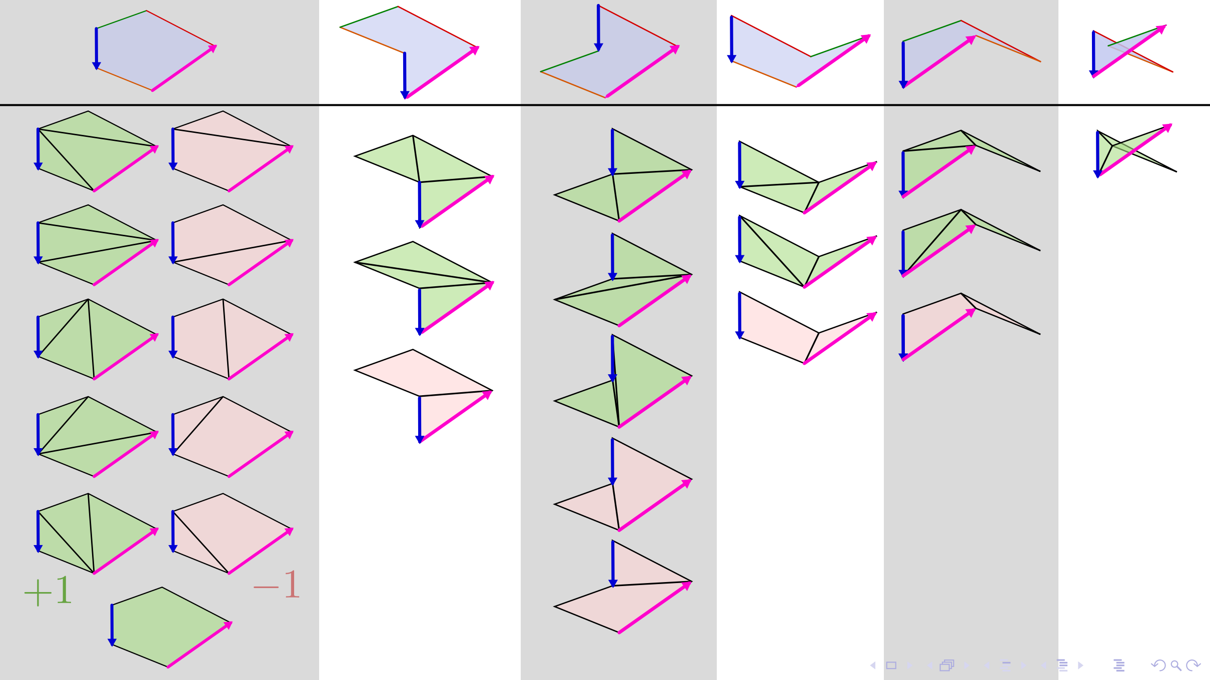


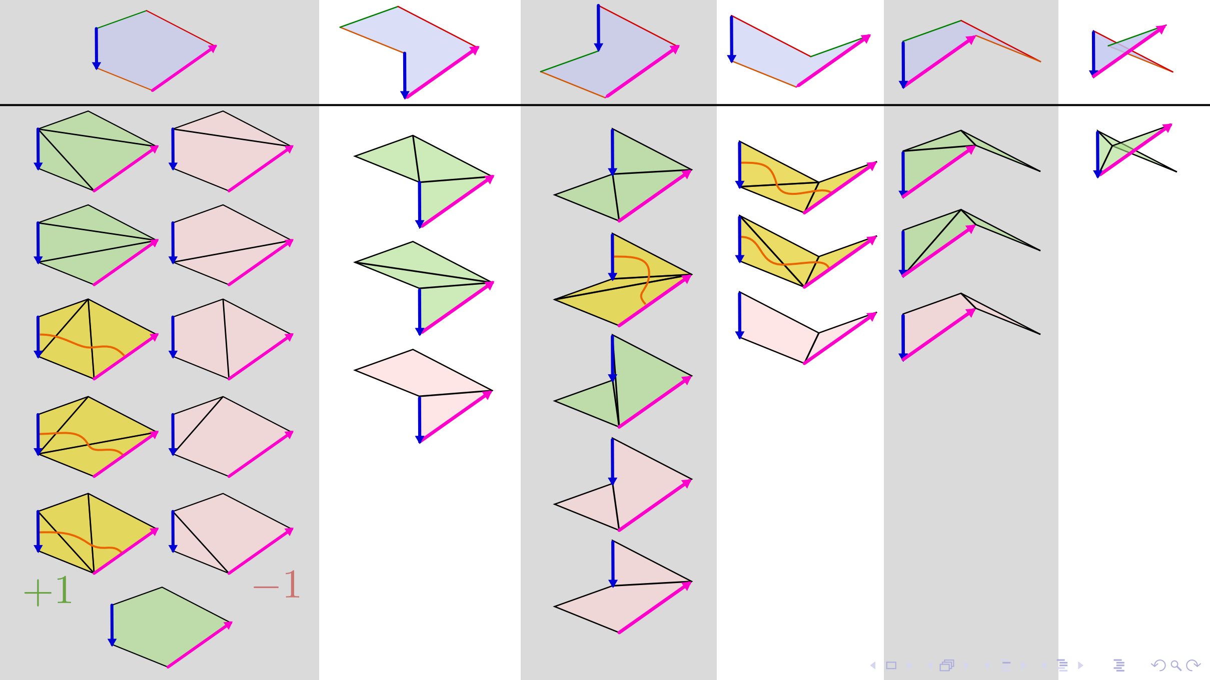


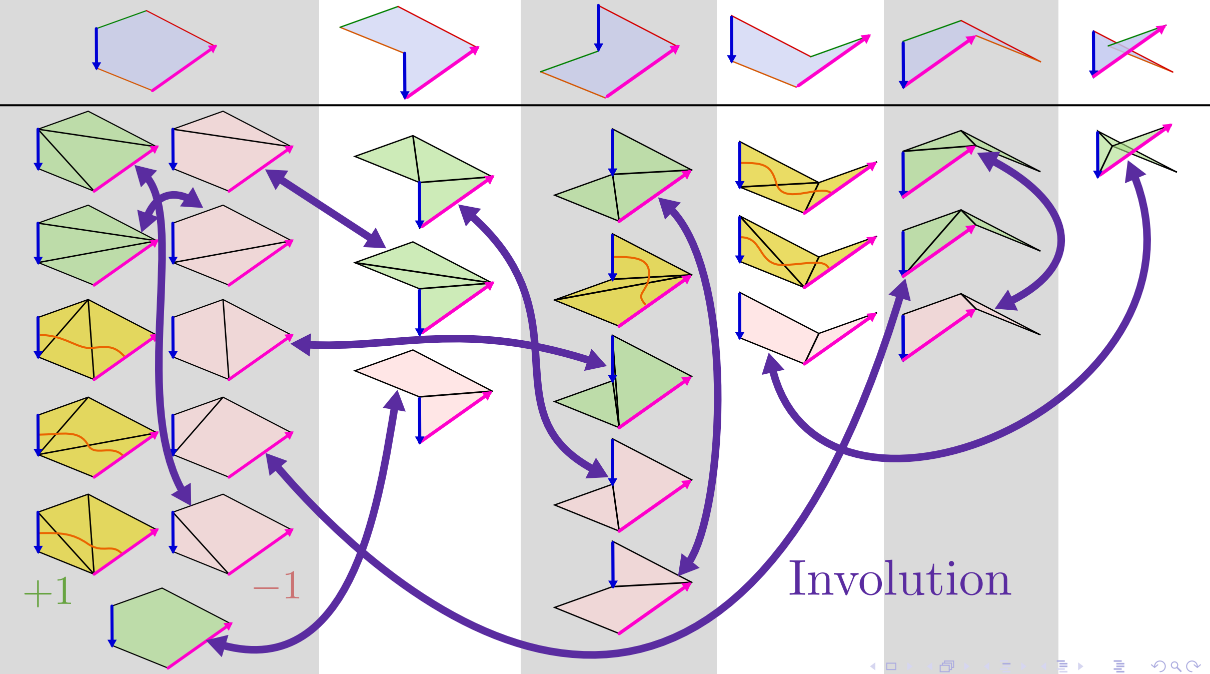
+1
−1



+1
−1

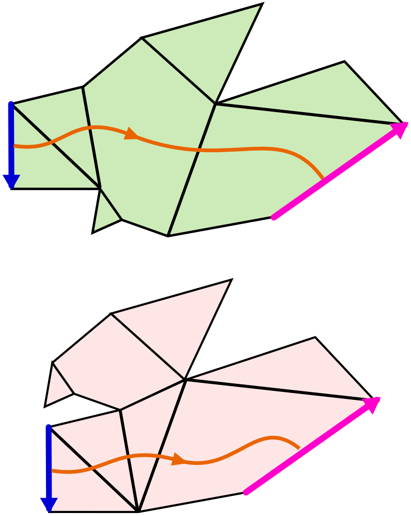

+1
−1


+1
−1

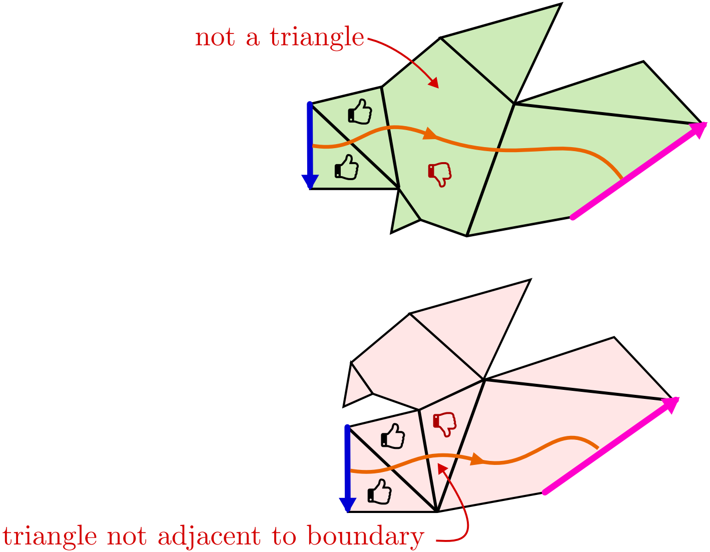


+1
−1
Involution

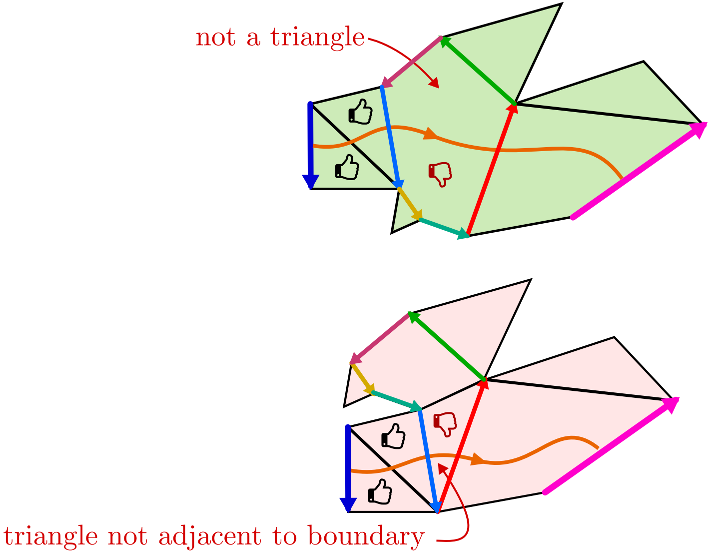
Involution?



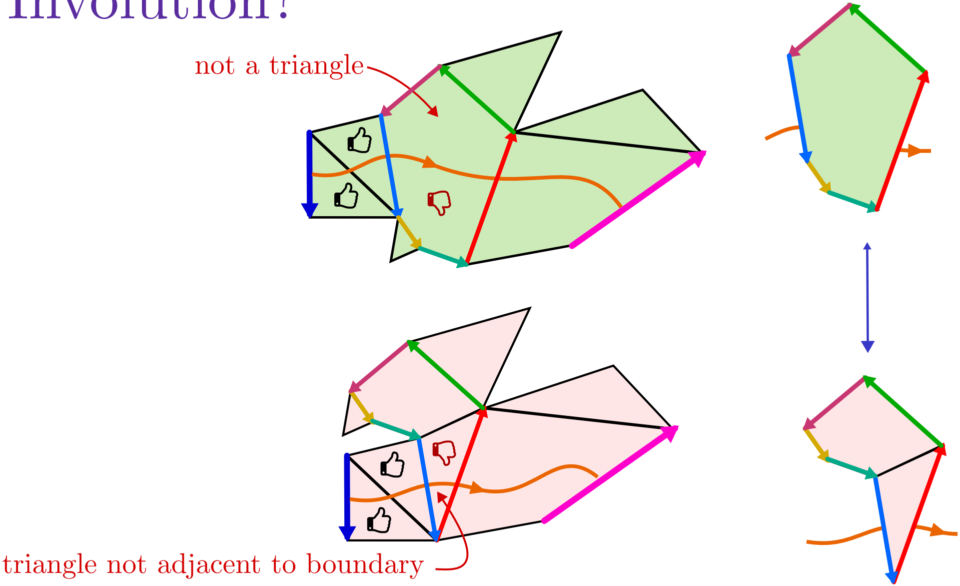
Involution?



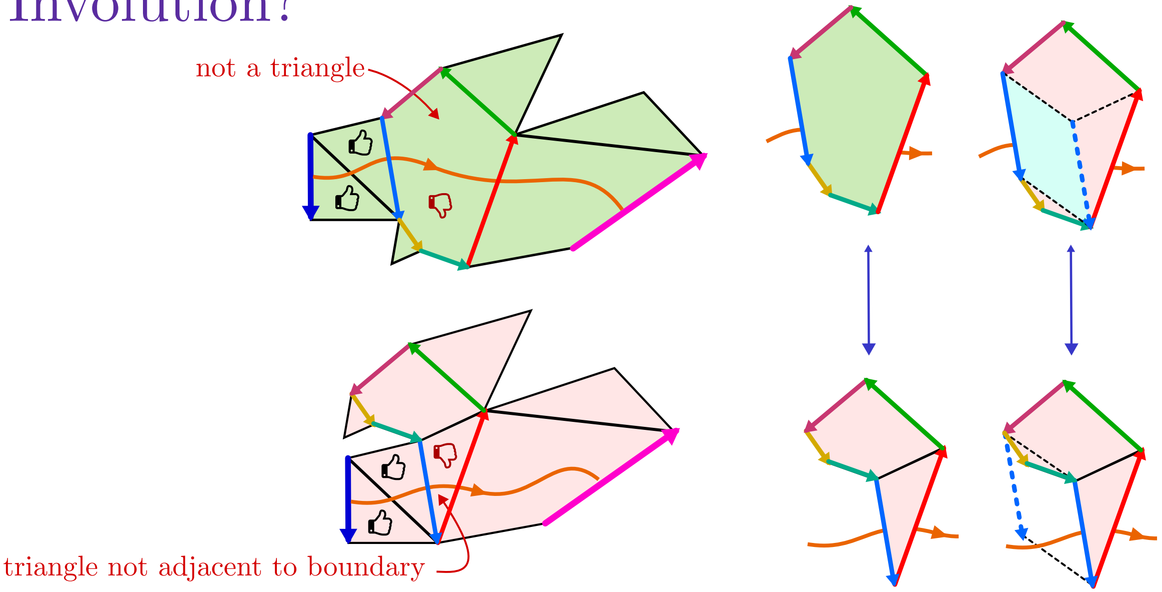
Involution?



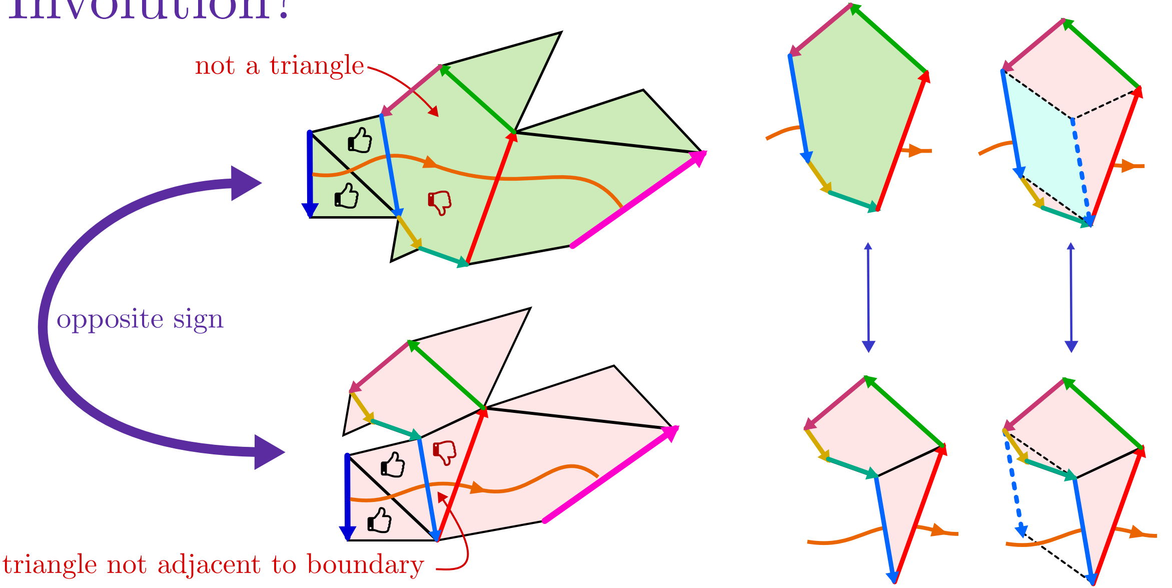
Involution?



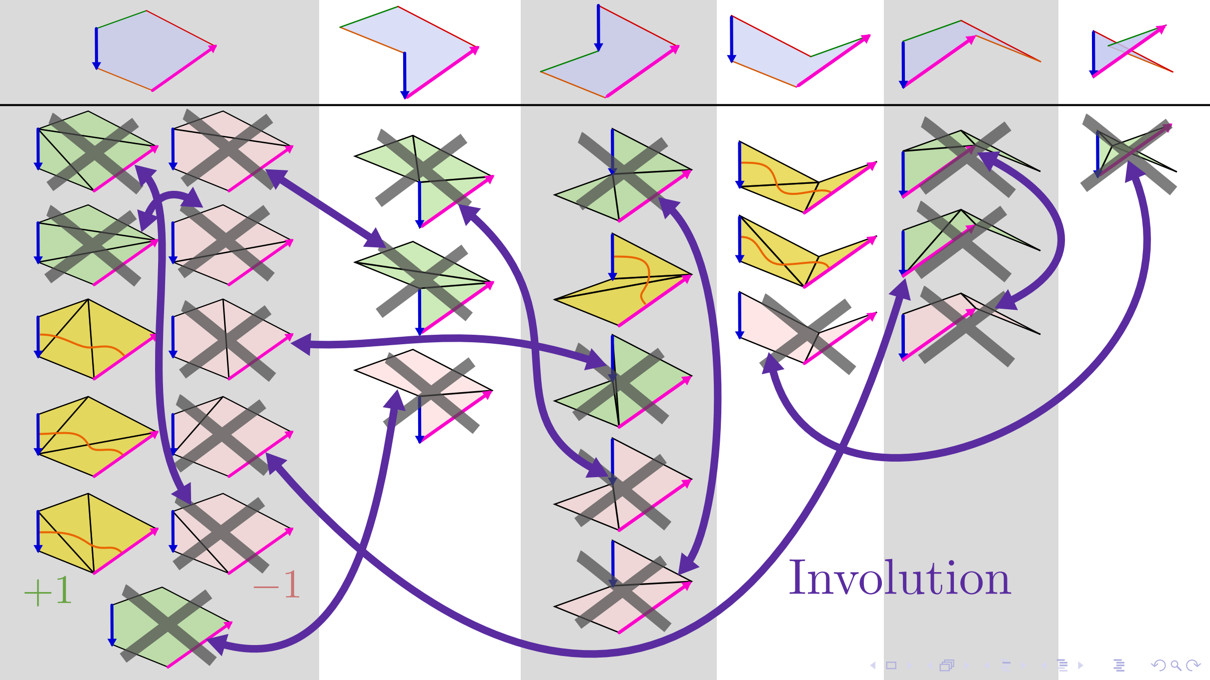
Involution?



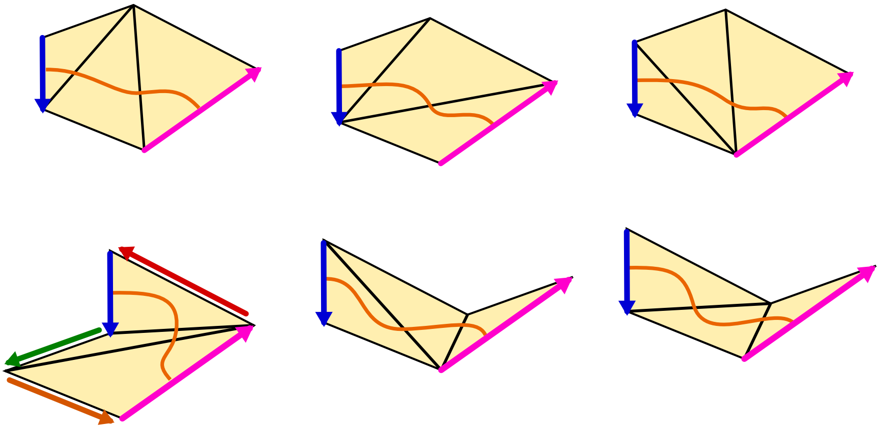
Involution?

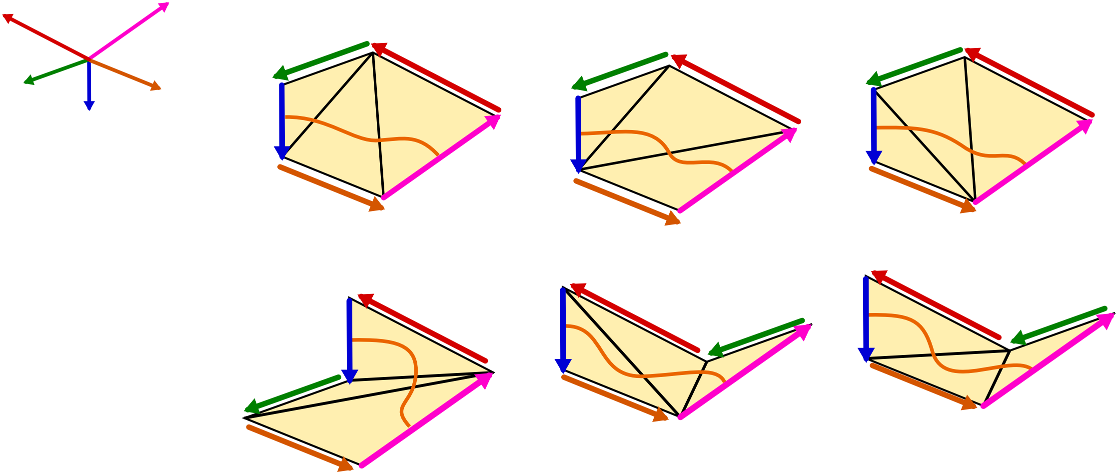


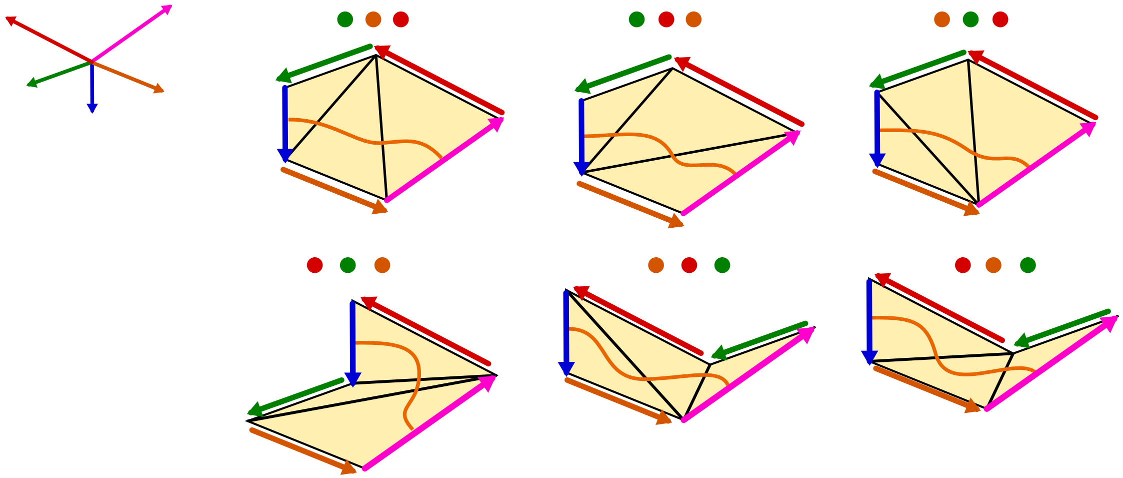
not a triangle
opposite sign
triangle not adjacent to boundary

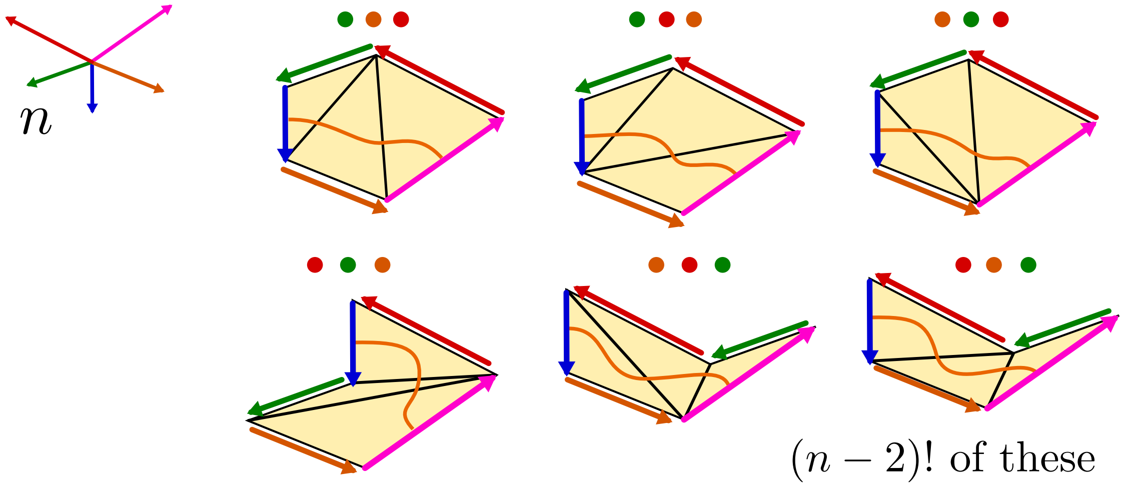


+1
−1
Involution

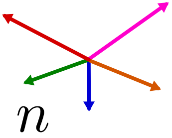


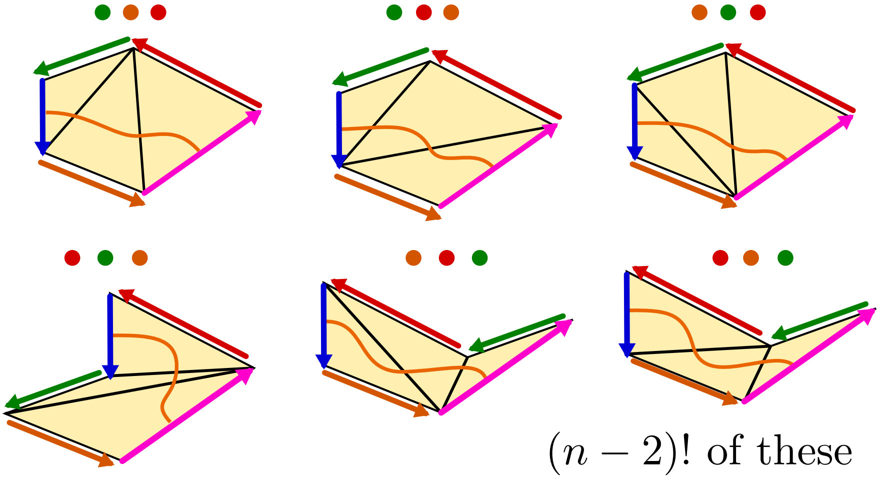






$(n-2)!$ of these





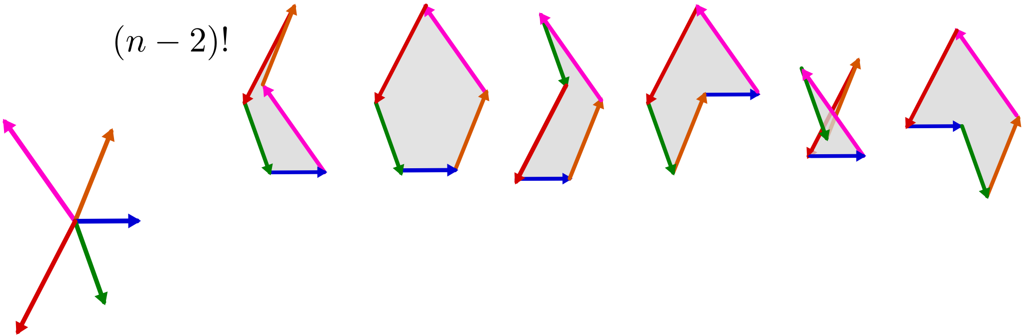
$(n-2)!$ of these

Theorem (TB, '24+)

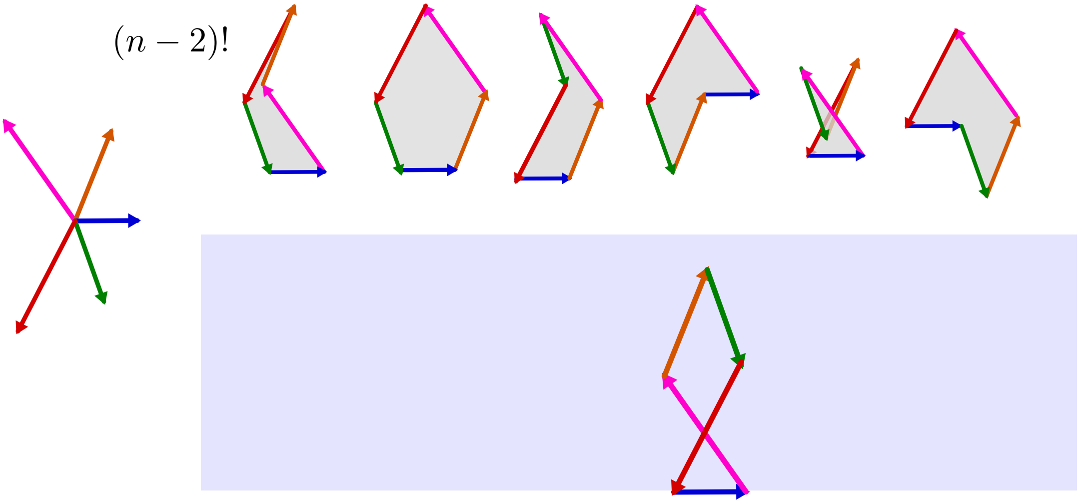
The number of n-sided disks with sides in a fixed generic[*] zero-sum set $Z \subset \mathbb{R}^2$ is $(n-2)!$.

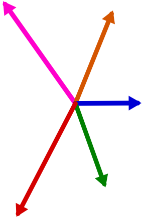
* Z is *generic* if each non-trivial pair of subsets $\subset Z$ has linearly independent sums.
(In particular, all angles distinct!)

$$(n-2)!$$

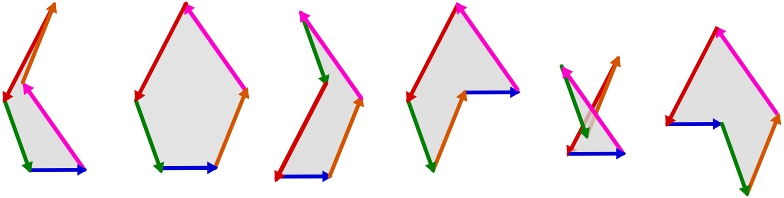


$$(n-2)!$$





$(n-2)!$



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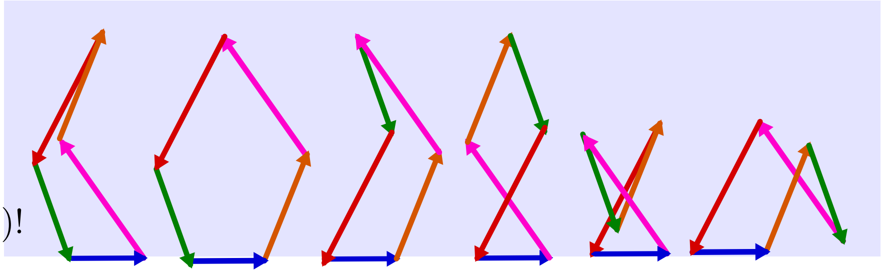
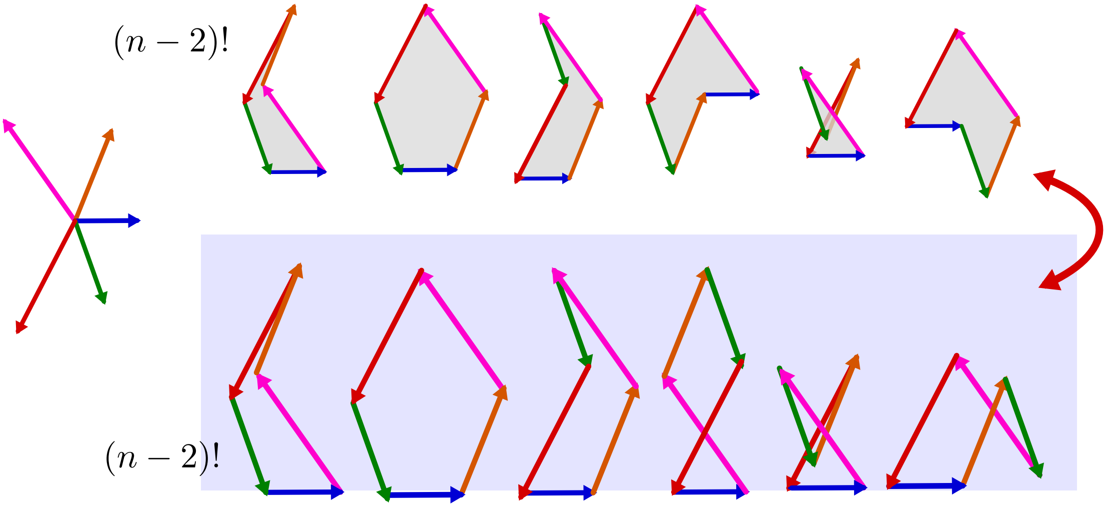


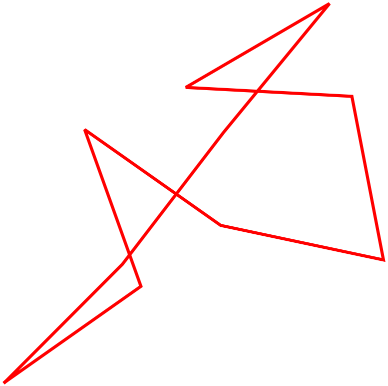


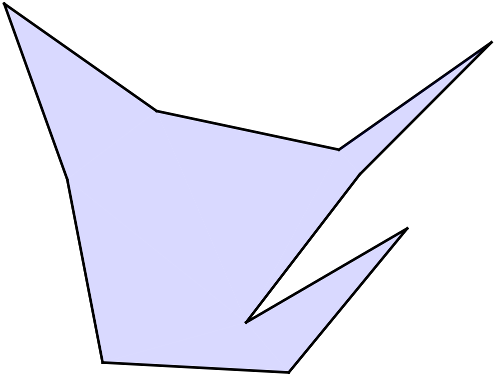
Theorem (TB, '24+)

If Z is generic, there is an explicit bijection between flat disks and excursions in the half-plane.

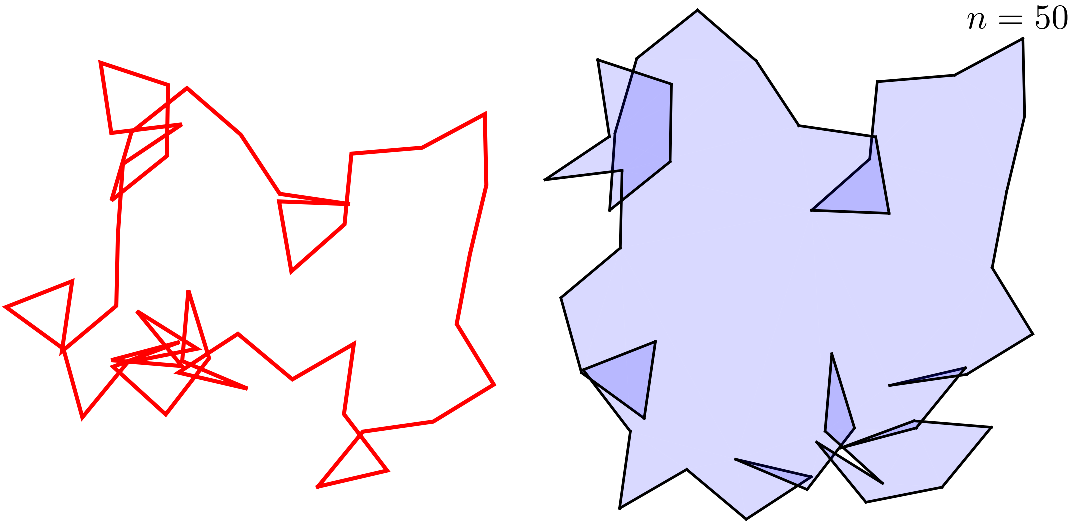
Show me da random diskz!

$n = 10$





Show me da random diskz!



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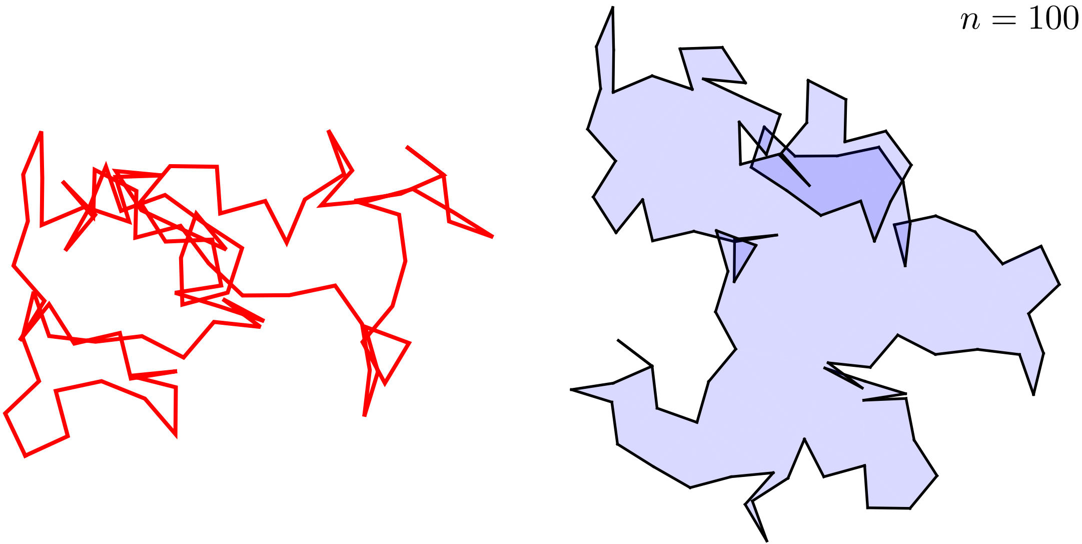
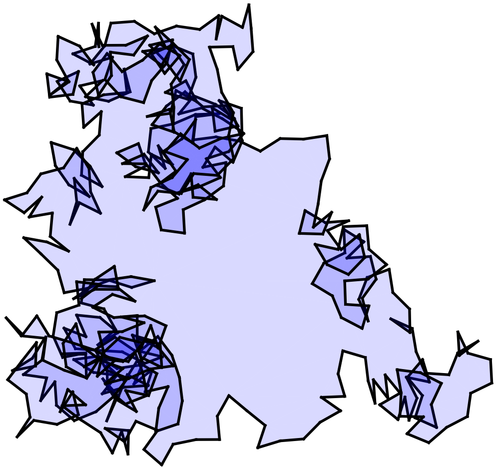
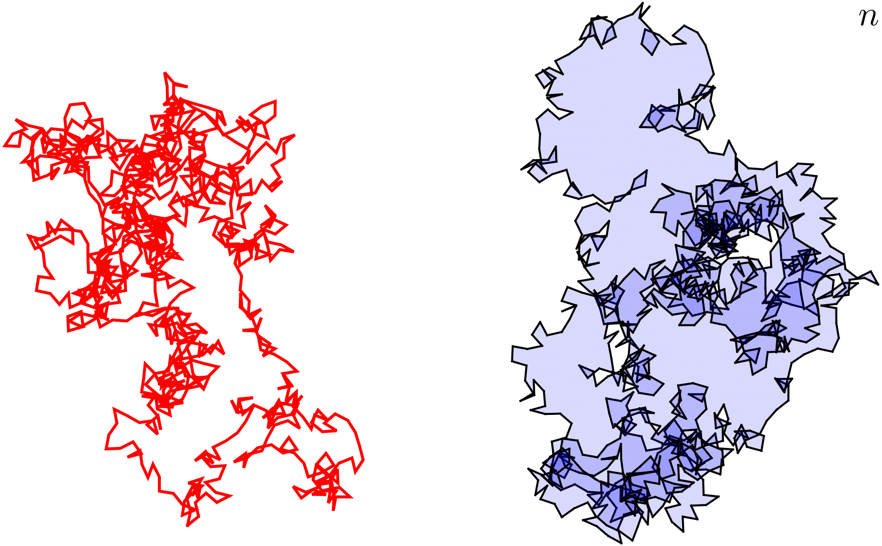


Show me da random diskz!

$n = 500$

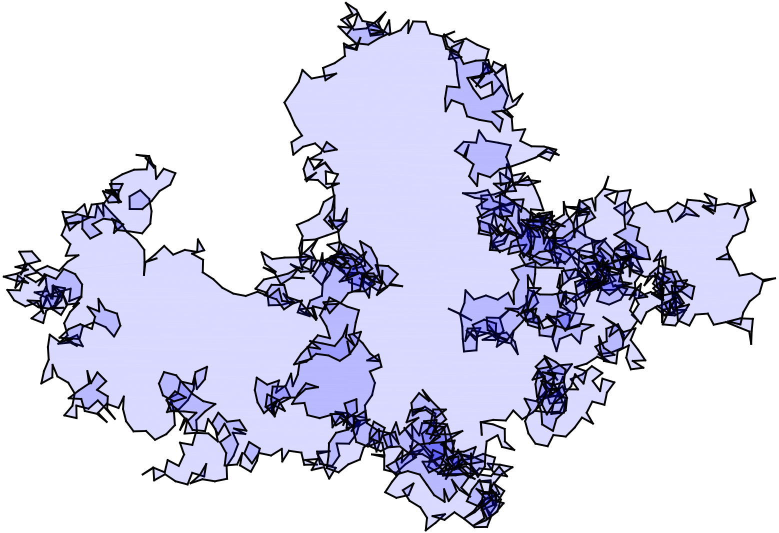


Show me da random diskz!



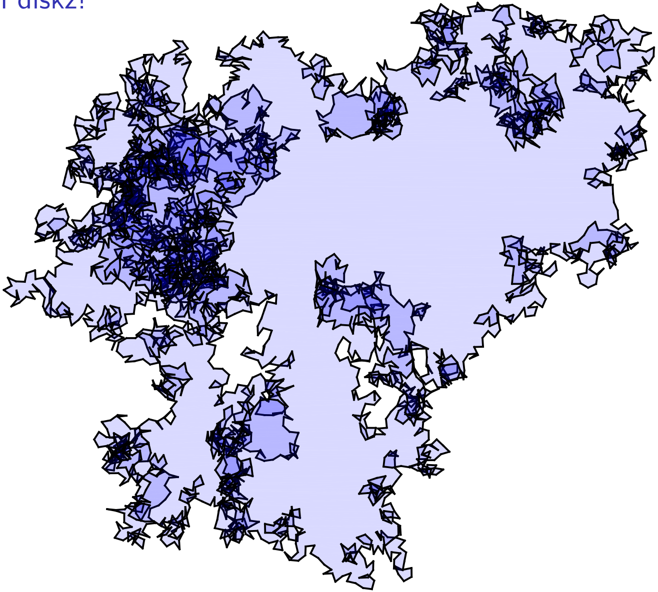
$n = 1000$

Show me da random diskz!



$n = 2000$

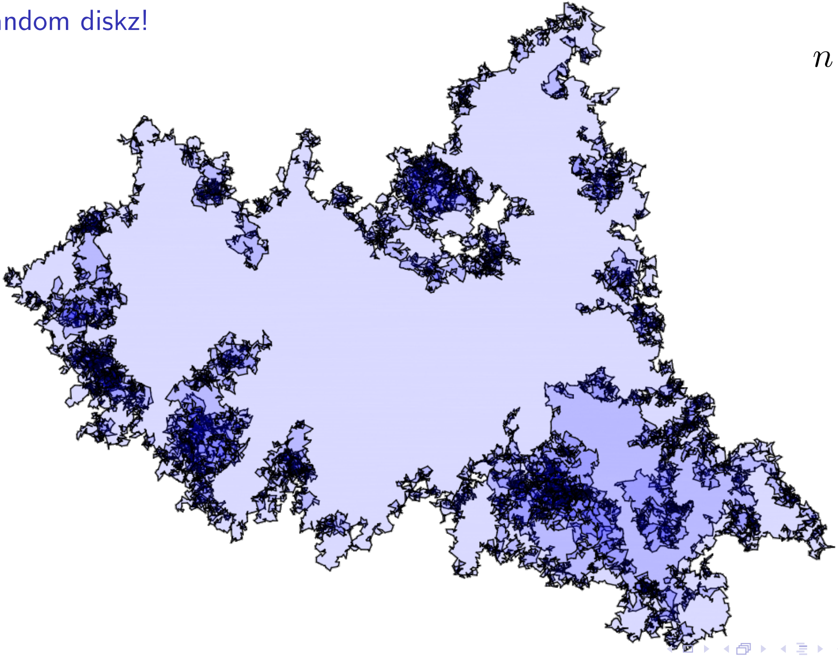
Show me da random diskz!



$n = 5000$

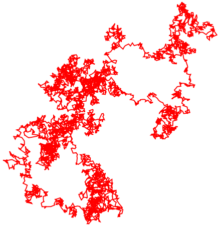
Show me da random diskz!

$n = 25000$



Area of uniform random flat disk

- Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge in $\mathbb{R}^2$



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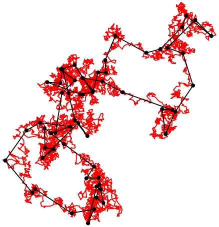
- Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge in $\mathbb{R}^2$ and $Z_n = \{B_{\frac{1}{n}}, B_{\frac{2}{n}-\frac{1}{n}}, B_{\frac{3}{n}-\frac{2}{n}}, \ldots, -B_{1-\frac{1}{n}}\}$.



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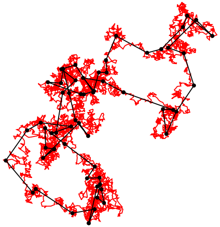

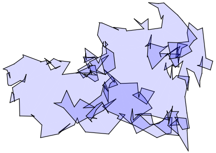

Theorem (TB, '24+)

The area of the uniform flat disk with sides Z_n satisfies

$$\mathbb{E}[\mathrm{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \textit{as } n \to \infty,$$

with $C = -\int_0^1 \Gamma'(x+1)/(4\pi\sqrt{x}\,\Gamma(x+1))\mathrm{d}x = 0.0285\ldots$

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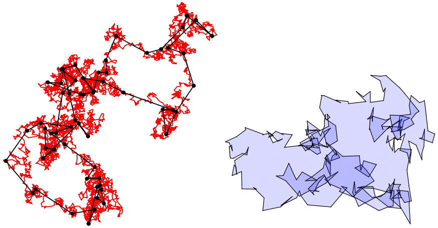

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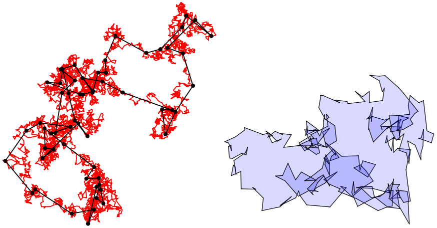
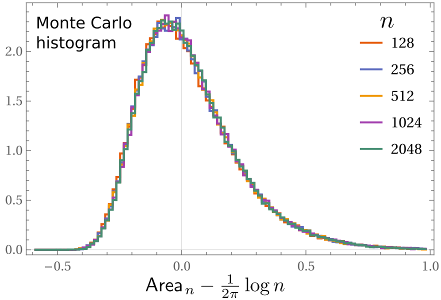
- Question: does $\mathrm{Area}_n - \frac{\log n}{2\pi}$ converge in distribution?





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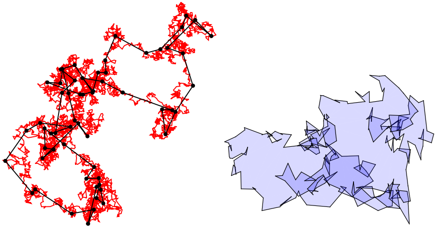
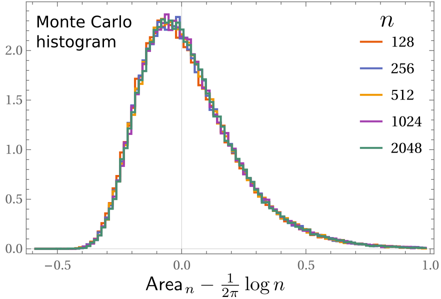
- Question: does $\mathrm{Area}_n - \frac{\log n}{2\pi}$ converge in distribution?
- Eerie similarity to N-winding area A_N of $(B_t)_t$ itself:

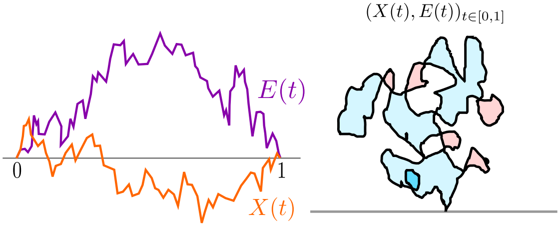
$$\sum_{N=1}^{n} N A_N - \frac{\log n}{2\pi} \quad \text{converges a.s.} \quad \text{[Werner, '94]}$$





Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a *Brownian flat disk*: Let $(X(t))_{t\in[0,1]}$ a Brownian bridge
$(E(t))_{t\in[0,1]}$ a Brownian excursion



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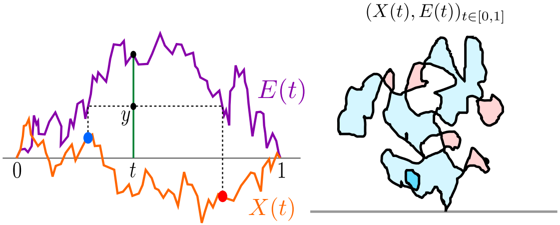
$$\text{and} \quad Y(t) := \int_0^{E(t)} \operatorname{sign}\Big[X\big(\min\{s \geq t : E(s) = y\}\big) - X\big(\min\{s \leq t : E(s) = y\}\big)\Big]\mathrm{d}y.$$



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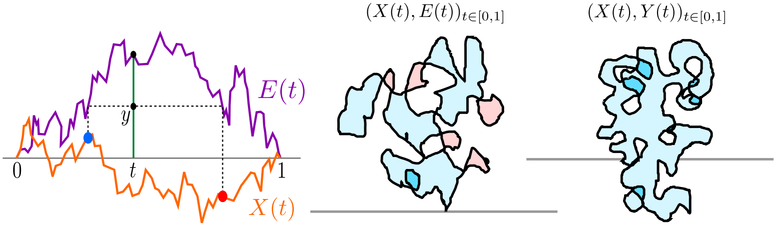

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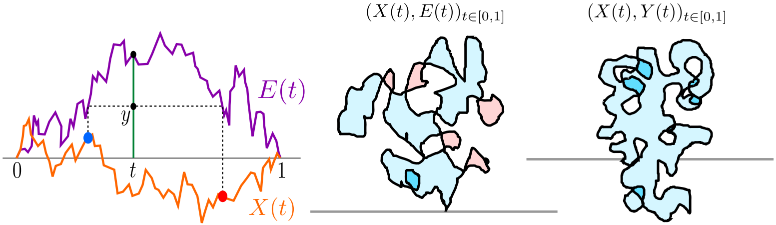

Conjecture (TB)

The law of $(X(t),Y(t))_{t\in[0,1]}$ is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides Z_n as $n\to\infty$.

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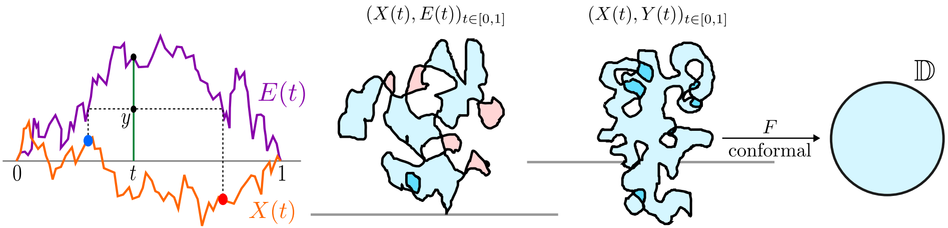

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Conjecture (F. Ferrari, '24 $\leftarrow$ my interpretation)

If F is conformal map to $\mathbb{D}$,

Conjectural scaling limit: Brownian flat disk?

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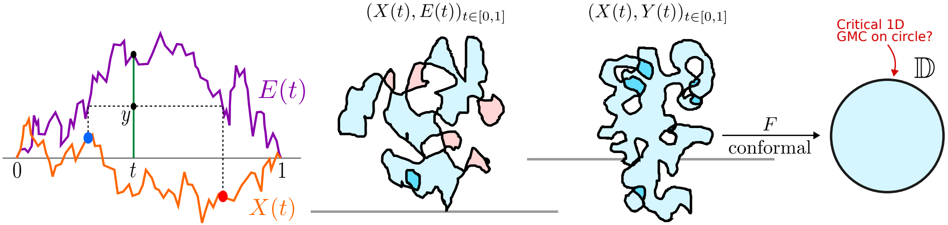

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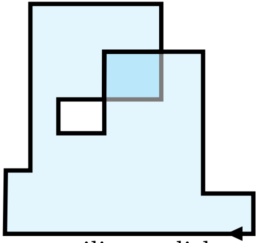
Conjecture (F. Ferrari, '24 $\leftarrow$ my interpretation)

If F is conformal map to $\mathbb{D}$, Lebesgue on $[0,1]$ is pushed by $F(X(\cdot), Y(\cdot))$ to critical GMC on $\partial\mathbb{D}$.

Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.



rectilinear disk

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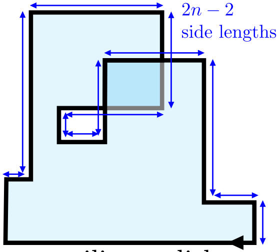

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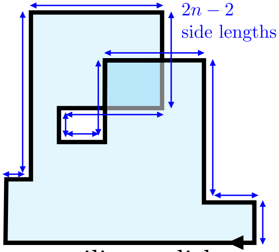

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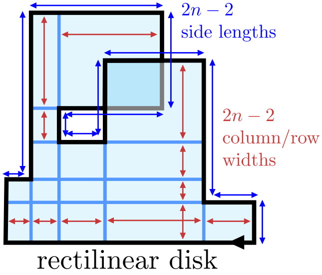

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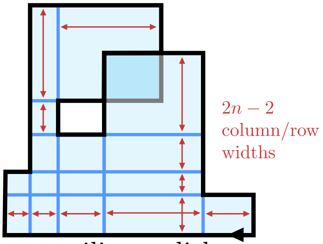

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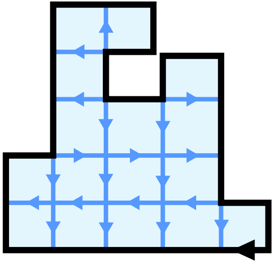

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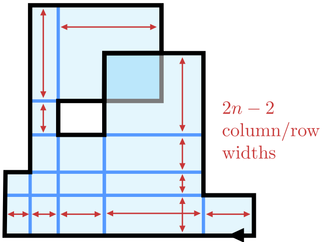

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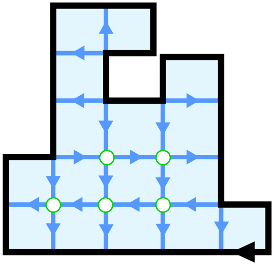

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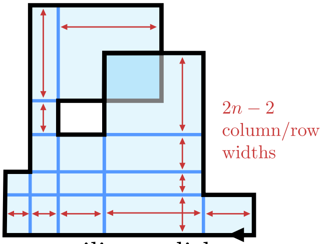

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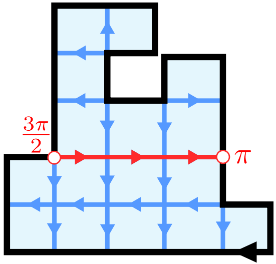

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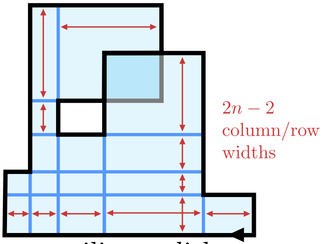
Theorem (TB, '24+)

The number of rigid quadrangulations is asymptotic to

$$\frac{(4\pi)^{n-1}}{16n^2\log^2 n}(1+o(1)) \quad \text{as } n\to\infty.$$



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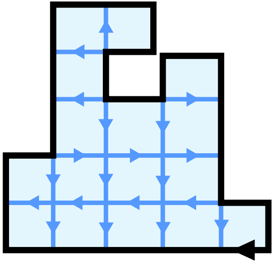

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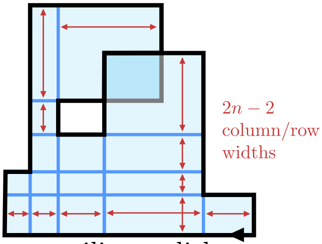
- Akin to random planar map models in universality class of $LQG_{\gamma=2}$...



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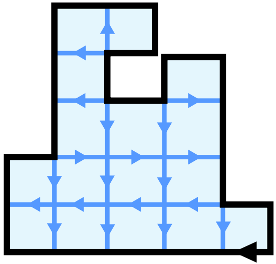
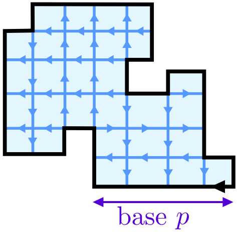
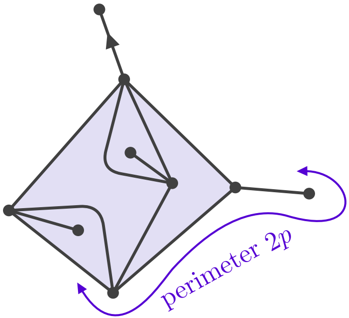

rigid quadrangulation

Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

rigid quadrangulation





quadrangulation

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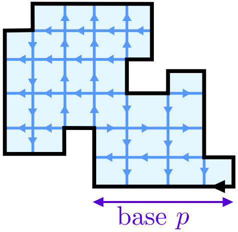

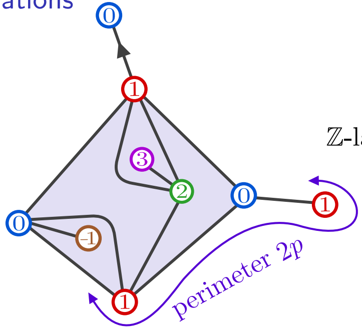

$\mathbb{Z}$-labeled quadrangulation

Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

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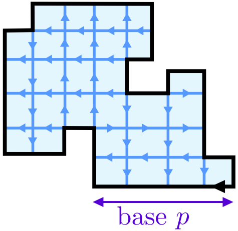
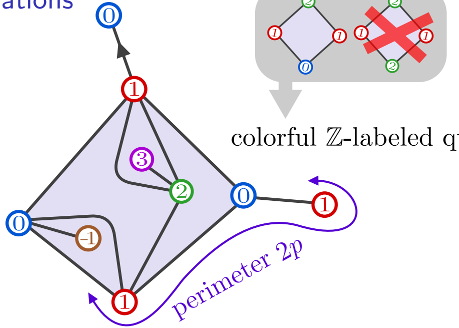
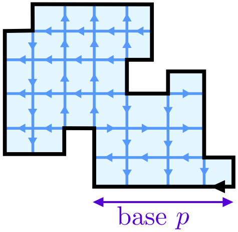


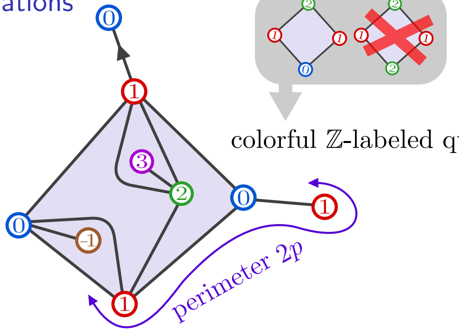


colorful $\mathbb{Z}$-labeled quadrangulation

Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

rigid quadrangulation





colorful $\mathbb{Z}$-labeled quadrangulation

Theorem (TB, '24+)

For $n \geq 2$ and $p \geq 1$ there exists a bijection

$$\left\{ \begin{array}{c} \textit{rigid quadrangulations with} \\ 2n \textit{ corners and base } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{colorful } \mathbb{Z}\textit{-labeled quadrangulations} \\ \textit{with } n \textit{ vertices and perimeter } 2p \end{array} \right\}$$

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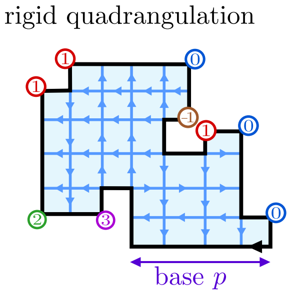

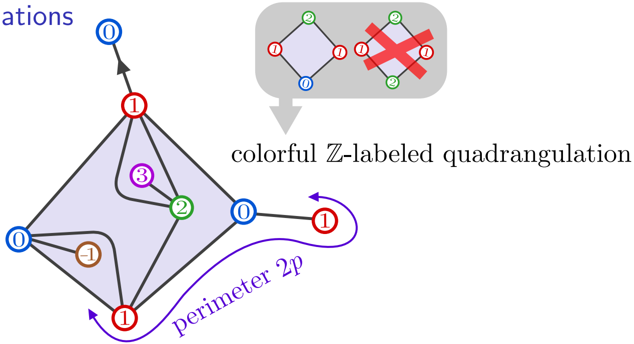

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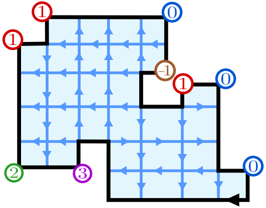
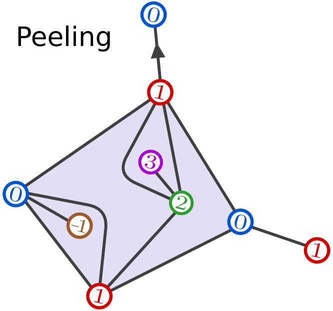
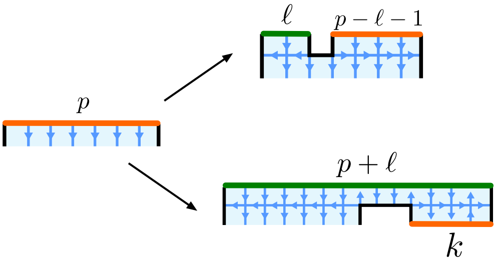
$$\tfrac{\pi}{2}\textit{-corner with } \underbrace{\textit{turning number } \ell}_{\#\textit{left}-\#\textit{right}} \longleftrightarrow \textit{vertex with label } \ell$$

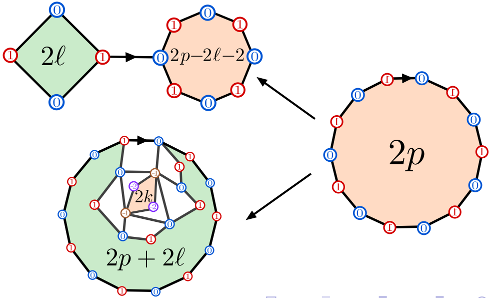
Bijection via exploration

Scanning vs Peeling









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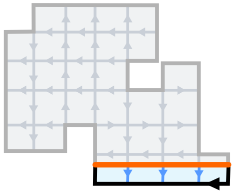

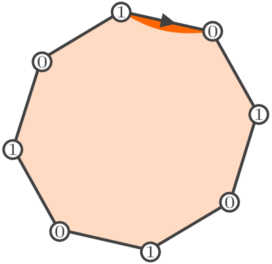

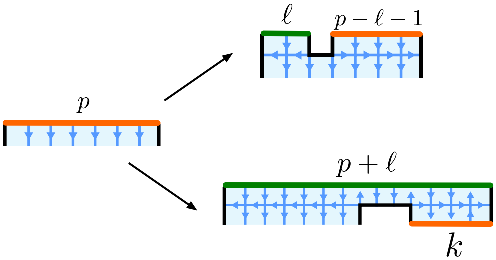

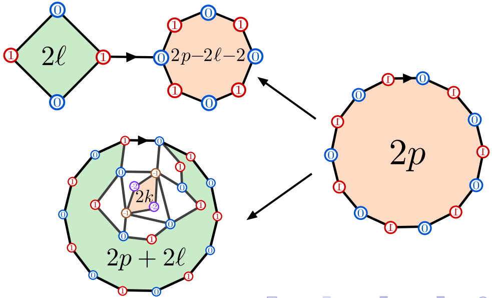

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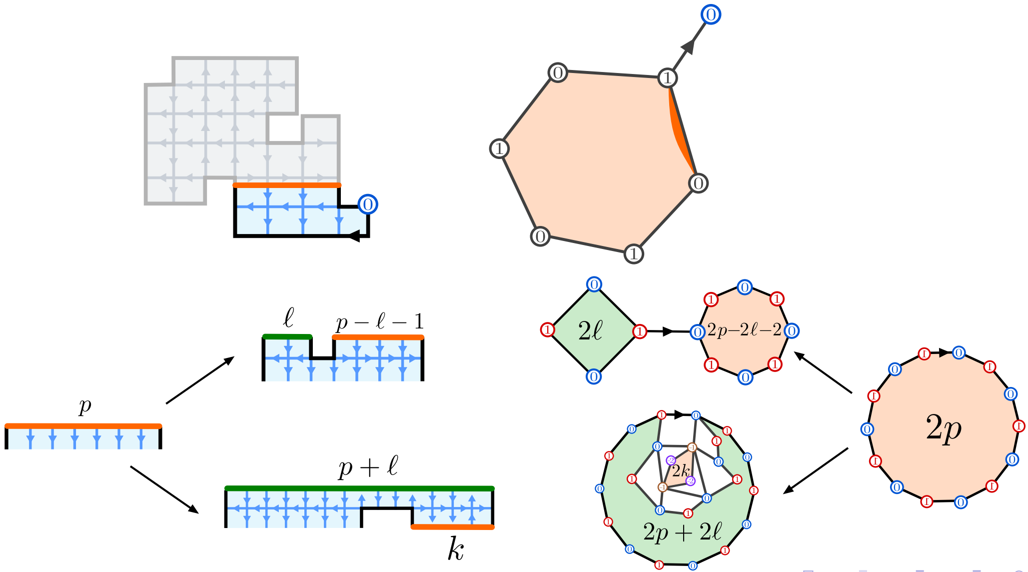

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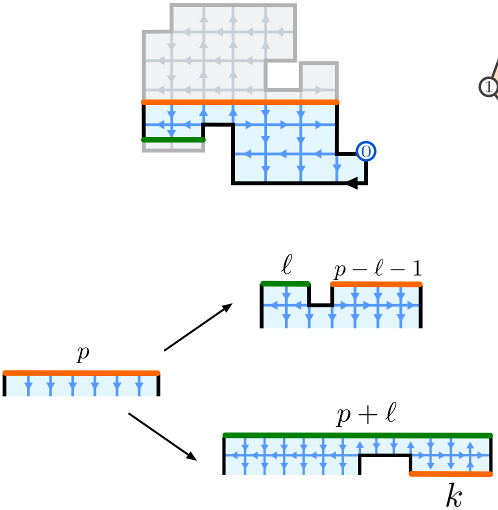

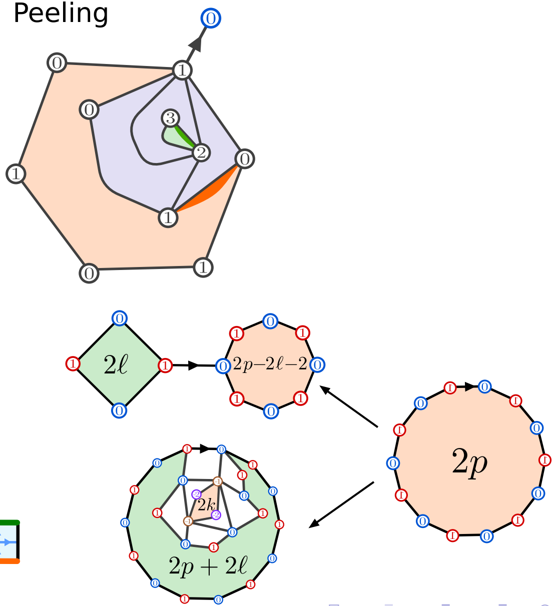

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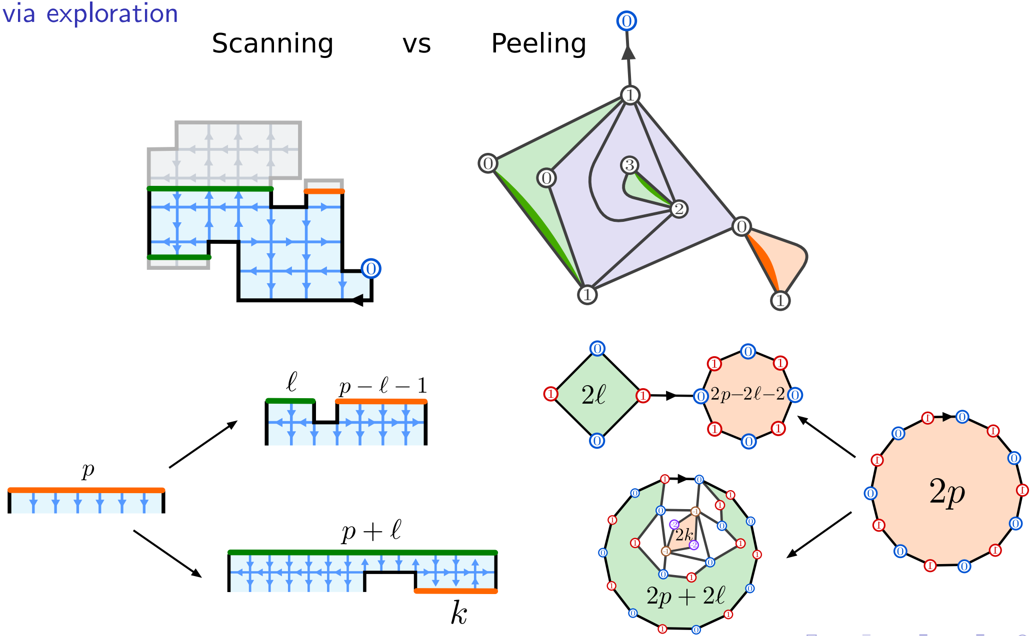

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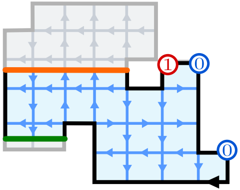

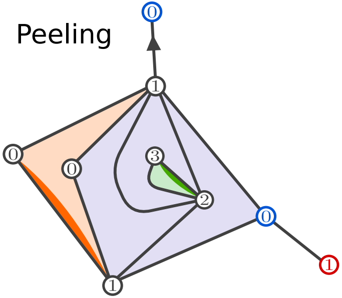

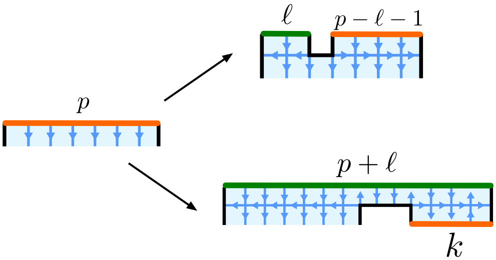

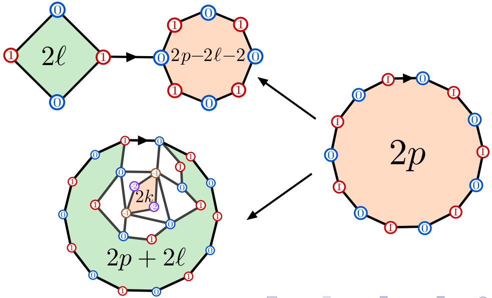

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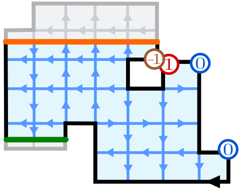

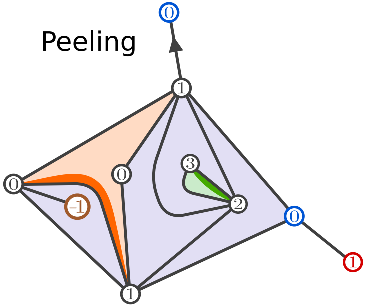

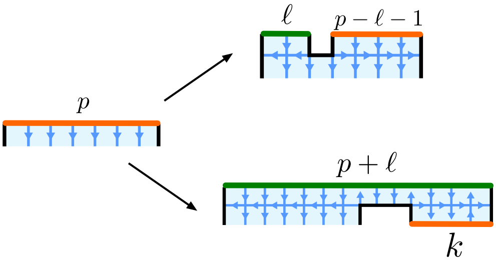

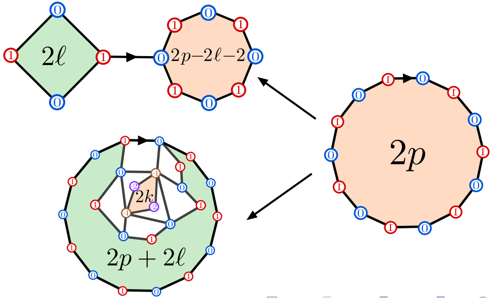

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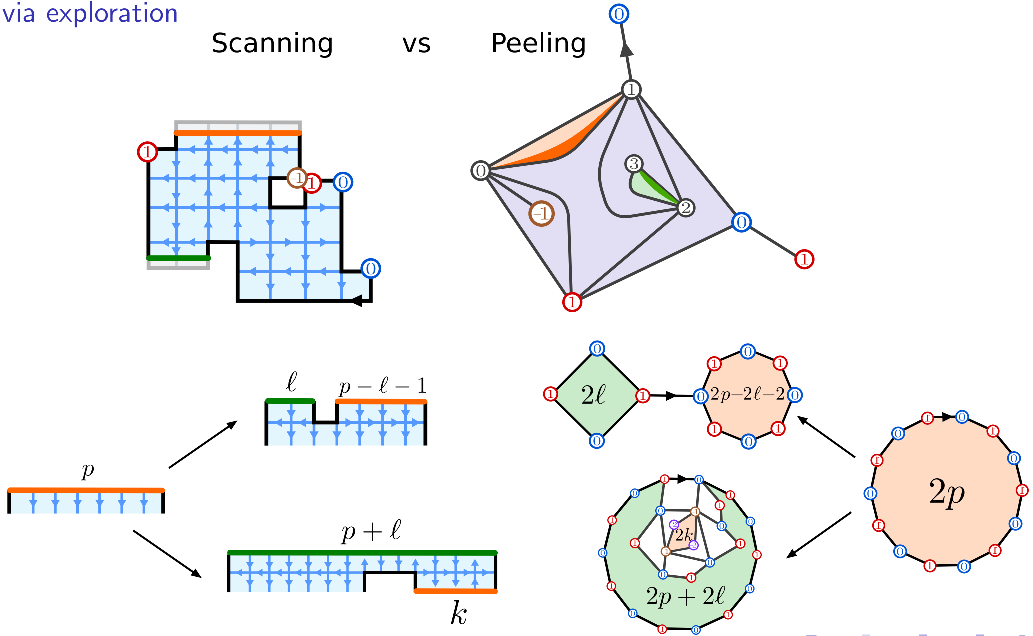

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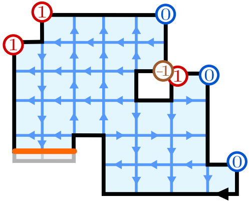

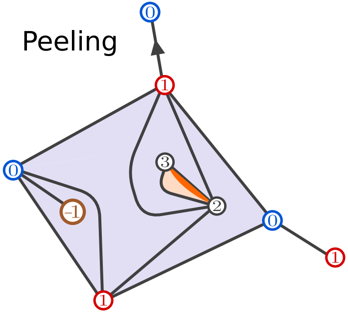

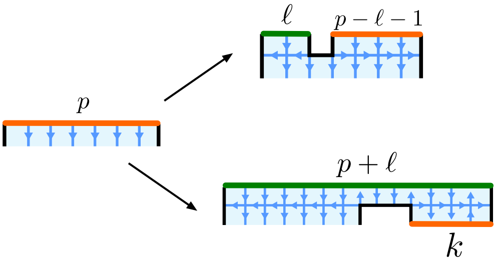

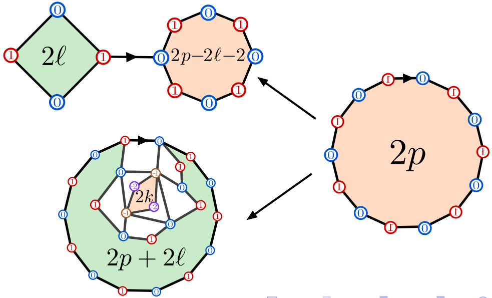

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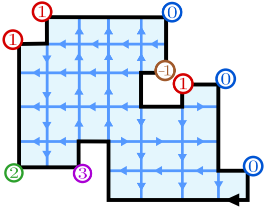
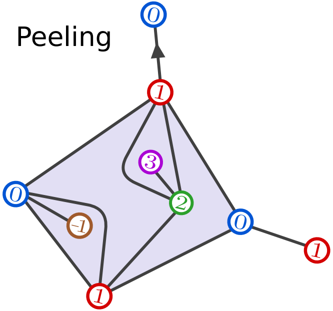




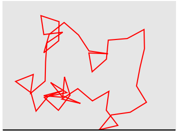
- Many guises of these labeled quadrangulations: planar Eulerian orientations, $\mathbb{Z}$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00][Zinn-Justin, '00][Elvey Price, Guttmann, '17]
- Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

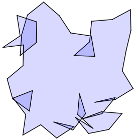
The generating function of perimeter-2p colorful $\mathbb{Z}$-labeled quadrangulations is

$$Q^{(p)}(x) = \sum_{k\geq p} \frac{1}{k+1}\binom{2k}{k}\binom{2k-p}{k} R(x)^{k+1}, \qquad \textit{when} \sum_{k\geq p} \frac{1}{k+1}\binom{2k}{k}^2 R(x)^{k+1} = x.$$

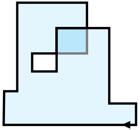
Status and outlook in pictures



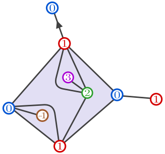
walks in half plane



polygonal flat disks



rectilinear flat disks



labeled quadrangulations

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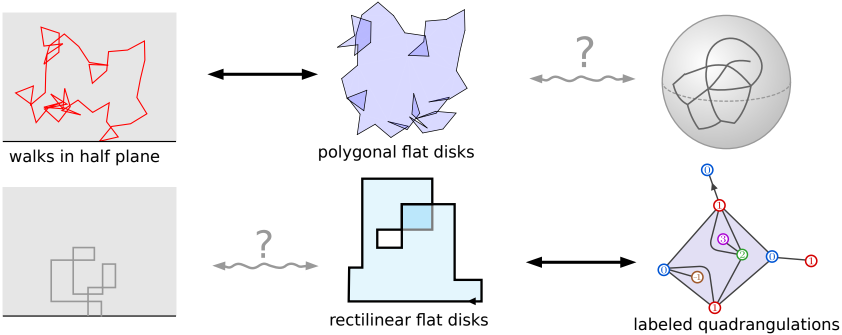

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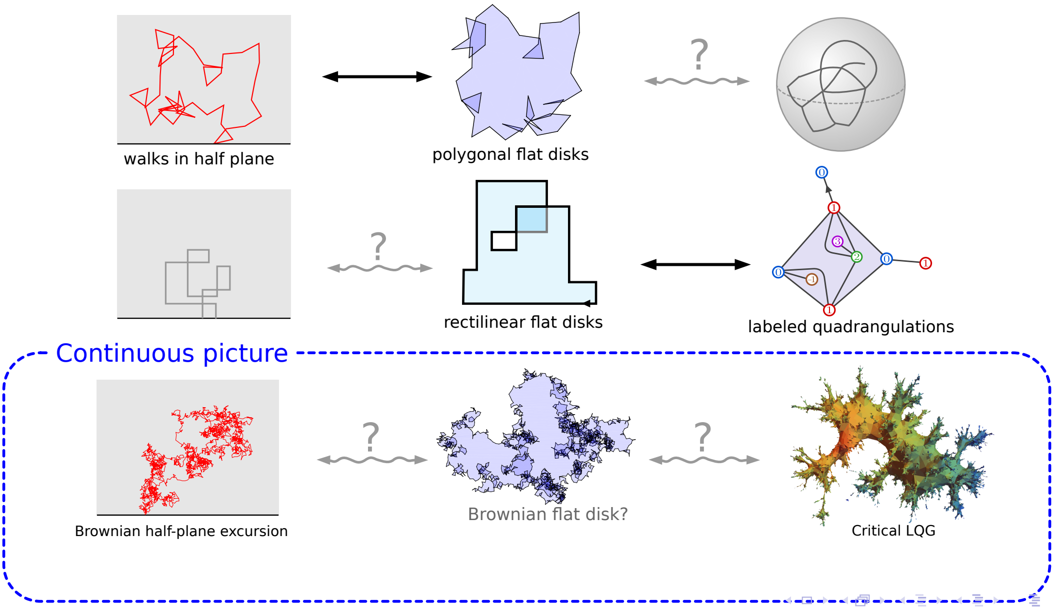

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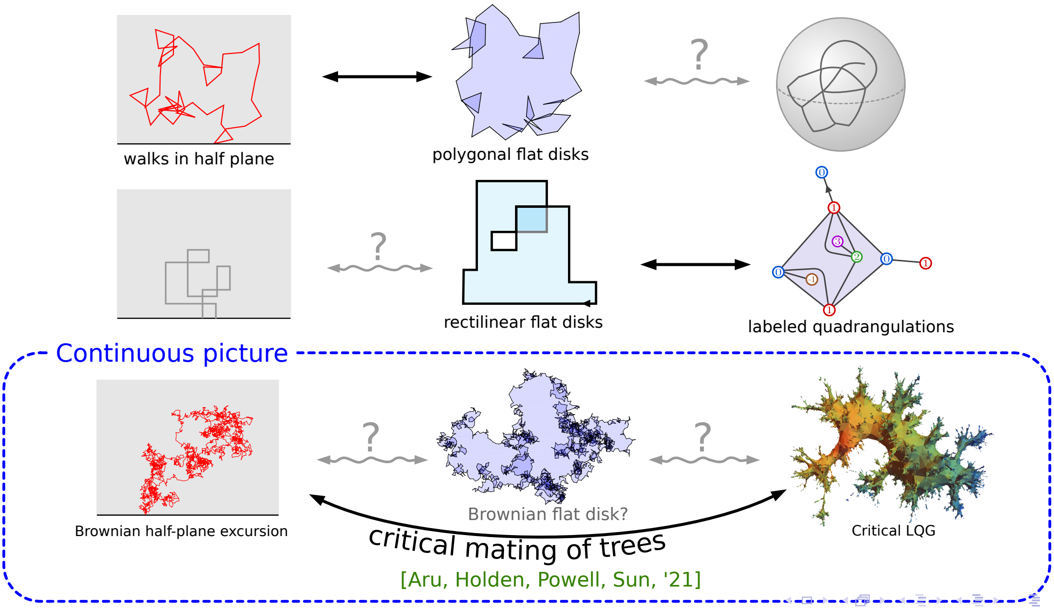

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