Uniform random flat disks

Photo credit: By OIST from Onna Village, Japan - Aerial View of Campus Buildings
16-07-2024 Self-similar Markov Trees mini-workshop @ CIRM Marseille
Timothy Budd
T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/
Motivation: Quantum JT gravity

Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[
Z_{\text{EQG}}(\Lambda, L) = \int \text{dg } e^{-\Lambda \text{Area}_{\{\text{all metrics with bdry length L}\}}},
\]

\[
Z_{\text{QJT}}(L, \Lambda) = \int \text{dg } e^{-\Lambda \text{Area}_{\{\text{const. curvature metrics}\}}}
\]
Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[
Z_{\text{EQG}}(\Lambda, L) = \int \mathcal{D}g \ e^{-\Lambda \text{Area}}_{\{\text{all metrics with bdry length L}\}},
\]

\[
Z_{\text{QJT}}(L, \Lambda) = \int \mathcal{D}g \ e^{-\Lambda \text{Area}}_{\{\text{const. curvature metrics}\}}.
\]

[Jackiw, Teitelboim, '80s]

- Activity in Quantum JT gravity due to holographic correspondence.

[Stanford, Yang, Turiaci, Verlinde, Griguolo, Panerai, Papalini, Seminara, . . .]

- But predominantly boundaries are asymptotic / treated perturbatively. Is there a finite boundary equivalent?

[Ferrari: should allow disks to self-overlap.]

[Ferrari, '24]

Is there a tractable model of uniform random discrete flat disks?
Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[
Z_{\text{EQG}}(\Lambda, L) = \int_{\{\text{all metrics with bdry length } L\}} \text{d} \varrho \, e^{-\Lambda \text{Area}}, \\
Z_{\text{QJT}}(L, \Lambda) = \int_{\{\text{const. curvature metrics}\}} \text{d} \varrho \, e^{-\Lambda \text{Area}}.
\]

[Jackiw, Teitelboim, '80s]

- Activity in Quantum JT gravity due to holographic correspondence.
  [Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Verlinde, Witten, Yang, .............]

- But predominantly boundaries are asymptotic / treated perturbatively. Is there a finite boundary equivalent? [Stanford, Yang, Turiaci, Verlinde, Griguolo, Panerai, Papalini, Seminara, . . . . . . . . . . . .

- Ferrari: should allow disks to self-overlap. [Ferrari, '24]

Is there a tractable model of uniform random discrete flat disks?
Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[ Z_{\text{EQG}}(\Lambda, L) = \int_{\{\text{all metrics with bdry length } L\}} dg \, e^{-\Lambda \text{Area}}, \]
\[ Z_{\text{QJT}}(L, \Lambda) = \int_{\{\text{const. curvature metrics}\}} dg \, e^{-\Lambda \text{Area}}, \]

[Jackiw, Teitelboim, '80s]

- Activity in Quantum JT gravity due to holographic correspondence.

[Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Verlinde, Witten, Yang, ............]

- But predominantly boundaries are asymptotic / treated perturbatively. Is there a finite boundary equivalent?

[Ferrari, '24]
Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[
Z_{\text{EQG}}(\Lambda, L) = \int_{\{\text{all metrics with bdry length } L\}} d\mathcal{g} e^{-\Lambda \text{Area}},
\]

\[
Z_{\text{QJT}}(L, \Lambda) = \int_{\{\text{const. curvature metrics}\}} d\mathcal{g} e^{-\Lambda \text{Area}}.
\]

[Jackiw, Teitelboim, ‘80s]

- Activity in Quantum JT gravity due to holographic correspondence.

[Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Verlinde, Witten, Yang, ...]

- But predominantly boundaries are asymptotic / treated perturbatively. Is there a finite boundary equivalent?

[Stanford, Yang, Turiaci, Verlinde, Griguolo, Panerai, Papalini, Seminara, ...]
Motivation: Quantum JT gravity

- Two-dimensional quantum gravity on the disk:

\[
Z_{\text{EQG}}(\Lambda, L) = \int_{\{\text{all metrics with bdry length } L\}} \text{d}\gamma \ e^{-\Lambda \text{Area}},
\]

\[
Z_{\text{QJT}}(L, \Lambda) = \int_{\{\text{const. curvature metrics}\}} \text{d}\gamma \ e^{-\Lambda \text{Area}}.
\]

[Jackiw, Teitelboim, '80s]

- Activity in Quantum JT gravity due to holographic correspondence.
  [Kitaev, Maldacena, Maxfield, Mertens, Polchinski, Saad, Shenker, Stanford, Turiaci, Verlinde, Witten, Yang, . . . . . . . . . . . .]

- But predominantly boundaries are asymptotic / treated perturbatively. Is there a finite boundary equivalent?
  [Stanford, Yang, Turiaci, Verlinde, Griguolo, Panerai, Papalini, Seminara, . . .]

- Ferrari: should allow disks to self-overlap. [Ferrari, '24]

Is there a tractable model of uniform random discrete flat disks?
What is a discrete flat disks?

convex polygon

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]
What is a discrete flat disks?

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]

convex polygon

non-convex polygon
What is a discrete flat disks?

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]
What is a discrete flat disks? 

- convex polygon
- non-convex polygon
- self-overlapping polygons

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]
What is a discrete flat disks?

convex polygon

non-convex polygon

self-overlapping polygons
What is a discrete flat disks? [Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]

- **convex polygon**
- **non-convex polygon**
- **self-overlapping polygons**

$>2\pi$
What is a discrete flat disks?

convex polygon

non-convex polygon

self-overlapping polygons

modulo translation, not modulo rotation

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]

>2π

>2π
What is a discrete flat disks?

convex polygon
non-convex polygon
self-overlapping polygons

modolo translation, not modulo rotation

How many discrete flat disks with $n$ sides are there?

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]
First model: fix the sides!
First model: fix the sides!

* generic = each non-trivial pair of subsets $\subset Z$ has linearly independent sums.
  (In particular, all angles distinct!)
First model: fix the sides!

\[ \mathbb{F}_n \]

generic* zero-sum set \( Z \subset \mathbb{R}^2 \setminus \{0\} \) of \( n \) vectors

* \textit{generic} = each non-trivial pair of subsets \( \subset Z \) has linearly independent sums.
(In particular, all angles distinct!)
First model: fix the sides!

Generic* zero-sum set $Z \subset \mathbb{R}^2 \setminus \{0\}$ of $n$ vectors

how many disks are there?

* generic = each non-trivial pair of subsets $\subset Z$ has linearly independent sums.
(In particular, all angles distinct!)
Theorem (Devadoss, Shah, Shao, Winston, ’09 & Braun, Ehrenborg, ’10)

For any polygon \( P \), the complex \( K_P \) is contractible.

Hence, Euler characteristic \( \chi = \sum_{\sigma} (-1)^{\dim \sigma} \).

**Associahedron** \( K_n \)

[Tamari, ’51] [Stasheff, ’63]
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon $P$, the complex $K_P$ is contractible.

Hence, Euler characteristic $\chi = \sum_{\sigma} (-1)^{\dim \sigma} = 1$. 

Associahedron $K_n$

[Tamari, '51] [Stasheff, '63]
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon $P$, the complex $K_P$ is contractible. Hence, Euler characteristic $\chi = X_{\text{cells}} \sigma (-1)^{\dim \sigma} = 1$. 

**Associahedron $K_n$**

[Tamari, '51] [Stasheff, '63]
Theorem (Devadoss, Shah, Shao, Winston, ’09 & Braun, Ehrenborg, ’10)

For any polygon $P$, the complex $K_P$ is contractible.

Hence, Euler characteristic $\chi = \sum_{\sigma} (-1)^{\dim \sigma}$.

*Associahedron $K_n$*

[Tamari, ’51] [Stasheff, ’63]
Theorem (Devadoss, Shah, Shao, Winston, ’09 & Braun, Ehrenborg, ’10)

For any polygon $P$, the complex $K_P$ is contractible. Hence, Euler characteristic $\chi = \sum_{\sigma} (-1)^{\dim \sigma} = 1$. 

Associahedron $K_n$

[Tamari, ’51] [Stasheff, ’63]

Convex diagonalizations

[Devadoss, Shah, Shao, Winston, ’09]
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon $P$, the complex $K_P$ is contractible.

Hence, Euler characteristic $\chi = \sum_{\sigma} (-1)^{\dim \sigma} = 1$.

**Associahedron** $K_n$

[Tamari, '51] [Stasheff, '63]

**Convex diagonalizations** $K_P$

[Devadoss, Shah, Shao, Winston, '09]
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon $P$, the complex $K_{P}$ is contractible. Hence, Euler characteristic $\chi = X_{\text{cells}} \sigma (-1)^{\text{dim} \sigma} = 1$.
Theorem (Devadoss, Shah, Shao, Winston, ’09 & Braun, Ehrenborg, ’10)

For any polygon $P$, the complex $K_P$ is contractible. Hence, Euler characteristic $\chi = \sum_{\sigma} (-1)^{\dim \sigma} = 1$. 

**Associahedron $K_n$**

[Tamari, ’51] [Stasheff, ’63]

**Convex diagonalizations $K_P$**

[Devadoss, Shah, Shao, Winston, ’09]
Theorem (Devadoss, Shah, Shao, Winston, ’09 & Braun, Ehrenborg, ’10)

For any polygon $P$, the complex $\mathcal{K}_P$ is contractible.
Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon \( P \), the complex \( \mathcal{K}_P \) is contractible. Hence, Euler characteristic \( \chi = \sum_{\text{cells } \sigma} (-1)^{\dim \sigma} = 1 \).
Involution?
Involution?

not a triangle

triangle not adjacent to boundary
Involution?

not a triangle

triangle not adjacent to boundary
Involution?

not a triangle

triangle not adjacent to boundary
Involution?

not a triangle

triangle not adjacent to boundary
Involution?

not a triangle

opposite sign

triangle not adjacent to boundary
Theorem (TB, '24+)  

The number of \( n \)-sided disks with sides in a fixed generic zero-sum set \( Z \subset \mathbb{R}^2 \) is \((n - 2)!\).  

\( Z \) is generic if each non-trivial pair of subsets \( \subset Z \) has linearly independent sums. (In particular, all angles distinct!)
Theorem (TB, '24+): The number of \( n \)-sided disks with sides in a fixed generic zero-sum set \( Z \subset \mathbb{R}^2 \) is \( (n-2)! \).

\( Z \) is generic if each non-trivial pair of subsets \( \subset Z \) has linearly independent sums. (In particular, all angles distinct!)
Theorem (TB, '24+)

The number of \( n \)-sided disks with sides in a fixed generic zero-sum set \( Z \subset \mathbb{R}^2 \) is \( (n - 2)! \).

\( Z \) is generic if each non-trivial pair of subsets \( \subset Z \) has linearly independent sums.
(In particular, all angles distinct!)
Theorem (TB, '24+) 

The number of $n$-sided disks with sides in a fixed generic \* zero-sum set $\mathcal{Z} \subset \mathbb{R}^2$ is $(n-2)!$. 

\* $\mathcal{Z}$ is generic if each non-trivial pair of subsets $\subset \mathcal{Z}$ has linearly independent sums. (In particular, all angles distinct!)

$(n - 2)!$ of these
Theorem (TB, '24+)

The number of $n$-sided disks with sides in a fixed generic* zero-sum set $Z \subset \mathbb{R}^2$ is $(n - 2)!$.

* $Z$ is generic if each non-trivial pair of subsets $\subset Z$ has linearly independent sums. (In particular, all angles distinct!)
Theorem (TB, ’24+

If \( Z \) is generic, there is an explicit bijection between flat disks and excursions in the half-plane.

\((n - 2)!\)
Theorem (TB, '24+)

If $Z$ is generic, there is an explicit bijection between flat disks and excursions in the half-plane.

$$(n - 2)!$$
Theorem (TB, '24+): If $Z$ is generic, there is an explicit bijection between flat disks and excursions in the half-plane.
Theorem (TB, ’24+)

If $Z$ is generic, there is an explicit bijection between flat disks and excursions in the half-plane.
Idea of proof: an extended bijection

- An excursion is a walk $w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2$ such that $w_1, \ldots, w_{n-1}$ are above $w_0, w_n$. 
Idea of proof: an extended bijection

- An excursion is a walk \( w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2 \) such that \( w_1, \ldots, w_{n-1} \) are above \( w_0, w_n \).
- Call a disk unobstructed if no corner is visible horizontally from the base.
Idea of proof: an extended bijection

- An excursion is a walk \( w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2 \) such that \( w_1, \ldots, w_{n-1} \) are above \( w_0, w_n \).

- Call a disk unobstructed if no corner is visible horizontally from the base.
Idea of proof: an extended bijection

- An excursion is a walk $w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2$ such that $w_1, \ldots, w_{n-1}$ are above $w_0, w_n$.
- Call a disk unobstructed if no corner is visible horizontally from the base.

**Theorem**

If $Z$ is generic*, \{excursions with increments $Z$\}

\[ \uparrow \]

\{unobstructed disks with sides $Z$\}

*as before, but also no proper subset of $Z$ has horizontal sum
Idea of proof: an extended bijection

- An excursion is a walk \( w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2 \) such that \( w_1, \ldots, w_{n-1} \) are above \( w_0, w_n \).
- Call a disk unobstructed if no corner is visible horizontally from the base.

Theorem

If \( Z \) is generic*, \( \{ \text{excursions with increments } Z \} \)
\[ \uparrow \]
\( \{ \text{unobstructed disks with sides } Z \} \)
\[ \downarrow \]
\( \{ \text{unordered } Z\text{-leaf-labeled binary trees with up & down branch at every vertex} \} \)

*as before, but also no proper subset of \( Z \) has horizontal sum
Idea of proof: an extended bijection

- An excursion is a walk $w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2$ such that $w_1, \ldots, w_{n-1}$ are above $w_0, w_n$.
- Call a disk unobstructed if no corner is visible horizontally from the base.

**Theorem**

If $Z$ is generic*, $\{\text{excursions with increments } Z\}$

\[ \downarrow \]

$\{\text{unobstructed disks with sides } Z\}$

\[ \downarrow \]

\{ unordered $Z$-leaf-labeled binary trees with up & down branch at every vertex \}

*as before, but also no proper subset of $Z$ has horizontal sum

- Excursions fragment by heights.
Idea of proof: an extended bijection

- An excursion is a walk \( w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2 \) such that \( w_1, \ldots, w_{n-1} \) are above \( w_0, w_n \).
- Call a disk unobstructed if no corner is visible horizontally from the base.

Theorem

If \( Z \) is generic*, \( \{ \text{excursions with increments } Z \} \)

\( \updownarrow \)

\( \{ \text{unobstructed disks with sides } Z \} \)

\( \updownarrow \)

\( \{ \text{unordered } Z\text{-leaf-labeled binary trees with up & down branch at every vertex} \} \)

*as before, but also no proper subset of \( Z \) has horizontal sum

- Excursions fragment by heights.
Fragmentation of disks

widens
Fragmentation of disks
Fragmentation of disks
Fragmentation of disks
Fragmentation of disks

widens
base

base
Fragmentation of disks
Fragmentation of disks
Fragmentation of disks
Fragmentation of disks
Fragmentation of disks
Show me da random diskz!

\[ n = 10 \]
Show me da random diskz!

$n = 50$
Show me da random diskz!

$n = 100$
Show me da random diskz!

$n = 500$
Show me da random diskz!

$n = 1000$
Show me da random diskz!

$n = 2000$
Show me da random diskz!

$n = 5000$
Show me da random diskz!
Area of uniform random flat disk

- Let \((B_t)_{t \in [0,1]}\) be a Brownian bridge in \(\mathbb{R}^2\)

Theorem (TB, '24+)

The area of the uniform flat disk with sides \(Z_n\) satisfies

\[
E[\text{Area}_n] = \log n^{2/\pi} + C + o(n) \quad \text{as} \quad n \to \infty,
\]

with

\[
C = -\int_0^1 \frac{\Gamma'(x+1)}{4\pi \sqrt{x} \Gamma(x+1)} \, dx = 0.0285\ldots.
\]

Question: does \(\text{Area}_n - \log n^{2/\pi}\) converge in distribution?

Eerie similarity to \(N\)-winding area \(A_N\) of \((B_t)_{t it itself:}

\(\sum_{N=1}^n A_N - \log n^{2/\pi}\) converges a.s. [Werner, '94]
Let $(B_t)_{t \in [0,1]}$ be a Brownian bridge in $\mathbb{R}^2$ and $Z_n = \{ B_{\frac{1}{n}}, B_{\frac{2}{n} - \frac{1}{n}}, B_{\frac{3}{n} - \frac{2}{n}}, \ldots, -B_{1 - \frac{1}{n}} \}$. 

Theorem (TB, '24+)

The area of the uniform flat disk with sides $Z_n$ satisfies $E[\text{Area}_{n}] = \log n - \frac{1}{2} \pi + C + o(1)$ as $n \to \infty$, with $C = -\int_0^1 \frac{\Gamma'(x+1)}{4\pi \sqrt{x} \Gamma(x+1)} \, dx = 0.0285 \ldots$.

Question: does $\text{Area}_{n} - \log n - \frac{1}{2} \pi$ converge in distribution?

Eerie similarity to $N$-winding area $A_N$ of $(B_t)$ itself: $n \prod_{i=1}^{N} A_{N} - \log n - \frac{1}{2} \pi$ converges a.s. [Werner, '94]
Let \((B_t)_{t \in [0,1]}\) be a Brownian bridge in \(\mathbb{R}^2\) and 
\[Z_n = \{B_{\frac{1}{n}}, B_{\frac{2}{n} - \frac{1}{n}}, B_{\frac{3}{n} - \frac{2}{n}}, \ldots, -B_{1 - \frac{1}{n}}\}\].

**Theorem (TB, ’24+)**

The area of the uniform flat disk with sides \(Z_n\) satisfies

\[
\mathbb{E}[\text{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \text{as } n \to \infty,
\]

with \(C = -\int_0^1 \frac{\Gamma'(x + 1)}{(4\pi \sqrt{x} \Gamma(x + 1))} dx = 0.0285 \ldots\)
Let \((B_t)_{t \in [0,1]}\) be a Brownian bridge in \(\mathbb{R}^2\) and 
\[Z_n = \{B_{1/n}, B_{2/n-1/n}, B_{3/n-2/n}, \ldots, -B_{1-1/n}\}\].

**Theorem (TB, '24+)**

The area of the uniform flat disk with sides \(Z_n\) satisfies

\[
\mathbb{E}[\text{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \text{as } n \to \infty,
\]

with \(C = - \int_0^1 \Gamma'(x+1)/(4\pi \sqrt{x} \Gamma(x+1)) \, dx = 0.0285 \ldots\)
Let \((B_t)_{t \in [0,1]}\) be a Brownian bridge in \(\mathbb{R}^2\) and 
\(Z_n = \{B_{1/n}, B_{2/n-1/n}, B_{3/n-2/n}, \ldots, -B_{1-1/n}\}\).

**Theorem (TB, '24+)**

The area of the uniform flat disk with sides \(Z_n\) satisfies

\[
\mathbb{E}[\text{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \text{as } n \to \infty,
\]

with \(C = -\int_0^1 \Gamma'(x+1)/(4\pi \sqrt{x} \Gamma(x+1)) \, dx = 0.0285\ldots\)

Question: does \(\text{Area}_n - \frac{\log n}{2\pi}\) converge in distribution?
Let \((B_t)_{t \in [0,1]}\) be a Brownian bridge in \(\mathbb{R}^2\) and \(Z_n = \{B_{\frac{1}{n}}, B_{\frac{2}{n}-\frac{1}{n}}, B_{\frac{3}{n}-\frac{2}{n}}, \ldots, -B_{\frac{1}{n}}\}\).

**Theorem (TB, ’24+)**

The area of the uniform flat disk with sides \(Z_n\) satisfies

\[
\mathbb{E}[\text{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \text{as } n \to \infty,
\]

with

\[
C = -\int_0^1 \frac{\Gamma'(x+1)}{4\pi \sqrt{x} \Gamma(x+1)} \, dx = 0.0285 \ldots
\]

**Question:** does \(\text{Area}_n - \frac{\log n}{2\pi}\) converge in distribution?

**Eerie similarity to \(N\)-winding area \(A_N\) of \((B_t)_{t}\) itself:**

\[
\sum_{N=1}^{n} N A_N - \frac{\log n}{2\pi} \quad \text{converges a.s.} \quad \text{[Werner, ’94]}
\]
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let $(X(t))_{t \in [0,1]}$ a Brownian bridge $(E(t))_{t \in [0,1]}$ a Brownian excursion
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let \((X(t))_{t \in [0,1]}\) a Brownian bridge and \((E(t))_{t \in [0,1]}\) a Brownian excursion.

\[
Y(t) := \int_0^{E(t)} \text{sign} \left[ X(\min\{s \geq t : E(s) = y\}) - X(\max\{s \leq t : E(s) = y\}) \right] dy.
\]

Conjecture (TB)

The law of \((X(t), Y(t))_{t \in [0,1]}\) is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides \(Z_n\) as \(n \to \infty\).

Conjecture (F. Ferrari, '24 ← my interpretation)

If \(F\) is conformal map to \(D\), \(\text{Lebesgue on } [0,1]\) is pushed by \(F(X(\cdot), Y(\cdot))\) to critical GMC on \(\partial D\).
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let \((X(t))_{t \in [0,1]}\) a Brownian bridge and \((E(t))_{t \in [0,1]}\) a Brownian excursion.

and \(Y(t) := \int_0^{E(t)} \text{sign} \left[ X(\min\{s \geq t : E(s) = y\}) - X(\max\{s \leq t : E(s) = y\}) \right] dy.\)
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let \((X(t))_{t \in [0,1]}\) a Brownian bridge and \((E(t))_{t \in [0,1]}\) a Brownian excursion.

and \(Y(t) := \int_0^{E(t)} \text{sign} \left[ X(\min\{s \geq t : E(s) = y\}) - X(\max\{s \leq t : E(s) = y\}) \right] dy.\)

Conjecture (TB)

The law of \((X(t), Y(t))_{t \in [0,1]}\) is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides \(Z_n\) as \(n \to \infty.\)
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let \((X(t))_{t \in [0,1]}\) a Brownian bridge \((E(t))_{t \in [0,1]}\) a Brownian excursion

and \(Y(t) := \int_0^{E(t)} \text{sign} \left[ X(\min\{s \geq t : E(s) = y\}) - X(\max\{s \leq t : E(s) = y\}) \right] dy.\)

Conjecture (TB)

*The law of \((X(t), Y(t))_{t \in [0,1]}\) is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides \(Z_n\) as \(n \to \infty.\)*

Conjecture (F. Ferrari, ’24 ← my interpretation)

*If \(F\) is conformal map to \(\mathbb{D},\)*
Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a Brownian flat disk: Let \( (X(t))_{t \in [0,1]} \) a Brownian bridge \( (E(t))_{t \in [0,1]} \) a Brownian excursion

and \( Y(t) := \int_0^{E(t)} \text{sign} \left[ X(\min\{s \geq t : E(s) = y\}) - X(\max\{s \leq t : E(s) = y\}) \right] dy \).

Critical 1D GMC on circle?

Conjecture (TB)

The law of \( (X(t), Y(t))_{t \in [0,1]} \) is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides \( Z_n \) as \( n \to \infty \).

Conjecture (F. Ferrari, ’24 ← my interpretation)

If \( F \) is conformal map to \( \mathbb{D} \), Lebesgue on \( [0,1] \) is pushed by \( F(X(\cdot), Y(\cdot)) \) to critical GMC on \( \partial \mathbb{D} \).
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.

Theorem (TB, ’24+)

The number of rigid quadrangulations is asymptotic to

$$\left(\frac{4\pi}{n}\right)^{n-1} \frac{1}{16n^2 \log 2} (1 + o(1))$$

as $n \to \infty$.

- Akin to random planar map models in universality class of LQG $\gamma = 2$. . .
Another model of flat disks: a link with critical LQG?

- Consider **rectilinear disks**: piecewise-straight boundary with $2n$ right-angled corners.

- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.

> Theorem (TB, ’24+)
The number of rigid quadrangulations is asymptotic to

\[
(4\pi)^{n-1/2n^2\log_2 n} (1 + o(1))
\]

as $n \to \infty$.

- Akin to random planar map models in universality class of LQG $\gamma = 2$.
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.
- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.
- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.

Theorem (TB, '24+): The number of rigid quadrangulations is asymptotic to $\left(\frac{4\pi}{16}\right)^n$ as $n \to \infty$.

Akin to random planar map models in universality class of $\text{LQG}_{\gamma=2}$.
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with \(2n\) right-angled corners.
- Moduli space \(\mathcal{M}_n\) is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
- Combinatorial type is a uniform random *rigid quadrangulation* with \(2n\) corners.

Theorem (TB, '24+)

The number of rigid quadrangulations is asymptotic to

\[
\left(\frac{4\pi}{16}n^{-\frac{1}{2}}\log n\right)(1 + o(1))
\]

as \(n \to \infty\).

- Akin to random planar map models in universality class of LQG \(\gamma = 2\).
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.
- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
- Combinatorial type is a uniform random *rigid quadrangulation* with $2n$ corners.
  - Quadrangulation of disk with inner vertices of degree 4, and boundary vertices of angle $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
Another model of flat disks: a link with critical LQG?

- Consider \textit{rectilinear disks}: piecewise-straight boundary with \(2n\) right-angled corners.
- Moduli space \(\mathcal{M}_n\) is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
- Combinatorial type is a uniform random rigid quadrangulation with \(2n\) corners.
  - Quadrangulation of disk with inner vertices of degree 4, and boundary vertices of angle \(\frac{\pi}{2}, \pi, \frac{3\pi}{2}\).
  - Rays start at \(\frac{3\pi}{2}\) and end at \(\pi\)-angled boundary vertex.

**Theorem (TB, ’24+)**

The number of rigid quadrangulations is asymptotic to 
\[
(4 \pi)^{n - \frac{1}{16} n^2 \log 2 n} (1 + o(1))
\]
as \(n \to \infty\).

- Akin to random planar map models in universality class of LQG \(\gamma = 2\).
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.
- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
- Combinatorial type is a uniform random rigid quadrangulation with $2n$ corners.
  - Quadrangulation of disk with inner vertices of degree 4, and boundary vertices of angle $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
  - Rays start at $\frac{3\pi}{2}$- and end at $\pi$-angled boundary vertex.

**Theorem (TB, ’24+)**

*The number of rigid quadrangulations is asymptotic to

\[
\frac{(4\pi)^{n-1}}{16n^2 \log^2 n}(1 + o(1)) \quad \text{as } n \to \infty.
\]
Another model of flat disks: a link with critical LQG?

- Consider rectilinear disks: piecewise-straight boundary with $2n$ right-angled corners.
- Moduli space $\mathcal{M}_n$ is naturally equipped with Lebesgue measure on independent side lengths.
- Determines random rectilinear disk of fixed perimeter.
- Combinatorial type is a uniform random rigid quadrangulation with $2n$ corners.
  - Quadrangulation of disk with inner vertices of degree 4, and boundary vertices of angle $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
  - Rays start at $\frac{3\pi}{2}$- and end at $\pi$-angled boundary vertex.

Theorem (TB, ’24+)

The number of rigid quadrangulations is asymptotic to

$$\frac{(4\pi)^{n-1}}{16n^2 \log^2 n} (1 + o(1)) \text{ as } n \to \infty.$$ 

- Akin to random planar map models in universality class of $LQG_{\gamma=2\ldots}$.
Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

Theorem (TB, '24+)

For $n \geq 2$ and $p \geq 1$ there exists a bijection

rigid quadrangulations with $2n$ corners and base $p$ $\longleftrightarrow$ colorful $\mathbb{Z}$-labeled quadrangulations with $n$ vertices and perimeter $2p$ $\longleftrightarrow$ vertex with label $\ell$
Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

For $n \geq 2$ and $p \geq 1$ there exists a bijection

$$\text{rigid quadrangulations with } 2n \text{ corners and base } p \leftrightarrow \text{colorful } \mathbb{Z}-\text{labeled quadrangulations with } n \text{ vertices and perimeter } 2p$$

- corner with turning number $\ell | \{ \text{left} - \text{right} \} \leftrightarrow \text{vertex with label } \ell$
Theorem (TB, ’24+)

For \( n \geq 2 \) and \( p \geq 1 \) there exists a bijection

\[
\text{rigid quadrangulations with } 2^n \text{ corners and base } p \leftrightarrow \text{colorful } \mathbb{Z}\text{-labeled quadrangulations with } n \text{ vertices and perimeter } 2p
\]

\( \pi \text{-corner with turning number } \ell \) \( \leftrightarrow \) vertex with label \( \ell \).
Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

Theorem (TB, ’24+)

For $n \geq 2$ and $p \geq 1$ there exists a bijection

\[
\begin{align*}
\{ \text{rigid quadrangulations with } & 2n \text{ corners and base } p \} & \leftrightarrow \{ \text{colorful } \mathbb{Z} \text{-labeled quadrangulations} \\
& \text{with } n \text{ vertices and perimeter } 2p \}
\end{align*}
\]
Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

Theorem (TB, '24+)

For $n \geq 2$ and $p \geq 1$ there exists a bijection

\[
\begin{align*}
\{ \text{rigid quadrangulations with} & \quad 2n \text{ corners and base } p \} \\
\text{ } & \longleftrightarrow \text{ } \{ \text{colorful } \mathbb{Z}\text{-labeled quadrangulations} & \quad \text{with } n \text{ vertices and perimeter } 2p \} \\
\frac{\pi}{2}\text{-corner with turning number } \ell & \longleftrightarrow \text{vertex with label } \ell \\
\#\text{left} - \#\text{right} & \longleftrightarrow \text{vertex with label } \ell
\end{align*}
\]
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $Z$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00] [Zinn-Justin, '00] [Elvey Price, Guttmann, '17]

Enumeration finally settled in Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful $Z$-labeled quadrangulations is

$$Q(p)(x) = \sum_{k \geq p} \frac{k!}{2^k} \left( R(x)^{k+1} \right)_{2k - p}$$

when

$$X_k \geq \frac{1}{2^k} \sum_{j=0}^{2k} \frac{k!}{2^j} R(x)^{j+1}.$$
Bijection via exploration

Scanning vs Peeling

- Many guises of these labeled quadrangulations: planar Eulerian orientations, \(Z\)-labeled bipartite planar maps, special case of six-vertex model

- Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful \(Z\)-labeled quadrangulations is

\[
Q(p)(x) = \sum_{k \geq p} \frac{1}{k!} \left( \frac{2^k}{2 - p^{k!}} \right) R(x)_k^{k+1},
\]

when \(\sum_{k \geq p} \frac{1}{k!} \left( \frac{2^k}{2 - p^{k!}} \right) R(x)_k^{k+1} = x\).
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, \( Z \)-labeled bipartite planar maps, special case of six-vertex model \([Kostov, '00]\)[Zinn-Justin, '00][Elvey Price, Guttmann, '17] Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful \( Z \)-labeled quadrangulations is

\[
Q(p)(x) = \sum_{k \geq p} \frac{1}{k+1} \left( \frac{x}{2k} \right)^{k} \frac{R(x)^{k+1}}{k!},
\]

when

\[
x = \sum_{k \geq p} \frac{1}{k+1} \left( \frac{x}{2k} \right)^{k} \frac{R(x)^{k+1}}{k!}.
\]
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $\mathbb{Z}$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00] [Zinn-Justin, '00] [Elvey Price, Guttmann, '17]

Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful $\mathbb{Z}$-labeled quadrangulations is

$$Q(p)(x) = \sum_{k \geq p} \frac{1}{2k} \frac{k!}{2k - p k!} R(x)^{k+1},$$

when

$$R(x)^{k+1} = x.$$
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $\mathbb{Z}$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00] [Zinn-Justin, '00] [Elvey Price, Guttmann, '17]

Enumeration finally settled in Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful $\mathbb{Z}$-labeled quadrangulations is $Q(p)(x) = \sum_{k \geq p} \frac{1}{k!} 2^{k-p} k! R(x)^{k+1}$, when $X_k \geq p$.

\[ R(x)^{k+1} = x. \]
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $Z$-labeled bipartite planar maps, special case of six-vertex model

[Bousquet-Mélou, Elvey Price, '20]

The generating function of perimeter-$2$ colorful $Z$-labeled quadrangulations is

$$Q(p)(x) = \sum_{k \geq p} \frac{1}{k!} 2^k - p \frac{1}{k!} 2^{k+1},$$

when $X_k \geq p \frac{1}{k!} 2^k$.
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, \( \mathbb{Z} \)-labeled bipartite planar maps, special case of six-vertex model \cite{Kostov, Zinn-Justin, Elvey Price, Guttmann}

Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2 colorful \( \mathbb{Z} \)-labeled quadrangulations is

\[
Q(p)(x) = \sum_{k \geq p} \frac{1}{k+1} \frac{2k}{2k-p} R(x)^{k+1},
\]

when

\[
R(x)^{k+1} = x.
\]
Bijection via exploration

Many guises of these labeled quadrangulations: planar Eulerian orientations, Z-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00] [Zinn-Justin, '00] [Elvey Price, Guttmann, '17]

Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)
The generating function of perimeter-$2^{p}$-colorful Z-labeled quadrangulations is

$$Q(p)(x) = \sum_{k \geq p} \frac{1}{k+1} \frac{1}{2k} \frac{k!}{2k-pk!} R(x)^{k+1},$$

when

$$X_{k \geq p} \frac{1}{k+1} \frac{1}{2k} \frac{k!}{2k-pk!} R(x)^{k+1} = x.$$
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $Z$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00] [Zinn-Justin, '00] [Elvey Price, Guttmann, '17]

Enumeration finally settled in Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-$2$ colorful $Z$-labeled quadrangulations is

$$Q(p)(x) = \sum_{k \geq p} \frac{x^{k+1}}{2^k k!} R(x)^{k+1},$$

when

$$X_{k \geq p} \frac{x^{k+1}}{2^k k!} R(x)^{k+1} = x.$$
Bijection via exploration

Scanning vs Peeling

Many guises of these labeled quadrangulations: planar Eulerian orientations, $\mathbb{Z}$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00][Zinn-Justin, '00][Elvey Price, Guttmann, '17]

Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

The generating function of perimeter-2$p$ colorful $\mathbb{Z}$-labeled quadrangulations is

$$Q^{(p)}(x) = \sum_{k \geq p} \frac{1}{k+1} \binom{2k}{k} \binom{2k-p}{k} R(x)^{k+1} , \quad \text{ when } \sum_{k \geq p} \frac{1}{k+1} \binom{2k}{k}^2 R(x)^{k+1} = x.$$
walks in half plane

polygonal flat disks

rectilinear flat disks

labeled quadrangulations
walks in half plane polygonal flat disks
Status and outlook in pictures
rectilinear flat disks labeled quadrangulations
Status and outlook in pictures

walks in half plane polygonal flat disks

rectilinear flat disks labeled quadrangulations

Brownian half-plane excursion Critical LQG

Continuous picture

Brownian half-plane excursion
Brownian flat disk?
Critical LQG
Status and outlook in pictures

walks in half plane polygonal flat disks
polyangular flat disks
rectilinear flat disks labeled quadrangulations

Brownian half-plane excursion
Continuous picture
Brownian flat disk?
Critical LQG

Critical mating of trees
[Aru, Holden, Powell, Sun, '21]
walks in half plane polygonal flat disks
Status and outlook in pictures
rectilinear flat disks labeled quadrangulations
Brownian half-plane excursion
Critical LQG
Brownian flat disk?
Continuous picture
critical mating of trees
[Aru, Holden, Powell, Sun, '21]