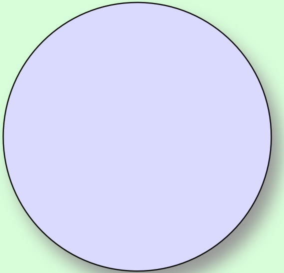
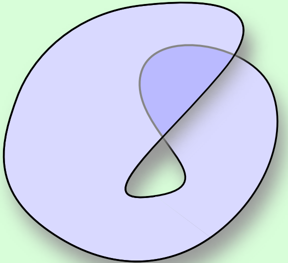
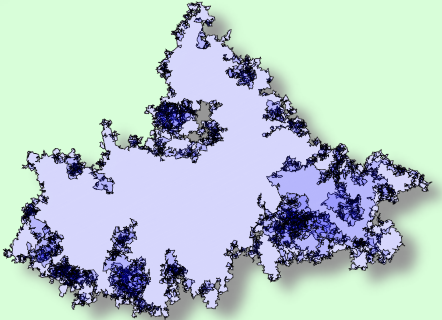
Timothy Budd

Uniform random flat disks







Radboud University

T.Budd@science.ru.nl
http://hef.ru.nl/~tbudd/

Motivation: Quantum JT gravity

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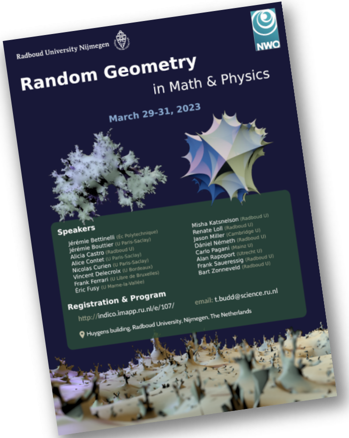

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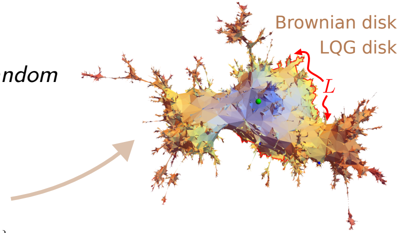

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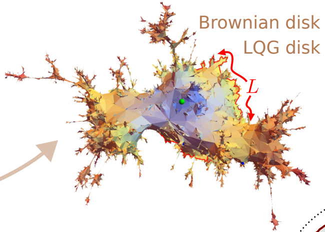

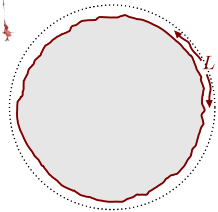

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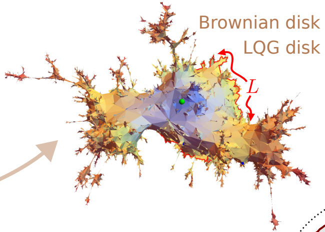

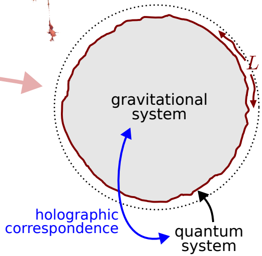

- Activity in Quantum JT gravity due to holographic correspondence.

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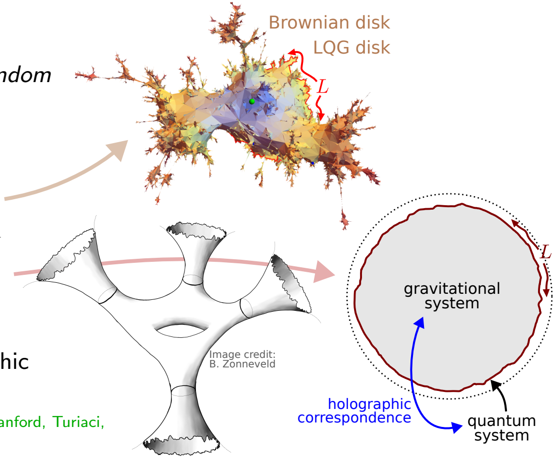

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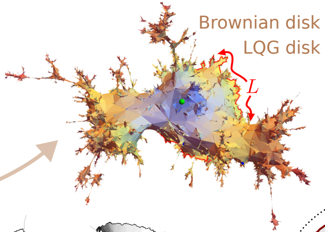

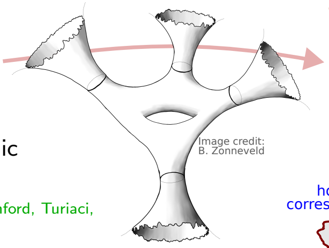

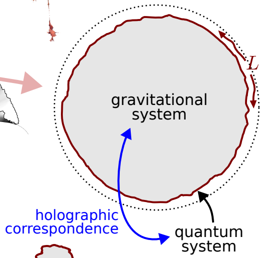

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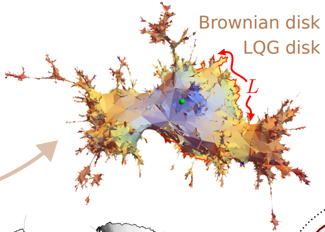
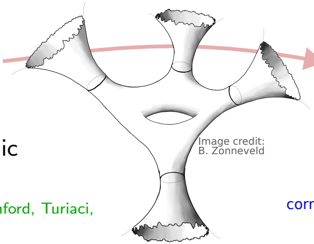
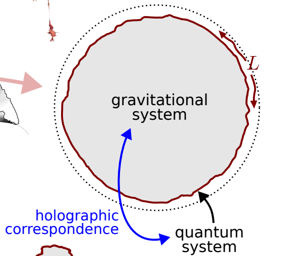
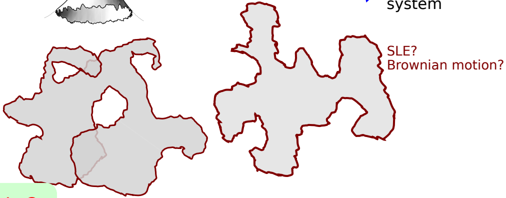
- Ferrari: should allow disks to self-overlap. [Ferrari, '24]





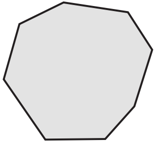




Is there a tractable model of uniform random discrete flat disks?

What is a discrete flat disks?

[Titus, Blank, Poénaru, Shor, van Wyk, Mukherjee, Graver, Cargo, Evans, Wenk, ...]



convex
polygon

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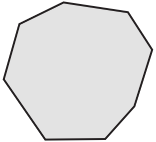

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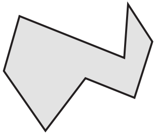

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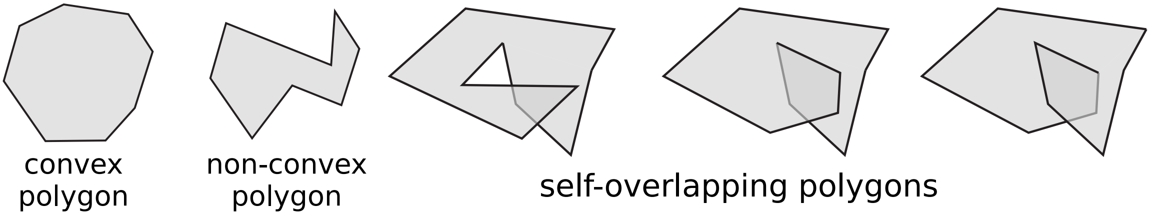

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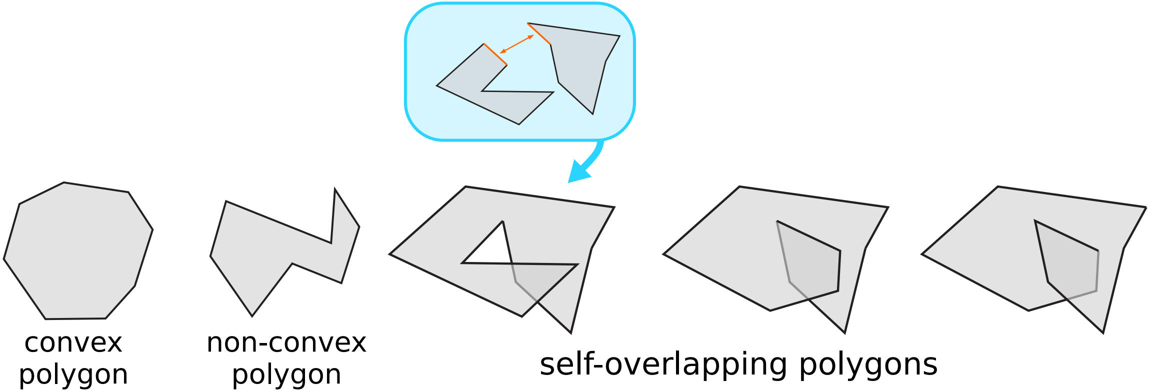

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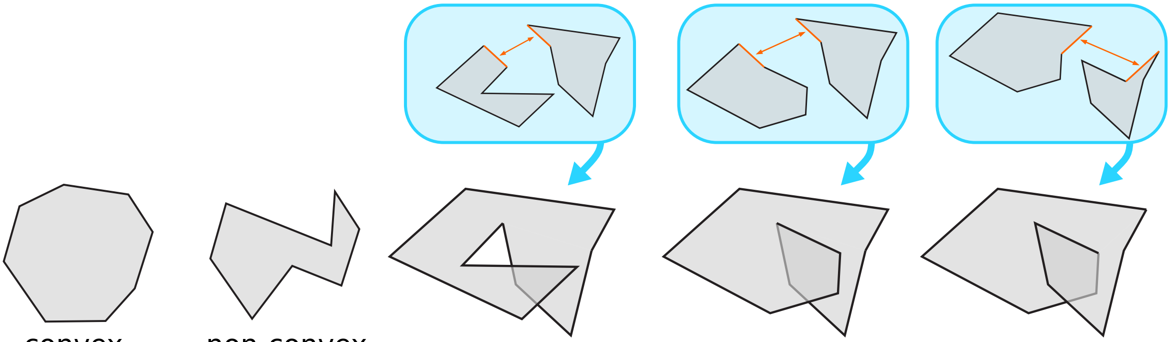

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self-overlapping polygons

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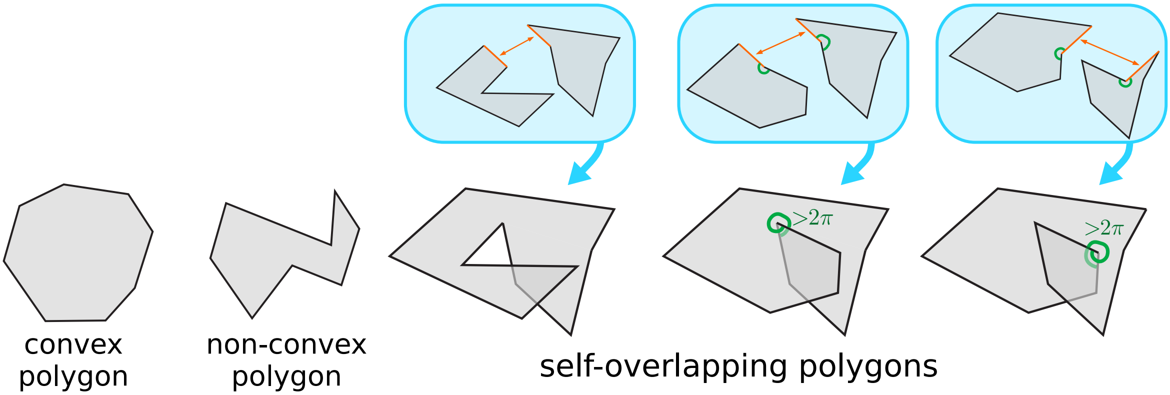

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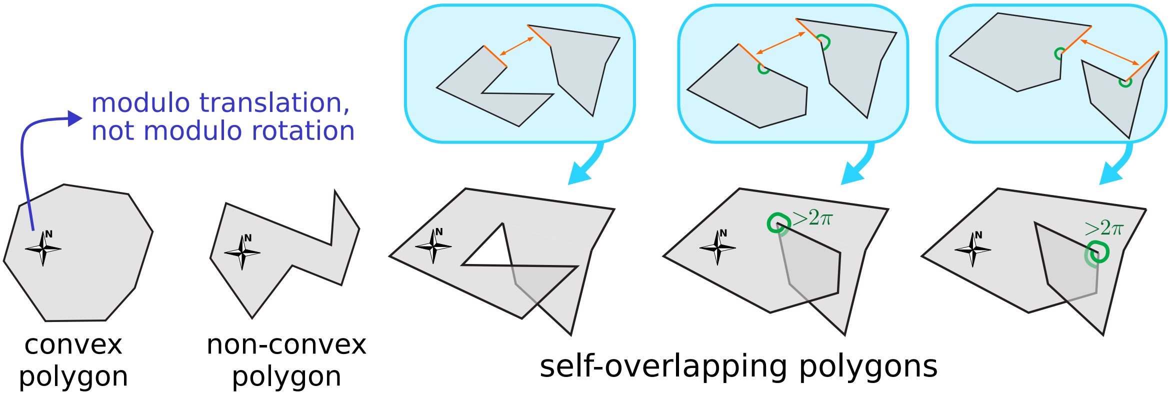

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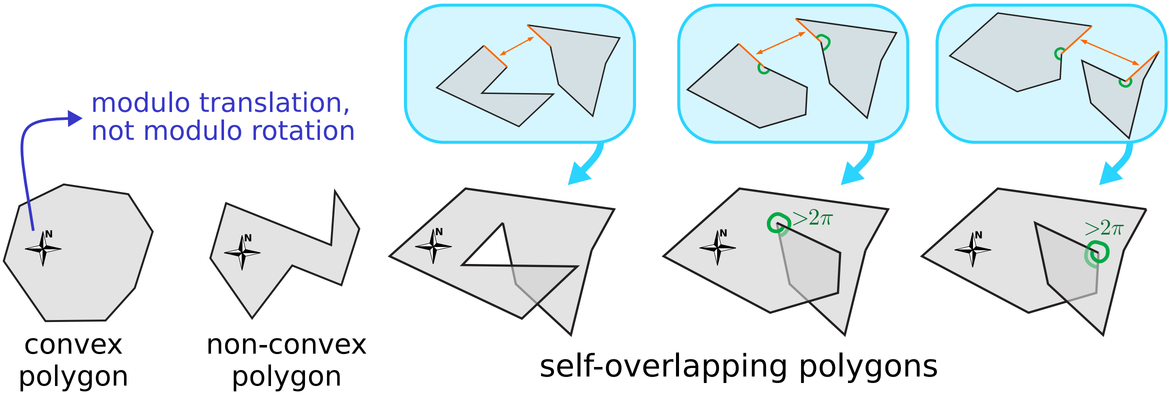
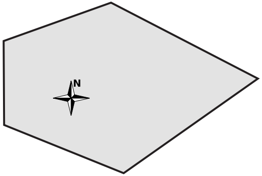


How many discrete flat disks with n sides are there?

First model: fix the sides!



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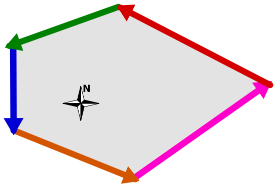

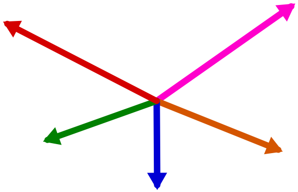

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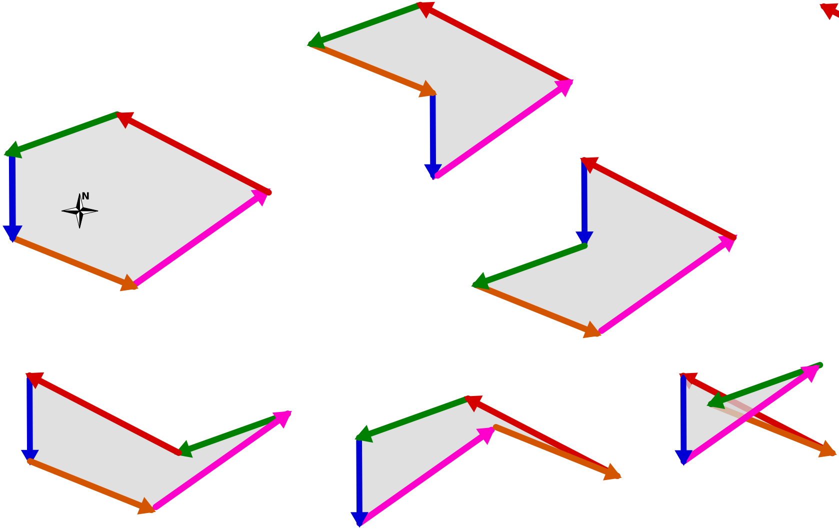

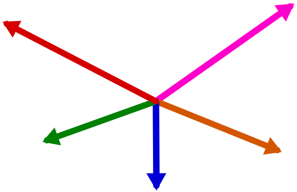

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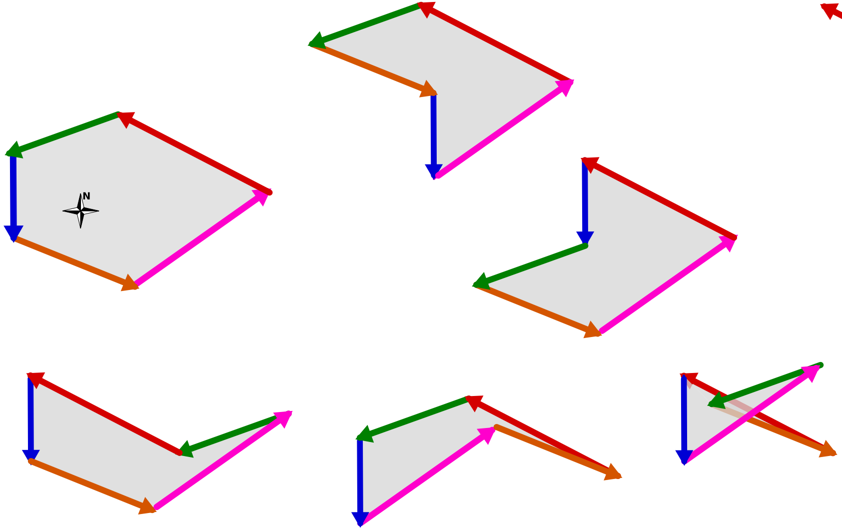

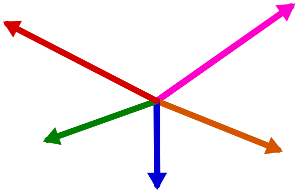

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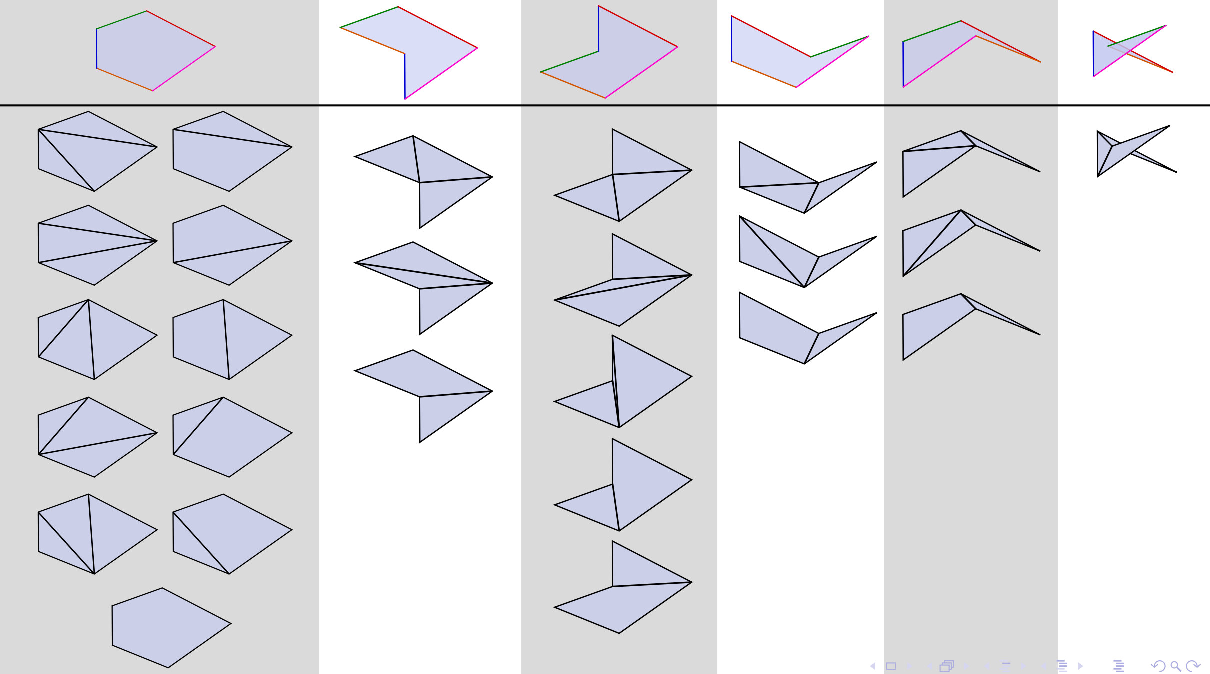
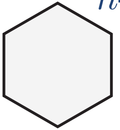
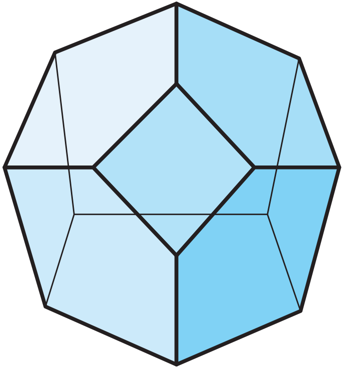




n-gon

Images adapted from Devadoss, Shah, Shao, Winston



Associahedron K_n

[Tamari, '51] [Stasheff, '63]

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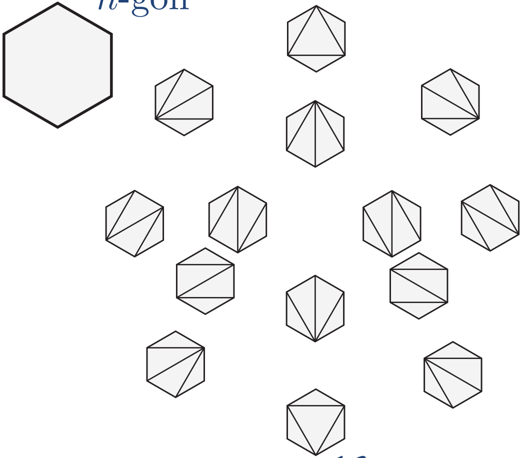
n-gon



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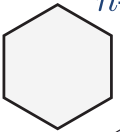

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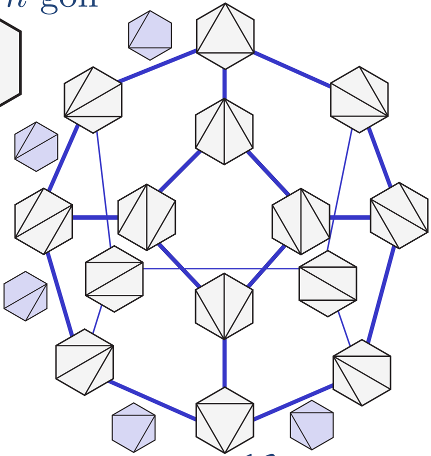

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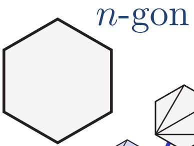

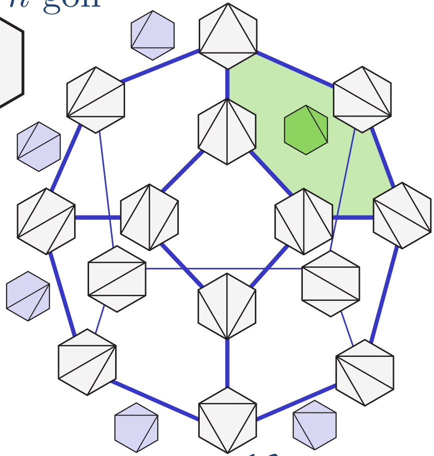

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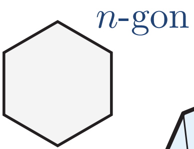

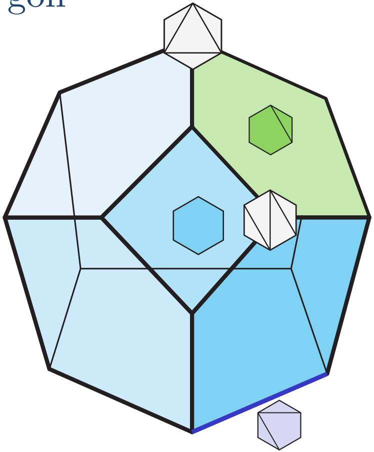

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Convex diagonalizations

[Devadoss, Shah, Shao, Winston, '09]

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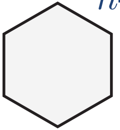

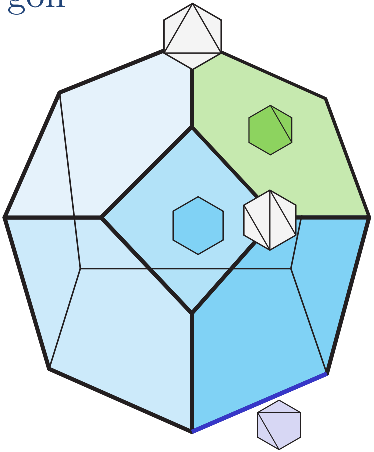

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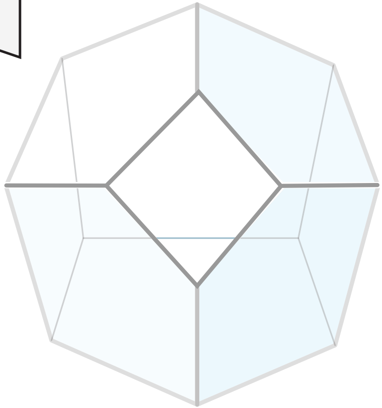

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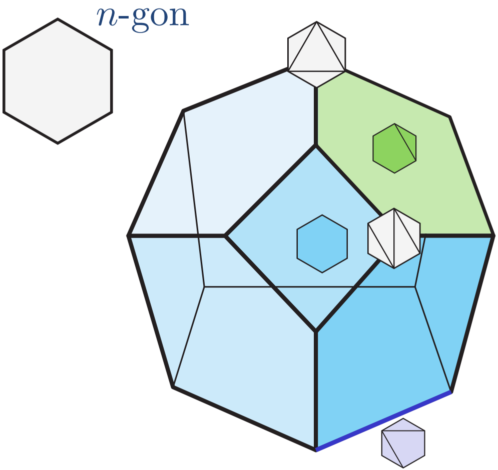

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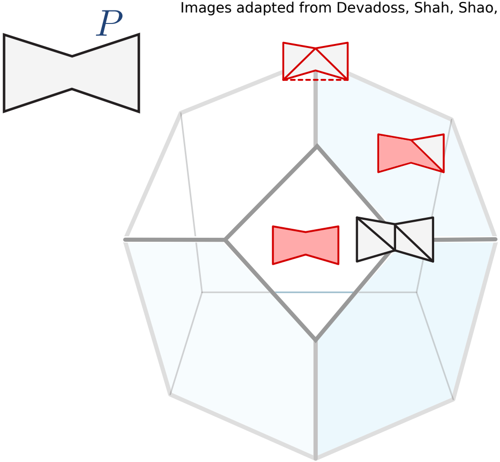

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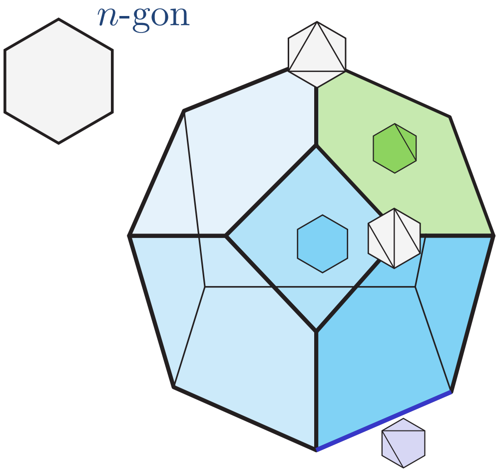

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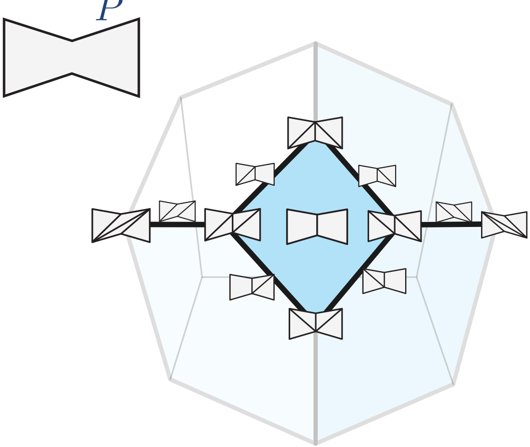

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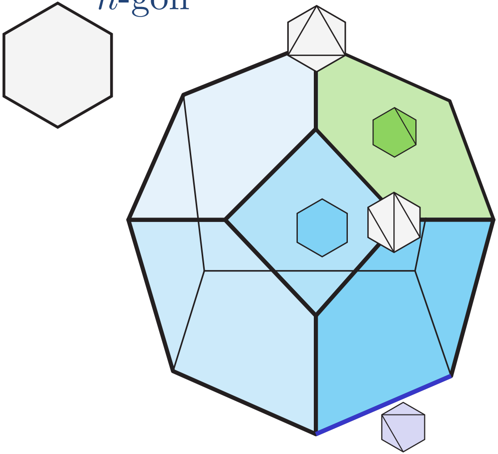
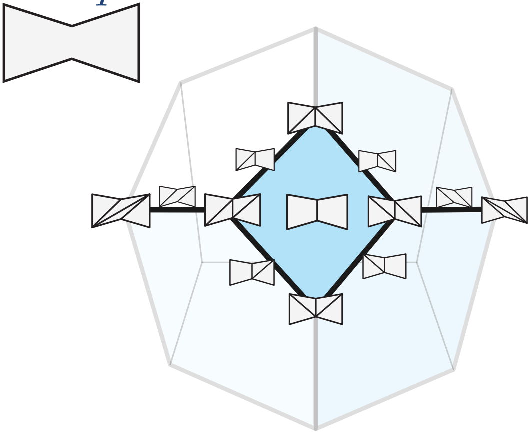
n-gon



Associahedron $\mathcal{K}_n$

[Tamari, '51] [Stasheff, '63]

P



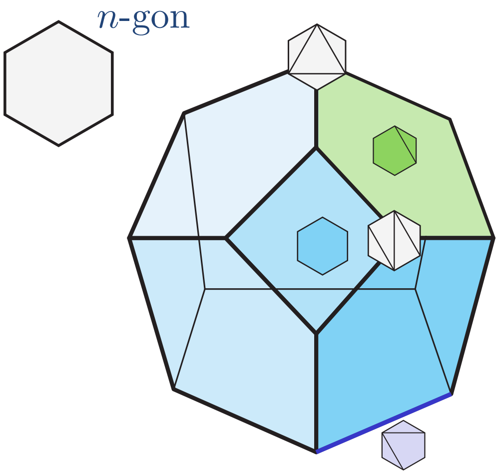
Convex diagonalizations $\mathcal{K}_P$

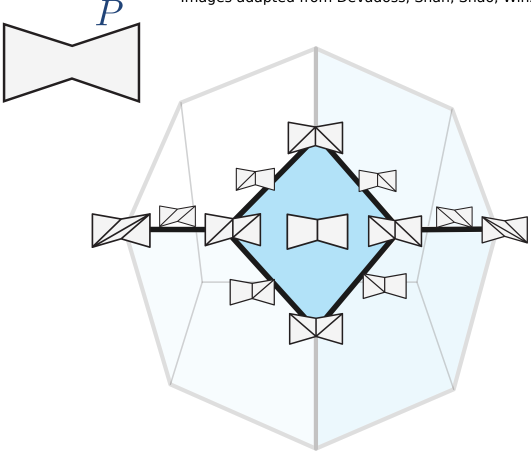
[Devadoss, Shah, Shao, Winston, '09]

Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon P, the complex $\mathcal{K}_P$ is contractible.

Images adapted from Devadoss, Shah, Shao, Winston





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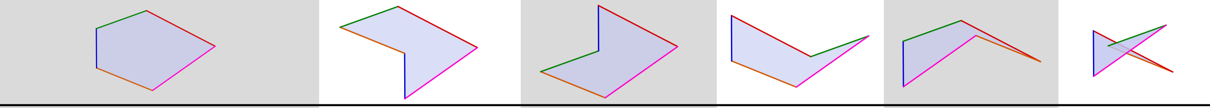
[Tamari, '51] [Stasheff, '63]

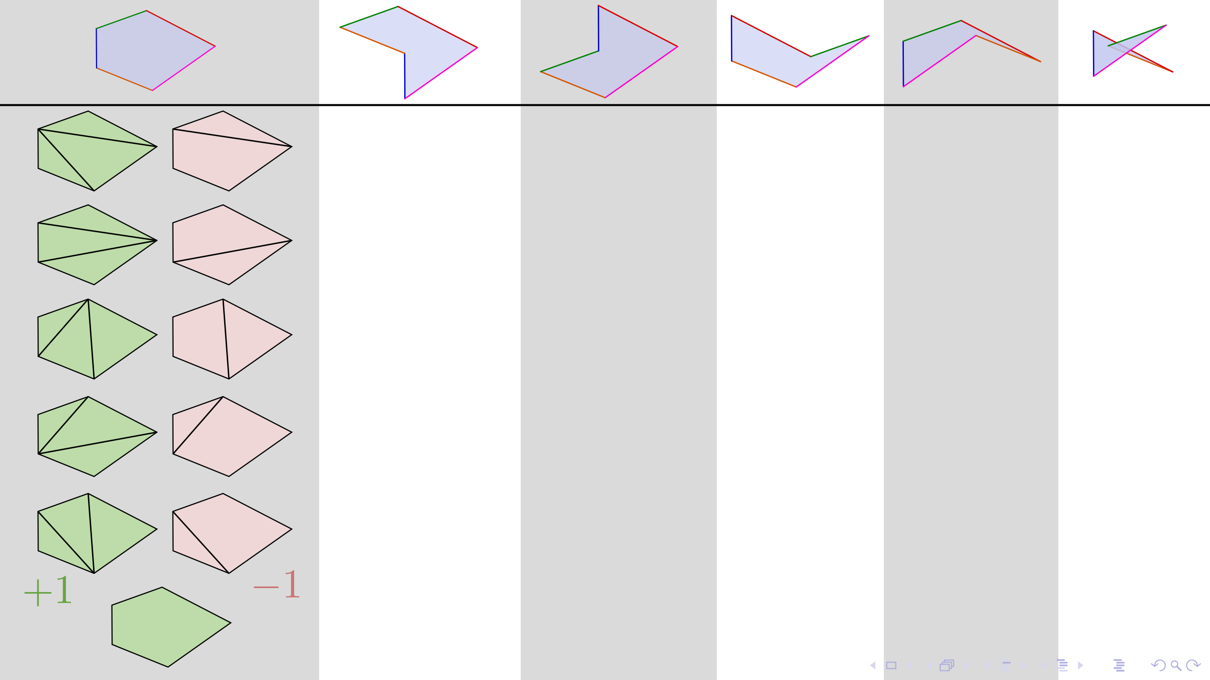
Convex diagonalizations $\mathcal{K}_P$

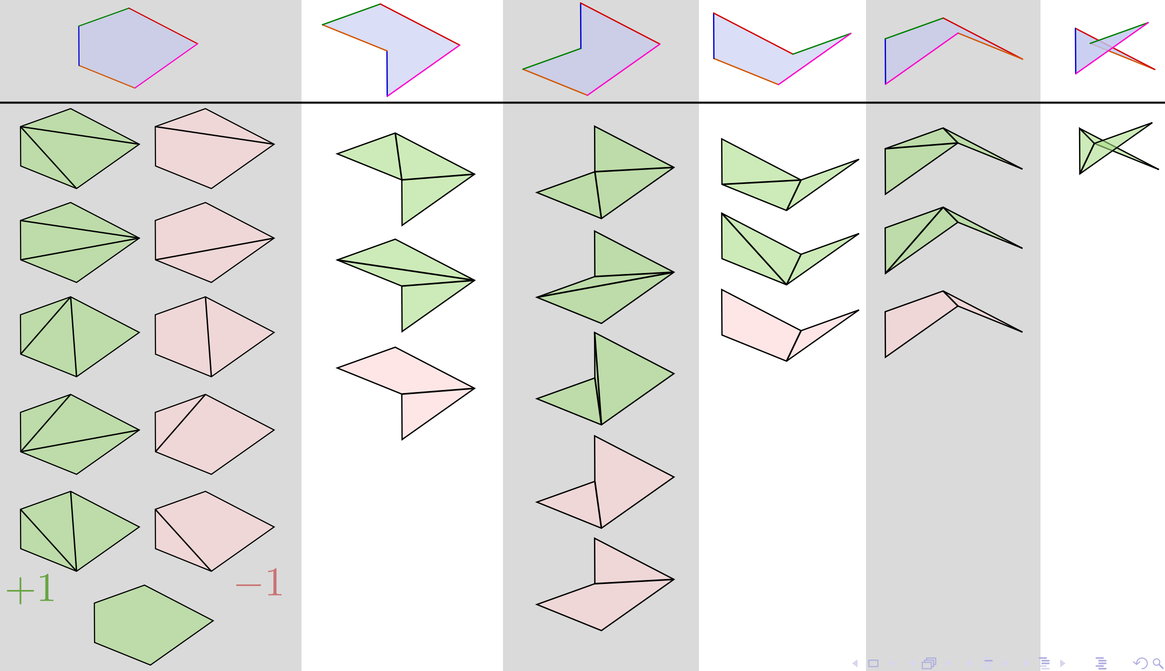
[Devadoss, Shah, Shao, Winston, '09]

Theorem (Devadoss, Shah, Shao, Winston, '09 & Braun, Ehrenborg, '10)

For any polygon P, the complex $\mathcal{K}_P$ is contractible. Hence, Euler characteristic $\chi = \sum_{\text{cells } \sigma} (-1)^{\dim \sigma} = 1.$




+1
−1


+1
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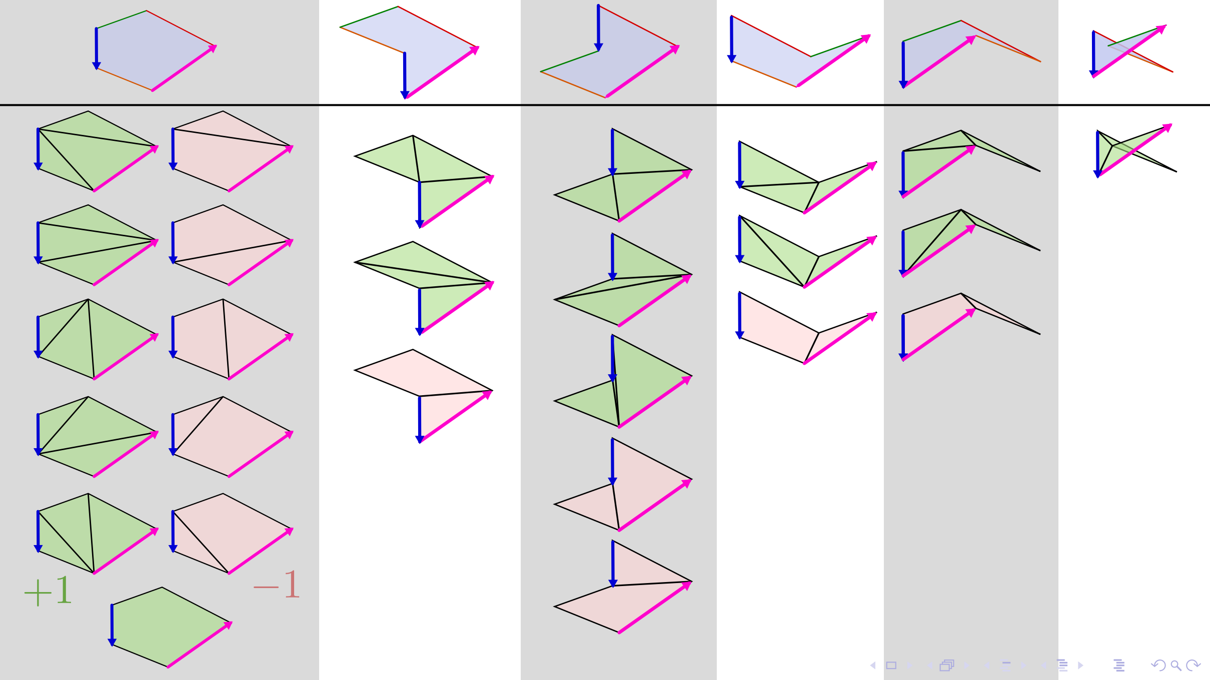
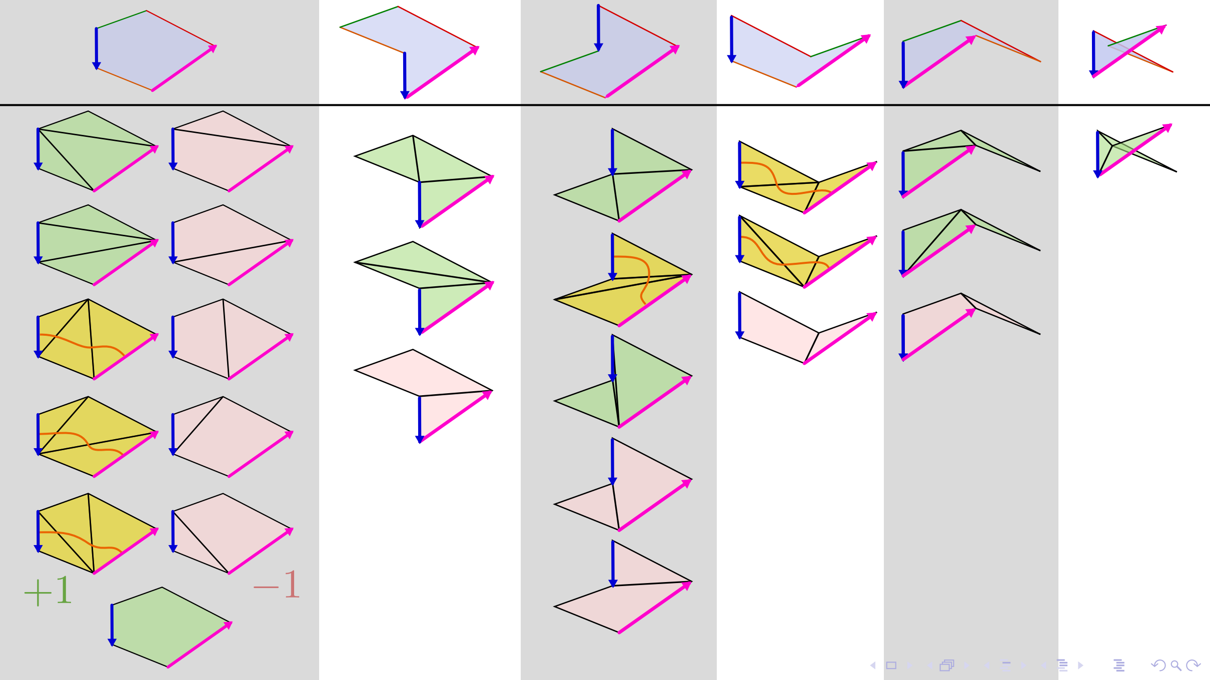
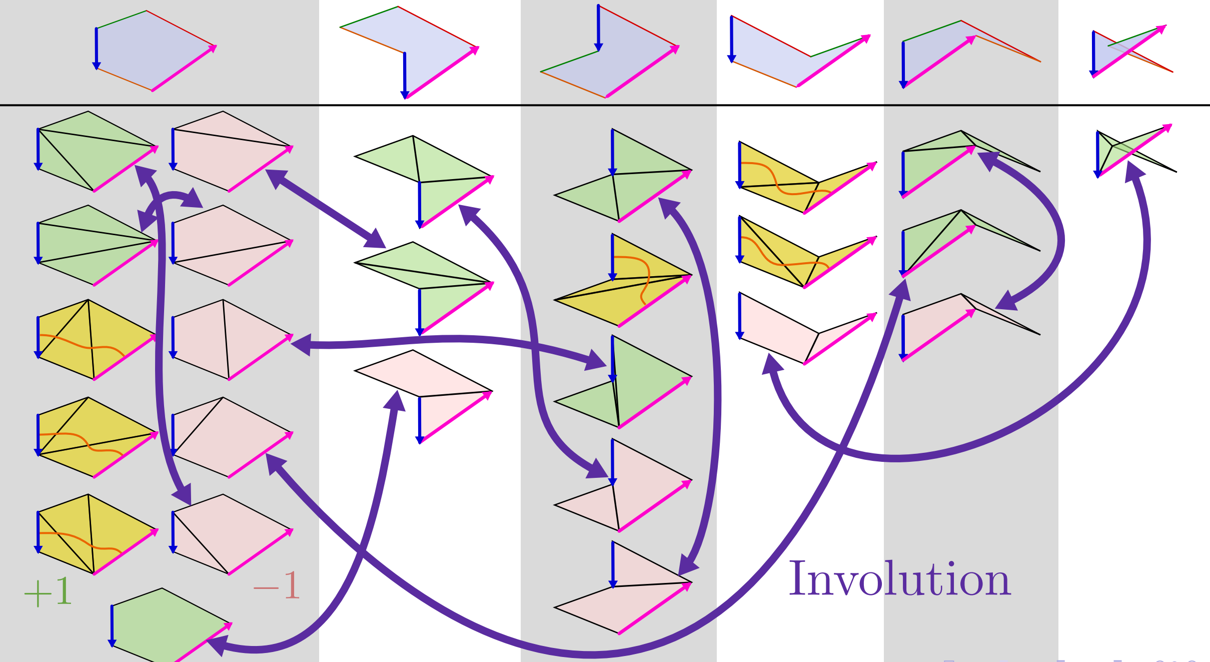
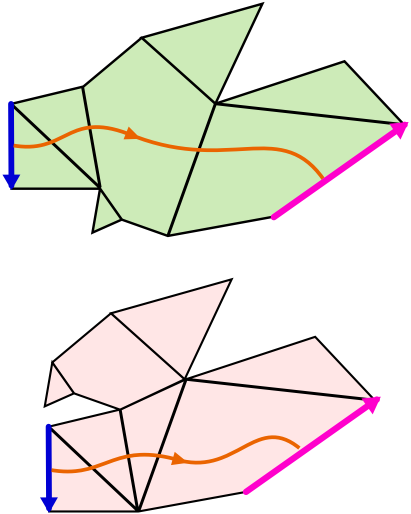


+1
−1

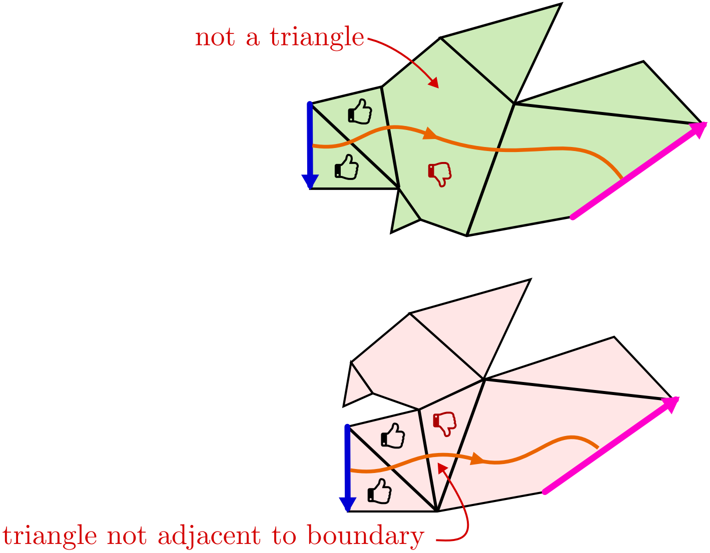

$+1$
-1


+1
−1
Involution

Involution?



Involution?



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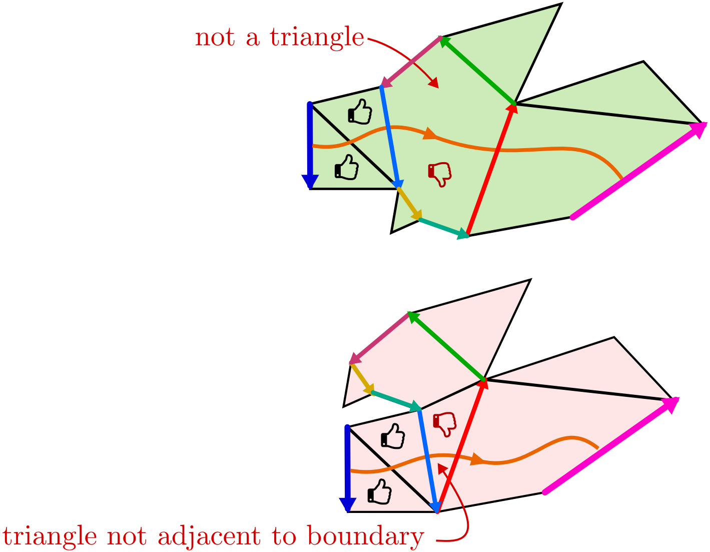

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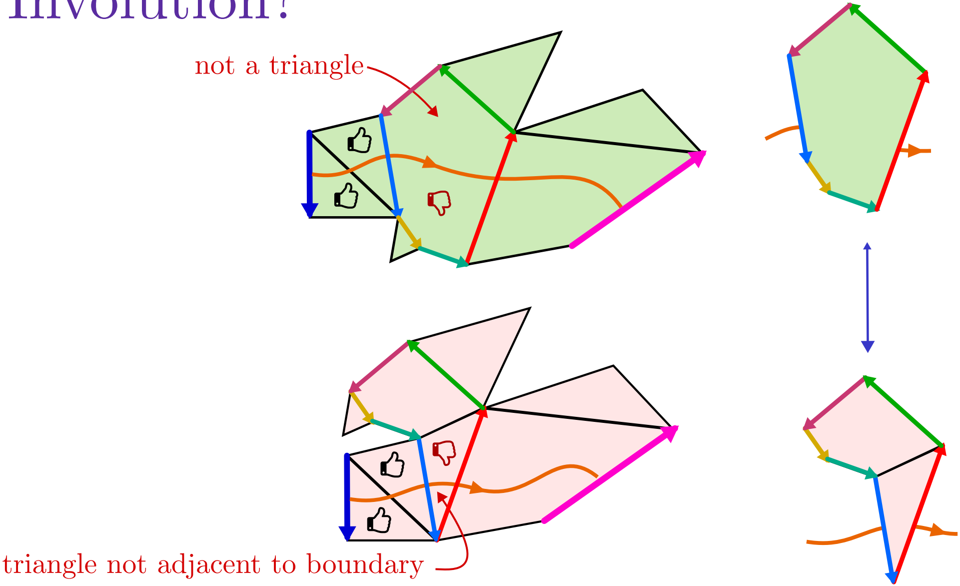
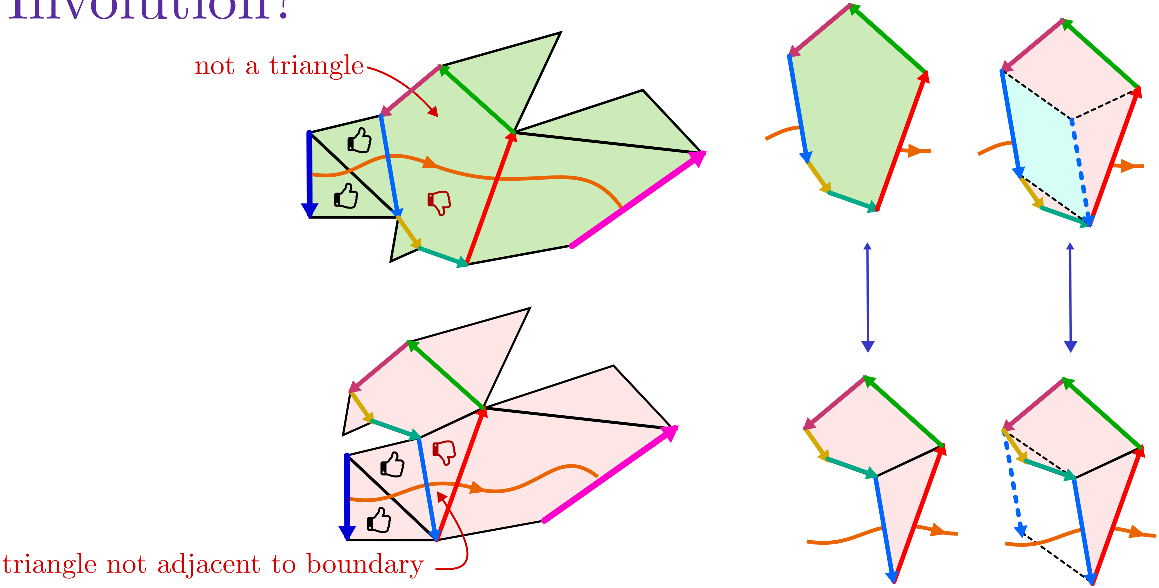
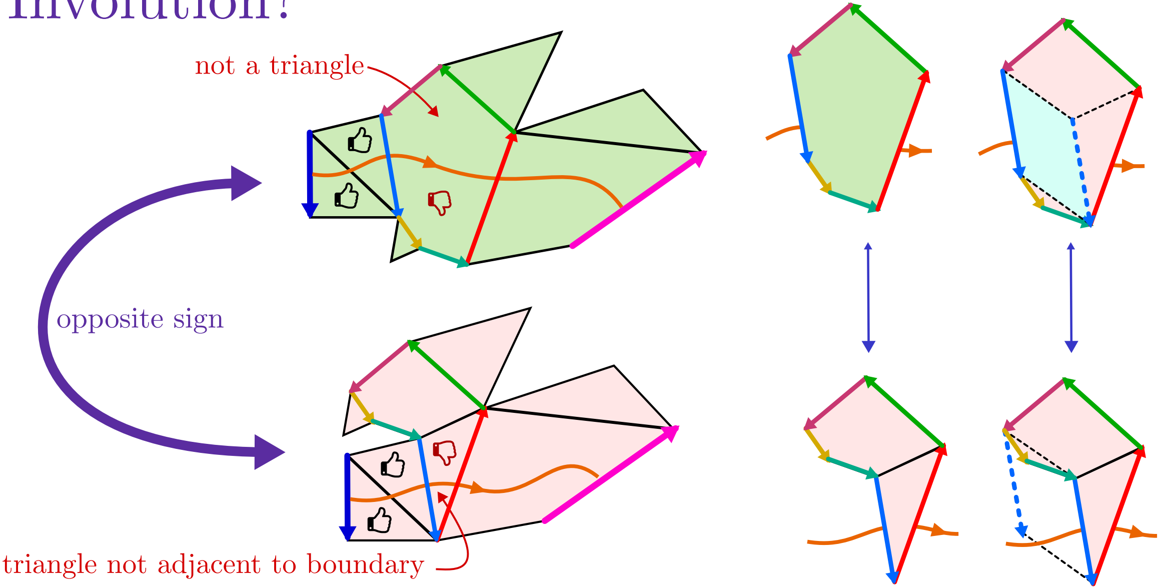
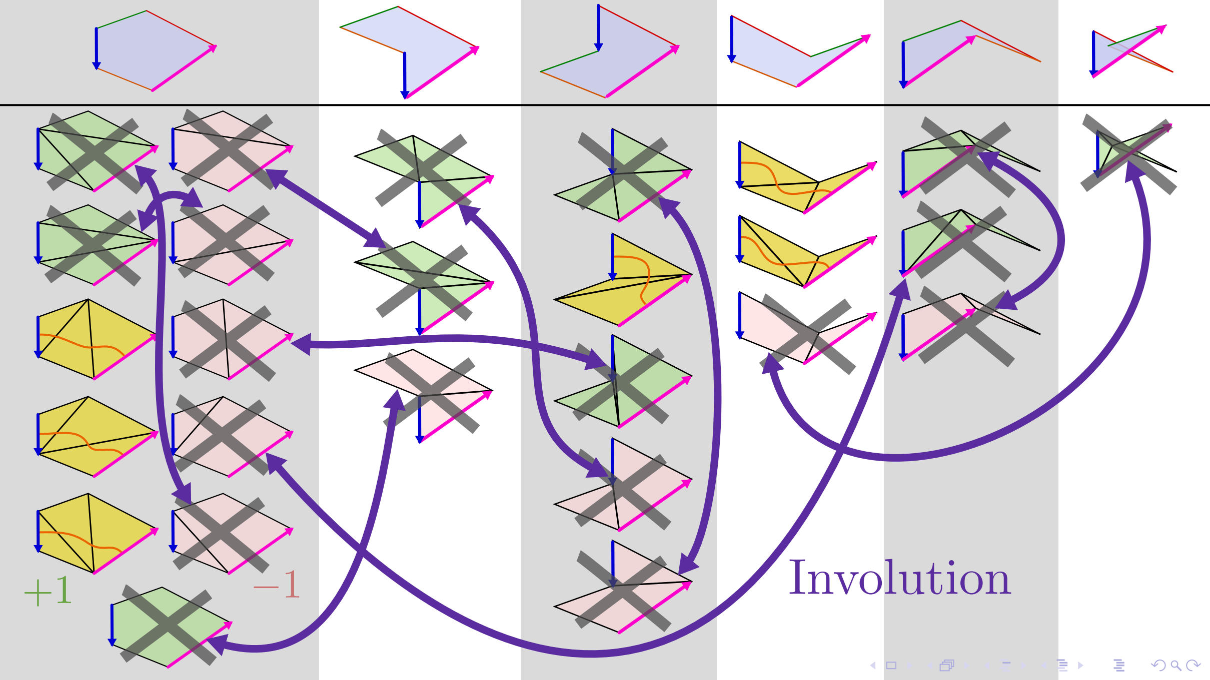


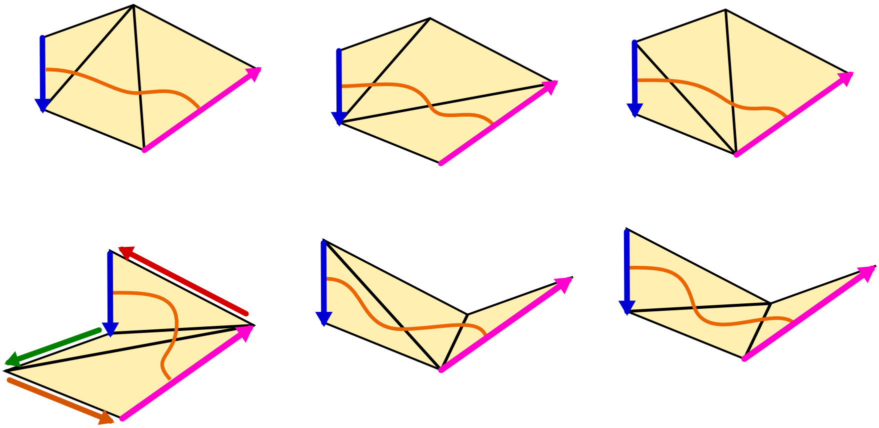
Involution?

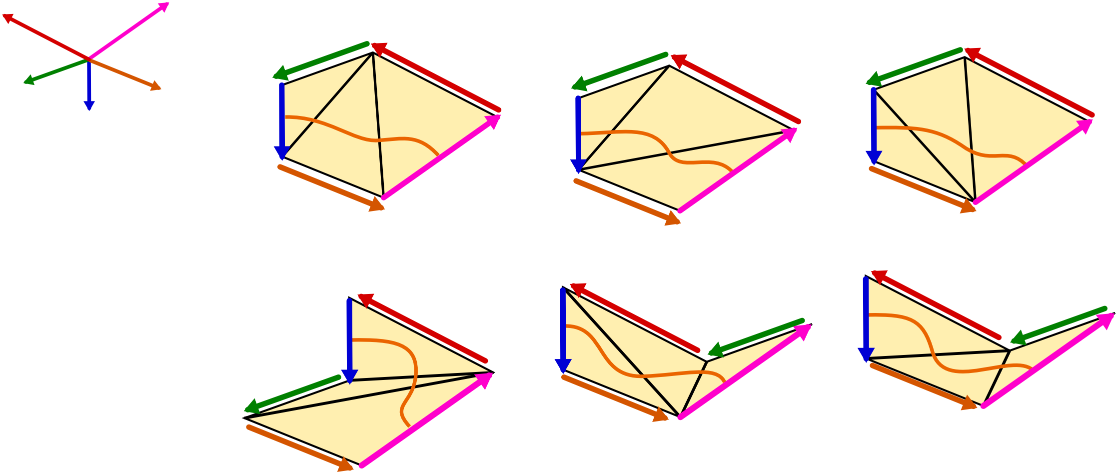


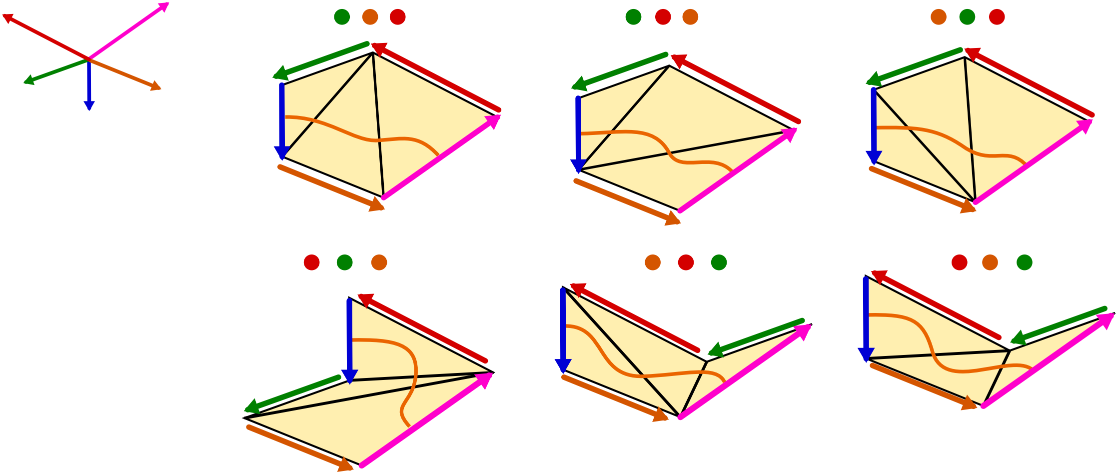
Involution?

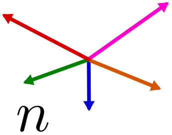

not a triangle
opposite sign
triangle not adjacent to boundary

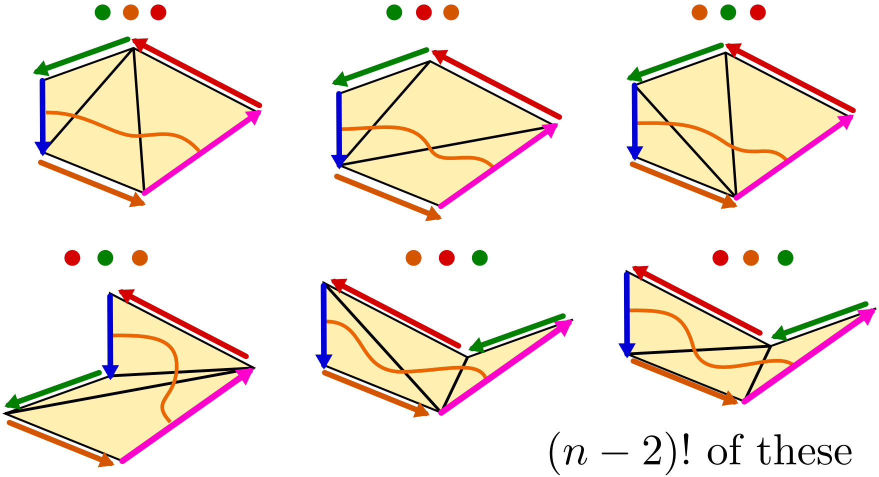

+1
−1
Involution



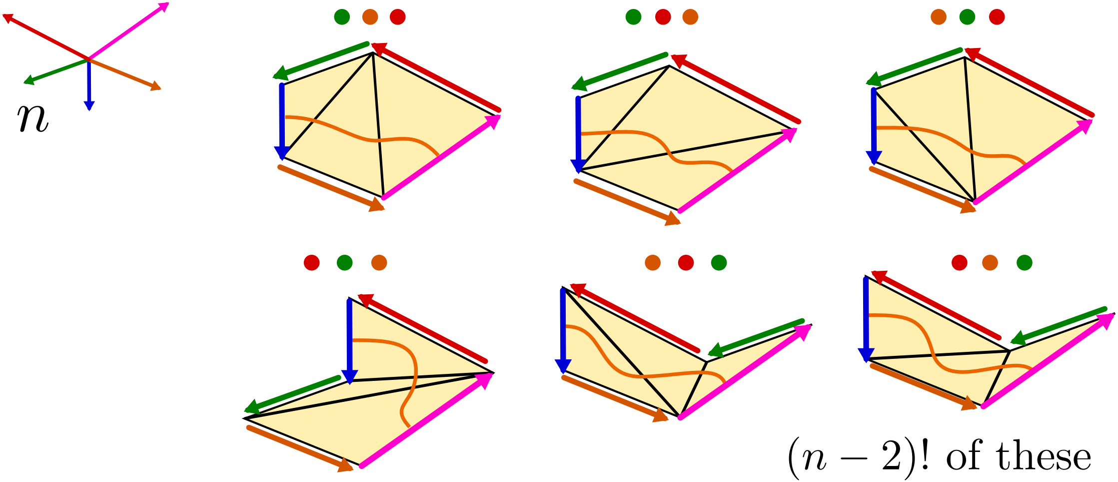








$(n-2)!$ of these



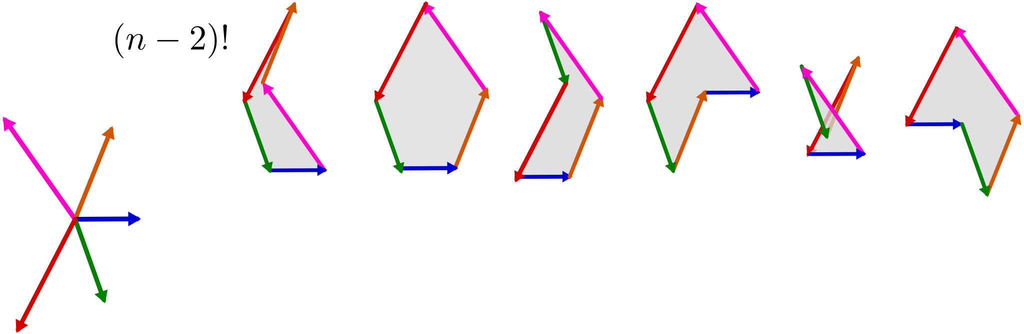
$(n-2)!$ of these

Theorem (TB, '24+)

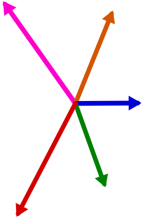
The number of n-sided disks with sides in a fixed generic zero-sum set $Z \subset \mathbb{R}^2$ is $(n-2)!$.*

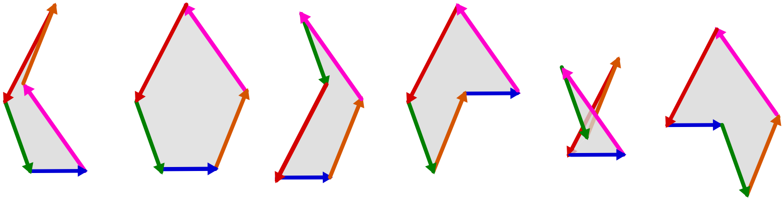
* Z is *generic* if each non-trivial pair of subsets $\subset Z$ has linearly independent sums.
(In particular, all angles distinct!)

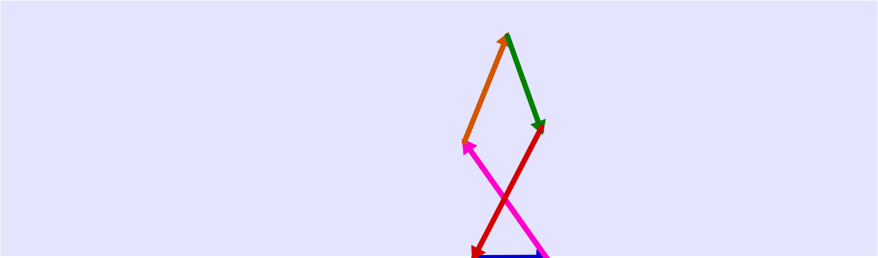
$(n-2)!$

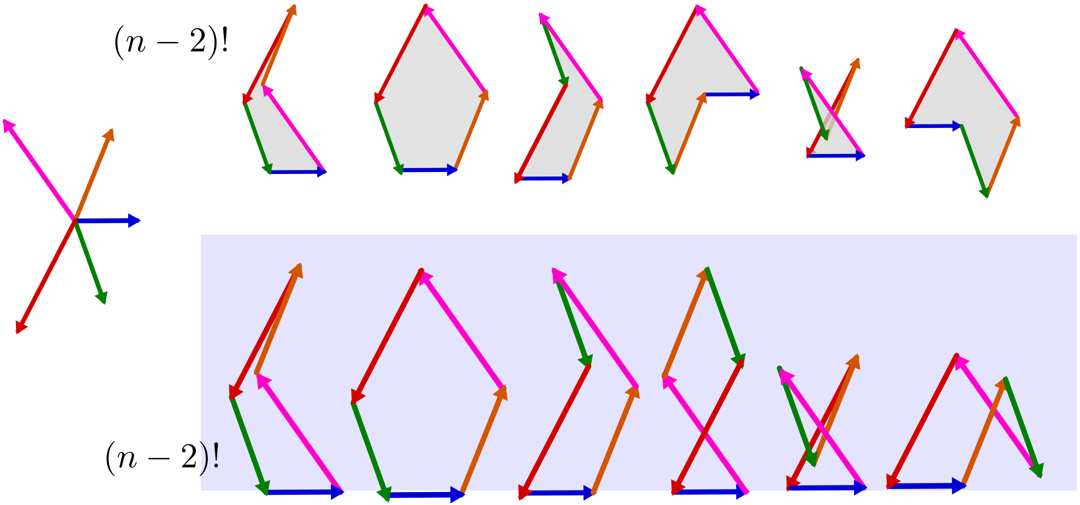


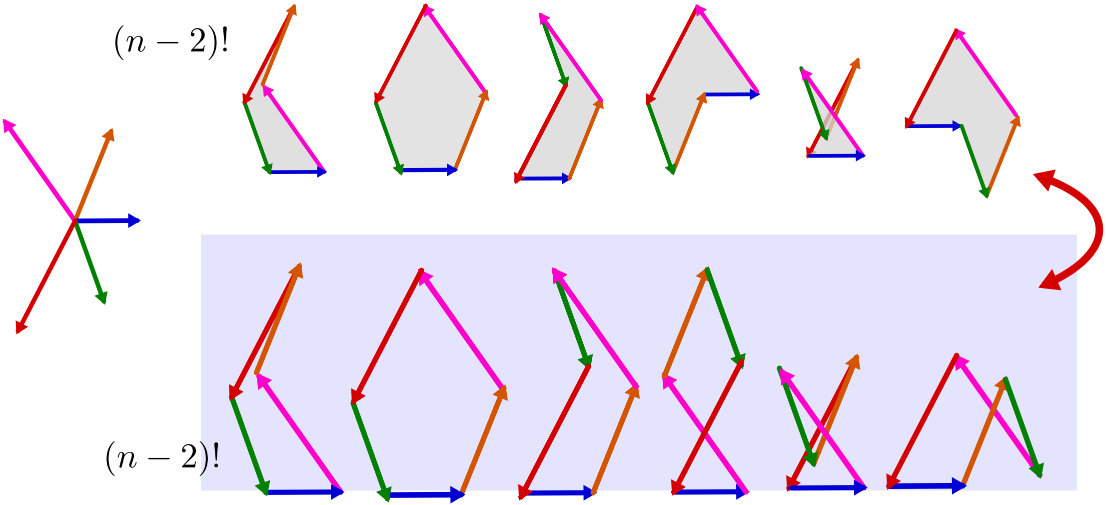
$$(n-2)!$$








$(n-2)!$
$(n-2)!$

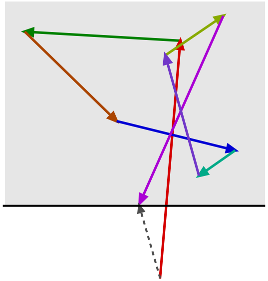


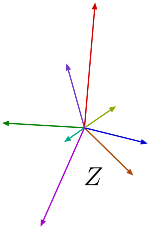
Theorem (TB, '24+)

If Z is generic, there is an explicit bijection between flat disks and excursions in the half-plane.

Idea of proof: an extended bijection

- An excursion is a walk $w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2$ such that $w_1, \ldots, w_{n-1}$ are above w_0, w_n.





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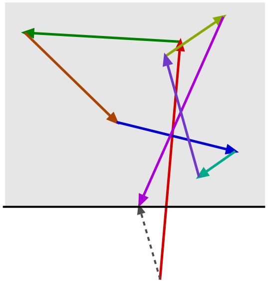
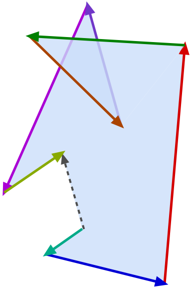
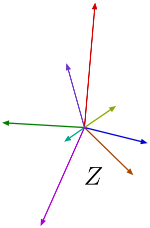
- An excursion is a walk $w_0 = 0, w_1, \ldots, w_n \in \mathbb{R}^2$ such that $w_1, \ldots, w_{n-1}$ are above w_0, w_n.
- Call a disk unobstructed if no corner is visible horizontally from the base.







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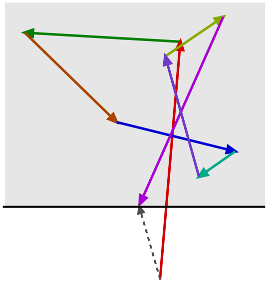

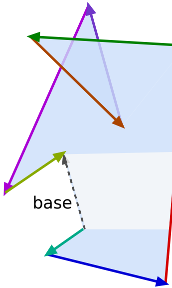

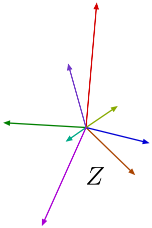

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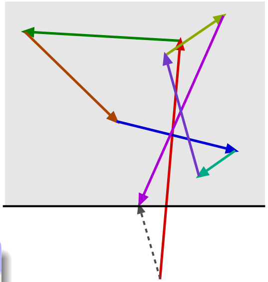
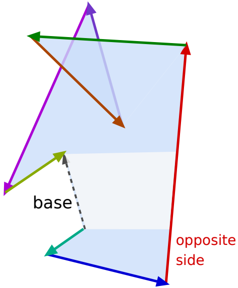


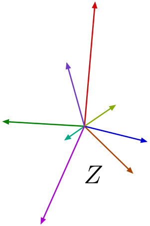


Theorem

If Z is generic,*

$$\{\textit{excursions with increments } Z\}$$
$$\updownarrow$$
$$\{\textit{unobstructed disks with sides } Z\}$$

*as before, but also no proper subset of Z has horizontal sum



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Theorem

If Z is generic,* $\{$*excursions with increments Z*$\}$

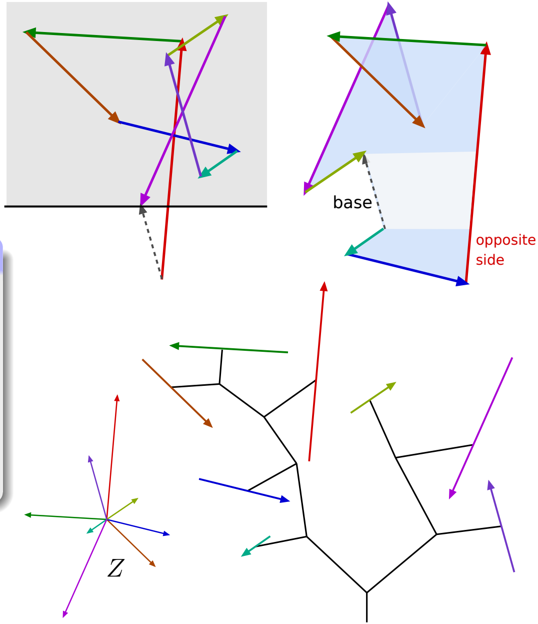
$$\updownarrow$$

$\{$*unobstructed disks with sides Z*$\}$

$$\updownarrow$$

$\left\{\begin{array}{c}\textit{unordered } Z\textit{-leaf-labeled binary trees} \\ \textit{with up \& down branch at every vertex}\end{array}\right\}$

*as before, but also no proper subset of Z has horizontal sum



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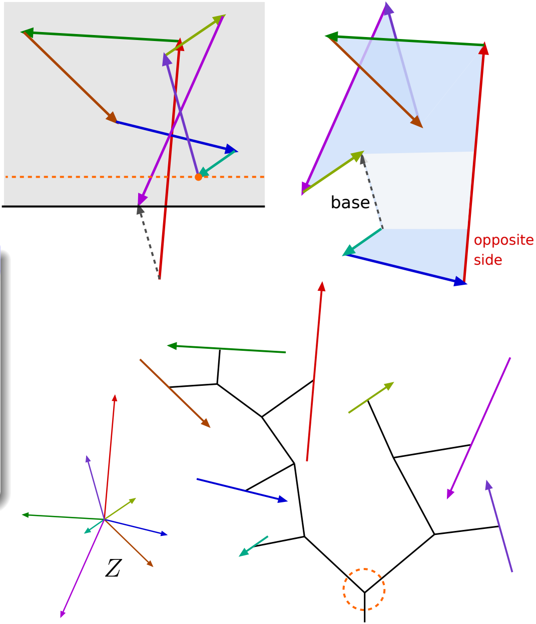
*as before, but also no proper subset of Z has horizontal sum

- Excursions fragment by heights.



Idea of proof: an extended bijection

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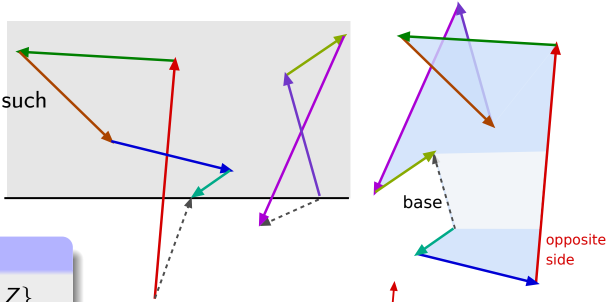


Theorem

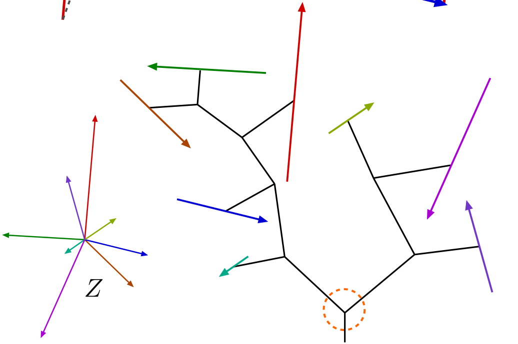
If Z is generic*, $\{$*excursions with increments Z*$\}$

$$\updownarrow$$

$\{$*unobstructed disks with sides Z*$\}$

$$\updownarrow$$

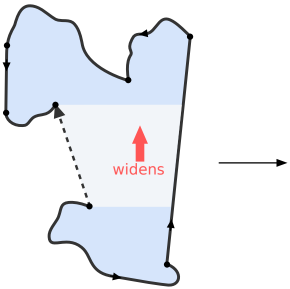
$\left\{ \begin{array}{c} \textit{unordered } Z\textit{-leaf-labeled binary trees} \\ \textit{with up \& down branch at every vertex} \end{array} \right\}$



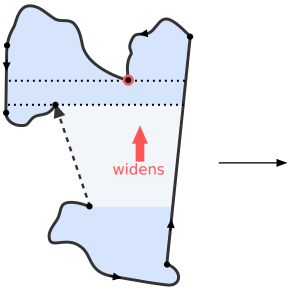
*as before, but also no proper subset of Z has horizontal sum

- Excursions fragment by heights.

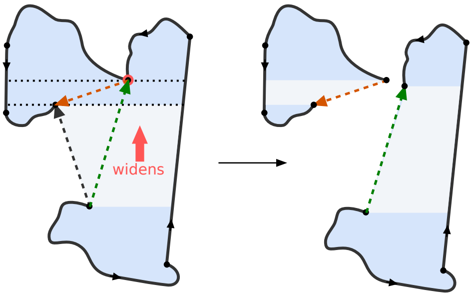
Fragmentation of disks



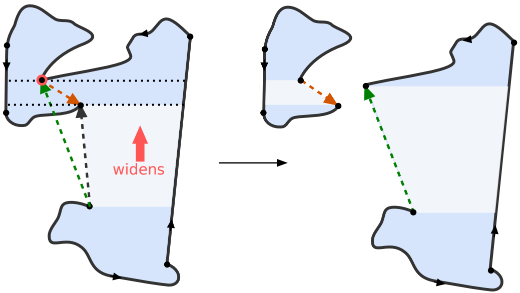
Fragmentation of disks



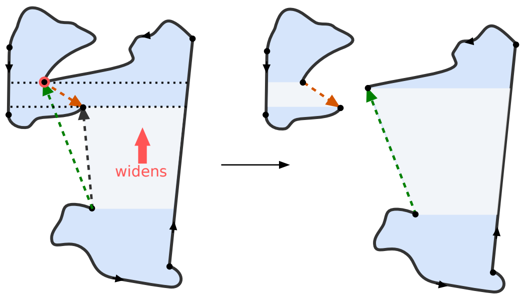
Fragmentation of disks

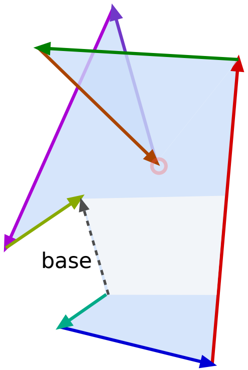


Fragmentation of disks

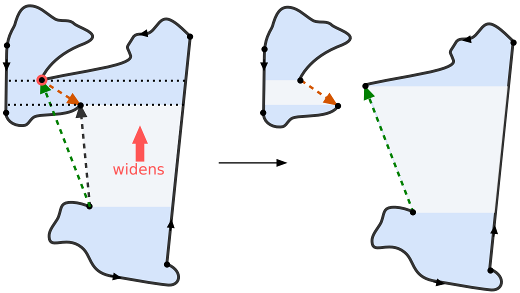


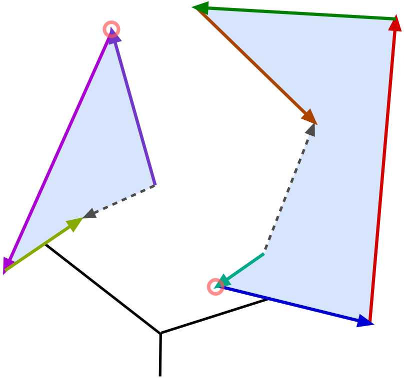
Fragmentation of disks



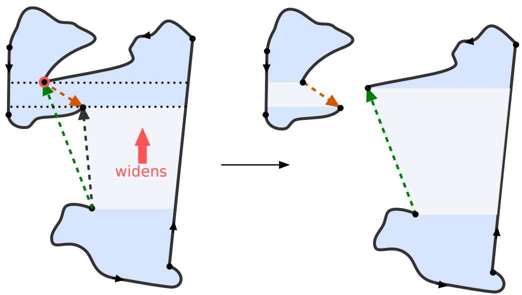


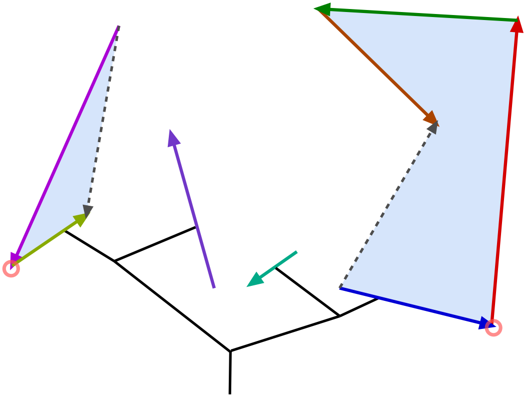
Fragmentation of disks



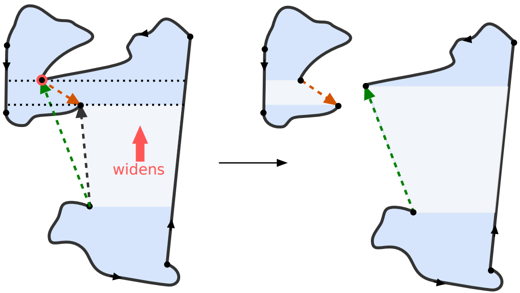


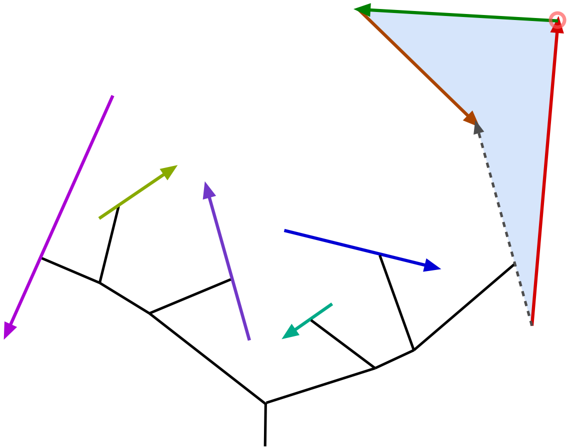
Fragmentation of disks



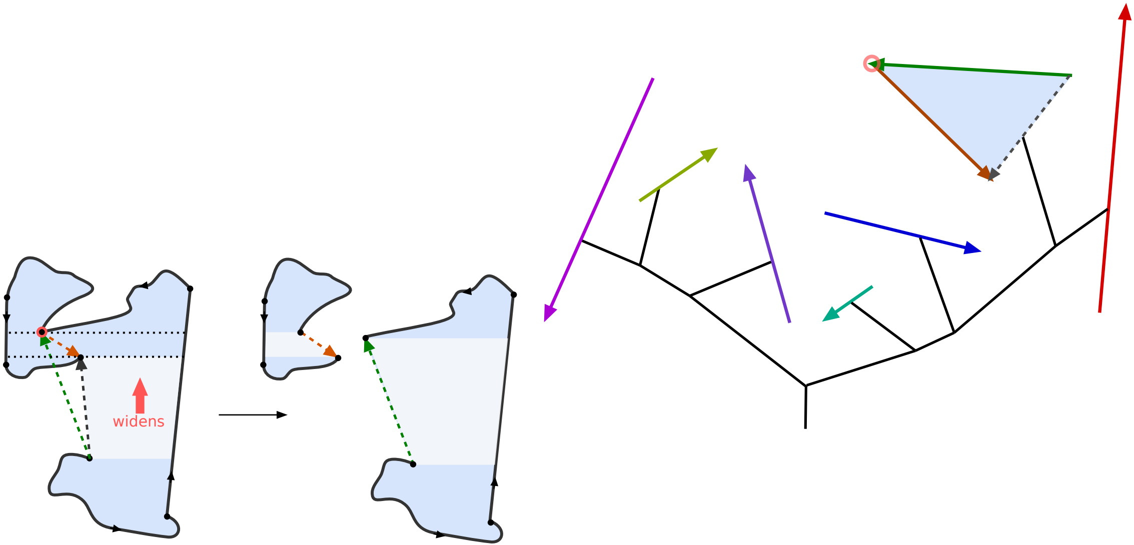


Fragmentation of disks

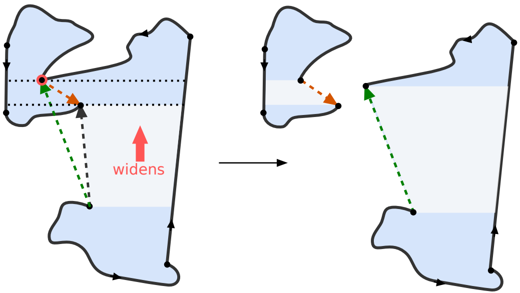


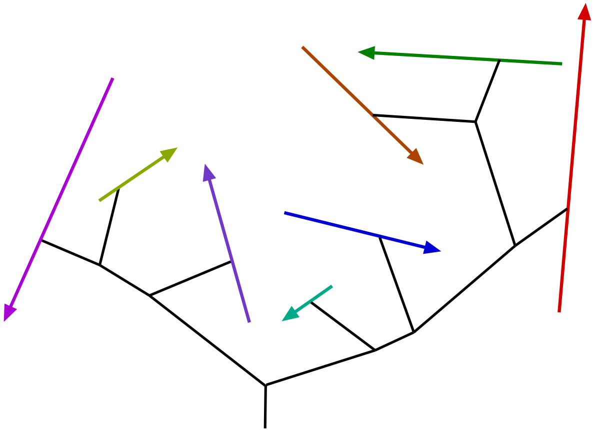


Fragmentation of disks



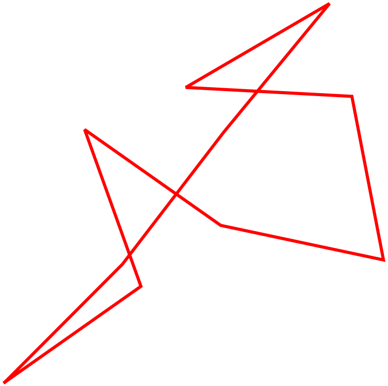
Fragmentation of disks

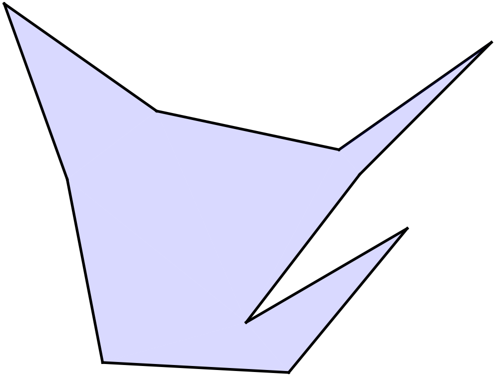




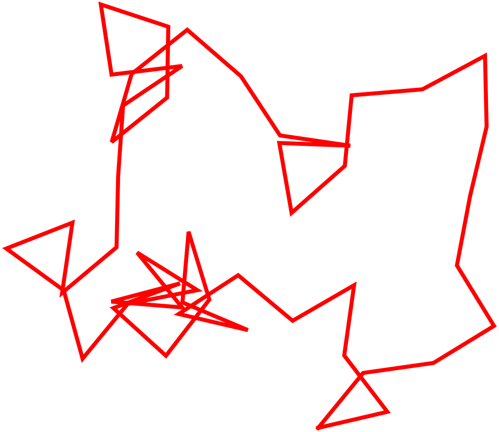
Show me da random diskz!

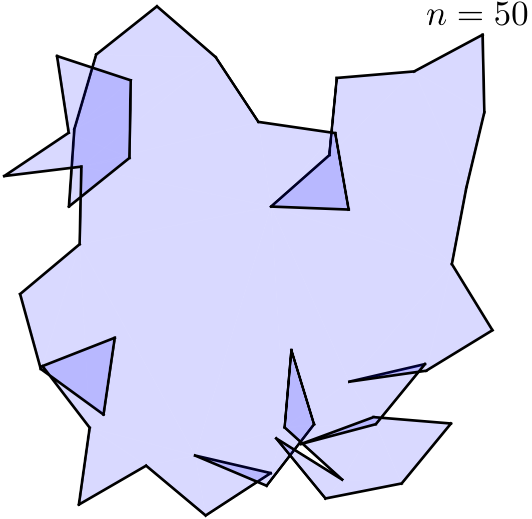
$n = 10$



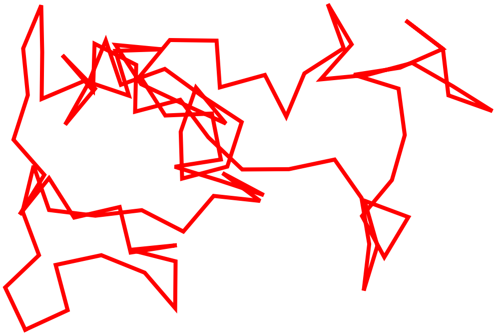


Show me da random diskz!

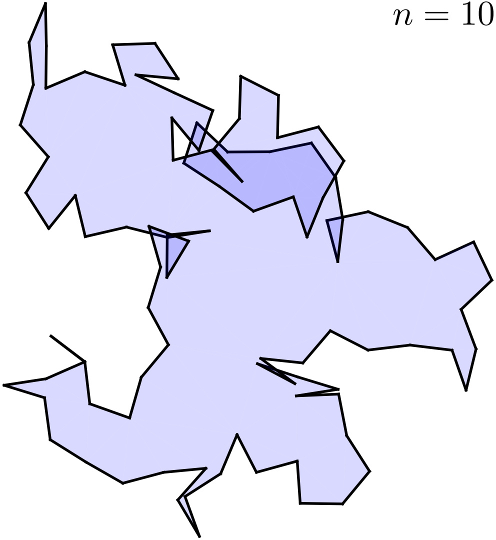




Show me da random diskz!

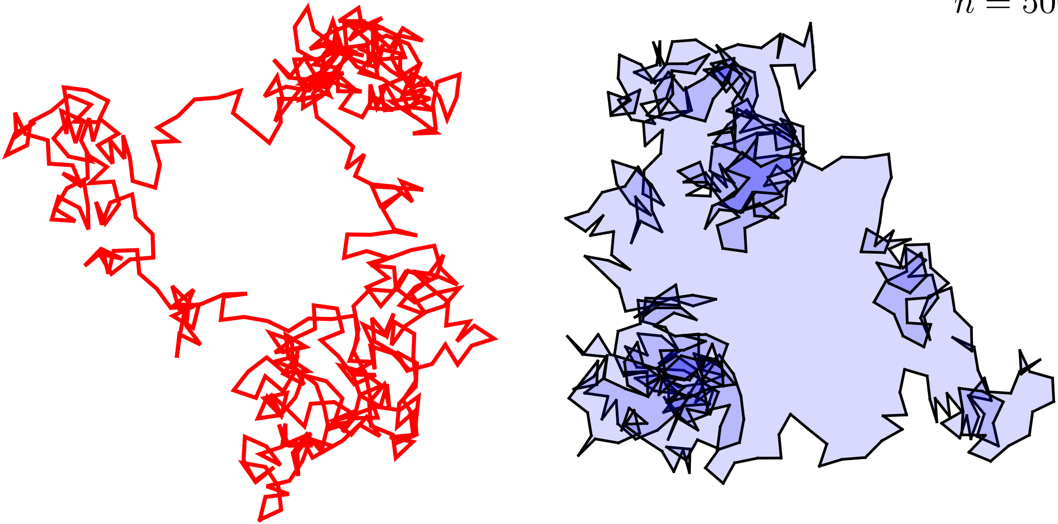


$n = 100$



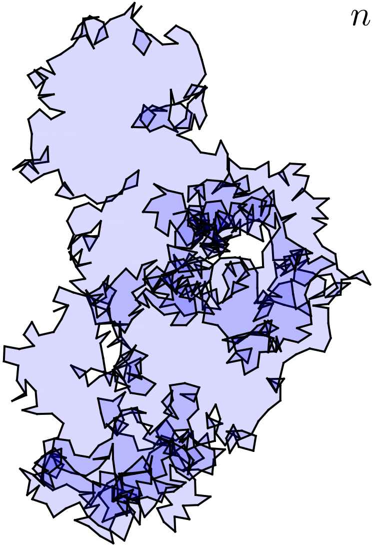
Show me da random diskz!

$n = 500$

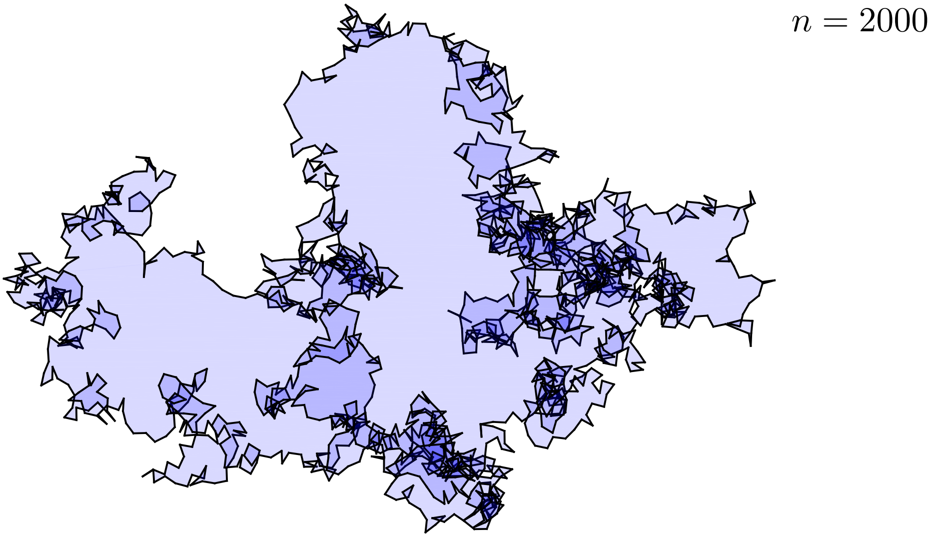


Show me da random diskz!

$n = 1000$



Show me da random diskz!



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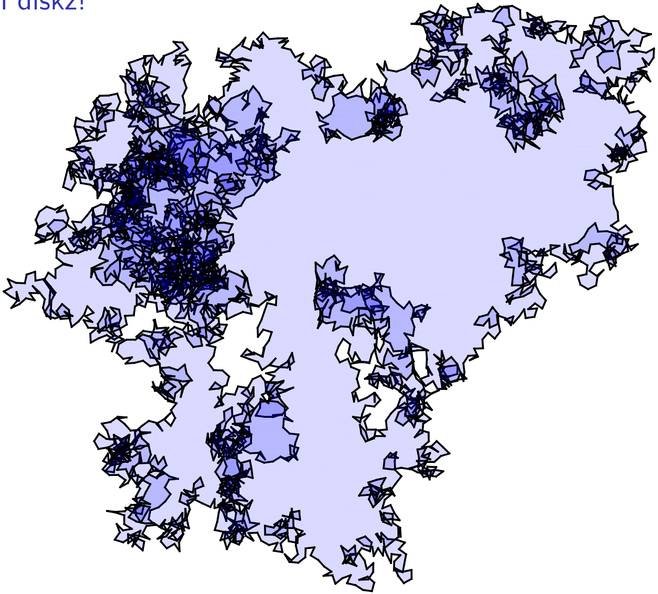
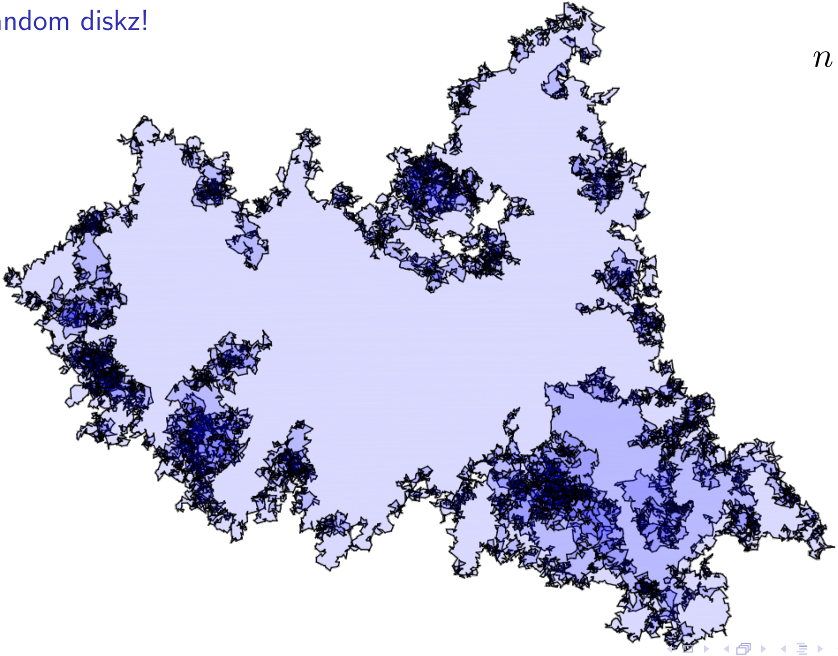


$n = 5000$

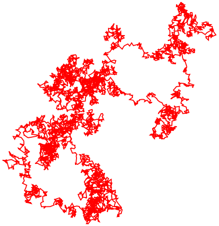
Show me da random diskz!

$n = 25000$



Area of uniform random flat disk

- Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge in $\mathbb{R}^2$



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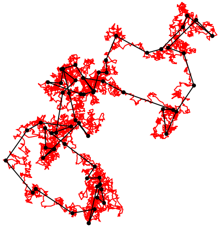

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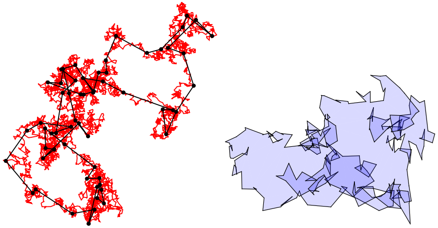

Theorem (TB, '24+)

The area of the uniform flat disk with sides Z_n satisfies

$$\mathbb{E}[\mathrm{Area}_n] = \frac{\log n}{2\pi} + C + o_n(1) \quad \text{as } n \to \infty,$$

with $C = -\int_0^1 \Gamma'(x+1)/(4\pi\sqrt{x}\,\Gamma(x+1))\mathrm{d}x = 0.0285\ldots$

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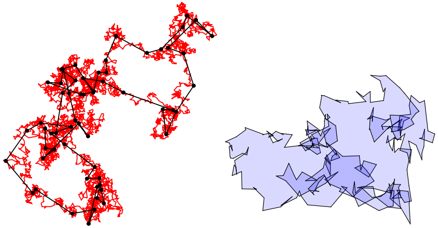

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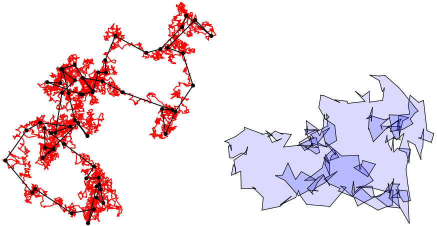

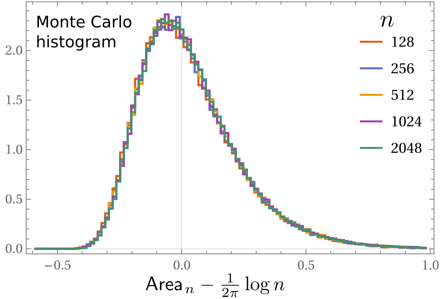

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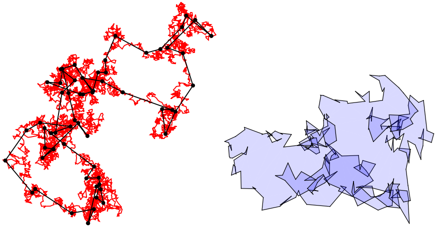

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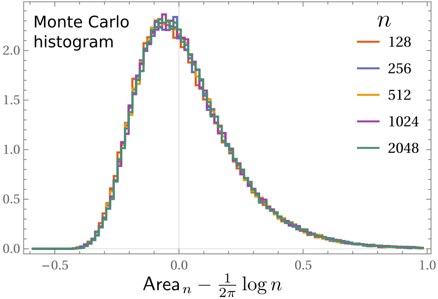

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- Eerie similarity to N-winding area A_N of $(B_t)_t$ itself:

$$\sum_{N=1}^{n} NA_N - \frac{\log n}{2\pi} \quad \text{converges a.s.} \quad \text{[Werner, '94]}$$

Conjectural scaling limit: Brownian flat disk?

The bijection suggests a construction of a *Brownian flat disk*: Let $(X(t))_{t\in[0,1]}$ a Brownian bridge
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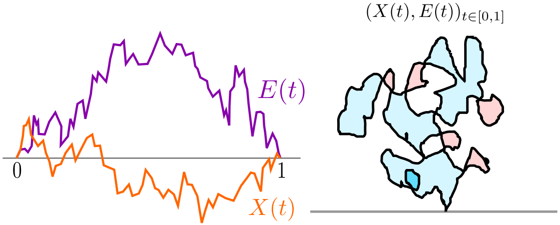

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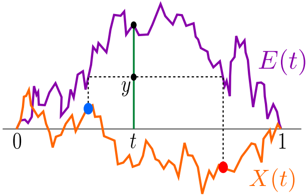

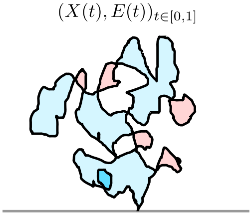

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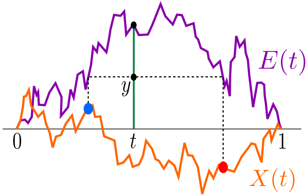

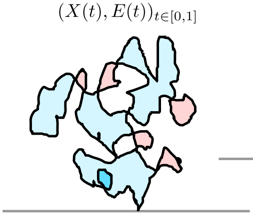

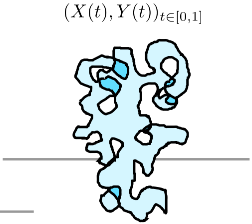

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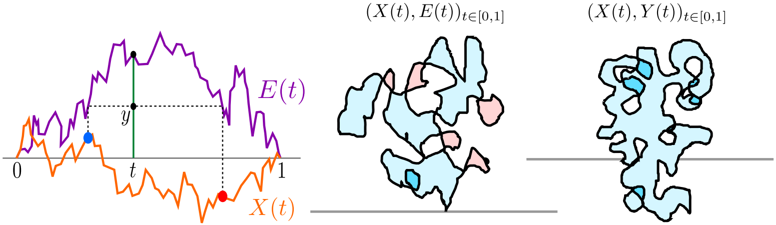

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The law of $(X(t), Y(t))_{t\in[0,1]}$ is rotationally invariant and is the weak limit of the boundary of the uniform disk with sides Z_n as $n \to \infty$.

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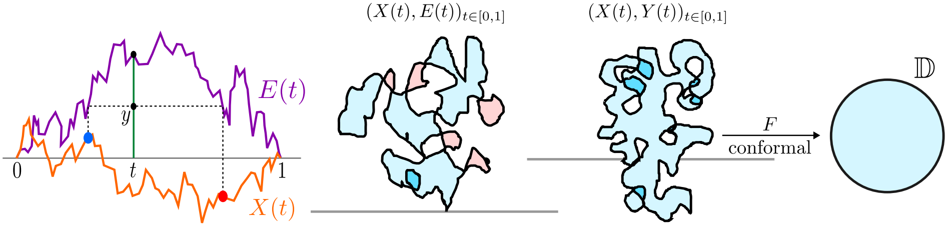

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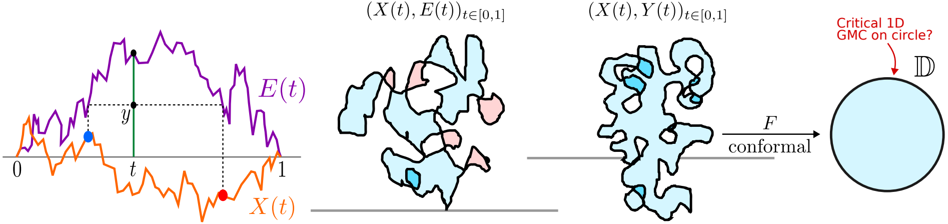

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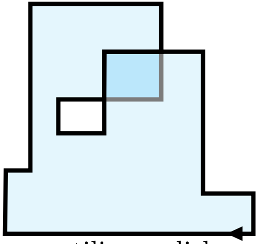
Another model of flat disks: a link with critical LQG?

- Consider *rectilinear disks*: piecewise-straight boundary with $2n$ right-angled corners.



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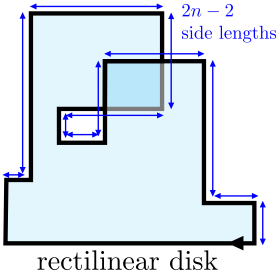

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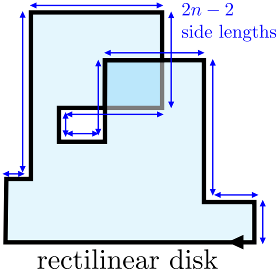

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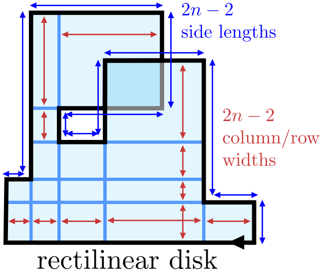

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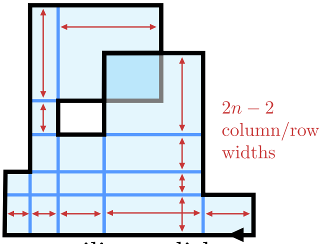
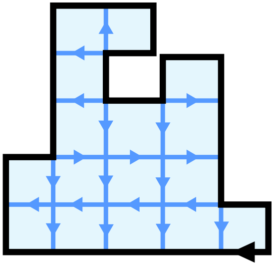


rectilinear disk



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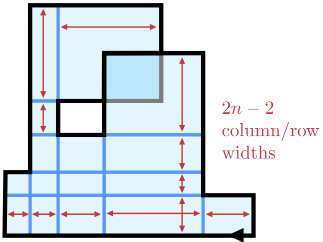

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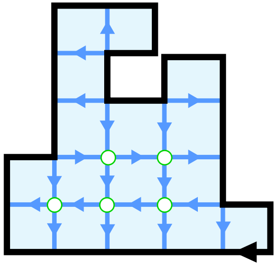
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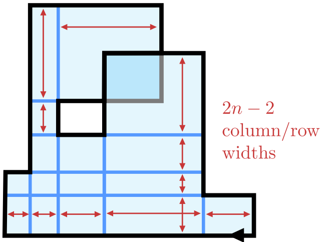

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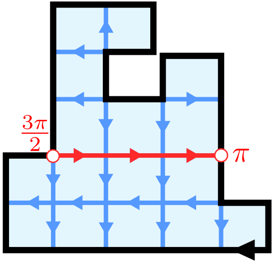

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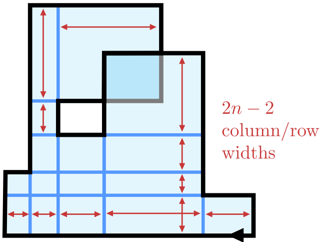

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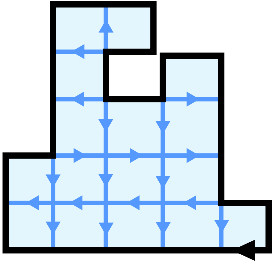
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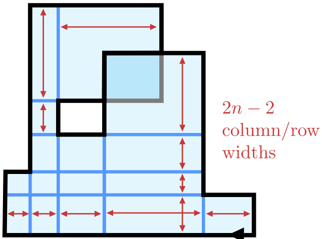
- Akin to random planar map models in universality class of $LQG_{\gamma=2}\dots$



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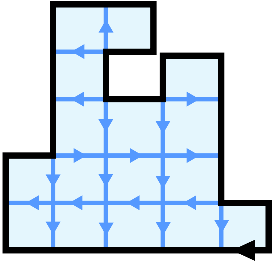
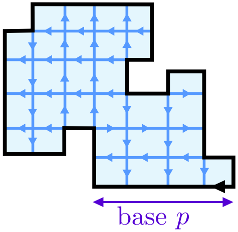
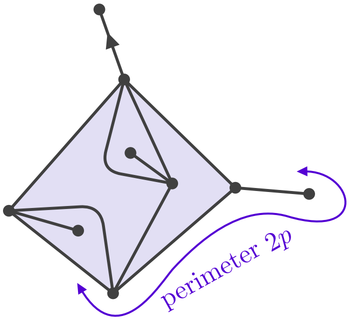

rigid quadrangulation

Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

rigid quadrangulation





quadrangulation

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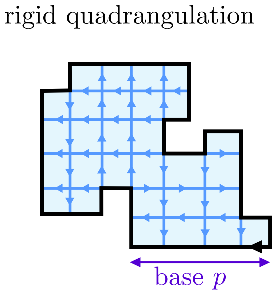
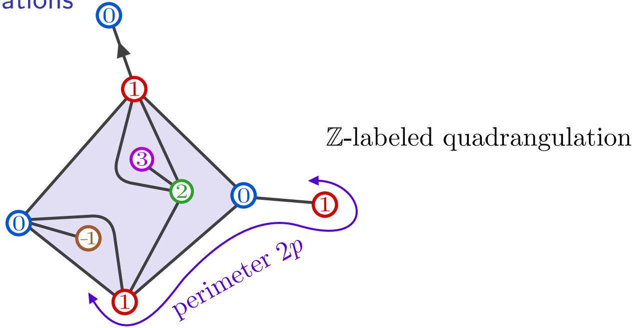
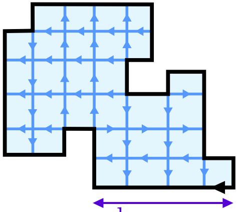
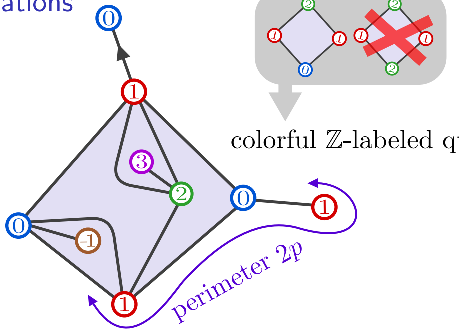




Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

rigid quadrangulation





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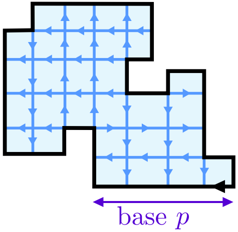
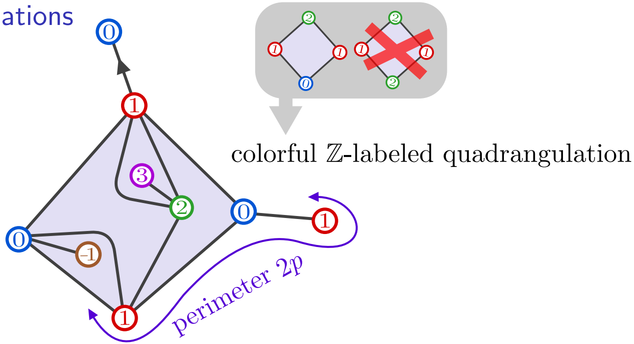
rigid quadrangulation





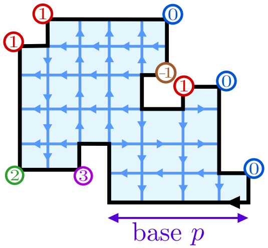
Theorem (TB, '24+)

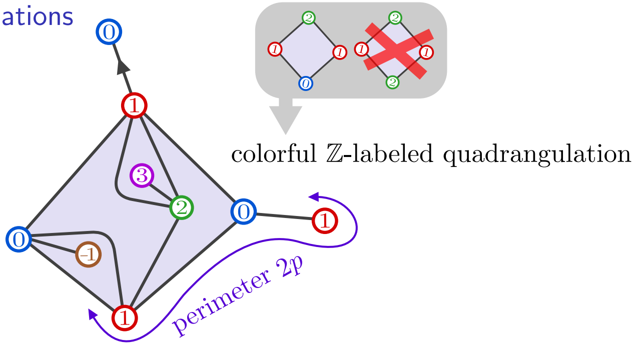
For $n \geq 2$ and $p \geq 1$ there exists a bijection

$$\left\{ \begin{array}{c} \textit{rigid quadrangulations with} \\ 2n \textit{ corners and base } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{colorful } \mathbb{Z}\textit{-labeled quadrangulations} \\ \textit{with } n \textit{ vertices and perimeter } 2p \end{array} \right\}$$

Bijection with colorful $\mathbb{Z}$-labeled quadrangulations

rigid quadrangulation





Theorem (TB, '24+)

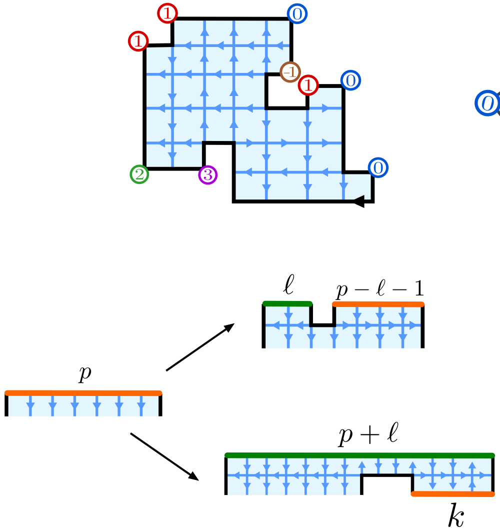
For $n \geq 2$ and $p \geq 1$ there exists a bijection

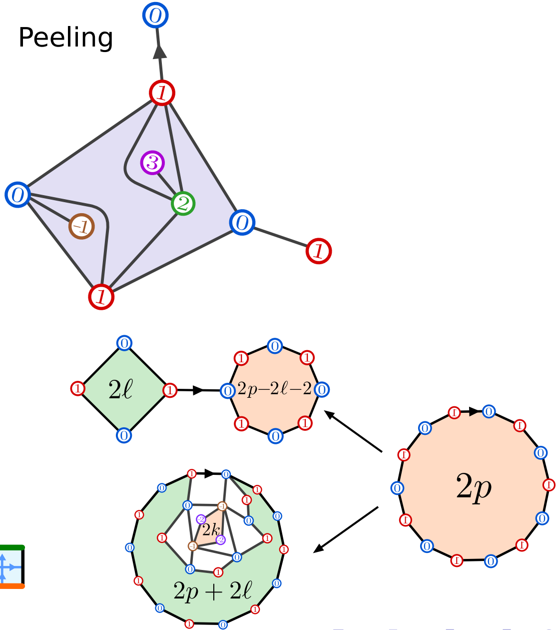
$$\left\{ \begin{array}{c} \textit{rigid quadrangulations with} \\ 2n \textit{ corners and base } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{colorful } \mathbb{Z}\textit{-labeled quadrangulations} \\ \textit{with } n \textit{ vertices and perimeter } 2p \end{array} \right\}$$

$$\tfrac{\pi}{2}\textit{-corner with } \underbrace{\textit{turning number } \ell}_{\#\textit{left}-\#\textit{right}} \longleftrightarrow \textit{vertex with label } \ell$$

Bijection via exploration

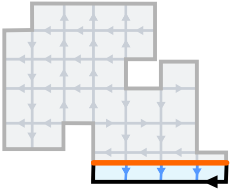
Scanning vs Peeling

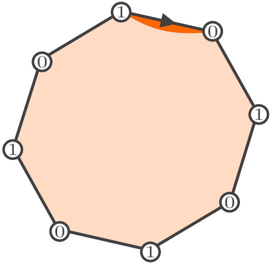


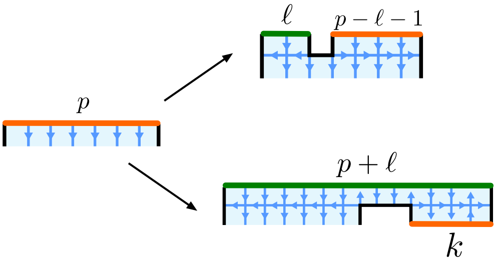


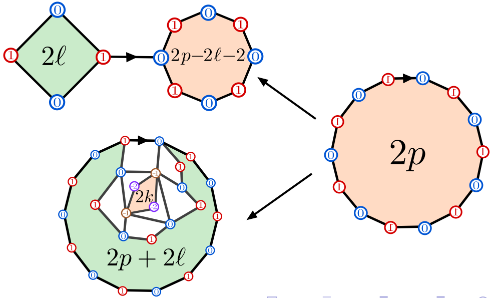
Bijection via exploration

Scanning vs Peeling



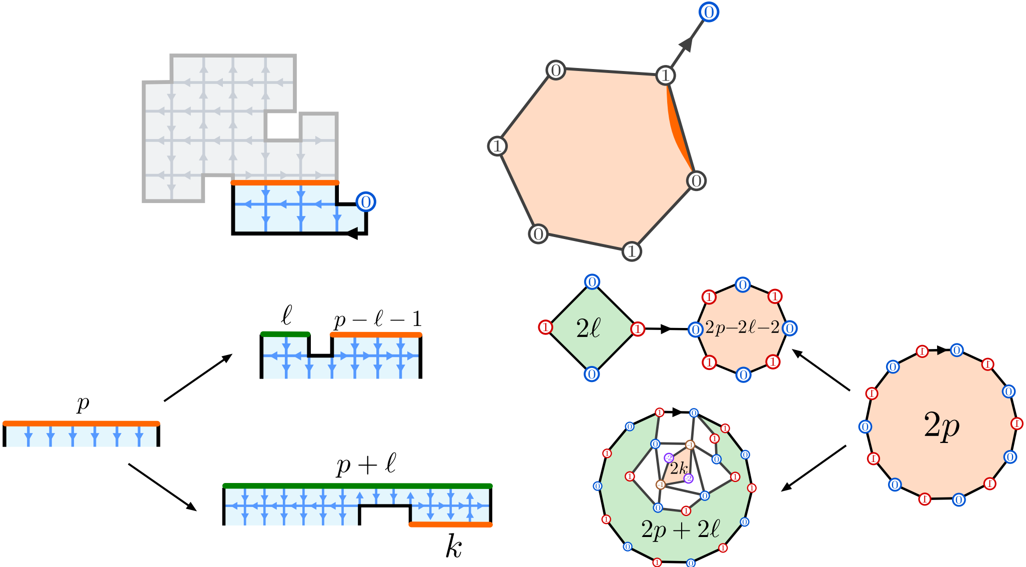






Bijection via exploration

Scanning vs Peeling



Bijection via exploration

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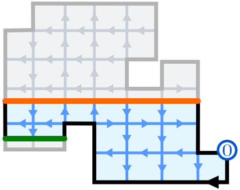

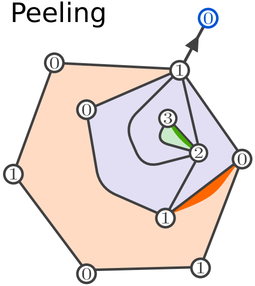

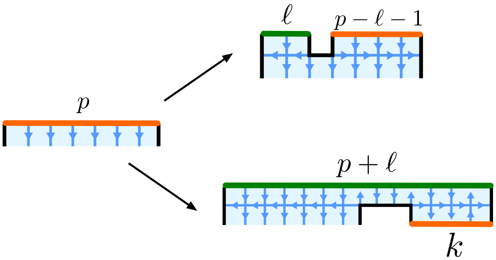

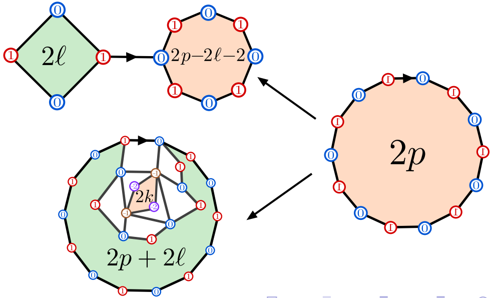

Bijection via exploration

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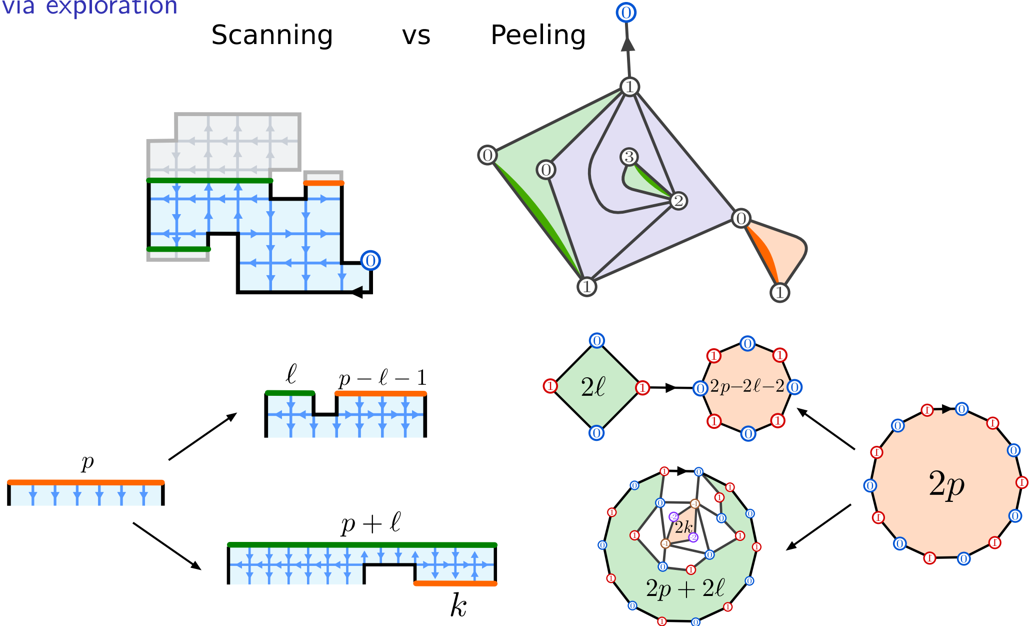

Bijection via exploration

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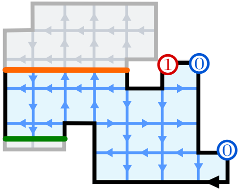

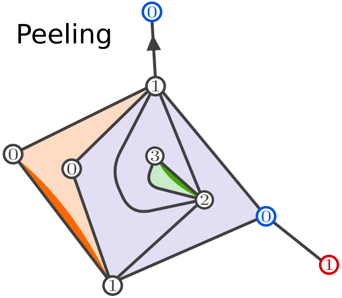

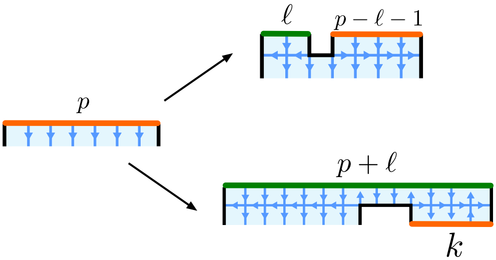

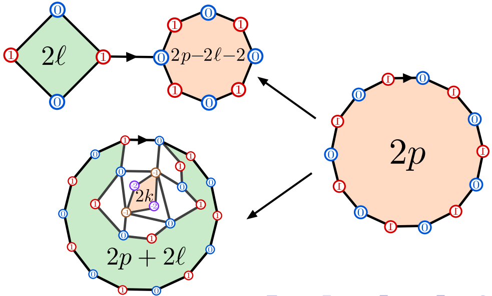

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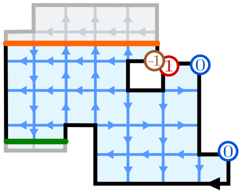

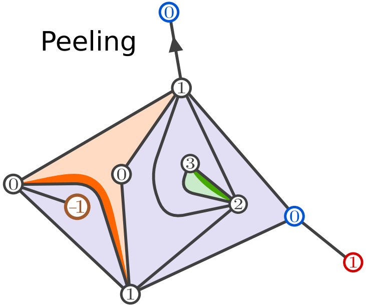

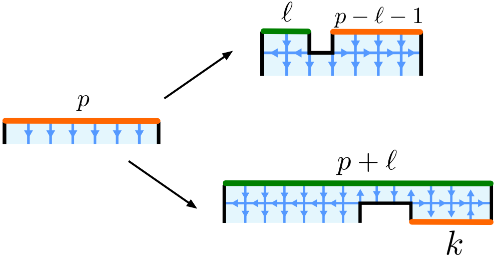

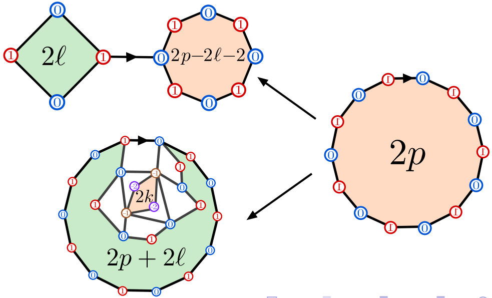

Bijection via exploration

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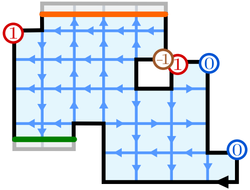
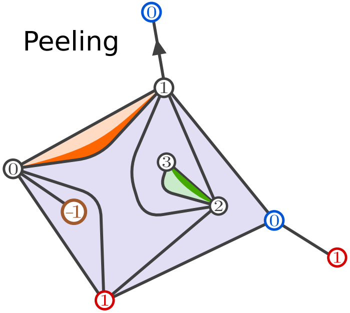
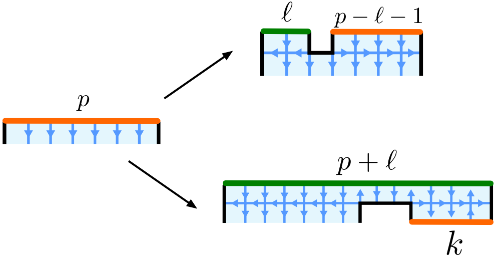
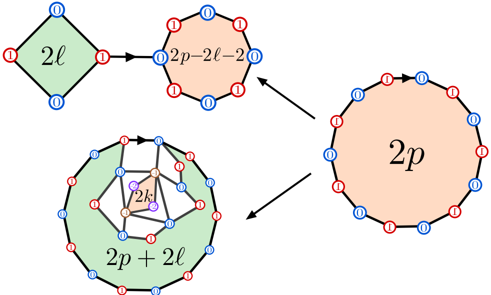


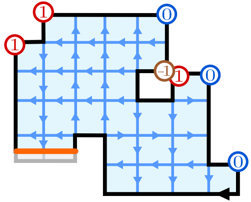


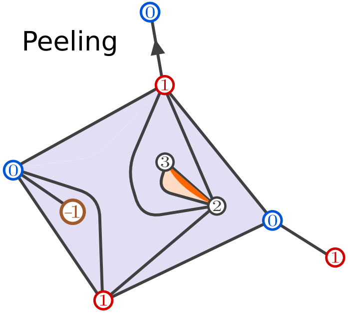


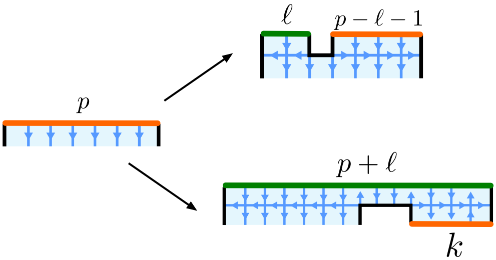


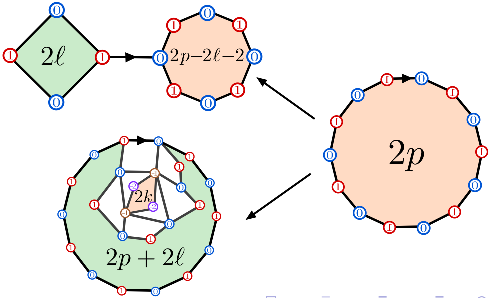
Bijection via exploration

Scanning vs Peeling









Bijection via exploration

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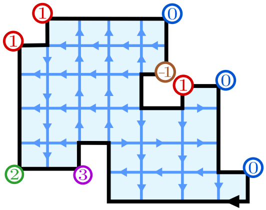
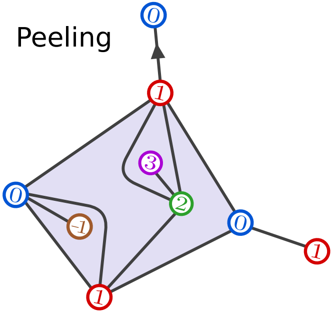




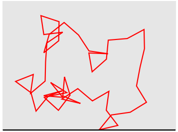
- Many guises of these labeled quadrangulations: planar Eulerian orientations, $\mathbb{Z}$-labeled bipartite planar maps, special case of six-vertex model [Kostov, '00][Zinn-Justin, '00][Elvey Price, Guttmann, '17]
- Enumeration finally settled in

Theorem (Bousquet-Mélou, Elvey Price, '20)

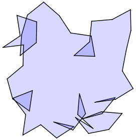
The generating function of perimeter-2p colorful $\mathbb{Z}$-labeled quadrangulations is

$$Q^{(p)}(x) = \sum_{k \geq p} \frac{1}{k+1} \binom{2k}{k} \binom{2k-p}{k} R(x)^{k+1}, \qquad \textit{when} \sum_{k \geq p} \frac{1}{k+1} \binom{2k}{k}^2 R(x)^{k+1} = x.$$

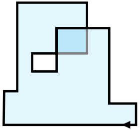
Status and outlook in pictures



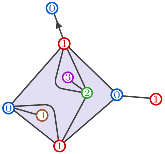
walks in half plane



polygonal flat disks

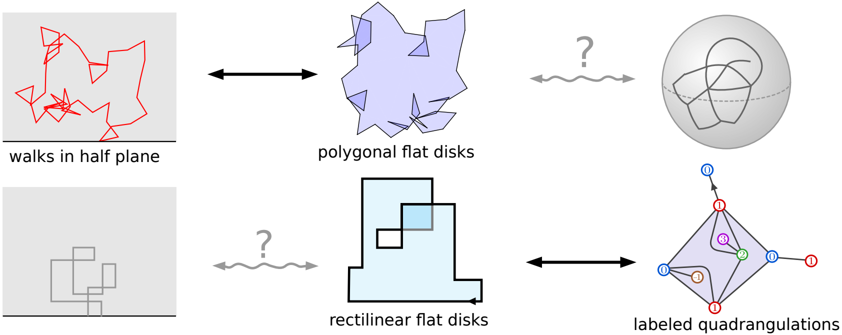


rectilinear flat disks



labeled quadrangulations

Status and outlook in pictures



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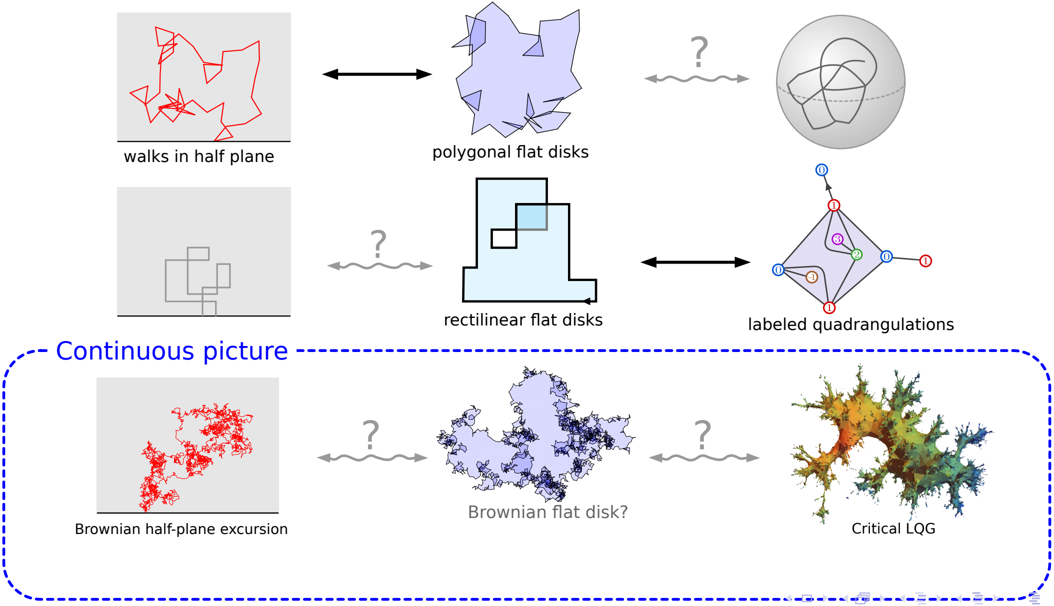

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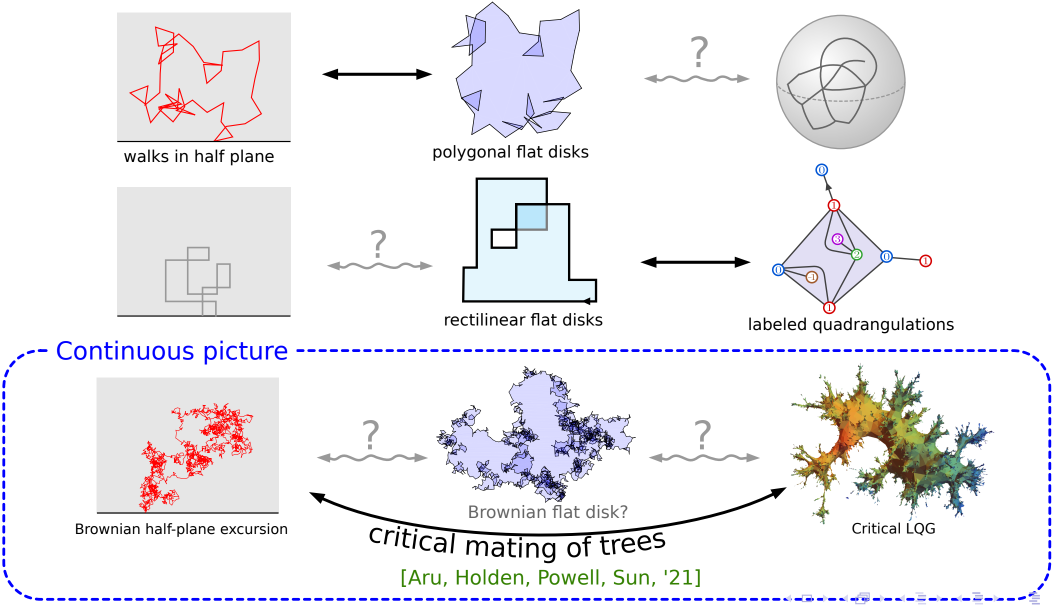

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