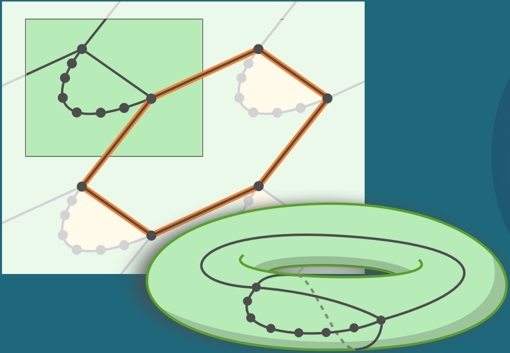
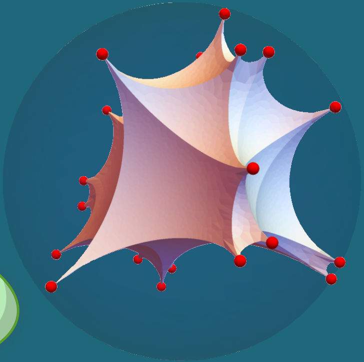
08-10-2020, Webinaire Équations Fonctionnelles et Interactions

Essentially irreducible maps and Weil-Petersson volumes

Timothy Budd





based on: arXiv:2006.15701 & w.i.p.

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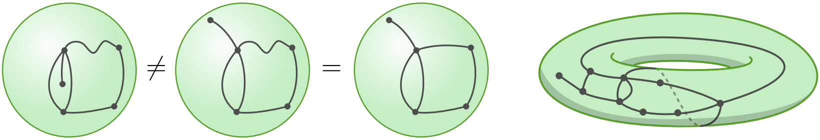
Genus-g maps

- A genus-g map is a connected graph that is properly embedded in a surface of genus g modulo orientation-preserving homeomorphisms.



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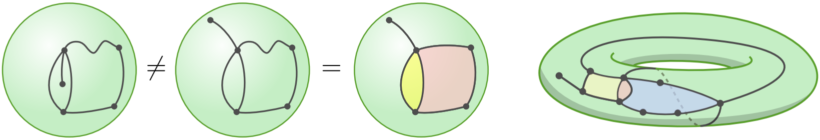

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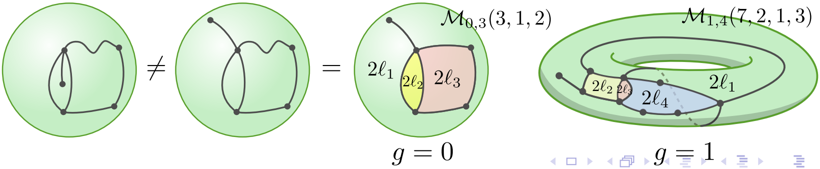
- For $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}_+^n$ let

$$\mathcal{M}_{g,n}(\boldsymbol{\ell}) = \{\text{genus-}g \text{ maps with } n \text{ labeled faces of degrees } 2\ell_1, \ldots, 2\ell_n\}$$



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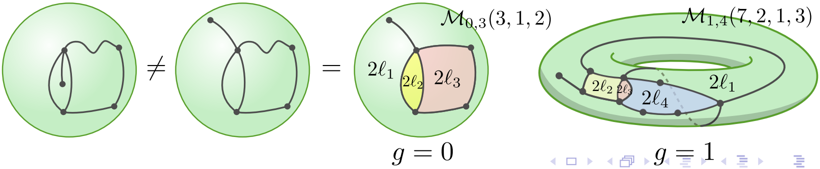
- Classical enumeration problem:

$$\|\mathcal{M}_{g,n}(\boldsymbol{\ell})\| := \sum_{\mathfrak{m} \in \mathcal{M}_{g,n}(\boldsymbol{\ell})} \frac{1}{|\mathrm{Aut}(\mathfrak{m})|} = \frac{|\mathcal{M}_{g,n}^{\text{rooted}}(\boldsymbol{\ell})|}{2|\boldsymbol{\ell}|} \qquad (= |\mathcal{M}_{g,n}(\boldsymbol{\ell})| \text{ in most cases})$$



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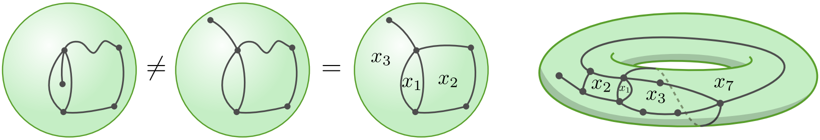
- Generating function of even maps: genus-g partition function

$$F_g(\mathrm{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\boldsymbol{\ell} \in \mathbb{Z}_+^n} \|\mathcal{M}_{g,n}(\boldsymbol{\ell})\| \prod_{i=1}^{n} x_{\ell_i}, \quad \mathrm{x} = (x_1, x_2, \ldots)$$



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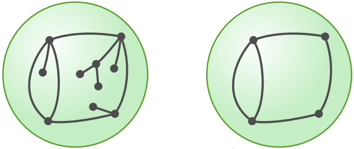

$\mathcal{M}_{0,3}(2,2,7)$ $\quad$ $\hat{\mathcal{M}}_{0,3}(2,1,2)$

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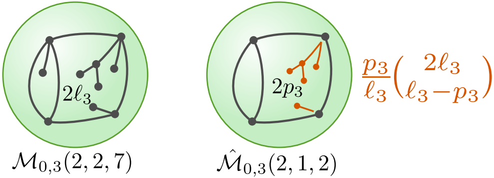

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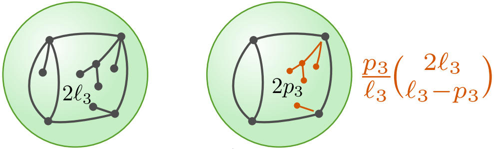
- Similarly $\hat{F}_g(\hat{\mathrm{x}}) = F_g(\mathrm{x})$ by the substitution $\hat{x}_p = \sum_{\ell \geq p} \frac{p}{\ell}\binom{2\ell}{\ell - p} x_\ell$.



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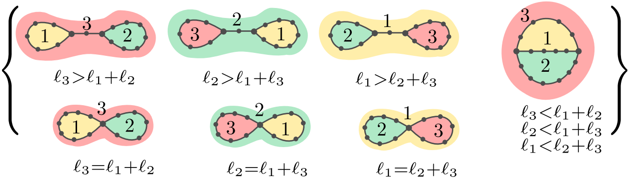
$$\hat{\mathcal{M}}_{0,3}(\boldsymbol{\ell}) =$$



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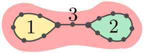
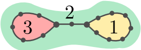


$$\|\hat{\mathcal{M}}_{0,3}(\boldsymbol{\ell})\| = \mathbf{1}_{\{\ell_3>\ell_1+\ell_2\}}$$



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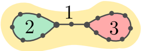

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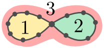

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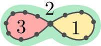

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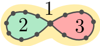

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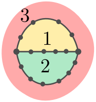

$$+\,\mathbf{1}_{\left\{\substack{\ell_3<\ell_1+\ell_2 \\ \ell_2<\ell_1+\ell_3 \\ \ell_1<\ell_2+\ell_3}\right\}} = 1$$

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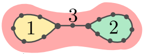

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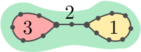

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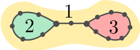

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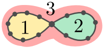

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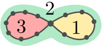

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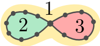

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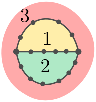

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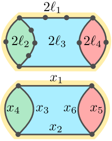
- More generally,

$$\|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\| = \sum_{\text{skeleton } \mathfrak{s}} \frac{|\{\mathbf{x} \in \mathbb{Z}_+^{\#\text{edges}(\mathfrak{s})} : \mathbf{A}_{\mathfrak{s}}\mathbf{x} = 2\boldsymbol{\ell}\}|}{|\mathrm{Aut}(\mathfrak{s})|}$$



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with $\mathbf{A}_{\mathfrak{s}}$ the face-edge incidence matrix of $\mathfrak{s}$.

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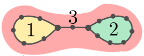

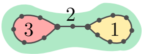

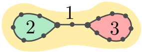

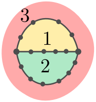

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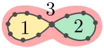

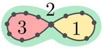

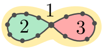

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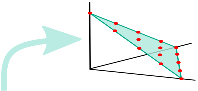

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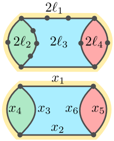

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with $\mathsf{A}_{\mathfrak{s}}$ the face-edge incidence matrix of $\mathfrak{s}$.

- Problem of counting integer points in convex polyhedra:
$|\{\mathsf{x} \in \mathbb{Z}_+^{\#\text{edges}(\mathfrak{s})} : \mathsf{A}_{\mathfrak{s}}\mathsf{x} = 2\boldsymbol{\ell}\}|$ is piecewise $\underbrace{\text{quasi-polynomial}}_{\text{coeffs depend periodically on } \boldsymbol{\ell}}$ in $\boldsymbol{\ell}$ [Ehrhart, '62] [Sturmfels, '95]

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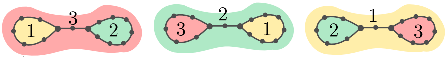

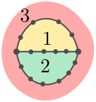

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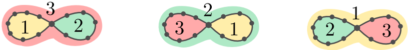

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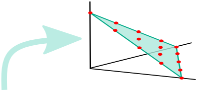

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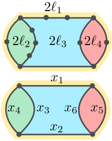

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- Finitely many skeletons $\implies \|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\|$ is piecewise quasi-polynomial in $\boldsymbol{\ell}$.

Theorem (Norbury, '08)

$N_{g,n}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\|$ is a (symmetric) polynomial in $\ell_1^2, \ldots, \ell_n^2$ of degree $3g-3+n$.

$$N_{0,3} = 1, \qquad N_{0,4} = \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 1$$

$$N_{1,1} = \tfrac{1}{12}\ell_1^2 - \tfrac{1}{12}, \qquad N_{1,2} = \tfrac{1}{24}(\ell_1^4 + \ell_2^4) + \tfrac{1}{12}\ell_1^2\ell_2^2 - \tfrac{1}{8}(\ell_1^2 + \ell_2^2) + \tfrac{1}{12}.$$

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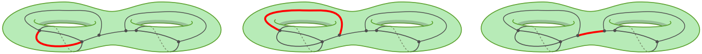
- Proof: There exists a topological recursion / Tutte equation / loop equation for $N_{g,n}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\|$,

$$\text{“}N_{g,n} = \sum N_{g,n-1} + \sum N_{g-1,n+1} + \sum N_{g',n'} N_{g-g',n-n'+1}\text{”}$$



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$N_{g,n}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\|$ is a (symmetric) polynomial in $\ell_1^2, \dots, \ell_n^2$ of degree $3g-3+n$.

$$N_{0,3} = 1, \qquad N_{0,4} = \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 1$$

$$N_{1,1} = \tfrac{1}{12}\ell_1^2 - \tfrac{1}{12}, \qquad N_{1,2} = \tfrac{1}{24}(\ell_1^4 + \ell_2^4) + \tfrac{1}{12}\ell_1^2\ell_2^2 - \tfrac{1}{8}(\ell_1^2 + \ell_2^2) + \tfrac{1}{12}.$$

- Proof: There exists a topological recursion / Tutte equation / loop equation for $N_{g,n}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}(\boldsymbol{\ell})\|$,

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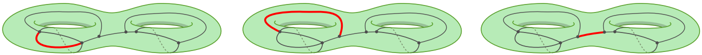

Theorem (Norbury, '09)

The polynomials satisfy “string” and “dilaton” equations,

$$N_{g,n+1}(\boldsymbol{\ell}, 1) = \sum_{j=1}^{n} \sum_{k=1}^{\ell_j} 2k\, N_{g,n}(\boldsymbol{\ell})|_{\ell_j = k} - \sum_{j=1}^{n} \ell_j N_{g,n}(\boldsymbol{\ell}), \qquad (\text{“string”})$$

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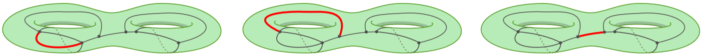

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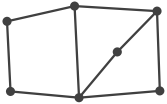
- They completely determine $N_{g,n}(\boldsymbol{\ell})$ for $g = 0, 1$ once $N_{0,3}$ and $N_{1,1}$ are known. For $g \geq 2$ more equations are necessary.

Essentially irreducible maps [Bouttier, Guitter, '13] [Bonichon, Fusy, Lévêque, '19]

- A planar map is *d-irreducible* if $\overbrace{\text{every simple cycle has length} \geq d}^{\text{has girth at least } d}$ with equality only if the cycle bounds a face of degree d.

$\hat{\mathcal{M}}_{0,4}(3,2,2,2)$



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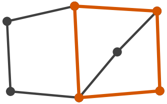

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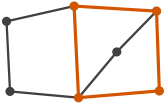

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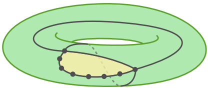

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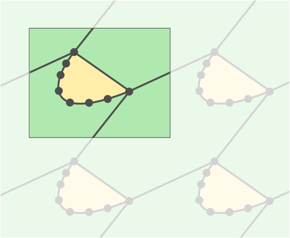

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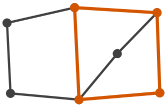

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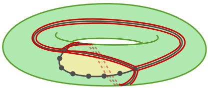

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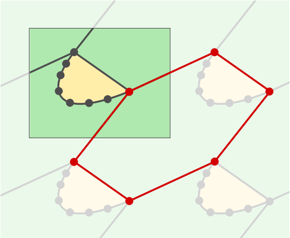

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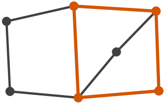

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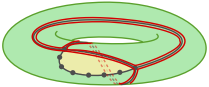

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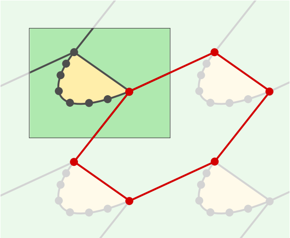

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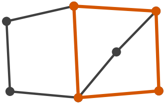
- For even maps only need to consider $d = 2b$ even.

$\hat{\mathcal{M}}_{0,4}(3,2,2,2)$



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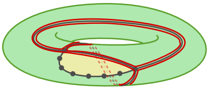

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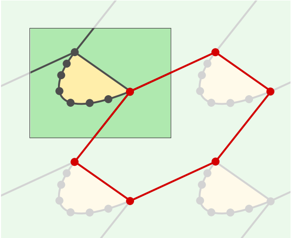

- Enumeration again amounts to counting integer points in convex polyhedra:

$$k = \#\mathrm{edges}(\mathfrak{s}) + \#\mathrm{cycles}(\mathfrak{s})$$

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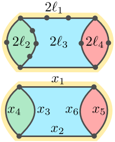

skeleton $\mathfrak{s}$

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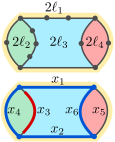

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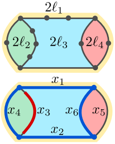

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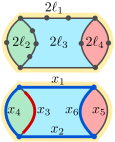

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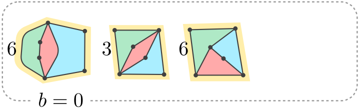

b	#
0	15
1	
2	

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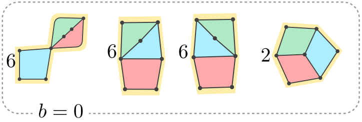

b	#
0	20
1	
2	

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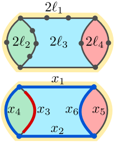

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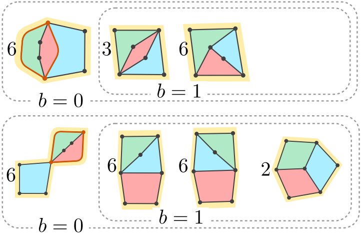

b	#
0	15
1	9
2	

b	#
0	20
1	14
2	

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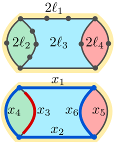

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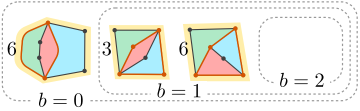

b	#
0	15
1	9
2	0

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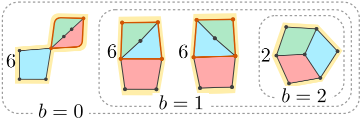

b	#
0	20
1	14
2	2

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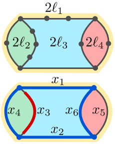

skeleton $\mathfrak{s}$

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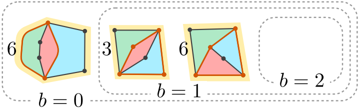
$$\mathbf{B}_\mathfrak{s} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\mathsf{A}_\mathfrak{s}$ the face-edge matrix and $\mathsf{B}_\mathfrak{s}$ a cycle-edge enclosure matrix.

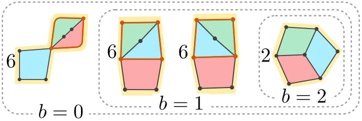
- $\|\hat{\mathcal{M}}^{(2b)}_{g,n}(\boldsymbol{\ell})\|$ is piecewise quasi-polynomial in $b, \boldsymbol{\ell}$. [Sturmfels, '95]
- Example:

$$\hat{\mathcal{M}}^{(2b)}_{0,4}(2,2,2,2) =$$



b	#
0	15
1	9
2	0

$$\hat{\mathcal{M}}^{(2b)}_{0,4}(3,2,2,2) =$$



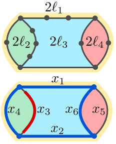
b	#
0	20
1	14
2	2

- How about $\|\hat{\mathcal{M}}^{(2b)}_{0,4}(\boldsymbol{\ell})\| = \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 3b^2 - 3b - 1$?

- Enumeration again amounts to counting integer points in convex polyhedra:

$$k = \#\mathrm{edges}(\mathfrak{s}) + \#\mathrm{cycles}(\mathfrak{s})$$

$$\|\hat{\mathcal{M}}^{(2b)}_{g,n}(\boldsymbol{\ell})\| = \sum_{\text{skeleton } \mathfrak{s}} \frac{\left|\left\{\mathbf{x}\in\mathbb{Z}^k_+ : \begin{pmatrix}\mathbf{A}_\mathfrak{s} & 0\\ \mathbf{B}_\mathfrak{s} & \mathbb{I}\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\boldsymbol{\ell} \\ \sum\boldsymbol{\ell}-b\end{pmatrix}\right\}\right|}{|\mathrm{Aut}(\mathfrak{s})|}$$



skeleton $\mathfrak{s}$

$$\mathbf{A}_\mathfrak{s} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 1\end{pmatrix} \qquad \mathbf{B}_\mathfrak{s} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

with $\mathsf{A}_\mathfrak{s}$ the face-edge matrix and $\mathsf{B}_\mathfrak{s}$ a cycle-edge enclosure matrix.

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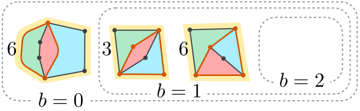
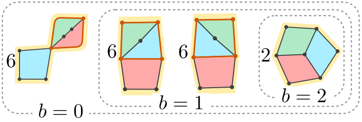


b	#	
0	15	15
1	9	9
2	0	−3

$$\hat{\mathcal{M}}^{(2b)}_{0,4}(3,2,2,2) =$$



b	#	
0	20	20
1	14	14
2	2	2

- How about $\|\hat{\mathcal{M}}^{(2b)}_{0,4}(\boldsymbol{\ell})\| = \ell_1^2+\ell_2^2+\ell_3^2+\ell_4^2 - 3b^2 - 3b - 1$? Almost. . .

Theorem (TB, '20)

$$N_{g,n}^{(2b)}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}^{(2b)}(\boldsymbol{\ell})\| - 1_{\{g=0,\, n\geq 4,\, \ell_1=\cdots=\ell_n=b\}} \frac{(n-1)!}{2}(-1)^n \qquad (\ell_i \geq b \geq 0)$$

is polynomial in $b, \ell_1^2, \ldots, \ell_n^2$ *of degree* $3g-3+n$ *in* $\ell_1^2, \ldots, \ell_n^2$.

$$N_{0,3}^{(2b)} = 1, \qquad N_{0,4}^{(2b)} = \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 3b^2 - 3b - 1$$

$$N_{0,5}^{(2b)} = \tfrac{1}{2}(\ell_1^4 + \cdots) + 2(\ell_1^2\ell_2^2 + \cdots) - \big(6b^2 + 6b + \tfrac{5}{2}\big)(\ell_1^2 + \cdots) + \big(10b^4 + 20b^3 + 20b^2 + 10b + 2\big)$$

$$N_{1,1}^{(2b)} = \tfrac{1}{12}\ell_1^2 - \tfrac{1}{12}, \qquad N_{1,2}^{(2b)} = \tfrac{1}{24}(\ell_1^4 + \ell_2^4) + \tfrac{1}{12}\ell_1^2\ell_2^2 - \tfrac{1}{8}(\ell_1^2 + \ell_2^2) - \tfrac{1}{24}\big(b^4 + 2b^3 - b^2 - 2b - 2\big).$$

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The polynomials satisfy "string" and "dilaton" equations,

$$N_{g,n+1}^{(2b)}(\boldsymbol{\ell}, 1) = \sum_{j=1}^{n} \sum_{k=b+1}^{\ell_j} 2k\, N_{g,n}^{(2b)}(\boldsymbol{\ell})|_{\ell_j=k} - \sum_{j=1}^{n} \ell_j N_{g,n}^{(2b)}(\boldsymbol{\ell}), \qquad (\text{"string"})$$

$$N_{g,n+1}^{(2b)}(\boldsymbol{\ell}, 1) - N_{g,n+1}^{(2b)}(\boldsymbol{\ell}, 0) = (2g - 2 + n) N_{g,n}^{(2b)}(\boldsymbol{\ell}). \qquad (\text{"dilaton"})$$

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$$N^{(2b)}_{0,3} = 1, \qquad N^{(2b)}_{0,4} = \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 3b^2 - 3b - 1$$

$$N^{(2b)}_{0,5} = \tfrac{1}{2}(\ell_1^4 + \cdots) + 2(\ell_1^2\ell_2^2 + \cdots) - \left(6b^2 + 6b + \tfrac{5}{2}\right)(\ell_1^2 + \cdots) + \left(10b^4 + 20b^3 + 20b^2 + 10b + 2\right)$$

$$N^{(2b)}_{1,1} = \tfrac{1}{12}\ell_1^2 - \tfrac{1}{12}, \qquad N^{(2b)}_{1,2} = \tfrac{1}{24}(\ell_1^4 + \ell_2^4) + \tfrac{1}{12}\ell_1^2\ell_2^2 - \tfrac{1}{8}(\ell_1^2 + \ell_2^2) - \tfrac{1}{24}\left(b^4 + 2b^3 - b^2 - 2b - 2\right).$$

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- As before they uniquely determine $N^{(2b)}_{g,n}$ for $g = 0, 1$.

Theorem (TB, '20)

$$N_{g,n}^{(2b)}(\boldsymbol{\ell}) = \|\hat{\mathcal{M}}_{g,n}^{(2b)}(\boldsymbol{\ell})\| - 1_{\{g=0,\, n\geq 4,\, \ell_1=\cdots=\ell_n=b\}} \frac{(n-1)!}{2}(-1)^n \qquad (\ell_i \geq b \geq 0)$$

is polynomial in $b, \ell_1^2, \ldots, \ell_n^2$ of degree $3g-3+n$ in $\ell_1^2, \ldots, \ell_n^2$.

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- As before they uniquely determine $N_{g,n}^{(2b)}$ for $g = 0, 1$.
- Note that when $b > 1$ a combinatorial interpretation of $\ell_{n+1} = 0, 1$ is problematic.

Proof outline: "generatingfunctionology"

1. $\|\hat{\mathcal{M}}_{g,n}^{(2b)}(\boldsymbol{\ell})\| = \frac{\partial^n}{\partial \hat{x}_{\ell_1} \cdots \partial \hat{x}_{\ell_n}} \hat{F}_g^{(2b)}(\hat{\mathrm{x}})\Big|_{\hat{\mathrm{x}}=0}$

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2. Relate partition functions with/without degree-1 vertices via $F_g^{(2b)}(\mathrm{x}(\hat{\mathrm{x}})) = \hat{F}_g^{(2b)}(\hat{\mathrm{x}})$.

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3. Substitution approach: relate $F_g^{(2b)}(\mathrm{x})$ to partition function of arbitrary maps $F_g^{(0)}(\mathrm{x})$.
   [Bouttier, Guitter, '13]

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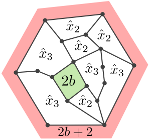
4. Obtain fixed-point equation for special series $\hat{R}^{(2b)}(\mathbf{x}) = \sum$



$$J(b;r) = \sum_{p\geq 1} \frac{(-1)^{p+1} r^p}{p!(p-1)!} \prod_{0\leq m<p-1} (b-m)(b-m-1)$$
$$I(b,\ell;r) = \sum_{p\geq 0} \frac{r^p}{(p!)^2} \prod_{0\leq m<p} (\ell^2-(b-m)^2)$$

$$J(b;\hat{R}^{(2b)}) = \sum_{\ell\geq b} \hat{x}_\ell \, I(b,\ell;\hat{R}^{(2b)}).$$

Proof outline: "generatingfunctionology"

1. $\|\hat{\mathcal{M}}_{g,n}^{(2b)}(\boldsymbol{\ell})\| = \frac{\partial^n}{\partial \hat{x}_{\ell_1} \cdots \partial \hat{x}_{\ell_n}} \hat{F}_g^{(2b)}(\hat{x})\big|_{\hat{x}=0}$
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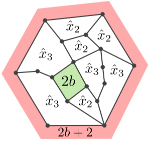

$$J(b;r) = \sum_{p\geq 1} \frac{(-1)^{p+1} r^p}{p!(p-1)!} \overbrace{\prod_{0\leq m<p-1} (b-m)(b-m-1)}^{\text{polynomial in } b}$$

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$$J(b;\hat{R}^{(2b)}) = \sum_{\ell\geq b} \hat{x}_\ell \, I(b,\ell;\hat{R}^{(2b)}).$$

5. $\frac{\partial^n}{\partial \hat{x}_{\ell_1} \cdots \partial \hat{x}_{\ell_n}} \hat{R}^{(2b)}\big|_{\hat{x}=0}$ is polynomial in ℓ_i, b and $\hat{R}^{(2b)}$ satisfies differential "string and dilation" identities.

Proof outline: "generatingfunctionology"

1. $\|\hat{\mathcal{M}}_{g,n}^{(2b)}(\boldsymbol{\ell})\| = \frac{\partial^n}{\partial \hat{x}_{\ell_1}\cdots\partial\hat{x}_{\ell_n}} \hat{F}_g^{(2b)}(\hat{\mathrm{x}})\big|_{\hat{\mathrm{x}}=0}$
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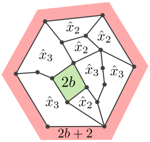
4. Obtain fixed-point equation for special series $\hat{R}^{(2b)}(\mathbf{x}) = \sum$ 

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5. $\frac{\partial^n}{\partial \hat{x}_{\ell_1}\cdots\partial\hat{x}_{\ell_n}} \hat{R}^{(2b)}\big|_{\hat{\mathrm{x}}=0}$ is polynomial in ℓ_i, b and $\hat{R}^{(2b)}$ satisfies differential "string and dilation" identities.
6. Use topological recursion [Eynard, '16] on $F_g^{(0)}(\mathrm{x})$ to express ($\partial \equiv \frac{\partial}{\partial \hat{x}_b}$)

$$\hat{F}_1^{(2b)}(\hat{\mathrm{x}}) = -\frac{1}{12}\log\frac{(\hat{R}^{(2b)})^{1-b}}{\partial \hat{R}^{(2b)}}, \quad \hat{F}_{g\geq 2}^{(2b)}(\hat{\mathrm{x}}) = \underbrace{\mathcal{F}_g^{(2b)}}_{\text{rational}}(\hat{R}^{(2b)}, \partial\hat{R}^{(2b)}, \cdots, \partial^{3g-2}\hat{R}^{(2b)}),$$

Proof outline: "generatingfunctionology"

1. $\|\hat{\mathcal{M}}^{(2b)}_{g,n}(\boldsymbol{\ell})\| = \frac{\partial^n}{\partial \hat{x}_{\ell_1}\cdots\partial \hat{x}_{\ell_n}} \hat{F}^{(2b)}_g(\hat{\mathrm{x}})\big|_{\hat{\mathrm{x}}=0}$
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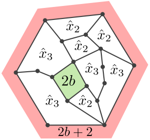

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7. Conclude that $\frac{\partial^n}{\partial \hat{x}_{\ell_1}\cdots\partial \hat{x}_{\ell_n}} \hat{F}^{(2b)}_g(\hat{\mathrm{x}})\big|_{\hat{\mathrm{x}}=0}$ is polynomial in ℓ_i, b and satisfies string and dilaton equation.

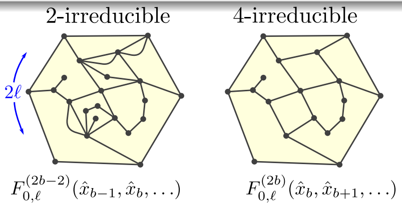
Substitution approach [Bouttier, Guitter, '13]

Proposition (Bouttier, Guitter, '13 & TB '20)

$\exists$ *formal power series* $X_i^{(2b)}(x_b, x_{b+1}, \ldots), i = 1, \ldots, b$, *such that* ($F_{0,\ell}^{(2b)} \equiv \frac{\partial}{\partial \hat{x}_\ell} F_0^{(2b)}$)

$$g = 0: \quad F_{0,\ell}^{(2b)}(x_b, x_{b+1}, \ldots) = F_{0,\ell}^{(0)}(X_1^{(2b)}, X_2^{(2b)}, \ldots, X_b^{(2b)}, x_{b+1}, x_{b+2}, \ldots)$$

$$g \geq 1: \quad F_g^{(2b)}(x_b, x_{b+1}, \ldots) = F_g^{(0)}(X_1^{(2b)}, X_2^{(2b)}, \ldots, X_b^{(2b)}, x_{b+1}, x_{b+2}, \ldots)$$



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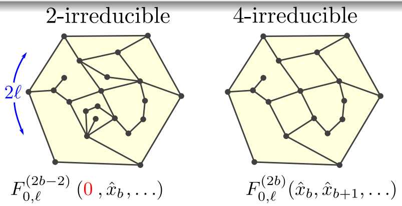

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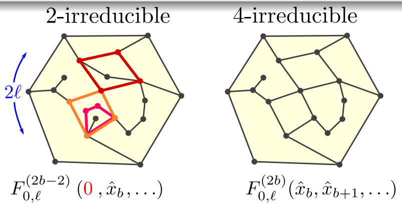

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$\exists$ *formal power series* $X_i^{(2b)}(x_b, x_{b+1}, \dots)$, $i = 1, \dots, b$, *such that* $(F_{0,\ell}^{(2b)} \equiv \frac{\partial}{\partial \hat{x}_\ell} F_0^{(2b)})$

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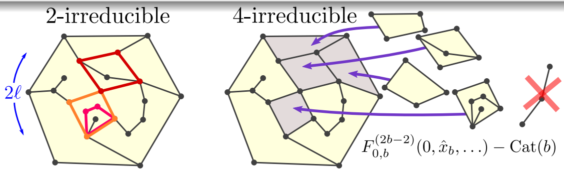

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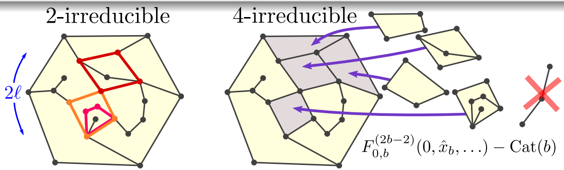

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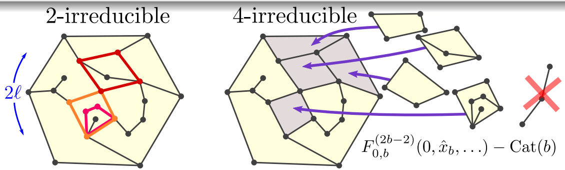

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- Substitution in higher genus is simpler: no need to distinguish a face to determine inside/outside of cycle,

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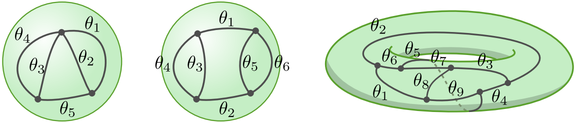
Irreducible metric maps: the limit $b \to \infty$

- A metric map is a map with vertices of degree ≥ 3 and positive real lengths $(\theta_e)_{e \in \mathrm{Edges}}$ associated to its edges.



$g = 0$ $\quad$ $g = 0$ $\quad$ $g = 1$

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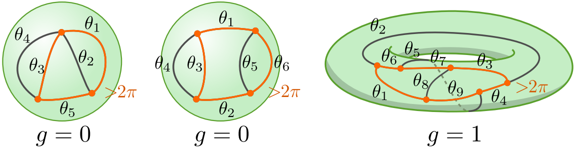

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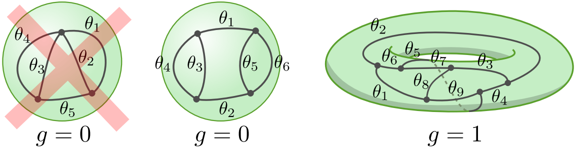

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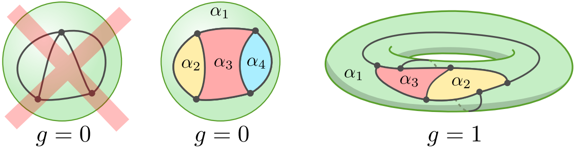
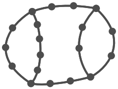
- If $\mathrm{Circ} : \mathcal{M}_{g,n}^{\mathrm{met}} \to [2\pi, \infty)^n$ denotes the face circumferences, then

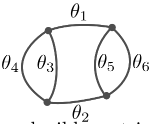
$$\mathrm{Circ}_* \mathrm{Leb} = \underbrace{V_{g,n}^{\mathrm{met}}(\alpha_1, \ldots, \alpha_n)}_{\text{Lebesgue volume subject to face constraints}} \, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n.$$



- Every $2b$-irreducible map naturally describes a 2π-irreducible metric map:



$2b$-irreducible map



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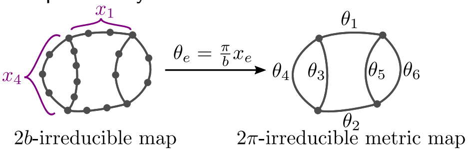

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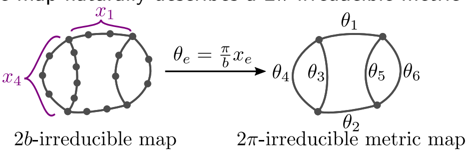


- As $b \to \infty$ the counting measure approaches the Lebesgue measure, therefore

Proposition (TB, '20)

$$V^{met}_{g,n}(\alpha_1, \ldots, \alpha_n) = \lim_{b \to \infty} \left(\tfrac{\pi}{b}\right)^{2n+6g-6} \hat{N}^{(2b)}_{g,n}\left(\tfrac{b}{2\pi}\alpha_1, \cdots, \tfrac{b}{2\pi}\alpha_n\right).$$

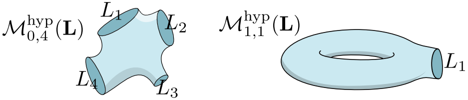
In particular, it is a polynomial of degree $n + 3g - 3$ *in* $\alpha_1^2, \ldots, \alpha_n^2$.

$$V^{\mathrm{met}}_{0,3} = 1, \qquad V^{\mathrm{met}}_{0,4} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 12\pi^2,$$
$$V^{\mathrm{met}}_{1,1} = \frac{1}{12}\alpha_1^2, \qquad V^{\mathrm{met}}_{1,2} = \tfrac{1}{24}(\alpha_1^2 + \alpha_2^2)^2 - \frac{2}{3}\pi^4.$$

Weil-Petersson volumes

- Consider the Moduli space

$$\mathcal{M}^{\mathrm{hyp}}_{g,n}(\mathsf{L}) = \left\{ \begin{array}{l} \text{hyperbolic metrics on genus-}g\text{ surface with } n \text{ geodesic} \\ \text{boundary components of lengths } L_1, \ldots, L_n \end{array} \right\} / \mathrm{Isom}.$$



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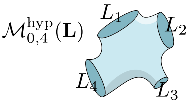

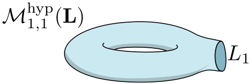

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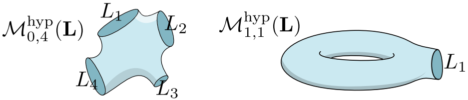

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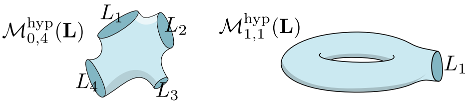

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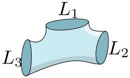
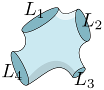
- Fully settled by Mirzakhani in '05:

Theorem (Mirzakhani, '05)

$V^{hyp}_{g,n}(\mathbf{L})$ satisfies a (topological) recursion formula. In particular, $V^{hyp}_{g,n}(\mathbf{L})$ is polynomial in $L_1^2,\ldots,L_n^2$ of degree $n+3g-3$.

$$V_{0,3}^{\mathrm{hyp}}(\mathbf{L}) = 1$$





$$V_{0,4}^{\mathrm{hyp}}(\mathbf{L}) = \tfrac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2) + 2\pi^2$$

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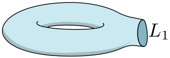
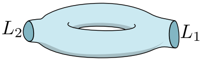
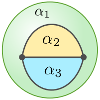




$$V_{1,2}^{\mathrm{hyp}}(\mathbf{L}) = \tfrac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2)$$



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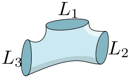

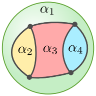

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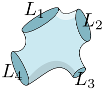

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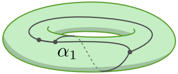

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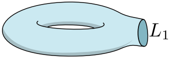

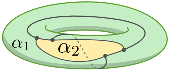

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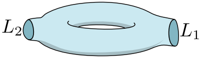

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- The Weil-Petersson volumes and 2π-irreducible metric map volumes look similar. Are they related?

- For $g = 0, 1$ the polynomials $V^{\mathrm{met}}_{g,n}$ are determined by string and dilaton equations:

$$V^{\mathrm{met}}_{g,n+1}(\alpha_1, \ldots, \alpha_n, 0) = \frac{1}{2} \sum_{j=1}^{n} \int_{2\pi}^{\alpha_j} \mathrm{d}\alpha_j \; \alpha_j V^{\mathrm{met}}_{g,n}(\alpha_1, \ldots, \alpha_n)$$

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- The same is true for Weil-Petersson volumes: [Do, Norbury, '09]

$$V^{\mathrm{hyp}}_{g,n+1}(L_1, \dots, L_n, 2\pi i) = \sum_{j=1}^{n} \int_{0}^{L_j} \mathrm{d}L_j \; L_j V^{\mathrm{hyp}}_{g,n}(L_1, \dots, L_n)$$

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- Conclusion:

Theorem (TB, '20)

For $g=0,1$, $\quad V^{met}_{g,n}(\alpha_1,\ldots,\alpha_n)=2^{3-2g-n}\,V^{hyp}_{g,n}\left(\sqrt{\alpha_1^2-4\pi^2},\cdots,\sqrt{\alpha_n^2-4\pi^2}\right).$

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$$V^{\text{met}}_{g,n+1}(\alpha_1, \dots, \alpha_n, 0) = \frac{1}{2} \sum_{j=1}^{n} \int_{2\pi}^{\alpha_j} \mathrm{d}\alpha_j \; \alpha_j V^{\text{met}}_{g,n}(\alpha_1, \dots, \alpha_n)$$

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- The same is true for Weil-Petersson volumes: [Do, Norbury, '09]

$$V^{\text{hyp}}_{g,n+1}(L_1, \dots, L_n, 2\pi i) = \sum_{j=1}^{n} \int_{0}^{L_j} \mathrm{d}L_j \; L_j V^{\text{hyp}}_{g,n}(L_1, \dots, L_n)$$

$$\frac{\partial V^{\text{hyp}}_{g,n+1}}{\partial L_{n+1}}(L_1, \dots, L_n, 2\pi i) = 2\pi i(2g - 2 + n) V^{\text{hyp}}_{g,n}(L_1, \dots, L_n).$$

- Conclusion:

Theorem (TB, '20)

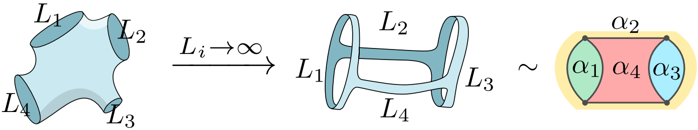
For $g = 0, 1$, $\quad V^{met}_{g,n}(\alpha_1, \dots, \alpha_n) = 2^{3-2g-n}\, V^{hyp}_{g,n}\left(\sqrt{\alpha_1^2 - 4\pi^2}, \cdots, \sqrt{\alpha_n^2 - 4\pi^2}\right).$

- Unfortunately: not valid for $g \geq 2$.
  Examining partition functions for both: tiny difference.

Bijective/geometric explanation?

$$V_{g,n}^{\text{met}}(\alpha_1,\ldots,\alpha_n) = 2^{3-2g-n}\, V_{g,n}^{\text{WP}}\left(\sqrt{\alpha_1^2-4\pi^2},\cdots,\sqrt{\alpha_n^2-4\pi^2}\right) \quad (g=0,1)$$

- Understood in $L_i = \sqrt{\alpha_i^2 - 4\pi^2} \to \infty$ limit. [Witten, '91] [Kontsevich, '95] [Do, '11]



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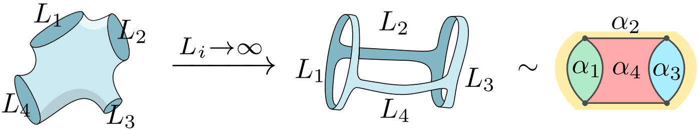
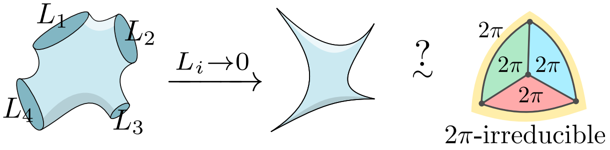


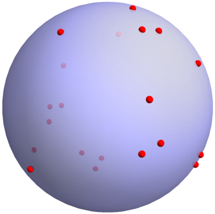
- How about $L_i \to 0$ and $g = 0$?



$$V^{\text{met}}_{0,n}(2\pi,\cdots,2\pi) = 2^{3-n}\, V^{\text{hyp}}_{0,n}(0,\cdots,0)$$

Bijection via ideal hyperbolic polyhedra [Rivin, '92 + '96] [Charbonnier, David, Eynard, '17]

- Combining two bijections of Rivin...



n points on sphere modulo
Möbius transformations

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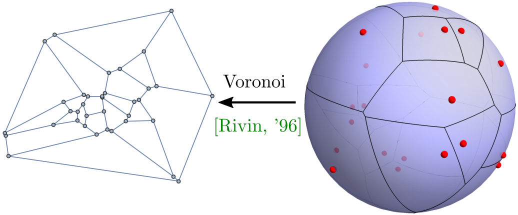

2π-irreducible map with n faces of circumference 2π

n points on sphere modulo Möbius transformations

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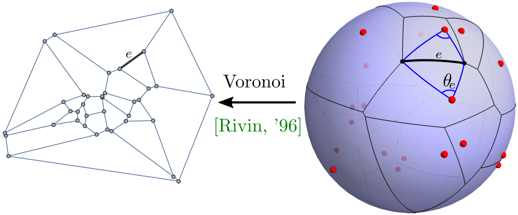

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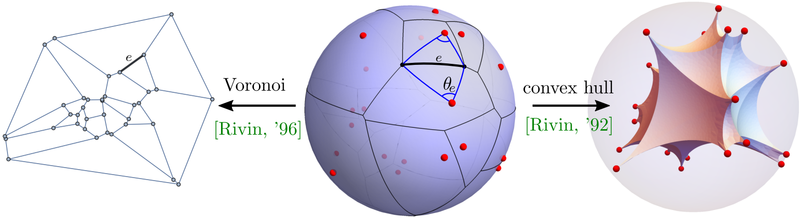

2π-irreducible map with
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hyperbolic surface with n cusps

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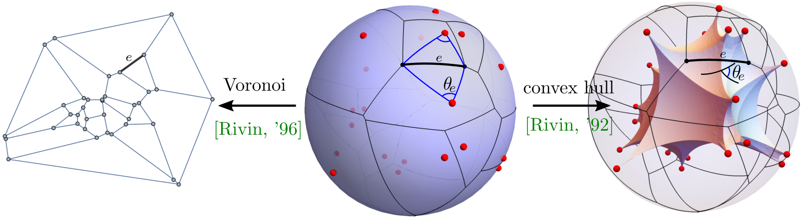

2π-irreducible map with n faces of circumference 2π

n points on sphere modulo Möbius transformations

hyperbolic surface with n cusps

- ...determines a bijection $\mathcal{M}^{\mathrm{met}}_{0,n}(2\pi,\dots,2\pi) \longleftrightarrow \mathcal{M}^{\mathrm{hyp}}_{0,n}(0,\dots,0)$.

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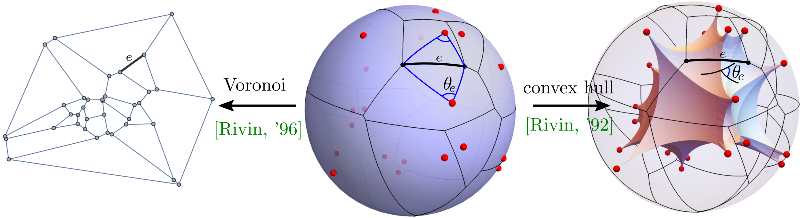

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- The Lebesgue measure on $\mathcal{M}_{0,n}^{\text{met}}(2\pi,\ldots)$ is proportional to the Weil-Petersson measure on $\mathcal{M}_{0,n}^{\text{hyp}}(0,\ldots)$. [Charbonnier, David, Eynard, '17] [TB, Charbonnier, '20+]

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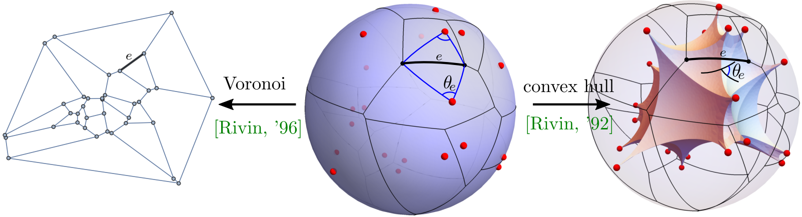

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- More to the story: tree bijections for irreducible planar maps [Bernardi, Fusy, '12] [Albenque, Poulalhon, '13] [Bouttier, Guitter, '13] have nice analogues for irreducible planar metric maps and hyperbolic punctured spheres [TB, Charbonnier, '20+].

Open questions

1. Both $V_{g,n}^{\text{hyp}}$ and $\hat{N}_{g,n}^{(0)}$ satisfy beautiful topological recursions [Mirzakhani, '05] [Eynard, Orantin, '07] [Norbury, '08]. Is there a topological recursion for $\hat{N}_{g,n}^{(2b)}$ or $V_{g,n}^{\text{met}}$?
2. Is there a bijective explanation for the relation between $V_{g,n}^{\text{met}}(\alpha)$ and $V_{g,n}^{\text{hyp}}(\mathrm{L})$ for $\mathrm{L} \in (0,\infty)^n$?
3. The coefficients of $V_{g,n}^{\text{hyp}}$ store intersection numbers on moduli spaces of curves. Is the same true for the coefficients of $V_{g,n}^{\text{met}}$?

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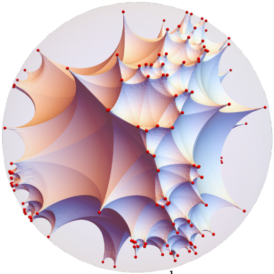
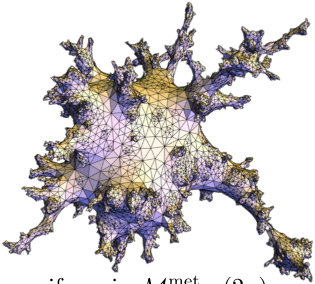
Thanks!



uniform in $\mathcal{M}^{\text{hyp}}_{0,241}(0)$



uniform in $\mathcal{M}^{\text{met}}_{0,9000}(2\pi)$
dual triangulation

Backup slides

- According to topological recursion $F_g^{(0)}(x)$ is expressed in terms of certain moments $M_p(x)$: [Ambjørn, Chekhov, Kristjansen, Makeenko, '93] [Eynard, '16]

$$F_1^{(0)} = -\frac{1}{12}\log M_0, \qquad F_g^{(0)} = \overbrace{P_g}^{\text{polynomial}}\left(\frac{1}{M_0}, \frac{M_1}{M_0}, \dots, \frac{M_{3g-3}}{M_0}\right) \quad (g \geq 2)$$

$$M_p = 1_{p=0} - \sum_{k\geq 1}\binom{2k+p+1}{2p+1} U_k R^{-k}, \quad U_k = \sum_{j>k}\binom{2j-1}{j+k} x_j R^{j+k}, \qquad U_0 \stackrel{!}{=} R - 1$$

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- Substituting $x \mapsto x(\hat{x})$ and $\hat{x}_i \mapsto X_i^{(2b)}$, $i = 1, \ldots, b$:

Proposition (TB, '20)

The partition functions of $2b$-irreducible maps with no degree-1 vertices are given by

$$\frac{\partial}{\partial \hat{x}_{\ell_1}}\frac{\partial}{\partial \hat{x}_{\ell_2}}\hat{F}_0^{(2b)} = \int_0^{\hat{R}^{(2b)}} \mathrm{d}r\, \frac{I(b,\ell_1,r)I(b,\ell_2,r)}{(1+r)^{2b+1}}$$

$$\hat{F}_1^{(2b)} = -\frac{1}{12}\log \hat{M}_0, \qquad \hat{F}_g^{(2b)} = \overbrace{P_g}^{\text{polynomial}}\left(\frac{1}{\hat{M}_0}, \frac{\hat{M}_1}{\hat{M}_0}, \ldots, \frac{\hat{M}_{3g-3}}{\hat{M}_0}\right) \quad (g \geq 2)$$

$$\hat{M}_p = \frac{\hat{R}^{1-b}}{(\partial\hat{R})^{2p+1}}\overbrace{T_p}^{\textit{polynomial}}(b, \hat{R}, \partial\hat{R}, \cdots, \partial^{p+1}\hat{R}), \qquad (\hat{R} \equiv \hat{R}^{(2b)},\ \partial \equiv \tfrac{\partial}{\partial \hat{x}_b})$$

$$T_0 = 1, \qquad T_1 = \tfrac{2}{3}b(b-1)\,(\partial\hat{R})^2 - \tfrac{2}{3}\hat{R}\,\partial^2\hat{R}, \qquad T_2 = \cdots$$

$$J(b;\hat{R}^{(2b)}) \overset{!}{=} \sum_{\ell\geq b}\hat{x}_\ell\, I(b,\ell;\hat{R}^{(2b)})$$

$$J(b;r) = \sum_{p\geq 1}\frac{(-1)^{p+1}r^p}{p!(p-1)!}\prod_{0\leq m<p-1}(b-m)(b-m-1)$$

$$I(b,\ell;r) = \sum_{p\geq 0}\frac{r^p}{(p!)^2}\prod_{0\leq m<p}(\ell^2-(b-m)^2)$$

String and dilaton equation

$$\hat{F}_1^{(2b)} = -\frac{1}{12}\log \hat{M}_0, \qquad \hat{F}_g^{(2b)} = \overbrace{P_g}^{\text{polynomial}}\left(\frac{1}{\hat{M}_0}, \frac{\hat{M}_1}{\hat{M}_0}, \dots, \frac{\hat{M}_{3g-3}}{\hat{M}_0}\right) \quad (g \geq 2)$$

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- May formally extend $\hat{R}^{(2b)}$, $\hat{M}_p$, $\hat{F}_g^{(2b)}$ to include variables $\hat{x}_0, \dots, \hat{x}_{b-1}$:

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- Then the string and dilaton equations (for $g \geq 1$) are equivalent to

$$D^{\text{str}} \hat{F}_g^{(2b)} = 0, \qquad (D^{\text{dil}} - 2g + 2)\hat{F}_g^{(2b)} = \text{const}$$

$$D^{\text{str}} := \frac{\partial}{\partial \hat{x}_1} - \sum_{\ell=0}^{\infty} \hat{x}_\ell \left(-\ell \frac{\partial}{\partial \hat{x}_\ell} + \sum_{k=1}^{\ell} 2k \frac{\partial}{\partial \hat{x}_k} - \sum_{k=1}^{b} 2k \frac{\partial}{\partial \hat{x}_k} \right),$$

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- This is proved by explicit computation:

$$D^{\text{str}} \hat{R}^{(d)} = \hat{R}^{(d)}, \qquad D^{\text{dil}} \hat{R}^{(d)} = 0, \qquad D^{\text{str}} \hat{M}_p = 0, \qquad D^{\text{dil}} \hat{M}_p = -\hat{M}_p.$$