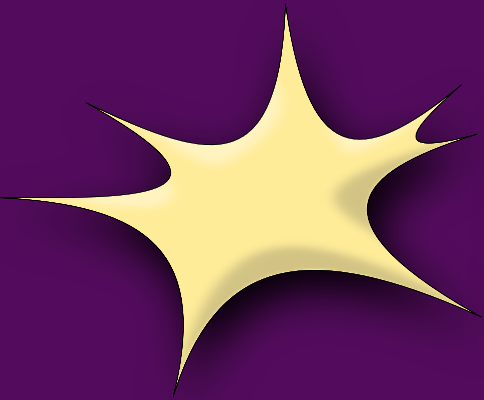
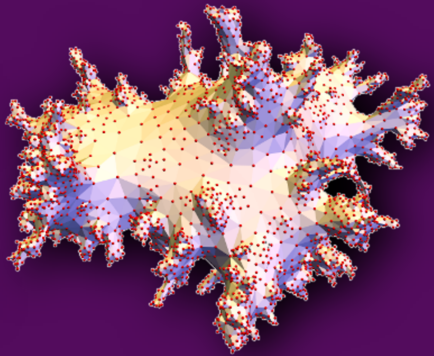
Random hyperbolic surfaces

Timothy Budd





in collaboration with Nicolas Curien

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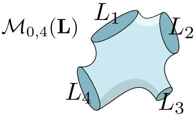
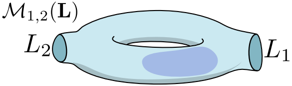
Moduli space of hyperbolic surfaces

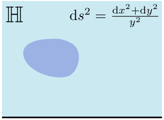
[Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

- Consider the Moduli space

$$\mathcal{M}_{g,n}(\mathsf{L}) = \left\{ \begin{array}{l} \text{hyperbolic metrics on genus-}g\text{ surface with } n \\ \text{geodesic boundaries of lengths } \mathsf{L} = (L_1, \ldots, L_n) \end{array} \right\} /\text{Isom.}$$







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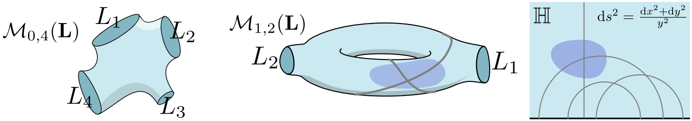

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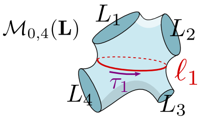

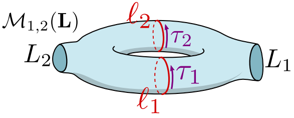

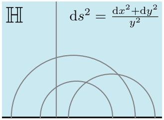

- Carries natural Weil-Petersson measure WP. In local Fenchel-Nielsen coordinates $\ell_1, \tau_1, \ldots, \ell_{3g-3+n}, \tau_{3g-3+n}$ it is

$$\mathrm{WP} = 2^{3-3g-n}\,\mathrm{d}\ell_1\mathrm{d}\tau_1 \cdots \mathrm{d}\ell_{3g-3+n}\mathrm{d}\tau_{3g-3+n}.$$ [Wolpert, '82]

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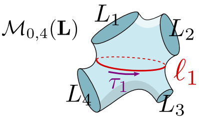

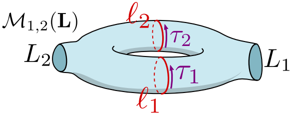

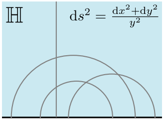

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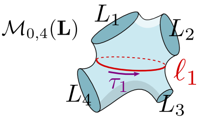

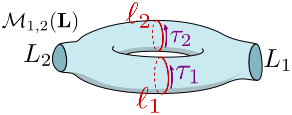

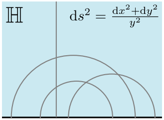

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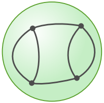
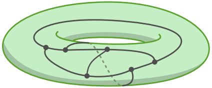
- Weil-Petersson volume: $V_{g,n}(\mathsf{L}) := \mathrm{WP}(\mathcal{M}_{g,n}(\mathsf{L})) < \infty$.
- Characterized in [Mirzakhani,'05]: $V_{g,n}(\mathsf{L})$ satisfies a (topological) recursion formula. In particular, $V_{g,n}(\mathsf{L})$ is polynomial in $L_1^2, \ldots, L_n^2$ of degree $3g-3+n$.

Relation to maps

- A (cubic) metric map is a map with vertices of degree 3 and positive real lengths $(x_e)_{e \in \text{Edges}}$ associated to its edges.





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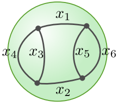

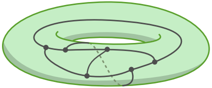

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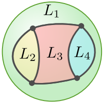
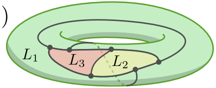
$\mathcal{M}^{\text{met}}_{0,4}(\mathbf{L})$ 

$\mathcal{M}^{\text{met}}_{1,3}(\mathbf{L})$ 

- The set of metric maps

$$\mathcal{M}^{\text{met}}_{g,n}(\mathsf{L}) = \left\{ \begin{array}{l} \text{genus-}g \text{ metric maps with } n \text{ labeled} \\ \text{faces of circumference } \mathsf{L} = (L_1, \dots, L_n) \end{array} \right\}$$

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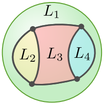

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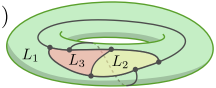

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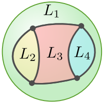
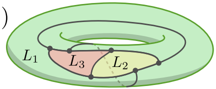
$\mathcal{M}^{\mathrm{met}}_{0,4}(\mathbf{L})$ 

$\mathcal{M}^{\mathrm{met}}_{1,3}(\mathbf{L})$ 

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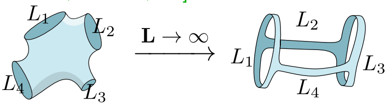

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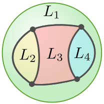
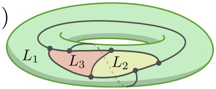
$\mathcal{M}^{\mathrm{met}}_{0,4}(\mathbf{L})$ 

$\mathcal{M}^{\mathrm{met}}_{1,3}(\mathbf{L})$ 

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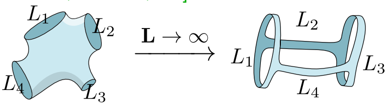
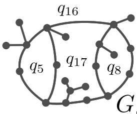


- How about finite L?

- Analog of map gen. fun. $G_g(q) = \sum_{\substack{\text{genus-}g \\ \text{maps } m}} \frac{1}{|\operatorname{Aut}(m)|} \prod_f q_{\deg(f)}$



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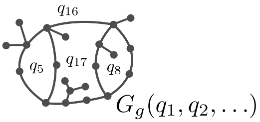
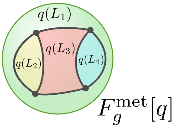
$$F_g^{\mathrm{met}}[q] = \sum_{n \geq 1} \frac{1}{n!} \int_0^\infty \left[\prod_{i=1}^n \mathrm{d}q(L_i) \right] V_{g,n}^{\mathrm{met}}(\mathrm{L}) \qquad .$$





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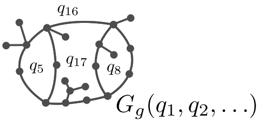

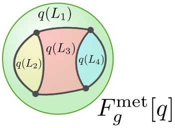

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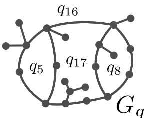
- $F_g(t_0, t_1, \ldots)$ is generating function of intersection numbers and $e^{\sum_g \lambda^g F_g}$ a τ-function of the KdV hierarchy. [Witten, '91], [Kontsevich, '92]



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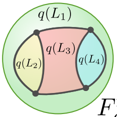

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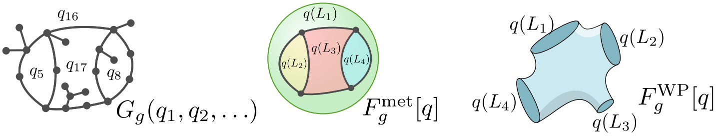
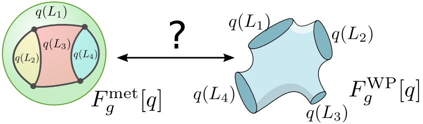
- The generating functionals of WP volumes are obtained by a shift [Kaufmann, Manin, Zagier, '96] [Zograf, '98]

$$F_g^{\mathrm{WP}}[q] := \sum_{n \geq 1} \frac{1}{n!} \int_0^\infty \Big[\prod_{i=1}^{n} \mathrm{d}q(L_i)\Big] V_{g,n}^{\mathrm{WP}}(\mathrm{L}) = F_g(t_0, t_1, t_2 + \pi^2, t_3 - \tfrac{1}{2}\pi^4, \ldots).$$



- Open problem: Is there a bijective/geometric interpretation?



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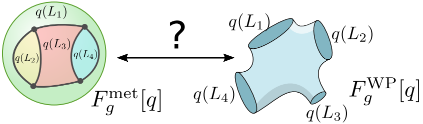

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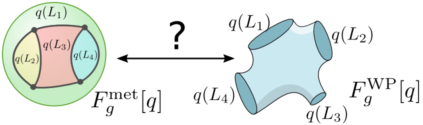

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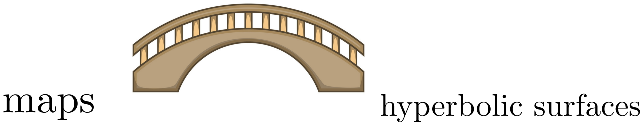
- Can we transfer methods between the two fields?



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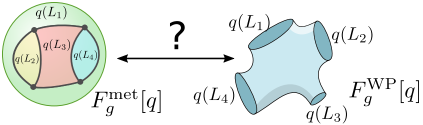

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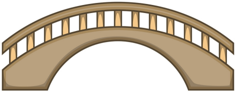

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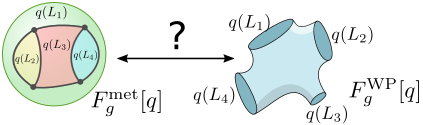

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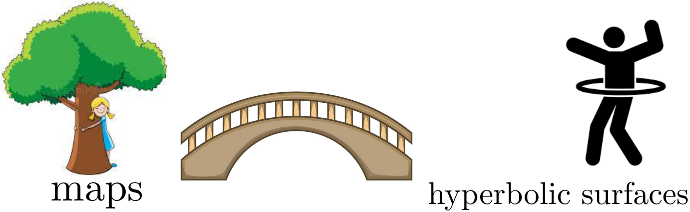

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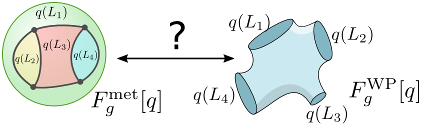

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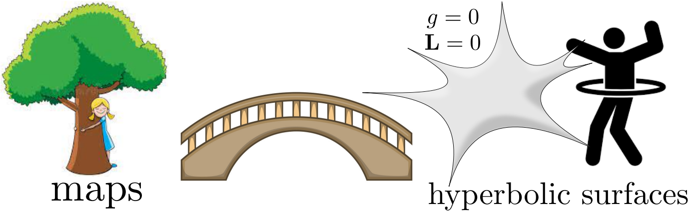

- Focus on genus 0 with cusps (= boundaries of length 0), $\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(\mathsf{L} = 0)$.

Recall Quadrangulations $\leftrightarrow$ labeled trees talk: Bettinelli, Dołęga

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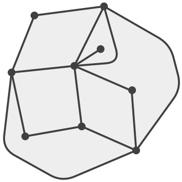

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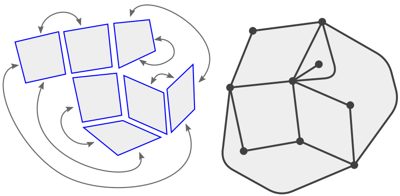

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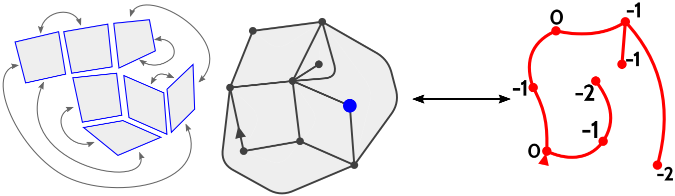

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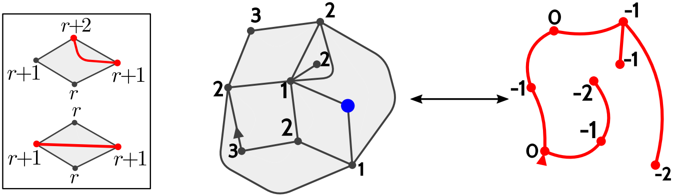

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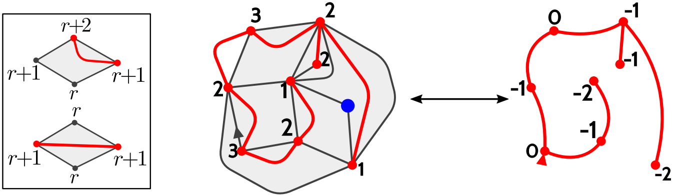

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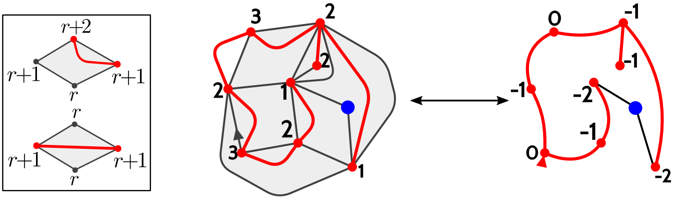

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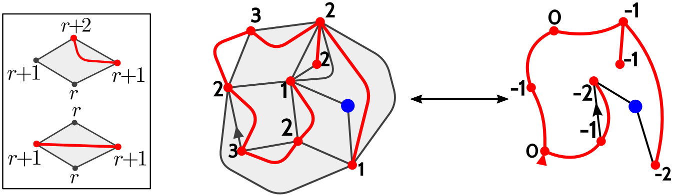

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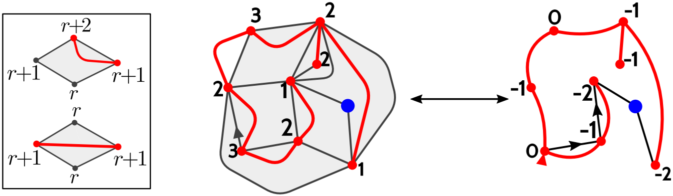

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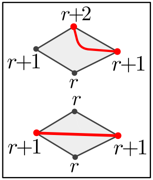

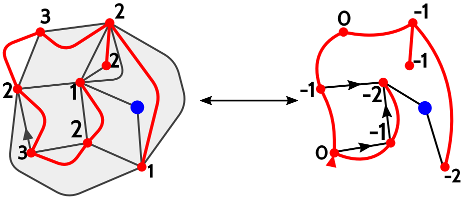

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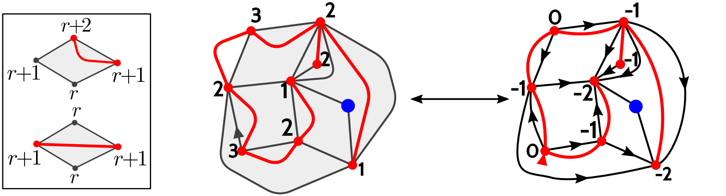

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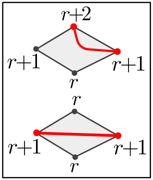
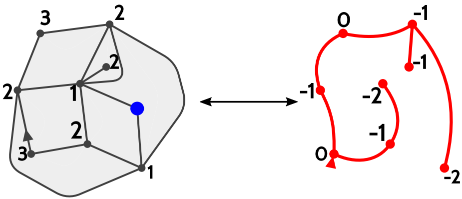
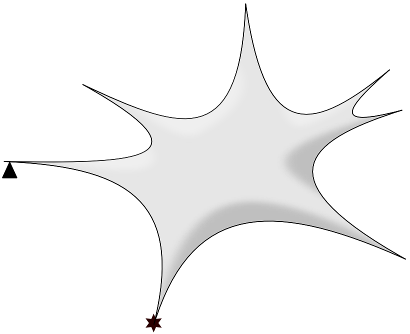
- The tree labels encode the distances to the distinguished vertex.





Where are the trees in a hyperbolic surface?

- Let $\mathcal{S}_n \in \mathcal{M}_{0,n}$ with two distinguished cusps $\star, \blacktriangle$ and determine cut locus / spine of $\star$: points with multiple shortest geodesics to $\star$.



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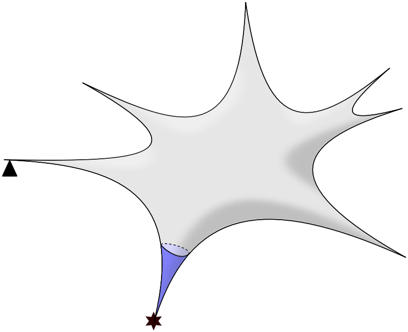

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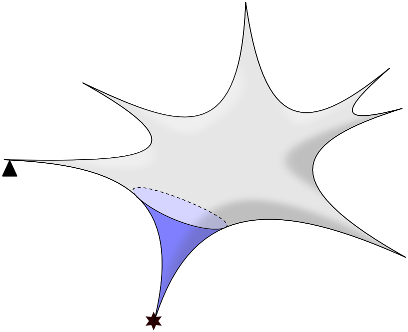

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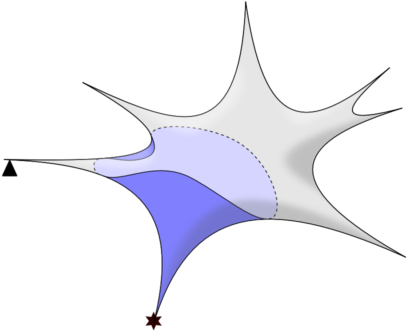

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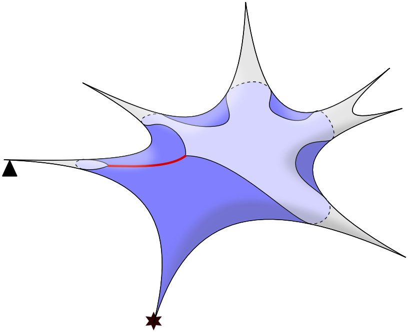

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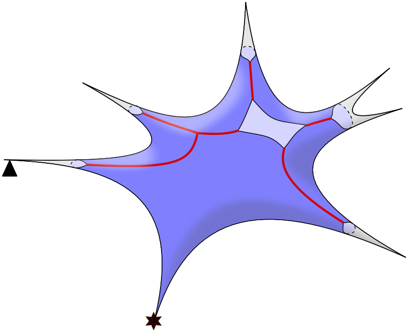

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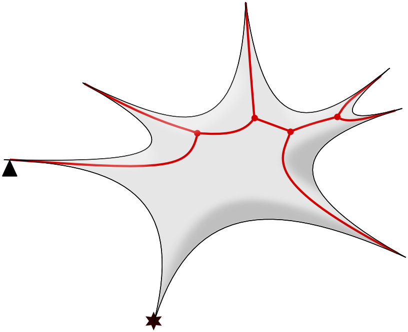

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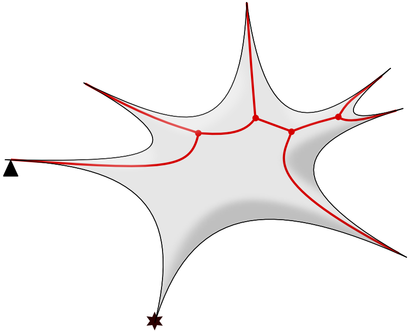

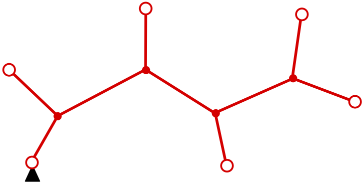

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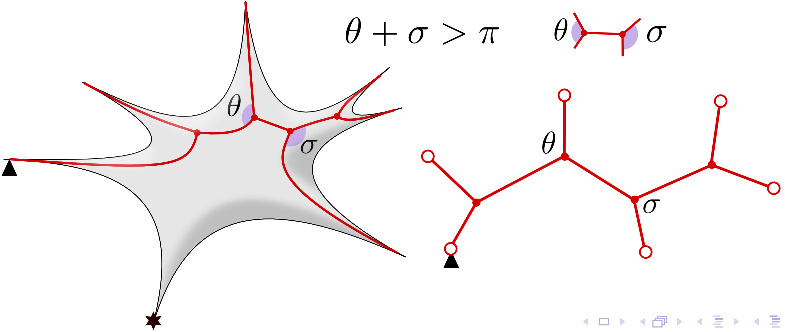

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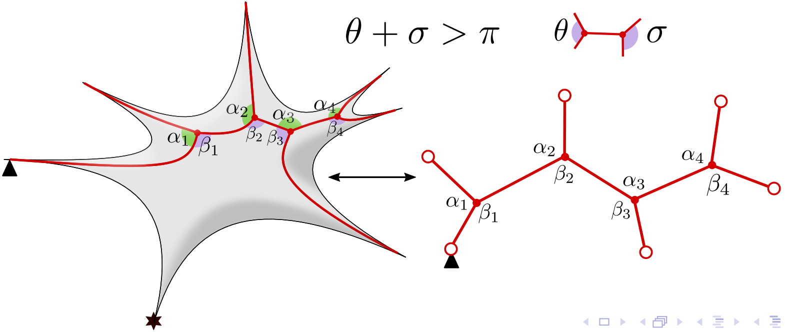
Theorem

There exists an open subset $\mathcal{M}^{\circ}_{0,n} \subset \mathcal{M}_{0,n}$ of full WP-measure, such that

$$\mathcal{M}^{\circ}_{0,n} \xleftrightarrow{\text{bijection}} \bigsqcup_{\mathcal{T} \in \mathrm{Bin}_n} \{(\alpha_i, \beta_i) \in (0,\pi)^{2n-6} : \alpha_i + \beta_i > \pi,\ \theta + \sigma > \pi\}.$$



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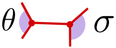
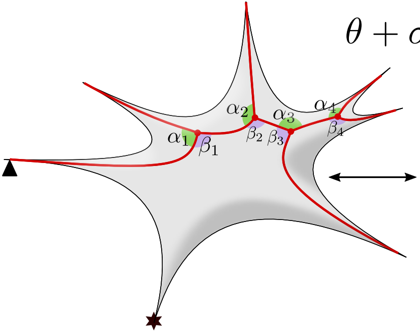
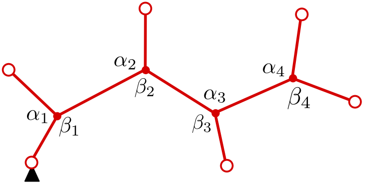
The WP measures is mapped to Lebesgue: $2^{n-3}\mathrm{d}\alpha_1\mathrm{d}\beta_1\cdots\mathrm{d}\alpha_{n-3}\mathrm{d}\beta_{n-3}$.

$\theta+\sigma>\pi$

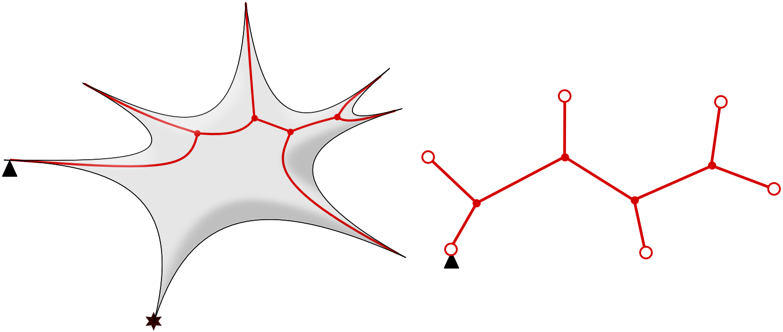






Proof: an associated ideal triangulation

- The cut locus determines a canonical ideal triangulation of $\mathcal{S}_n$.



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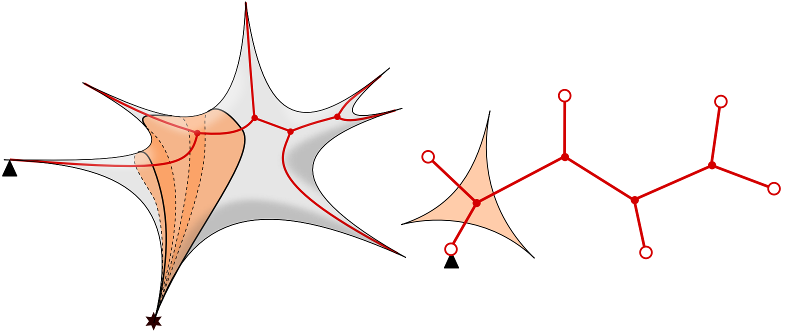

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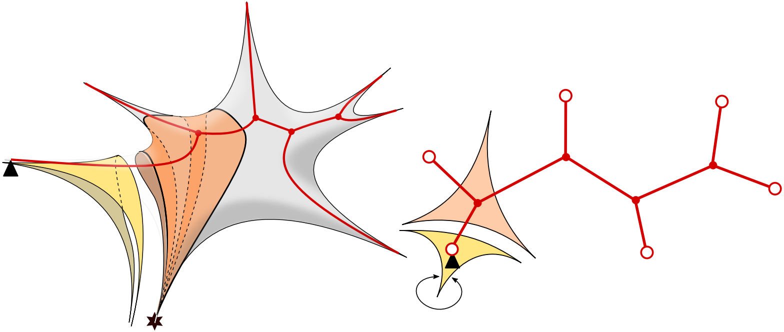

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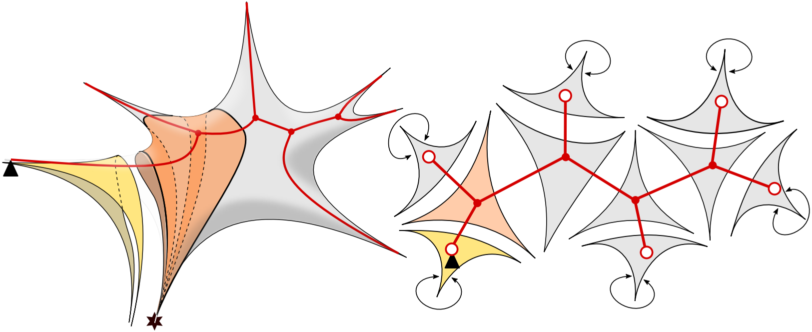

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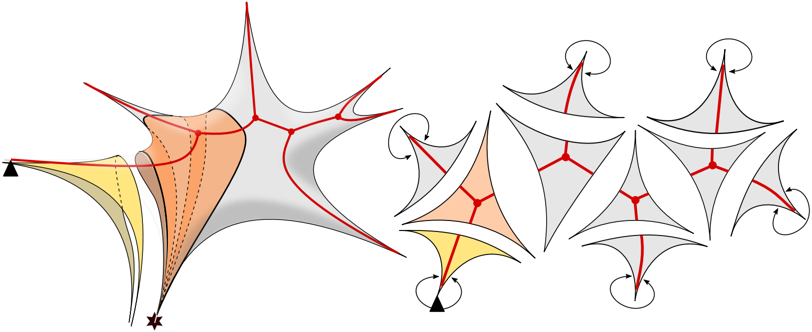

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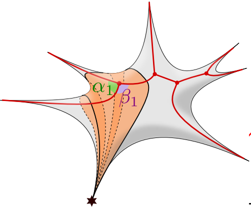

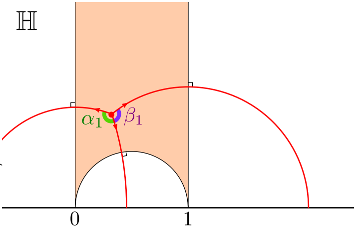

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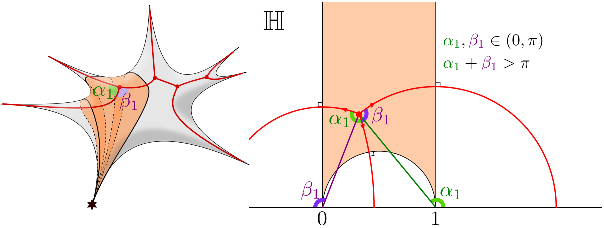

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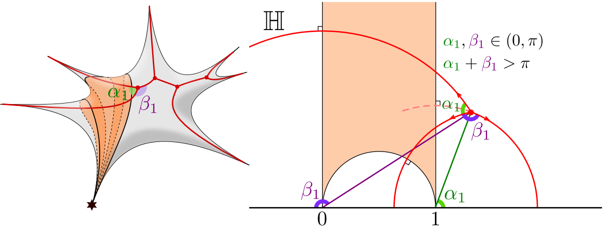

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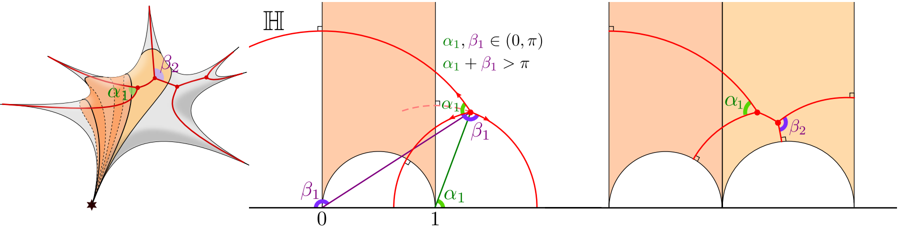

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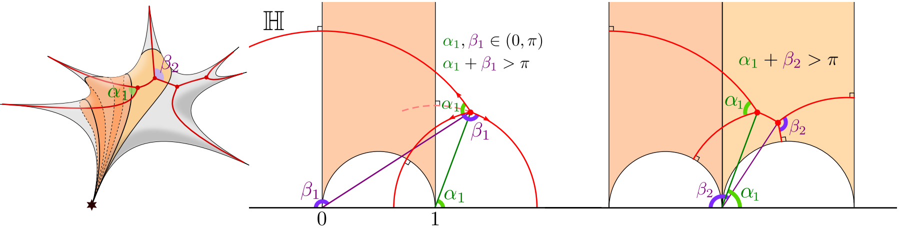

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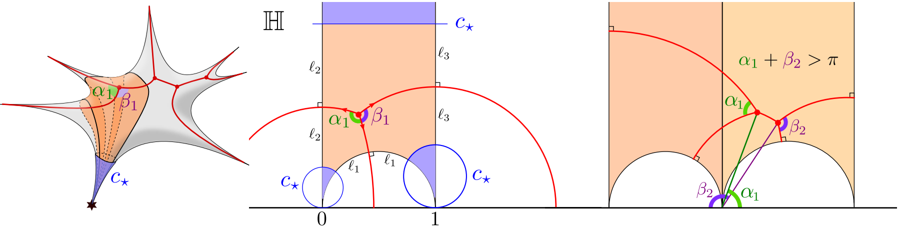

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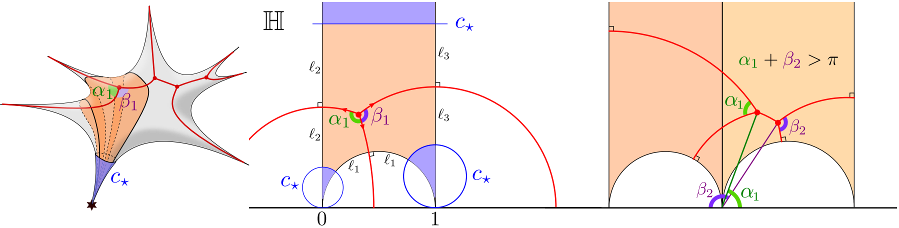

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- The Weil-Petersson measure is [Penner, '92]

$$WP = \frac{1}{(n-3)!}\Big(-2\sum_{\text{corners}} \mathrm{d}\ell_i \wedge \mathrm{d}\ell_j\Big)^{n-3} = 2^{n-3}\mathrm{d}\alpha_1 \mathrm{d}\beta_1 \cdots \mathrm{d}\alpha_{n-3}\mathrm{d}\beta_{n-3}.$$

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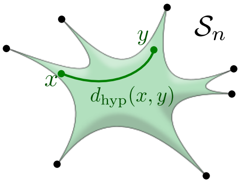

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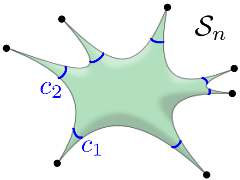

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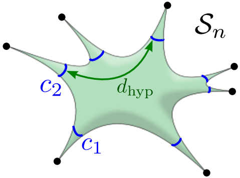

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We have

$$\left(\{c_1, \ldots, c_n\}, n^{-\frac{1}{4}} d_{hyp}\right) \xrightarrow[n\to\infty]{(d)} \mathrm{c}_{WP}(\mathrm{m}_\infty, D^*) \quad \textit{(Gromov-Hausdorff sense)}$$

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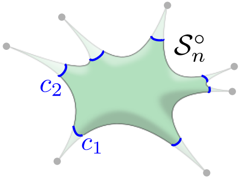

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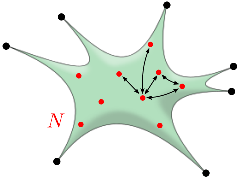

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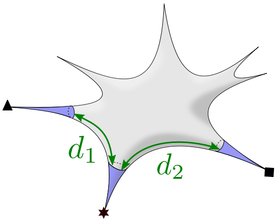

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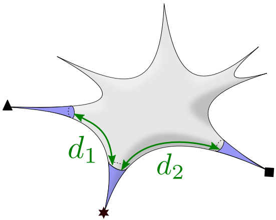

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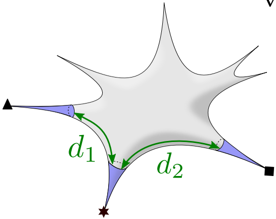

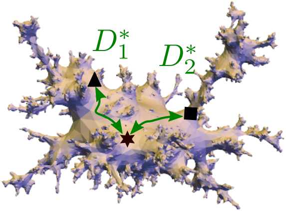

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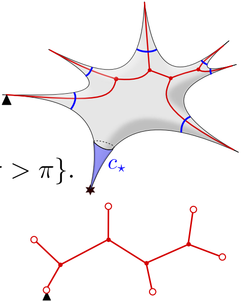

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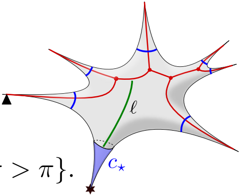

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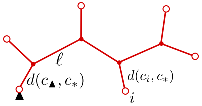

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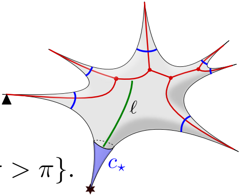

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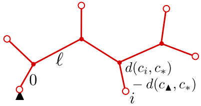

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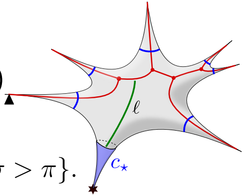

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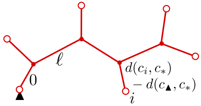

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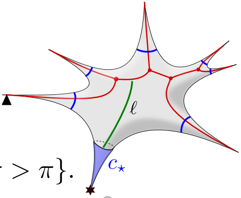

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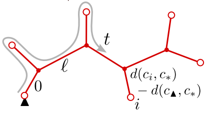

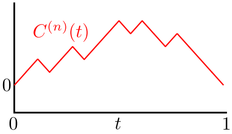

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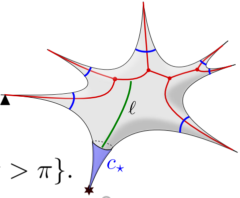

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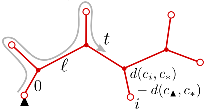

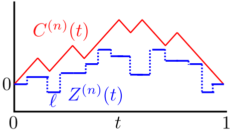

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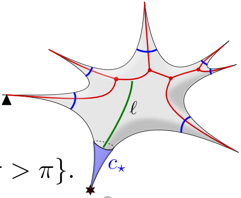

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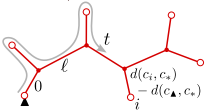

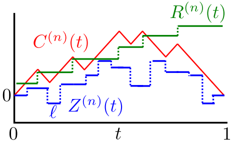

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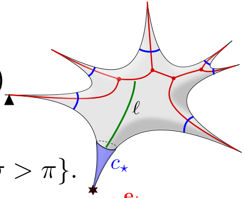

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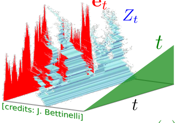

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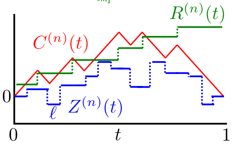

Proposition

$$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{R^{(n)}(t)}{n}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} (\mathrm{c}_1 e_t, \mathrm{c}_2 Z_t, t)_{0\le t\le 1}$$

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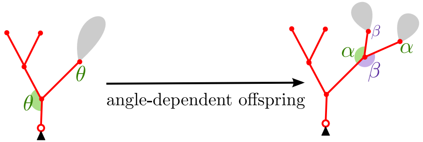
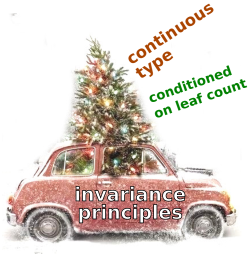


$\rightsquigarrow$ Continuous-type Galton-Watson tree?



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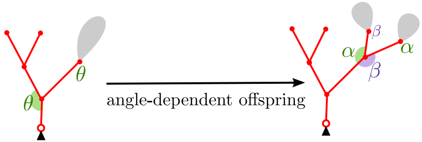
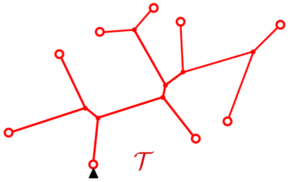
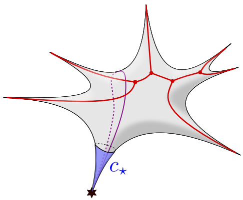


  $\rightsquigarrow$ Continuous-type Galton-Watson tree?
- Disassemble tree to fit better!

- Only some edges of $\mathcal{T}$ intersect their dual geodesic:





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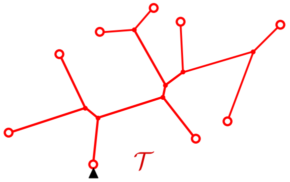

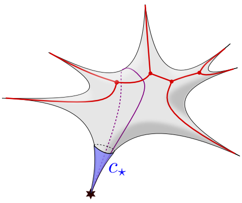

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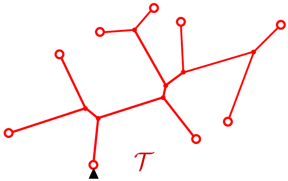

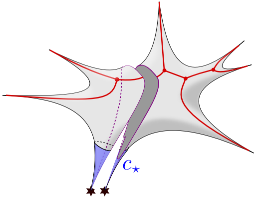

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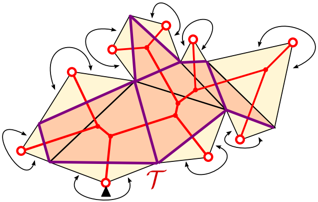

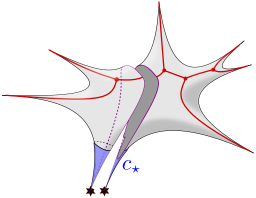

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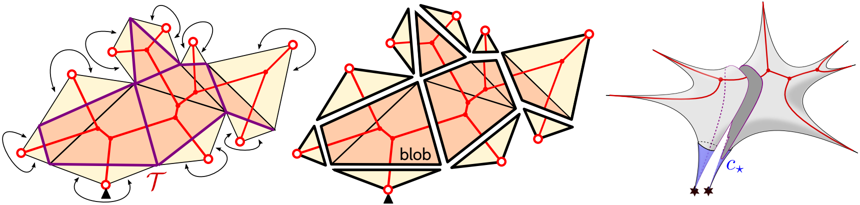

- Only some edges of $\mathcal{T}$ intersect their dual geodesic: canonical partition of the ideal triangulation into "blobs".
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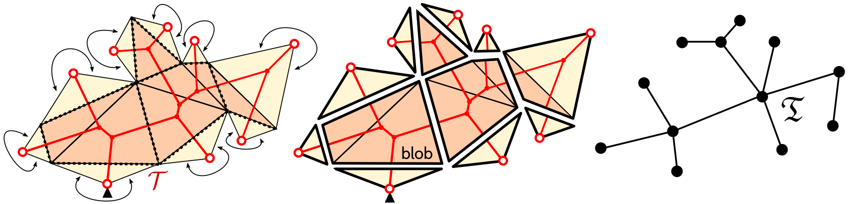

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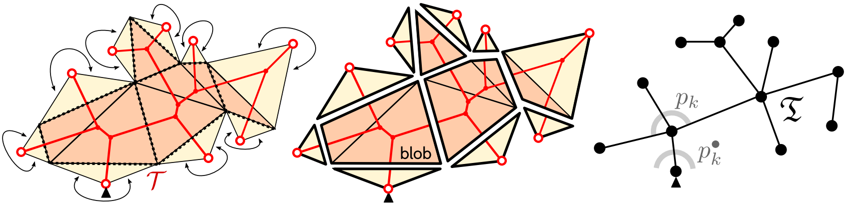

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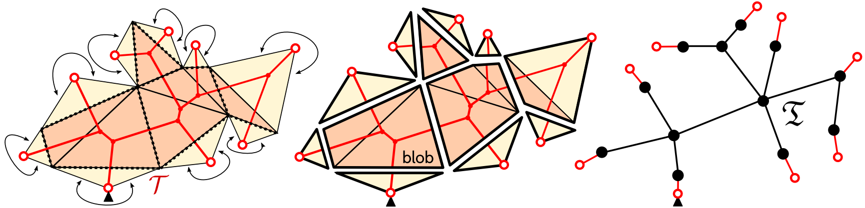

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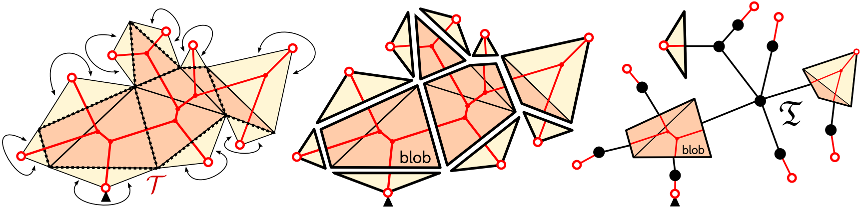

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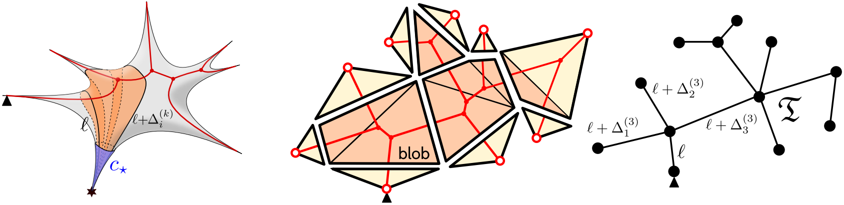
- Insert independent random blobs of appropriate degree (with or without leaf) sampled according to Leb.

Adding the labels



- Transfer the (distance) labels to the black tree.

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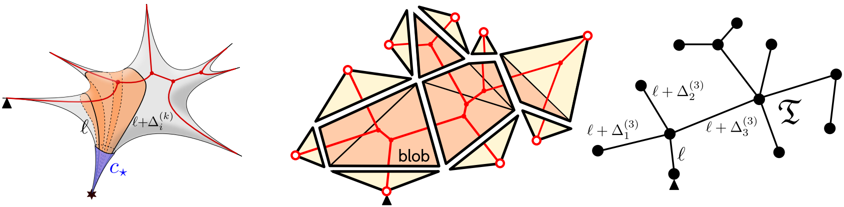

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- Conditionally on $\mathfrak{T}$, the increments $(\Delta_1^{(k)}, \dots, \Delta_k^{(k)})$ at a vertex of degree $k+1$ are independent of those at other vertices and

$$\mathbb{E}[\Delta_i^{(k)}] = 0, \quad \mathbb{E}[(\Delta_i^{(k)})^{4+\varepsilon}] < \infty, \quad i = 1, \dots, k.$$

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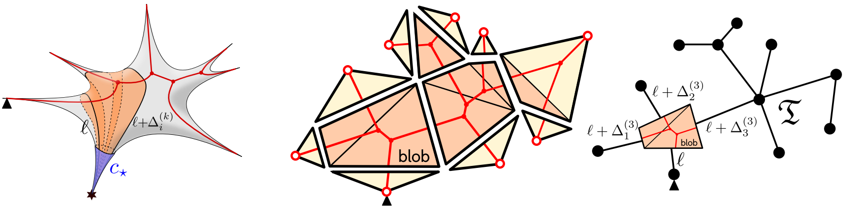

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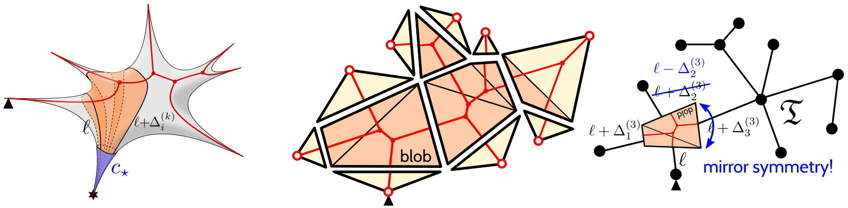

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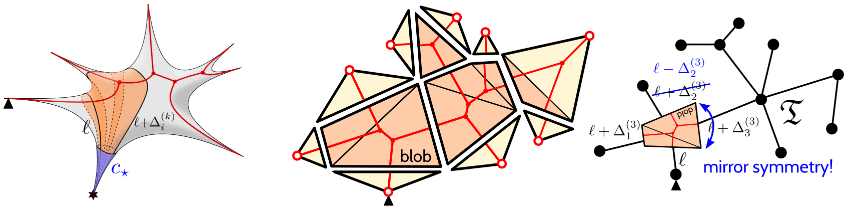

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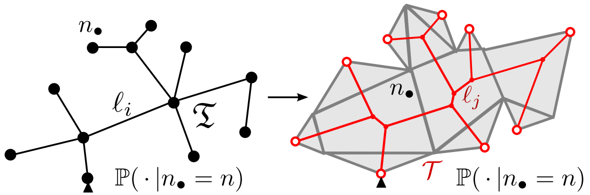
- [Marckert, Miermont, '07]: Conditioned on $n_\bullet$ the rescaled contour and label process of $\mathfrak{T}$ converges to Brownian snake $(e_t, Z_t)_{0 \le t \le 1}$ as $n_\bullet \to \infty$.

Proof of technical result



- Stretch to convergence on $\mathcal{T}$, still conditioning on $n_\bullet = n$,

$$\left(\frac{\tilde{C}^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{\tilde{Z}^{(n)}(t)}{n^{\frac{1}{4}}}, \frac{\tilde{R}^{(n)}(t)}{n}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\tilde{c}_1 e_t, \tilde{c}_2 Z_t, \tilde{c}_3 t)_{0\leq t\leq 1}.$$

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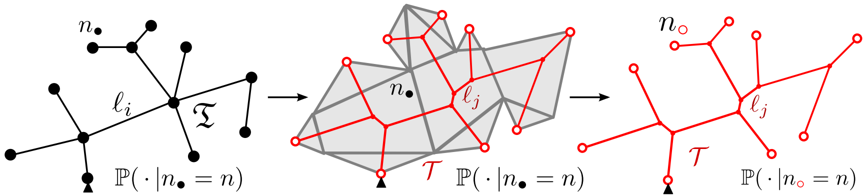

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- Change conditioning to fixed number $n_\circ = n$ of leaves,

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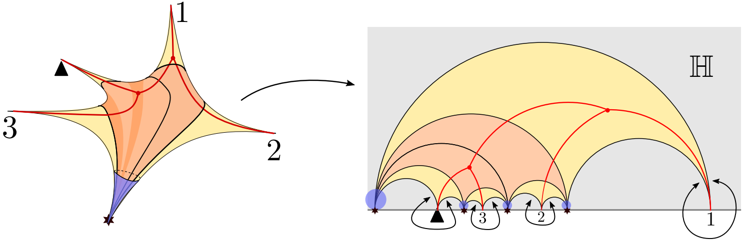
Bound on distances between arbitrary horocycles



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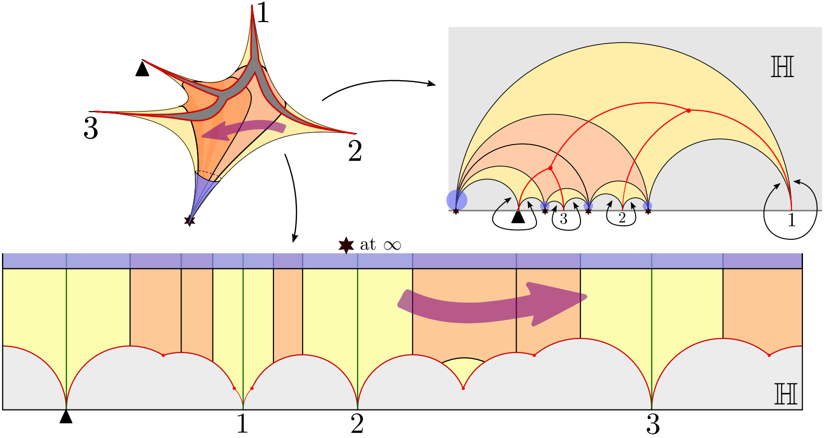

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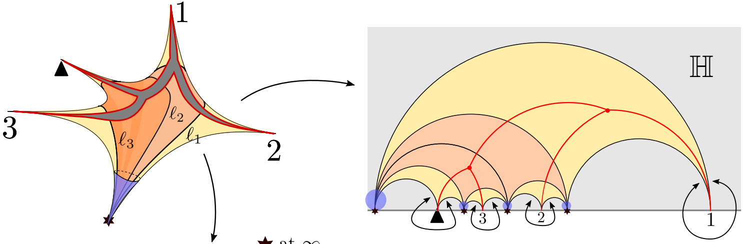

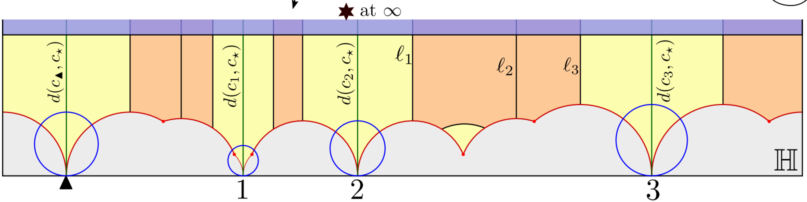

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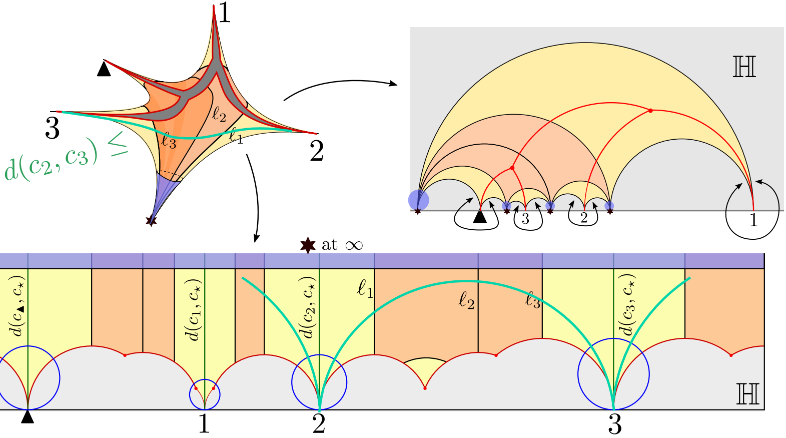

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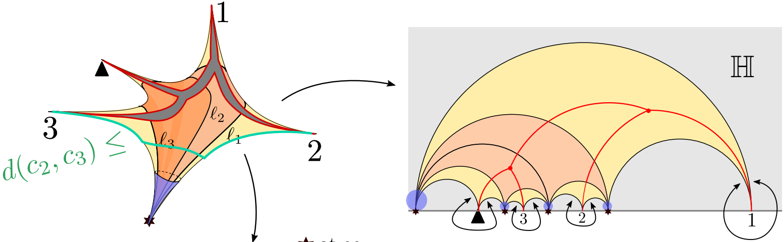

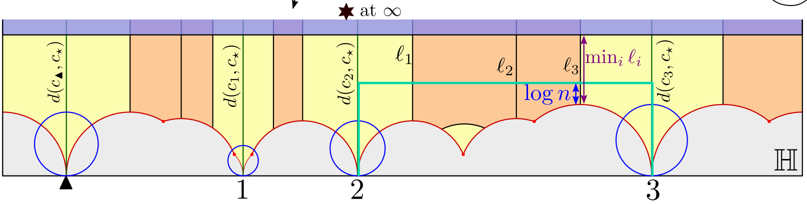

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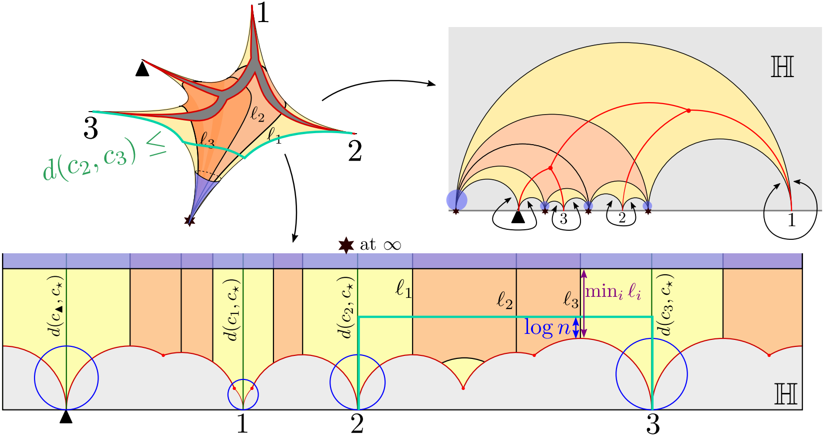

- Distances between arbitrary horocycles satisfy deterministic bound

$$d_{\text{hyp}}(c_i, c_j) \leq d_{\text{hyp}}(c_i, c_*) + d_{\text{hyp}}(c_j, c_*) - 2\min_k \ell_k + \underbrace{2\log n + 10}_{o(n^{\frac{1}{4}})}.$$

Convergence to the Brownian sphere

[Le Gall, '13] [Miermont, '13] [Addario-Berry, Albenque, '13] [Bettinelli, Jacob, Miermont, '14]

$$\left(\frac{C^{(n)}(t)}{n^{\frac{1}{2}}}, \frac{Z^{(n)}(t)}{n^{\frac{1}{4}}}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\mathrm{c}_1 e_t, \mathrm{c}_2 Z_t)_{0\leq t\leq 1}$$

$$+$$

$$d_{\mathrm{hyp}}^{(n)}(s,t) \leq Z^{(n)}(s) + Z^{(n)}(t) - 2\max\left\{\min_{[s,t]} Z^{(n)}, \min_{[t,s]} Z^{(n)}\right\} + o(n^{\frac{1}{4}})$$

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Invariance under rerooting

$\Downarrow$ [Le Gall, '13]'s rerooting trick

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$$\Downarrow$$

$$\left(\{c_1,\dots,c_n\}, n^{-\frac{1}{4}} d_{\mathrm{hyp}}\right) \xrightarrow[n\to\infty]{(d)} \mathsf{c}_{\mathrm{WP}}(\mathrm{m}_\infty, D^*)$$ (Gromov-Hausdorff sense)

Perspectives

- The tree bijection for hyperbolic surfaces in a sense simpler than maps: left-right symmetric!
- Benjamini-Schramm convergence to random hyperbolic surface of topology $\mathbb{R}^2 \setminus \mathbb{Z}^2$. [TB, Curien, '22+]
- Tree bijection extends to boundary lengths $\mathsf{L} > 0$ (natural analogue of BDG bijection).
- Another possible bridge: tight boundaries, see Miermont's talk tomorrow!

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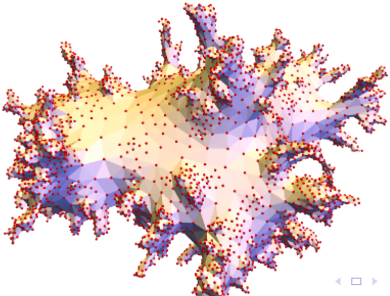

Backup slides

From labeled trees to the Brownian snake

- The labeled tree is encoded in contour process $C^{(n)}(t)$

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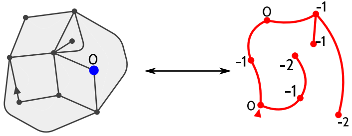

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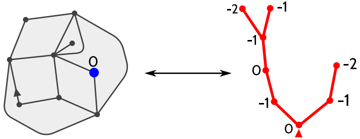

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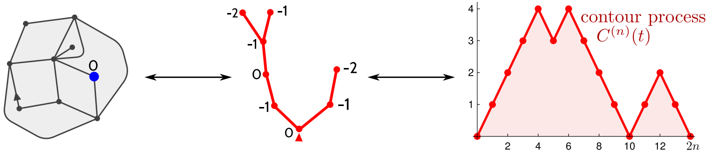

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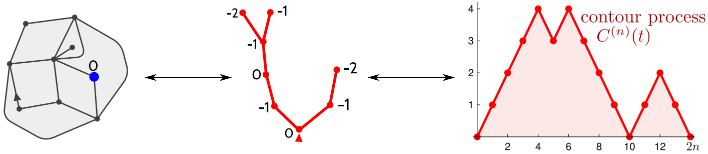

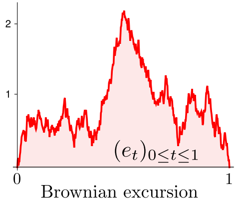

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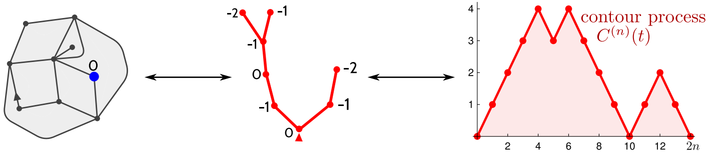

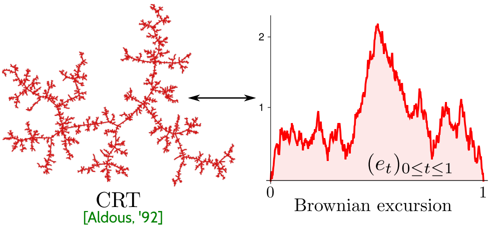

From labeled trees to the Brownian snake

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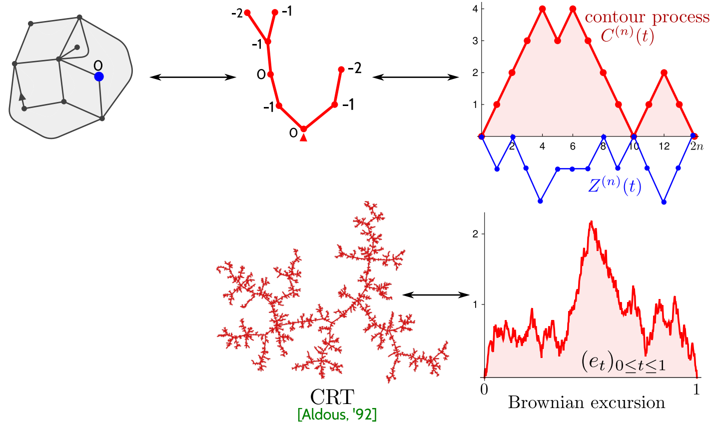

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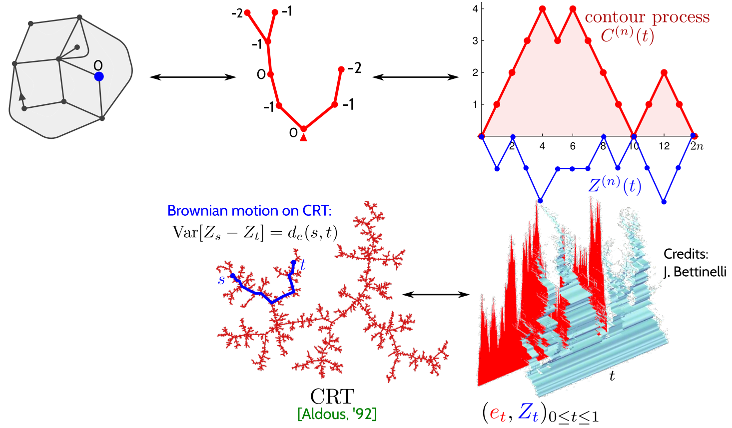

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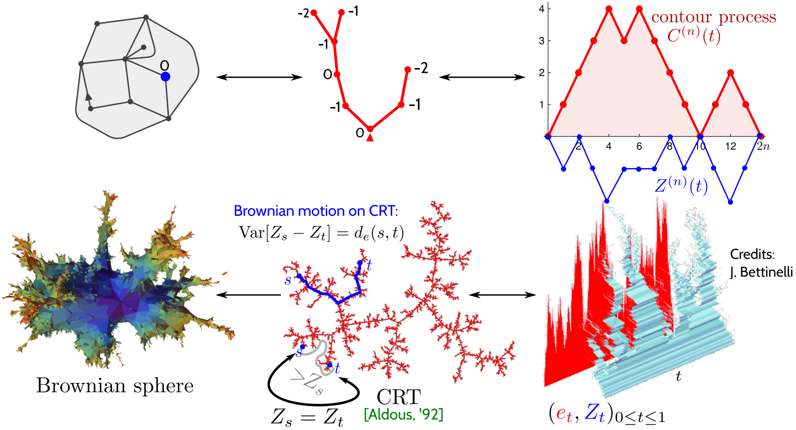

The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

- More precisely, given Brownian snake $(e_t, Z_t)_{0\le t\le 1}$, define pseudo-distance on $[0,1]$ via

$$D^\circ(s,t) = Z_s + Z_t - 2\max\left\{\min_{[s,t]} Z, \min_{[t,s]} Z\right\}, \quad s,t\in[0,1].$$

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- Gromov-Hausdorff convergence proven for many types of maps, including
  - p-angulations [Le Gall, '13][Miermont, '13][Addario-Berry, Albenque, '20]
  - Uniform (bipartite) maps [Bettinelli, Jacob, Miermont, '14][Abraham, '16]
  - Simple triangulations, quadrangulations [Addario-Berry, Albenque, '20]
  - Bipartite maps with prescribed degrees [Marzouk, '18], ...

The Brownian sphere [Marckert, Mokkadem, Le Gall, Miermont, ...]

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$$D^\circ(s,t) = Z_s + Z_t - 2\max\left\{\min_{[s,t]} Z, \min_{[t,s]} Z\right\}, \quad s,t\in[0,1].$$

- Writing $t \sim s$ if identified in CRT, introduce new pseudo-distance

$$D^*(s,t) = \inf\left\{D^\circ(s,t_1) + D^\circ(s_1,t_2) + \cdots + D^\circ(s_k,t) : t_i \sim s_i\right\}.$$

- Brownian sphere is defined as $(m_\infty = [0,1]/\{D^*=0\}, D^*)$.
- Gromov-Hausdorff convergence proven for many types of maps, including
  - p-angulations [Le Gall, '13][Miermont, '13][Addario-Berry, Albenque, '20]
  - Uniform (bipartite) maps [Bettinelli, Jacob, Miermont, '14][Abraham, '16]
  - Simple triangulations, quadrangulations [Addario-Berry, Albenque, '20]
  - Bipartite maps with prescribed degrees [Marzouk, '18], ...
- Can also be recovered from Liouville Quantum Gravity at $\gamma = \sqrt{\frac{8}{3}}$. [Miller, Sheffield]

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- Novelty of this work: Brownian sphere limit from continuous model!