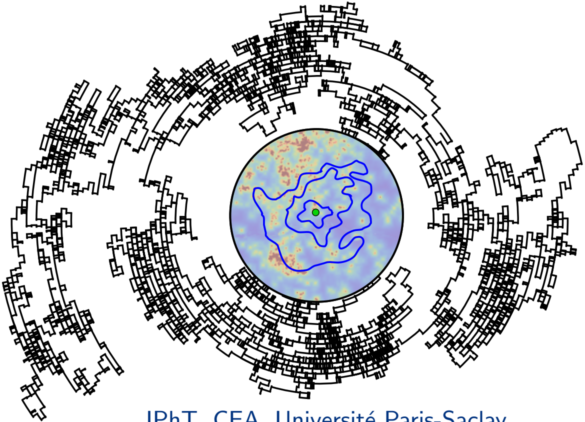
Dynamics on random graphs
CIRM, Marseille, France - October 22, 2017

Nesting of loops versus winding of walks

Timothy Budd



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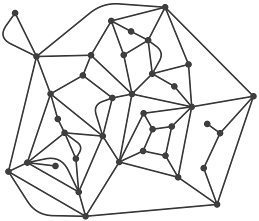
Planar maps coupled to a rigid $O(n)$ loop model

- Planar map: planar (multi)graph properly embedded in $\mathbb{R}^2$ viewed up to continuous deformations.



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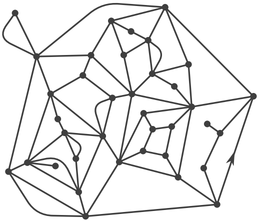

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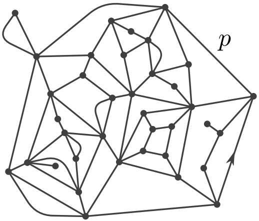

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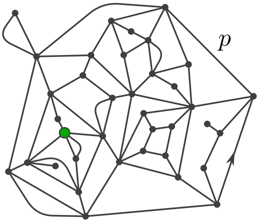

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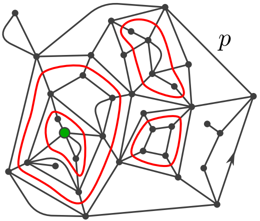
$$n^{\#\text{loops}} g^{\text{total loop length}} \prod_{\text{regular faces}} q_{\text{degree}}$$

for $n, g, q_2, q_4, q_6, \ldots \in \mathbb{R}_+$ fixed.



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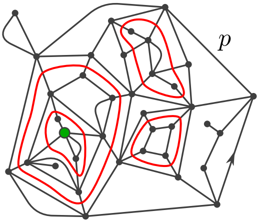
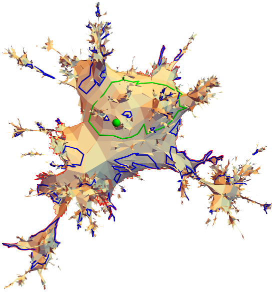
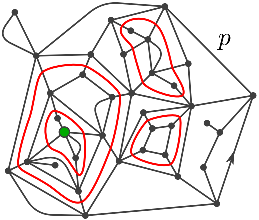
  for $n, g, q_2, q_4, q_6, \ldots \in \mathbb{R}_+$ fixed.
- For $n \in (0, 2]$ the model is critical iff:
  - $\#\text{faces} < \infty$ a.s., but $\mathbb{E}(\#\text{faces}) = \infty$,
  - supports loops of length $O(p)$ as $p \to \infty$.

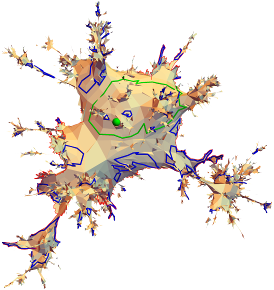




Loop nesting statistics

- Let N_p be the number of loops surrounding the marked vertex in a random map of perimeter p.





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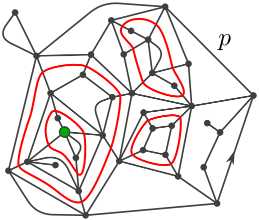
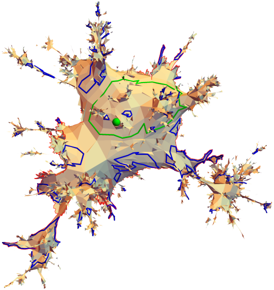
$$\frac{N_p}{\log p} \xrightarrow[p\to\infty]{\mathbb{P}} \frac{1}{\pi} \frac{n}{\sqrt{4-n^2}}.$$





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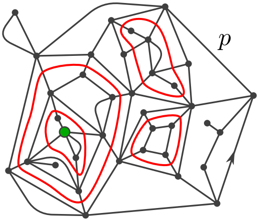
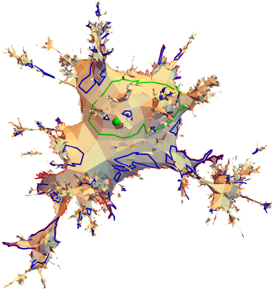
- Large deviation behaviour:

$$\frac{\log \mathbb{P}(N_p = \lfloor x \log p \rfloor)}{\log p} \longrightarrow x\Lambda_n^*(1/x)$$

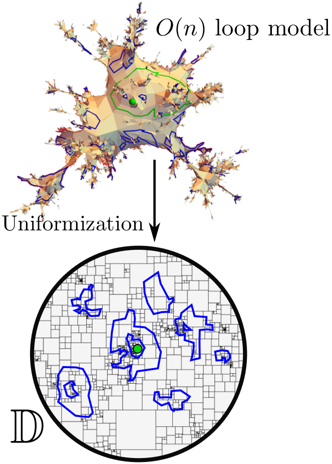
where $x\Lambda_n^*(1/x) = -\frac{1}{\pi} J(\pi x)$ and

$$J(x) = x \log\left(\frac{2}{n}\frac{x}{\sqrt{1+x^2}}\right) + \operatorname{arccot}(x) - \arccos(n/2).$$





Uniformization



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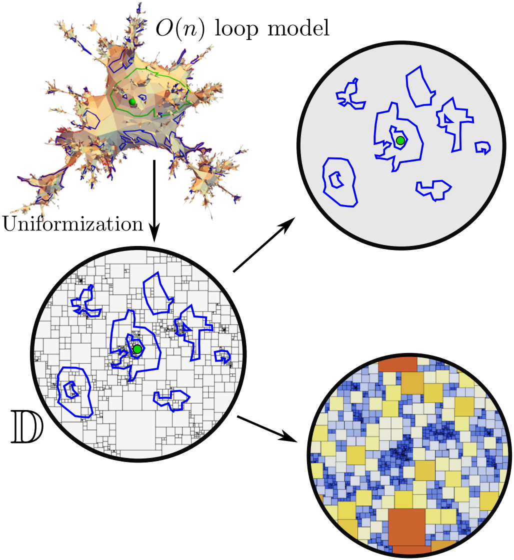

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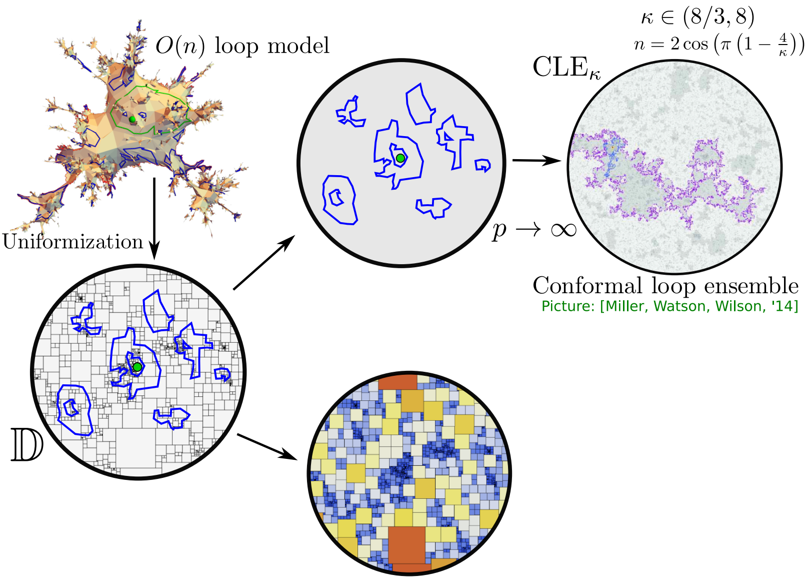

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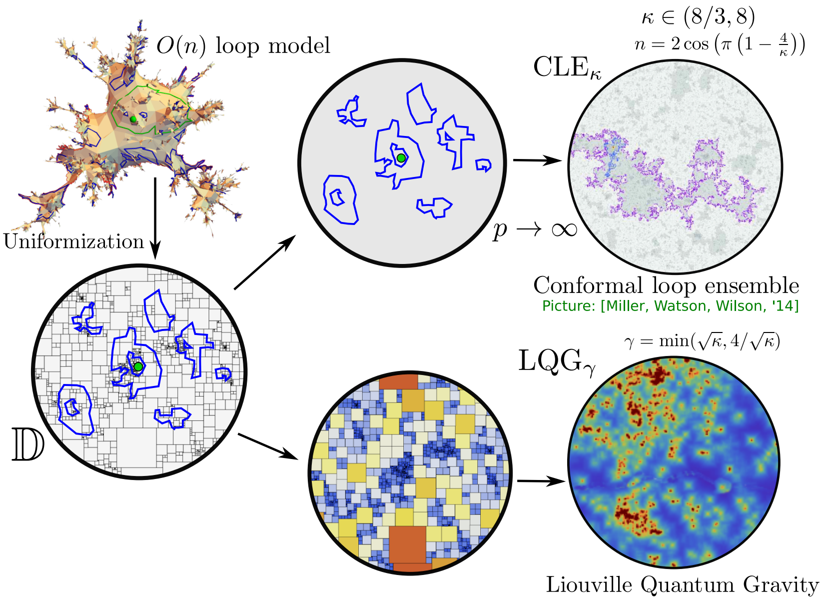
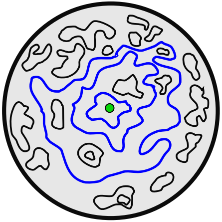
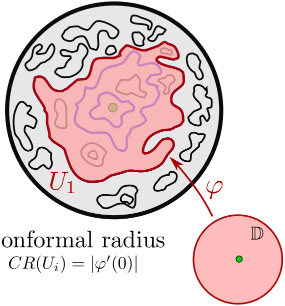


Nesting in CLE_κ



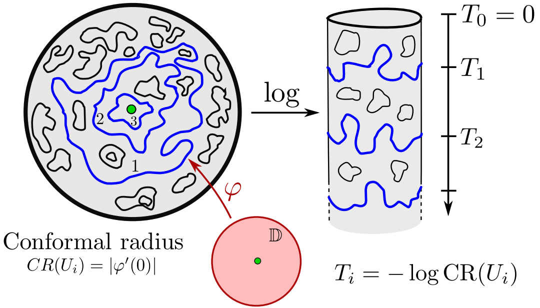
Nesting in CLE$_\kappa$



Conformal radius

$CR(U_i) = |\varphi'(0)|$

Nesting in CLE_κ



- The sequence (T_i) of log-conformal radii of the nested loops has i.i.d. increments and [Schramm, Sheffield, Wilson, '09]

$$\mathbb{E}\left[e^{-\lambda T_1}\right] = \frac{-\cos\left(\frac{4\pi}{\kappa}\right)}{\cos\left(\pi\sqrt{(1-4/\kappa)^2+8\lambda/\kappa}\right)} =: e^{\Lambda_\kappa(\lambda)}$$

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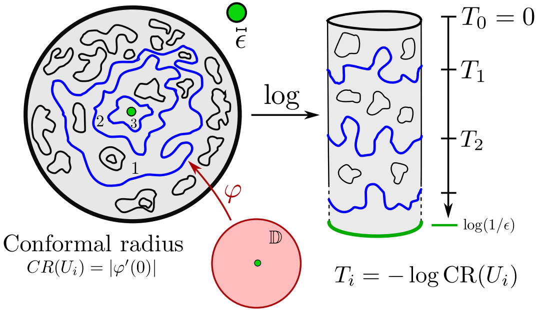

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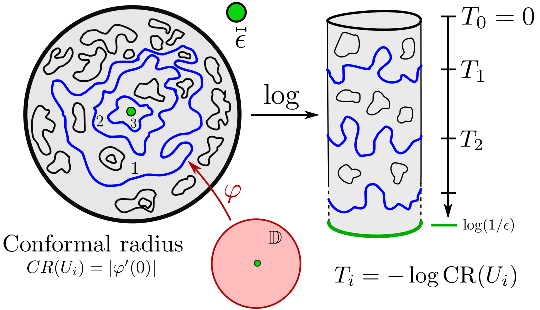

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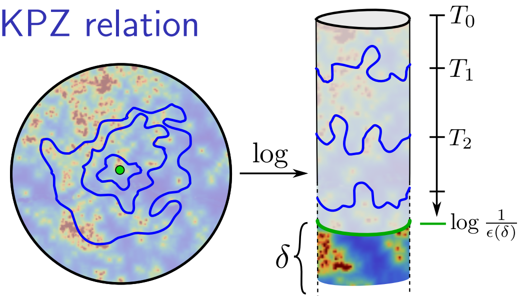
- Number of loops surrounding ϵ-disk: $\mathcal{N}_\epsilon \approx \sup\{i : T_i < \log(1/\epsilon)\}$
- Large deviation behaviour [Miller, Watson, Wilson, '14]:

$$\frac{\log \mathbb{P}(\mathcal{N}_\epsilon = \lfloor x \log(1/\epsilon) \rfloor)}{\log(1/\epsilon)} \xrightarrow{\epsilon\to 0} x\Lambda_\kappa^*(1/x)$$

LiouvilleQG: KPZ relation



- If we have a volume measure on $\mathbb{D}$ it is more natural to fix $\delta > 0$ and ask for nesting around $\epsilon(\delta)$-disk of volume δ.

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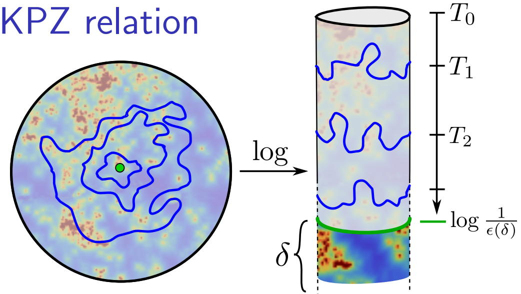

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- In LQG_γ the law of $\epsilon(\delta)$ as $\delta \to 0$ is well-understood [Duplantier, Sheffield, '08]: $\log(1/\epsilon(\delta)) \approx$ hitting time of $\log(1/\delta)/\gamma$ by a BM with drift $2/\gamma - \gamma/2$.

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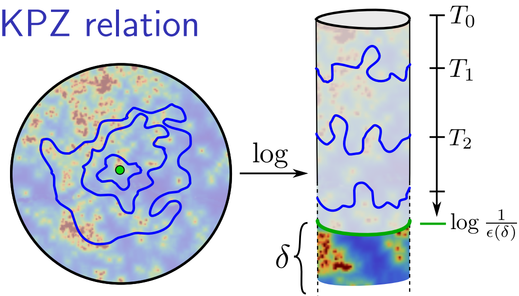

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$$\frac{\log \mathbb{P}(\mathcal{N}_{\epsilon(\delta)} = \lfloor x \log(1/\delta) \rfloor)}{\log(1/\delta)} \xrightarrow{\delta \to 0} x \underbrace{(\Lambda_\kappa \circ 2U_\gamma)^*}(1/x),$$

where U_γ is the famous KPZ formula [Knizhnik, Polyakov, Zamolodchikov, '88]

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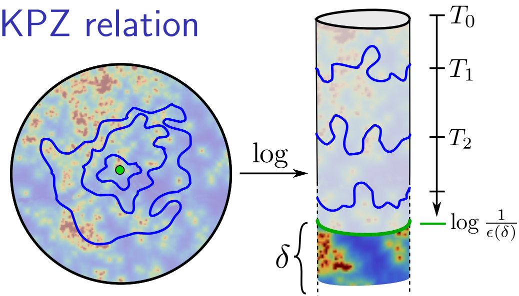

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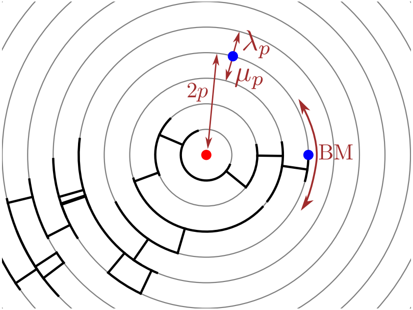
"Nesting in CLE_κ" + "KPZ" = "Nesting in $O(n)$ on planar maps"

Main question in this talk:

Can we disentangle the LHS starting from planar map combinatorics?

A Markov process on concentric circles

- Define the Markov process (X_u) on $\{x \in \mathbb{C} : |x| \in 2\mathbb{Z}\}$ such that
  - $|X_u| \arg X_u$ is standard Brownian motion;
  - $|X_u|/2$ is an independent birth-death process with birth rate $\lambda_p = \frac{1}{16}(2 + 1/p)$ and death rate $\mu_p = \frac{1}{16}(2 - 1/p)$;
  - (X_u) is trapped upon hitting 0.



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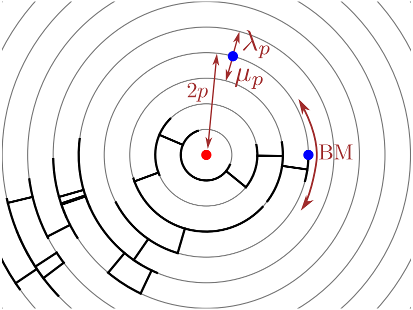

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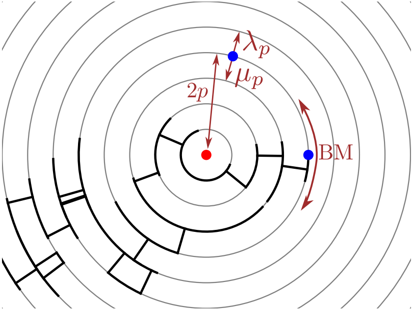
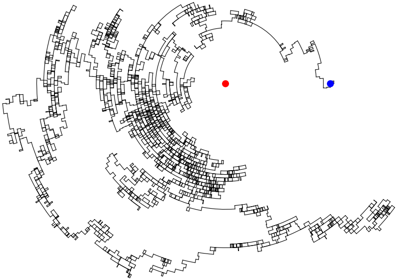
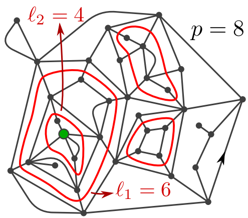
  - (X_u) is trapped upon hitting 0.
- It a.s. hits 0 in finite time.
- Far away from 0 it resembles 2D Brownian motion.





Loop length versus axis crossing

- Let $(\ell_1, \ell_2, \ldots \ell_N)$ be the sequence of lengths of loops surrounding the marked vertex in a critical $O(n)$ loop-decorated planar map with perimeter p.



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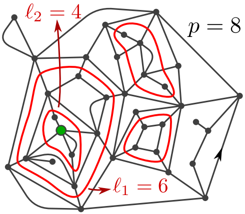
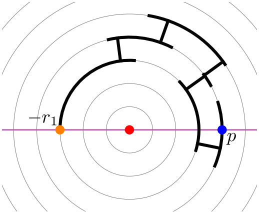
- Let $(r_1, r_2, \ldots, r_{\mathcal{N}})$ be the sequence of distances of the points at which (X_u) alternates between the two half x-axes before hitting 0 when started at $(p, 0)$.





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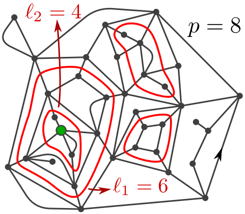

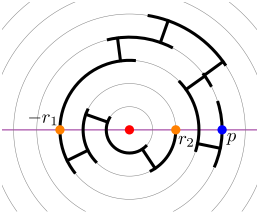

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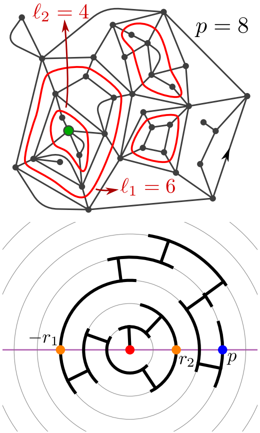

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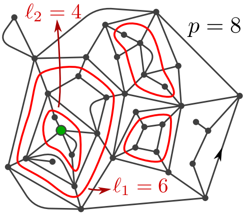

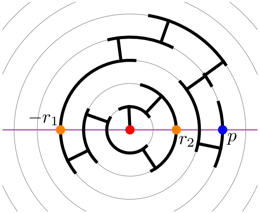

Theorem

If $n \in (0, 2]$ then $(\ell_1, \ell_2, \dots \ell_N) \overset{(d)}{=} (r_1, r_2, \dots, r_{\mathcal{N}})$ biased by $(n/2)^{\mathcal{N}}$.

- Can perform a time change $t(u) = \int_0^u |X_{u'}|^2 \mathrm{d}u'$, $X_u = 2R_{t(u)} e^{i\Theta_{t(u)}}$ such that (Θ_t) is standard Brownian motion and (R_t) is an independent birth-death process with rates $\hat{\lambda}_p = 4p^2\lambda_p$, $\hat{\mu}_p = 4p^2\mu_p$.

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- If $b = \frac{1}{\pi}\arccos(n/2)$, then there exists an $h_b : \mathbb{Z}_+ \to \mathbb{R}$ such that

$$H_b(\Theta, R) = \cos(b\,\Theta)\, h_b(R)$$

is harmonic w.r.t. the Markov process $(\Theta_t, R_t)_t$ until $\Theta_t = \pm\pi$.

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  is harmonic w.r.t. the Markov process $(\Theta_t, R_t)_t$ until $\Theta_t = \pm\pi$.
- Biasing by $(n/2)^{\mathcal{N}}$ is very similar to H_b-transforming $(\Theta_t, R_t)_t$ into $(\Theta_t^{(b)}, R_t^{(b)})_t$.

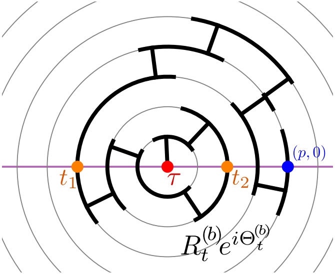
- Can perform a time change $t(u) = \int_0^u |X_{u'}|^2 \mathrm{d}u'$, $X_u = 2R_{t(u)} e^{i\Theta_{t(u)}}$ such that (Θ_t) is standard Brownian motion and (R_t) is an independent birth-death process with rates $\hat{\lambda}_p = 4p^2\lambda_p$, $\hat{\mu}_p = 4p^2\mu_p$.
- If $b = \frac{1}{\pi}\arccos(n/2)$, then there exists an $h_b : \mathbb{Z}_+ \to \mathbb{R}$ such that

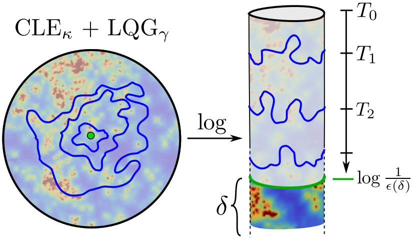
$$H_b(\Theta, R) = \cos(b\,\Theta)\, h_b(R)$$

  is harmonic w.r.t. the Markov process $(\Theta_t, R_t)_t$ until $\Theta_t = \pm\pi$.
- Biasing by $(n/2)^{\mathcal{N}}$ is very similar to H_b-transforming $(\Theta_t, R_t)_t$ into $(\Theta_t^{(b)}, R_t^{(b)})_t$.
- $\Theta_t^{(b)}$ and $R_t^{(b)}$ are still independent (as long as $R_t^{(b)} \neq 0$)!

Proposition

If $(t_i)_i$ are the half-axis alternation times of $R_t^{(b)} e^{i\Theta_t^{(b)}}$ and (T_i) are the log-conformal radii of CLE_κ with $\kappa = 4/(1 \pm b)$, then $(t_i)_i \overset{(d)}{=} (\kappa T_i)_i$.





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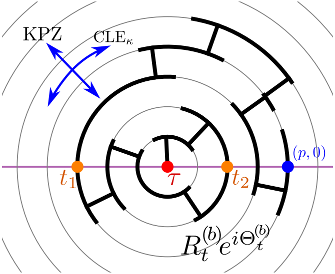
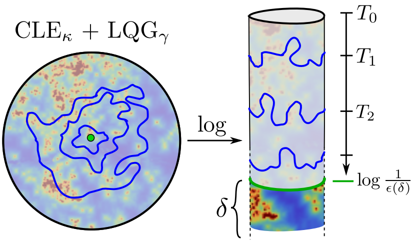
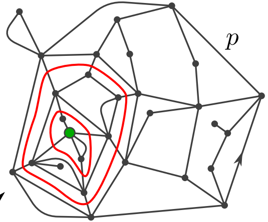
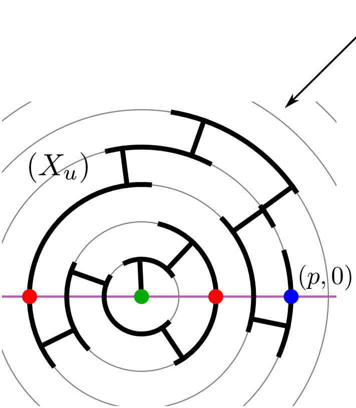




- Question: Are the distributions of $\tau := \inf\{t : R_t^{(b)} = 0\}$ and $\kappa \log \frac{1}{\epsilon(\delta)}$ identical in the limit $\log(1/\delta) \sim 2 \log p \to \infty$?

Connection: simple diagonal random walk on $\mathbb{Z}^2$





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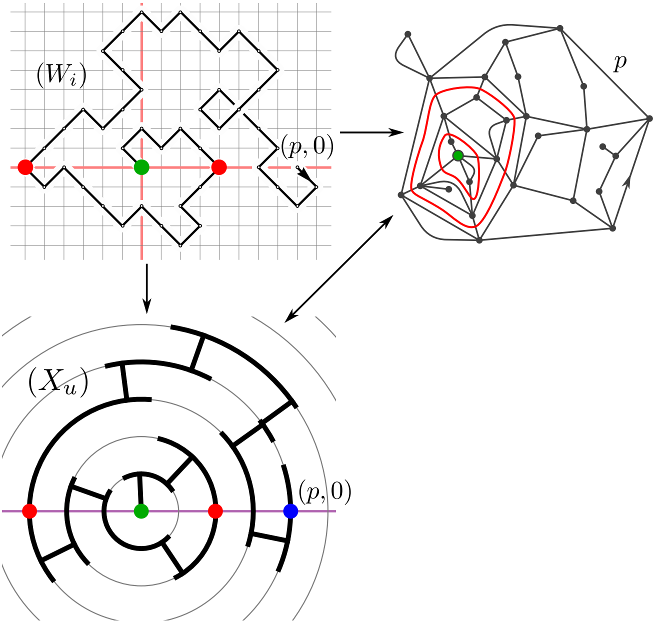

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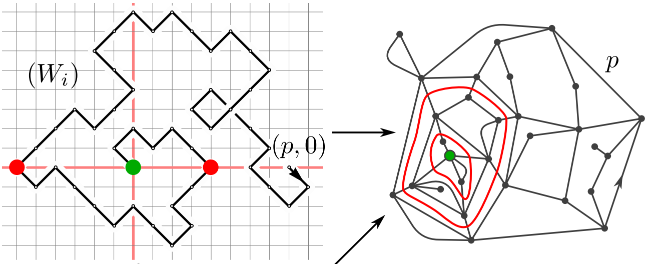

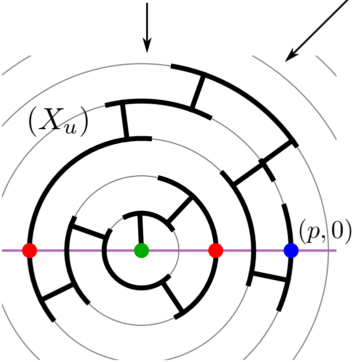

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$\Phi : \{\text{diagonal walks}\} \to \{\text{maps with nested loops}\}$
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$\Phi((W_i) : W_0 = (p,0))$ = critical map of perimeter p with nested $O(2)$ loops

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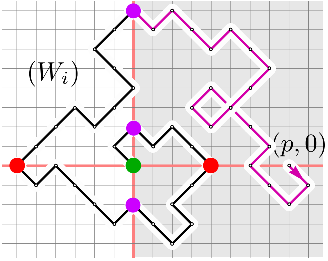

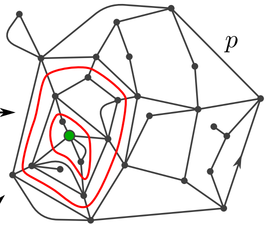

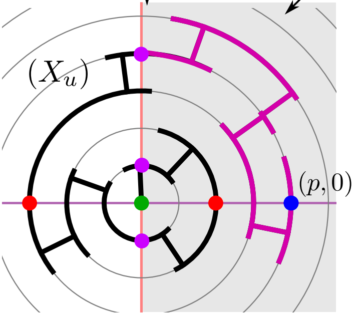

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(W_i) and (X_u) have same exit distribution from half-plane

Exit distribution from half plane [TB,'17]

- (W_i) exits at ℓ with prob $\sum_{n \text{ even}} \frac{p}{n}\binom{n}{\frac{n-p}{2}}\binom{n}{\frac{n-\ell}{2}}4^{-n}$.

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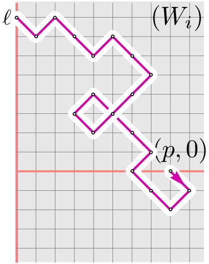

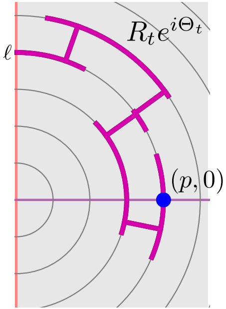

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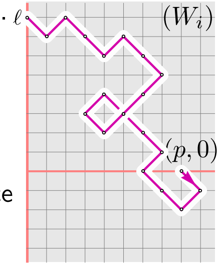

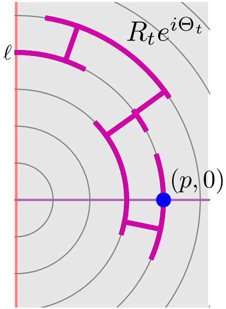

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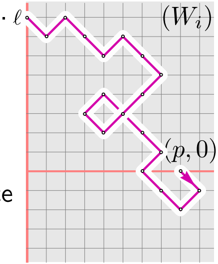

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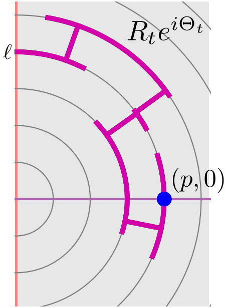

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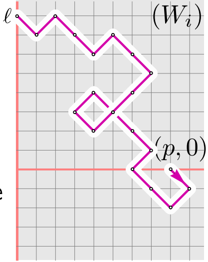
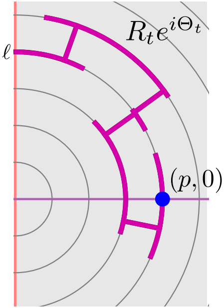
$$\int_0^\infty e^{-s\mathbf{K}} \mathrm{d}F(s)$$

where $F(s) = \frac{1}{2}\mathbb{P}(\sup_{t\in(0,s)} |\Theta_t| > \pi/2)$ and $\mathbf{K}$ is the generator $\mathbf{K}e_p = \lim_{t\to 0} \frac{1}{t}\mathbb{E}[e_p - e_{R_t}]$ of R_t.





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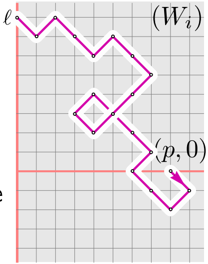

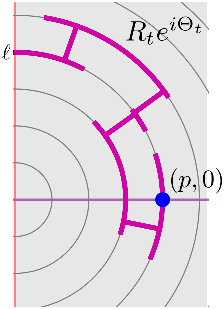

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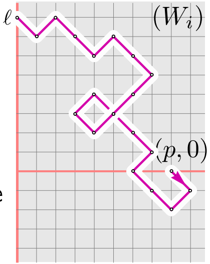

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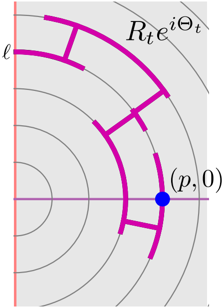

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- $\mathcal{D} = \mathcal{D}(\mathbb{D})$ is Hilbert space of analytic functions f on the unit disk $\mathbb{D} \subset \mathbb{C}$ with $f(0) = 0$ and finite norm w.r.t. $(\mathrm{d}A(x+iy) := \frac{1}{\pi}\mathrm{d}x\mathrm{d}y)$

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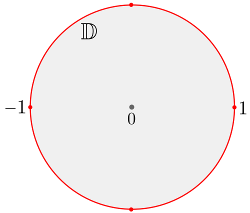

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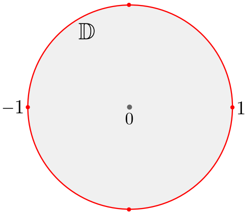

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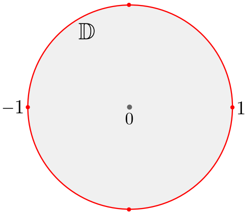

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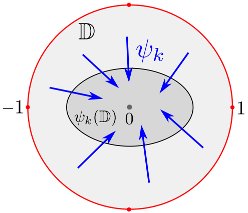

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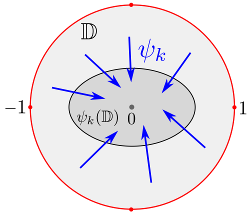
- By conformal invariance of the Dirichlet inner product,

$$\langle f, \mathbf{J}_k g \rangle_{\mathcal{D}} = \langle \mathbf{\Psi}_k f, \mathbf{\Psi}_k g \rangle_{\mathcal{D}} = \langle f \circ \psi_k, g \circ \psi_k \rangle_{\mathcal{D}} = \langle f, g \rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}.$$



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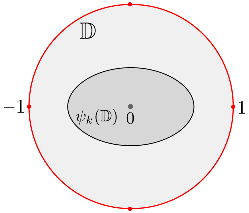

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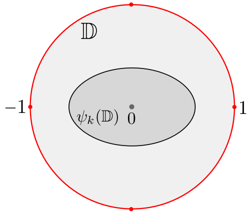

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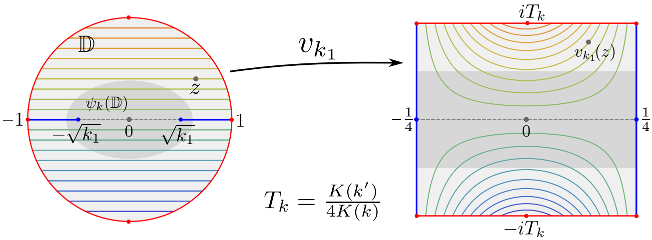
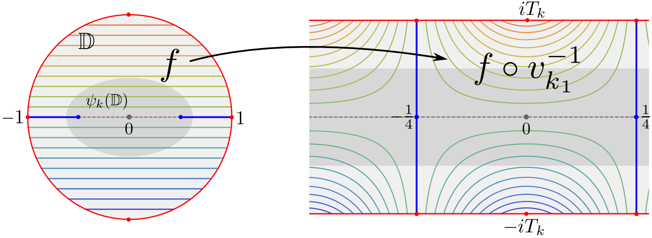
- Look for a nice conformal mapping.
- An elliptic integral does the job ($k' = \sqrt{1-k^2}$, $k_1 = \frac{1-k'}{1+k'}$)

$$v_{k_1}(z) = \frac{1}{4K(k_1)} \int_0^z \frac{\mathrm{d}x}{\sqrt{(k_1 - x^2)(1 - k_1 x^2)}} = \frac{\operatorname{arcsn}\left(\frac{z}{\sqrt{k_1}}, k_1\right)}{4K(k_1)}$$



- The push-forward of $f \in \mathcal{D}$ extends to an analytic function on the strip $\mathbb{R} + i(-T_k, T_k)$ that is even around $\pm 1/4$, hence 1-periodic.



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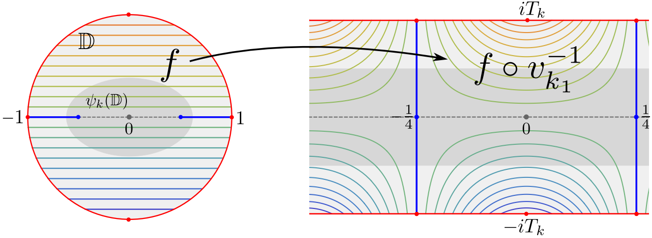
- Basis $\cos(2\pi m(\cdot + 1/4))$, $m \geq 1$, is orthogonal w.r.t. Dirichlet on strip of any height.



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$$f_m(z) = \cos(2\pi m(v_{k_1}(z) + 1/4)) - \cos(\pi m/2), \quad m \geq 1$$

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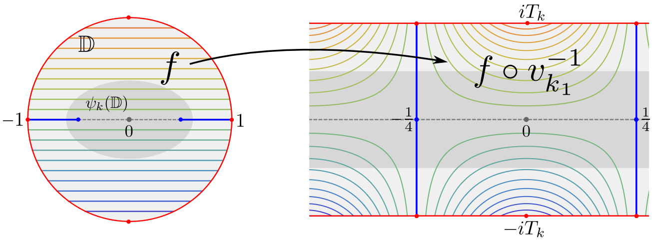

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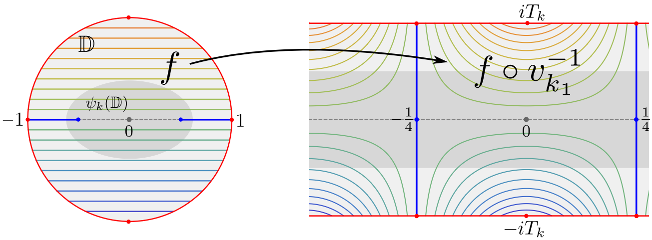
- Conclusion: $\mathbf{J}_k$ has eigenvectors $(f_m)_{m \geq 1}$ and eigenvalues

$$\frac{\langle f_m, f_m \rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}}{\langle f_m, f_m \rangle_{\mathcal{D}(\mathbb{D})}} = \frac{\sinh(2m\pi T_k)}{\sinh(4m\pi T_k)} = \frac{1}{2}\operatorname{sech}(2m\pi T_k), \qquad T_k = \frac{K(k')}{4K(k)}.$$



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$$\mathbf{K}_k e_p = \left(\frac{2K(k')}{\pi}\right)^2 \frac{p^2}{16}\left[\left(8-4k^2\right)e_p - \left(2 \pm \frac{1}{p}\right)k^2 e_{p\pm1}\right]$$

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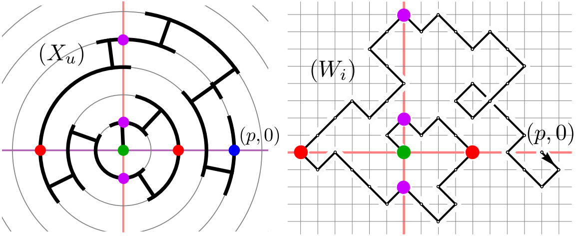
Proposition

The sequences of locations where the diagonal random walk (W_i) and the Markov process (X_u) alternate between the x- and y-axis are equal in law.



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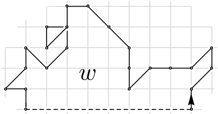

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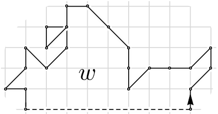

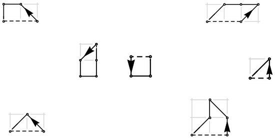

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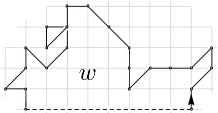

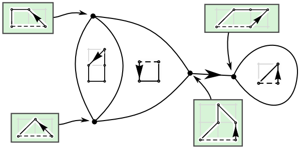

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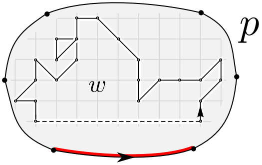

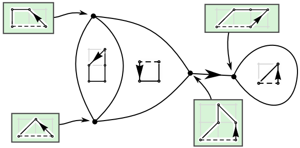

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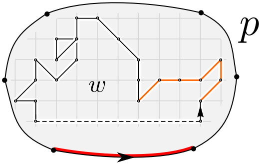

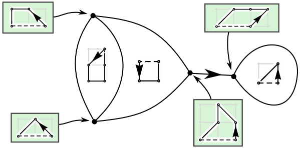

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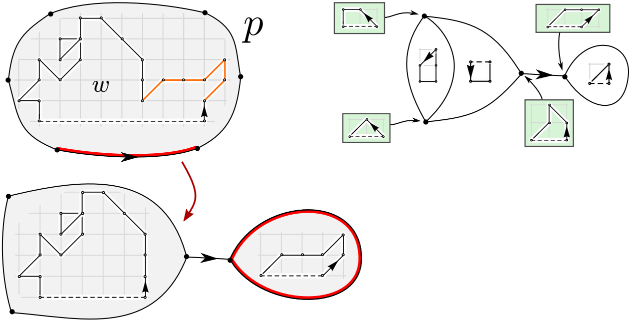

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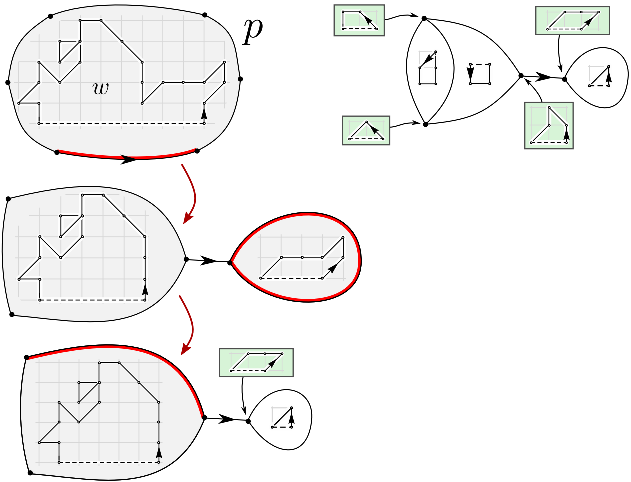

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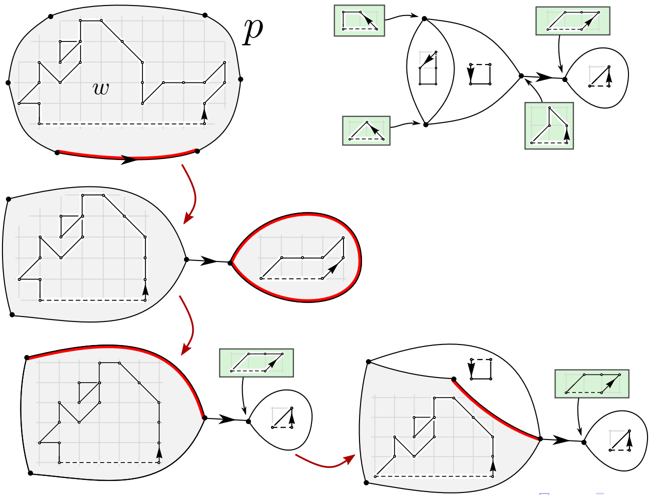

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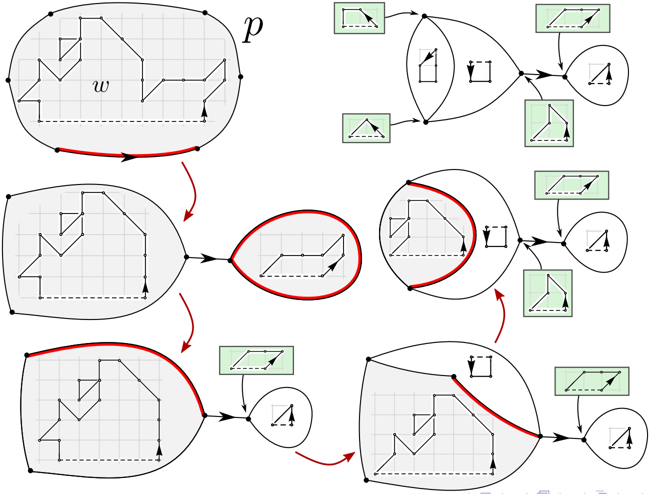

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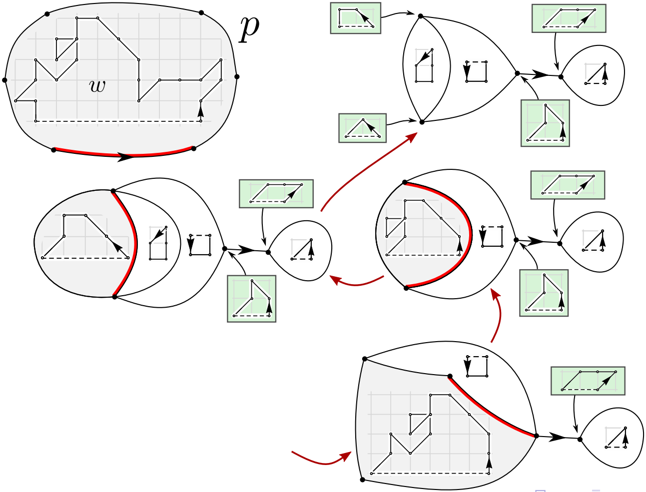

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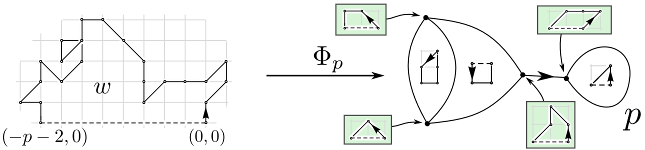

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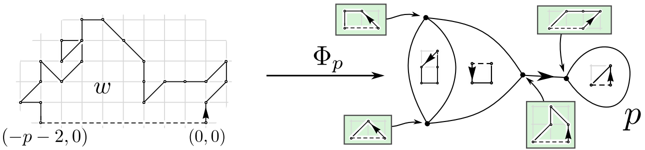

- Φ_p is a bijection with rooted planar maps of perimeter p with
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Walks on the slit plane

- This extends to a bijection $\Phi_{\ell,p}$ between walks on the slit plane from $(p,0)$ to $(-\ell,0)$ and rooted planar maps with perimeter p and
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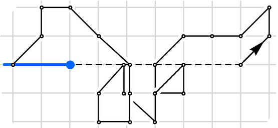

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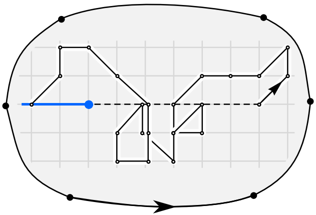

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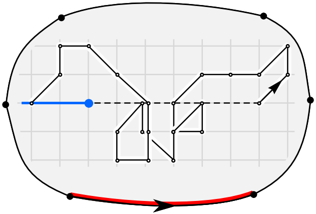

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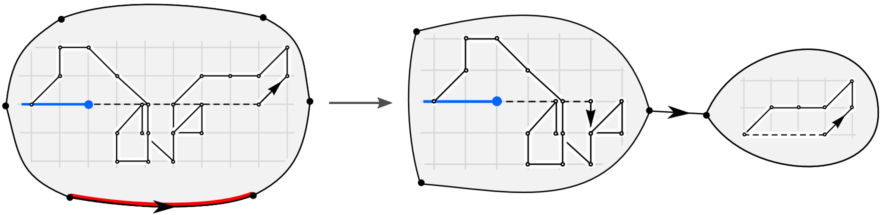

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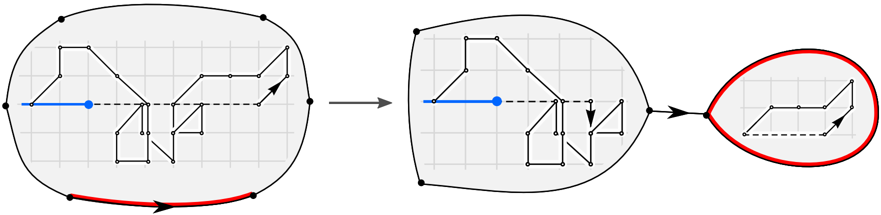

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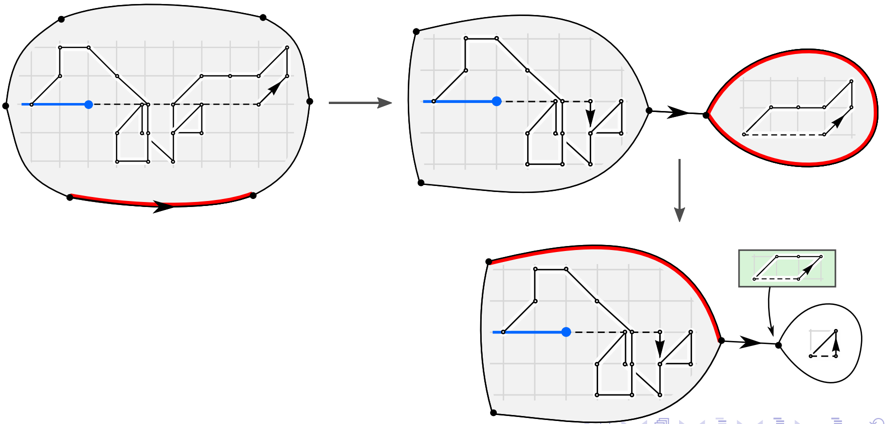

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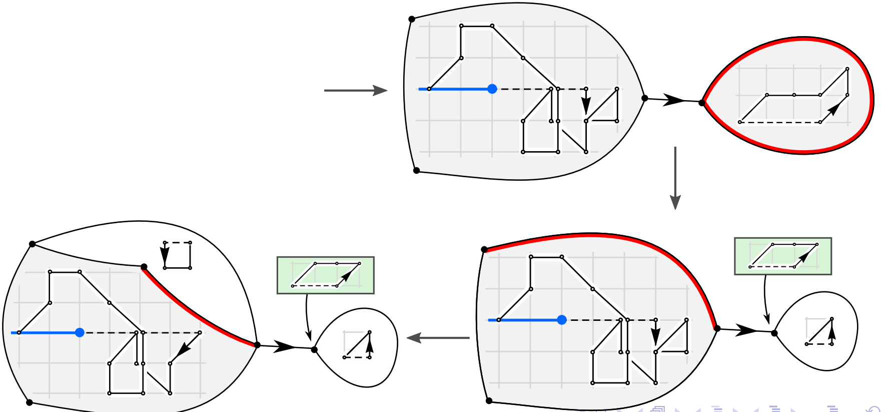

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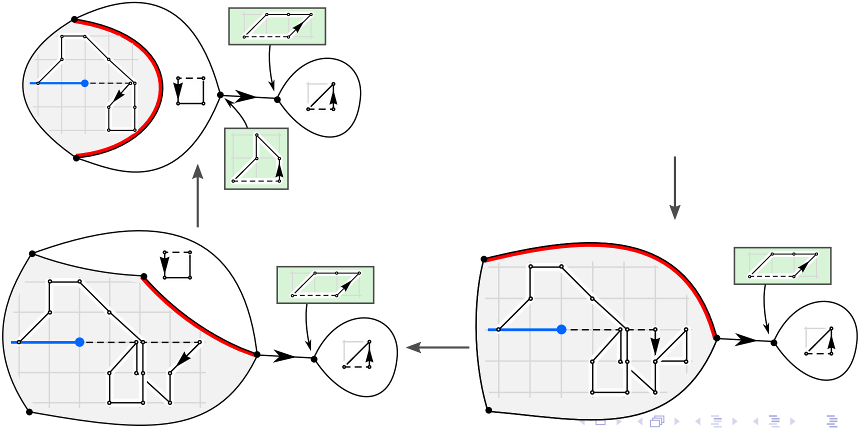

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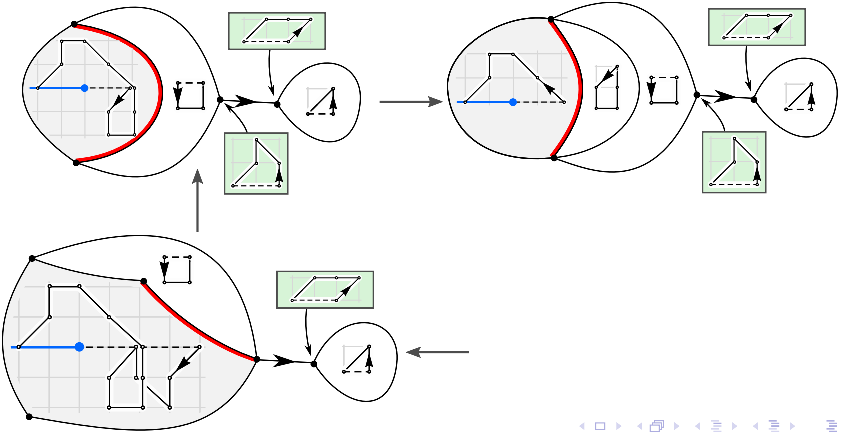

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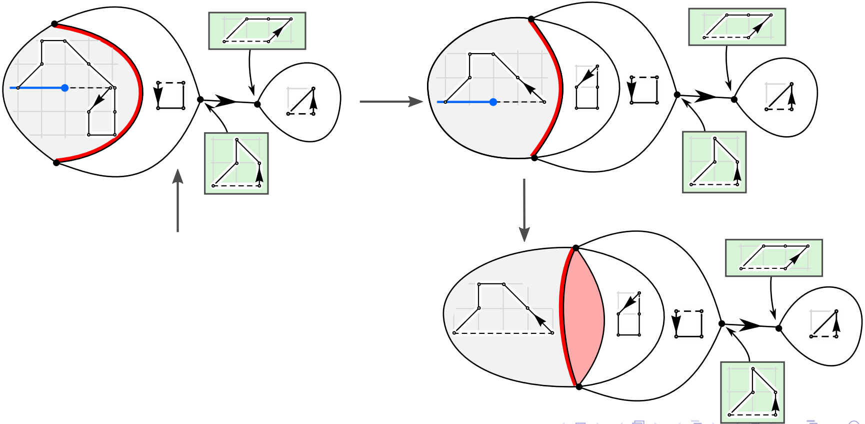

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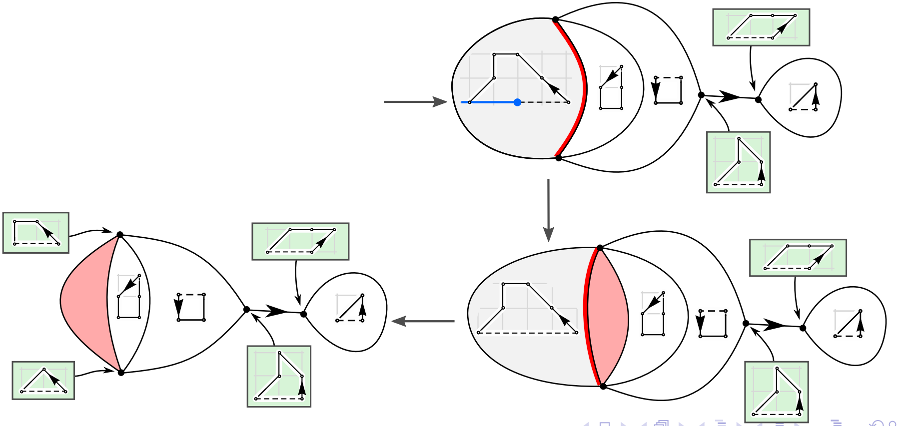

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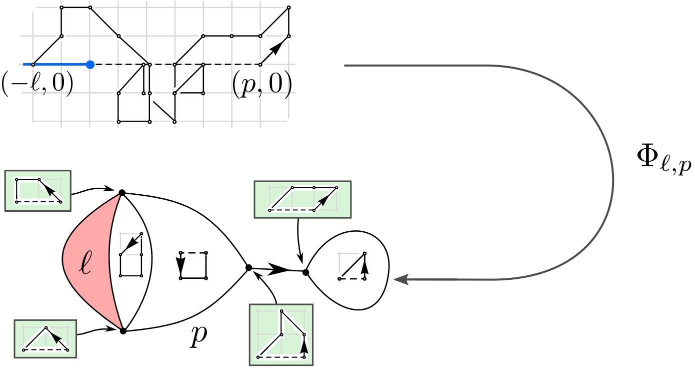
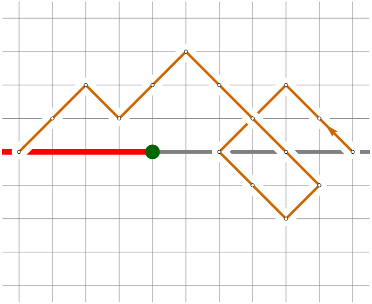
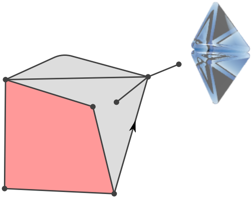


From walks to loop-decorated maps





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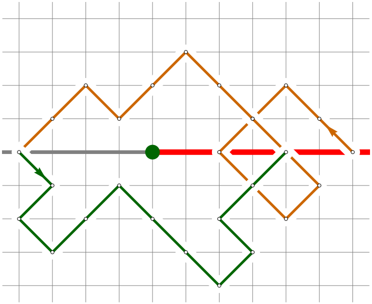
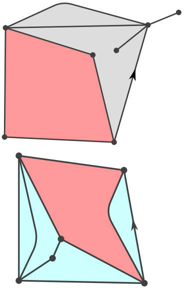
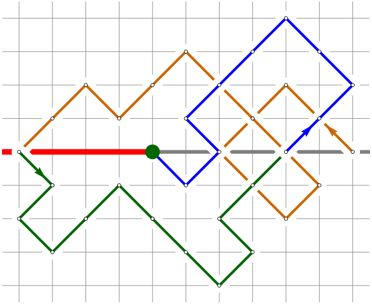
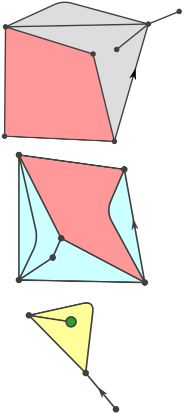


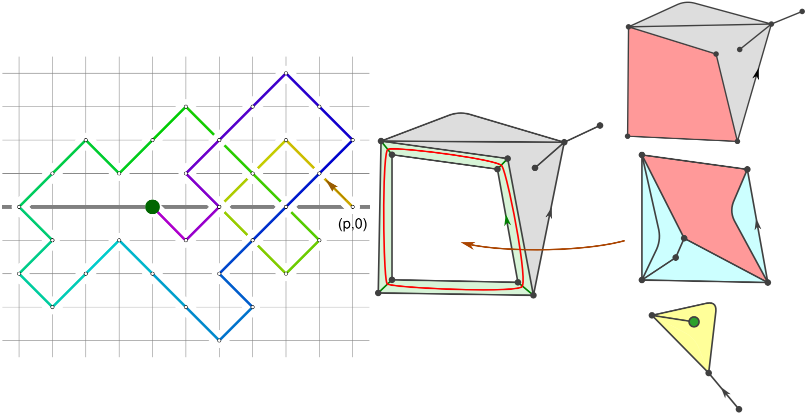


From walks to loop-decorated maps

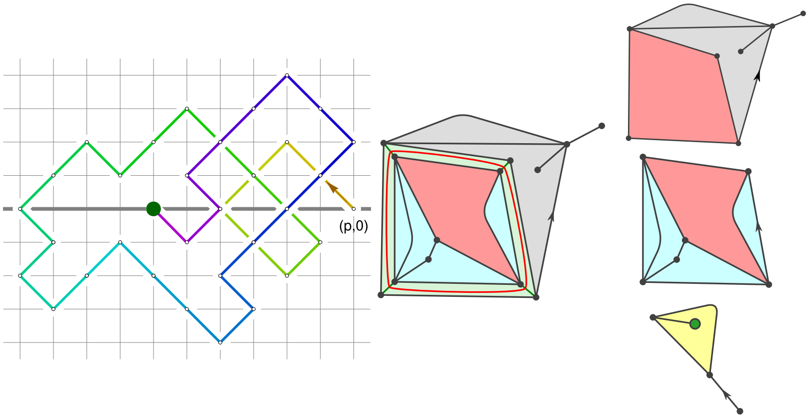




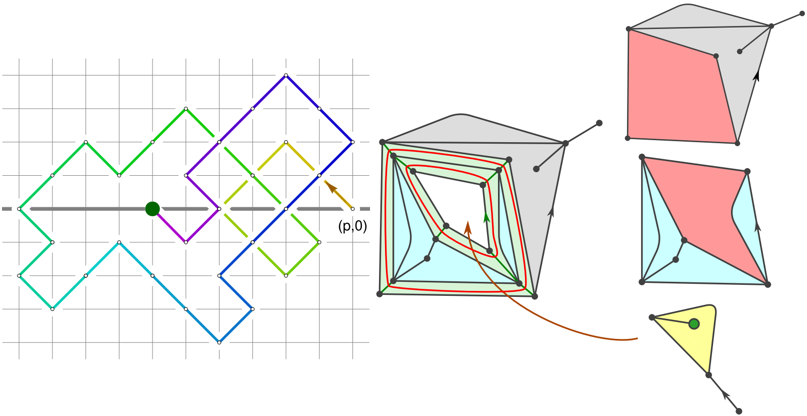
From walks to loop-decorated maps



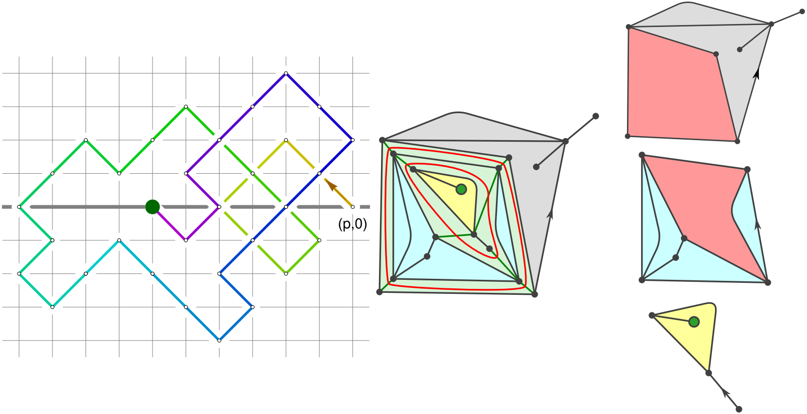
From walks to loop-decorated maps



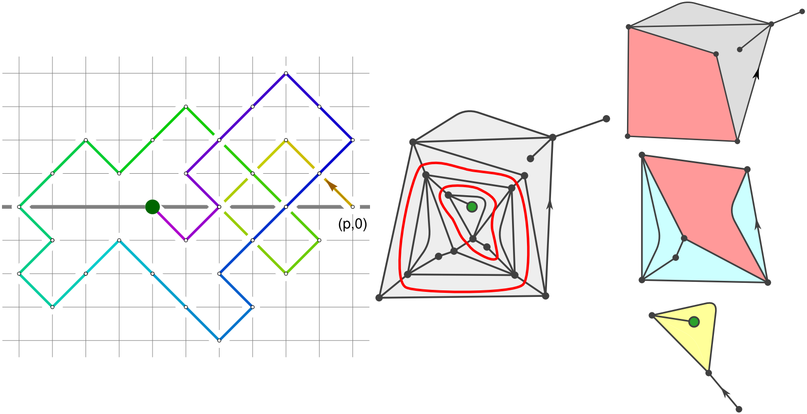
From walks to loop-decorated maps



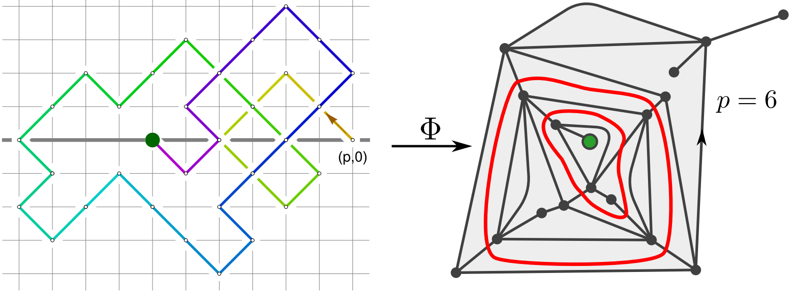
From walks to loop-decorated maps



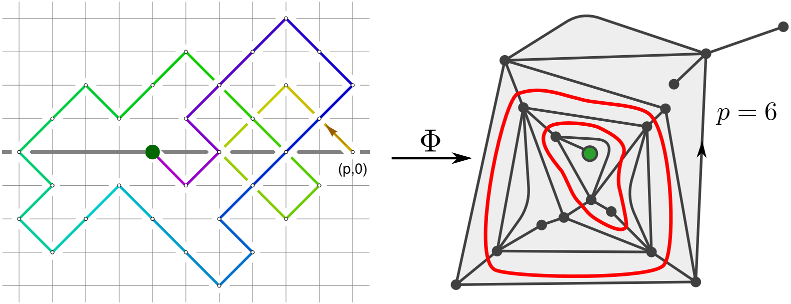
From walks to loop-decorated maps



From walks to loop-decorated maps



From walks to loop-decorated maps

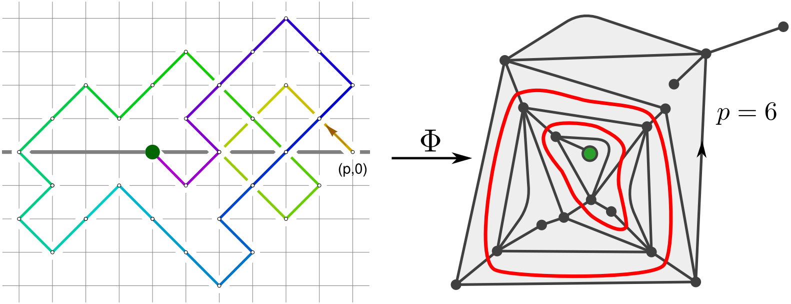


- If (W_i) is a simple diagonal random walk started at $(p, 0)$ and killed at $(0, 0)$,
  then $\Phi((W_i))$ is a rooted planar map with a marked vertex and rigid loops surrounding the marked vertex with probability proportional to

$$2^{\#\text{loops}} g^{\text{total loop length}} \prod_{\text{regular faces}} q_{\text{degree}}$$

for some $g, q_2, q_4, \ldots \in \mathbb{R}_+$.

From walks to loop-decorated maps

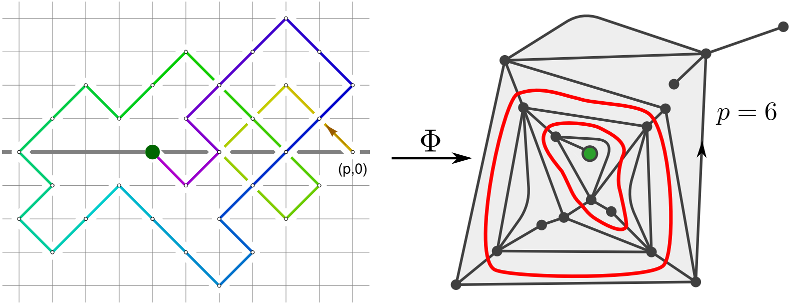


- If (W_i) is a simple diagonal random walk started at $(p, 0)$ and killed at $(0, 0)$,
  then $\Phi((W_i))$ is a rooted planar map with a marked vertex and rigid loops surrounding the marked vertex with probability proportional to

$$2^{\#\text{loops}} g^{\text{total loop length}} \prod_{\text{regular faces}} q_{\text{degree}}$$

  for some $g, q_2, q_4, \ldots \in \mathbb{R}_+$.
- $\#\text{loops} = \#\text{half-axis alternations of } (W_i)$.

From walks to loop-decorated maps



- If (W_i) is a simple diagonal random walk started at $(p, 0)$ and killed at $(0, 0)$, biased by $(n/2)^{\#\text{half-axis alternations}}$,
  then $\Phi((W_i))$ is a rooted planar map with a marked vertex and rigid loops surrounding the marked vertex with probability proportional to

$$n^{\#\text{loops}} g^{\text{total loop length}} \prod_{\text{regular faces}} q_{\text{degree}}$$

  for some $g, q_2, q_4, \ldots \in \mathbb{R}_+$.
- $\#\text{loops} = \#\text{half-axis alternations of } (W_i)$.

Thanks for you attention!
Comments?