In Search for Quantum Gravity: CDT & Friends,
Dec. 11-14, 2012, Nijmegen

CDT & Trees

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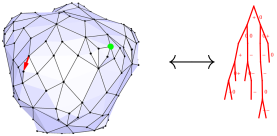
budd@nbi.dk

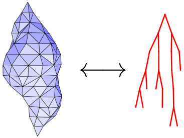
Collaboration with J. Ambjørn

Outline

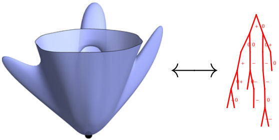
- Quadrangulations & Trees:



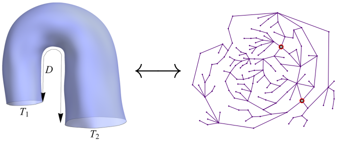
- CDT & Trees:



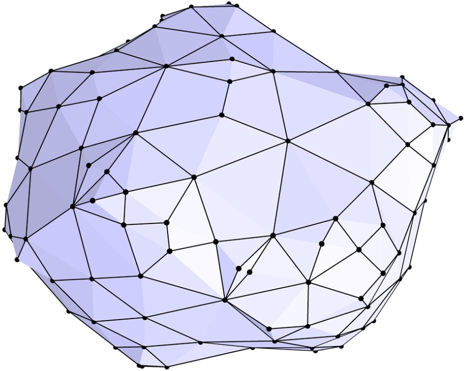
- Generalized CDT & Trees:



- Loop amplitudes & Planar maps:



The Cori–Vauquelin–Schaeffer bijection

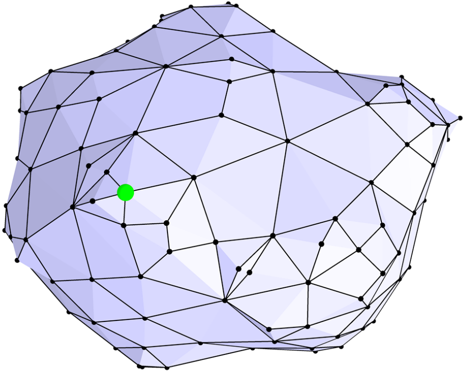


[Cori, Vauquelin, '81]

[Schaeffer, '98]

- Quadrangulation

The Cori–Vauquelin–Schaeffer bijection

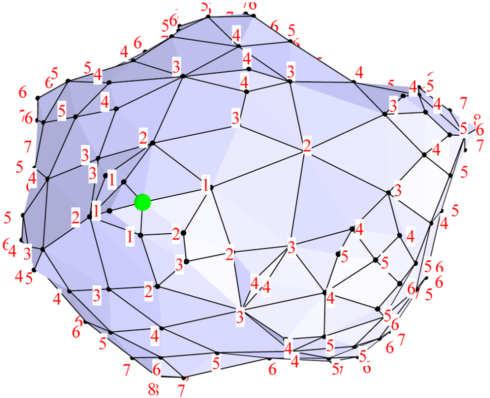


[Cori, Vauquelin, '81]

[Schaeffer, '98]

- Quadrangulation $\rightarrow$ mark a point

The Cori–Vauquelin–Schaeffer bijection



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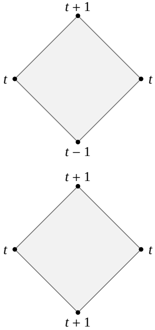
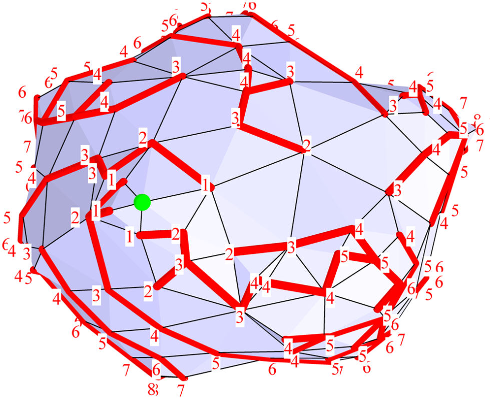


- Quadrangulation $\rightarrow$ mark a point $\rightarrow$ distance labeling

The Cori–Vauquelin–Schaeffer bijection



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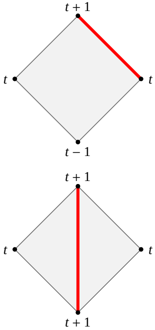
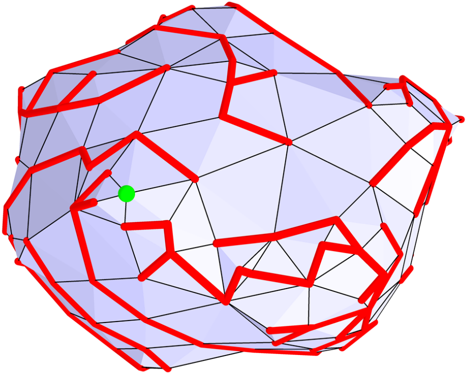


- Quadrangulation → mark a point → distance labeling → apply rules

The Cori–Vauquelin–Schaeffer bijection



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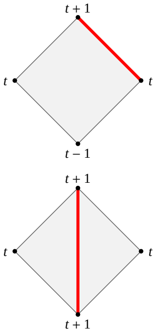

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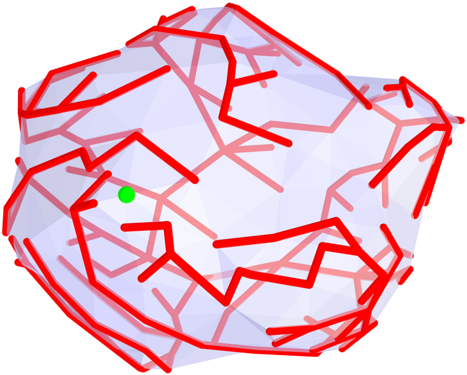

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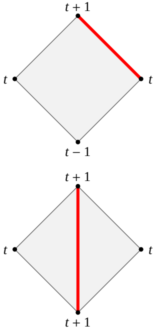

- Quadrangulation $\to$ mark a point $\to$ distance labeling $\to$ apply rules

The Cori–Vauquelin–Schaeffer bijection

[Cori, Vauquelin, '81]

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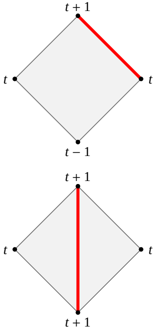
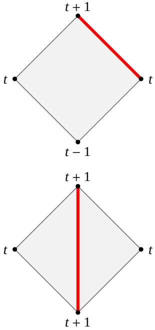


- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.

The Cori–Vauquelin–Schaeffer bijection

[Cori, Vauquelin, '81]

[Schaeffer, '98]



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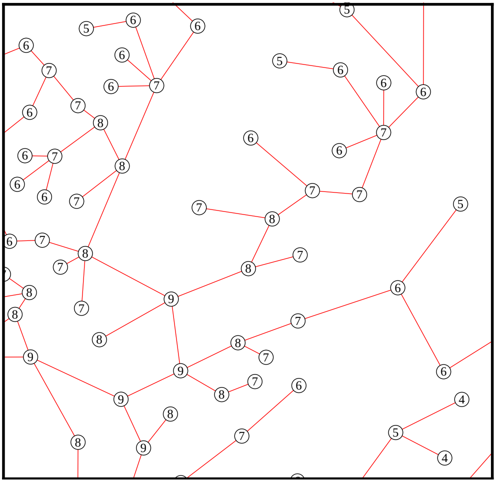

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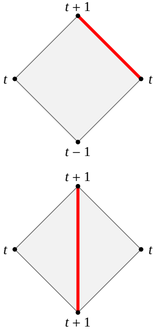
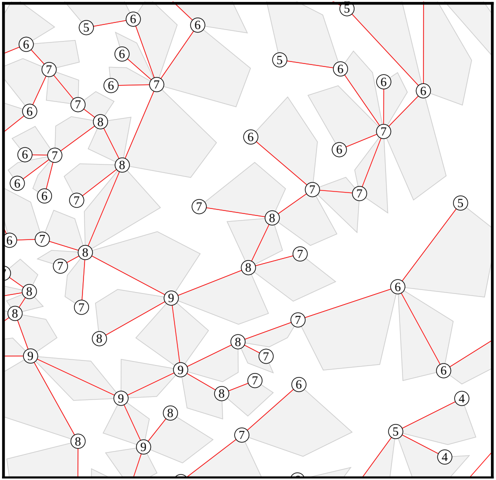


- Quadrangulation $\rightarrow$ mark a point $\rightarrow$ distance labeling $\rightarrow$ apply rules $\rightarrow$ labelled tree.
- Labelled tree

The Cori–Vauquelin–Schaeffer bijection



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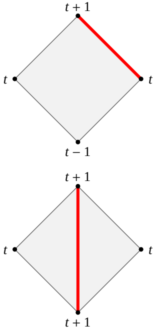
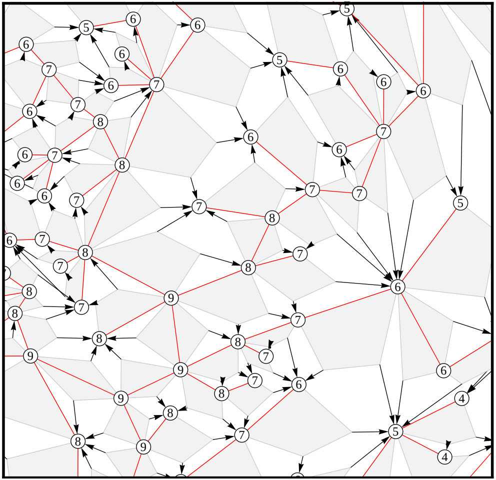


- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.
- Labelled tree → add squares

The Cori–Vauquelin–Schaeffer bijection



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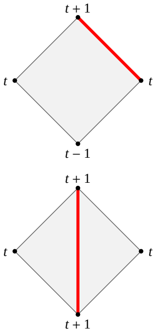
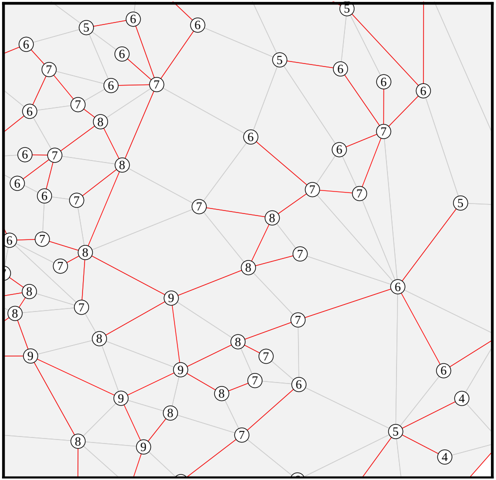


- Quadrangulation $\to$ mark a point $\to$ distance labeling $\to$ apply rules $\to$ labelled tree.
- Labelled tree $\to$ add squares $\to$ identify corners

The Cori–Vauquelin–Schaeffer bijection



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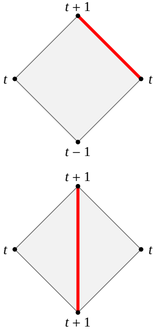

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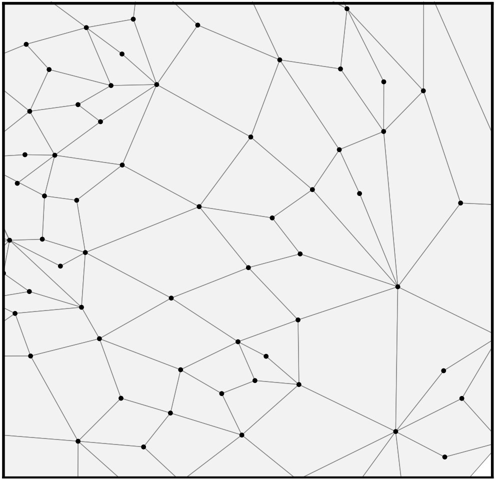
The Cori–Vauquelin–Schaeffer bijection



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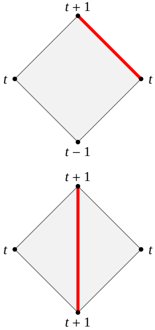
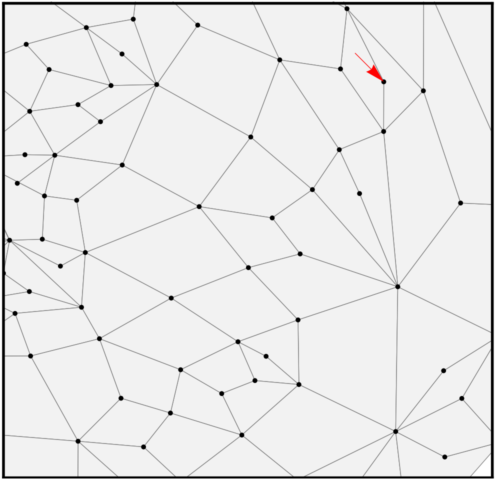
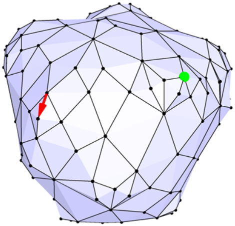


- Quadrangulation $\rightarrow$ mark a point $\rightarrow$ distance labeling $\rightarrow$ apply rules $\rightarrow$ labelled tree.
- Labelled tree $\rightarrow$ add squares $\rightarrow$ identify corners $\rightarrow$ quadrangulation.

Rooting the tree (Miermont, Bouttier, Guitter, Le Gall,...)





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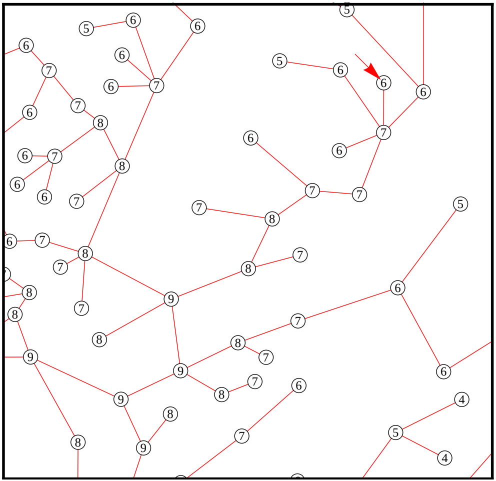
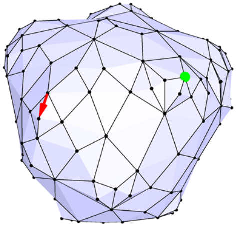
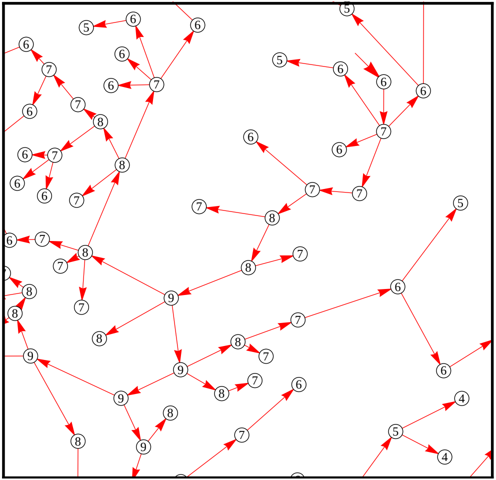
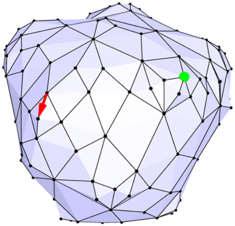




Rooting the tree (Miermont, Bouttier, Guitter, Le Gall,...)





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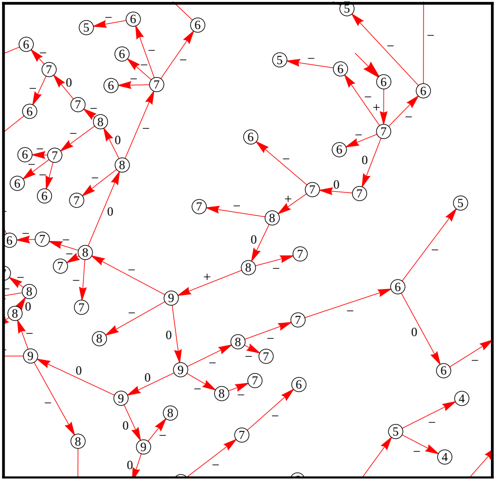

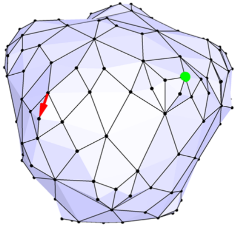

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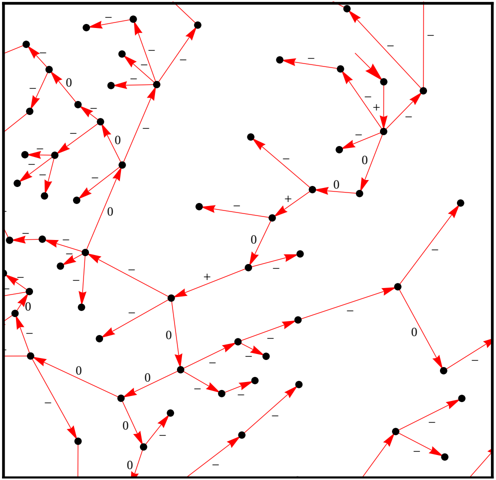
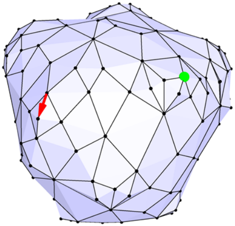
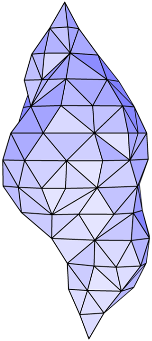




- We will be using the bijection:

$$\left\{ \begin{array}{l} \text{Quadrangulations with origin} \\ \text{and marked edge} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Rooted planar trees} \\ \text{labelled by } +, 0, - \end{array} \right\} \times \mathbb{Z}_2$$

Causal triangulations/quadrangulations

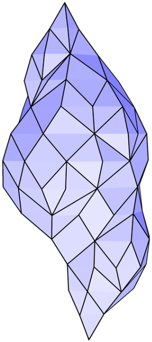


[Krikun, Yambartsev '08]

[Durhuus, Jonsson, Wheater '09]

(Wednesday's talks!)

Causal triangulations/quadrangulations



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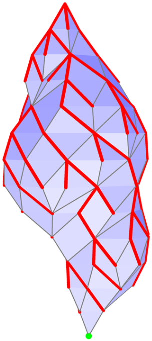
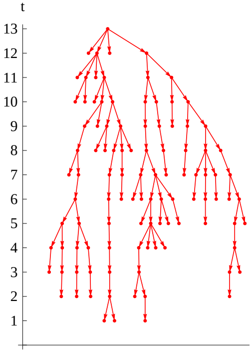
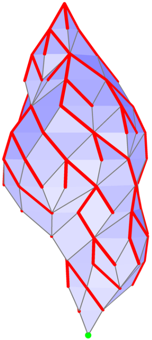


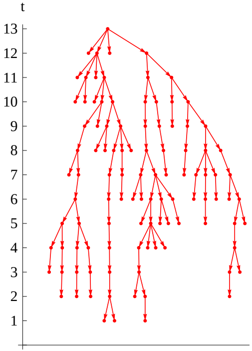


[Krikun, Yambartsev '08]
[Durhuus, Jonsson, Wheater '09]
(Wednesday's talks!)

- If we take the origin at $t = 0$ and the root at final time, all edges of the tree are labeled $-$.

Causal triangulations/quadrangulations





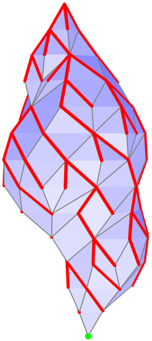
[Krikun, Yambartsev '08]

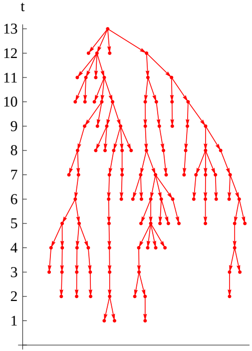
[Durhuus, Jonsson, Wheater '09]

(Wednesday's talks!)

- If we take the origin at $t = 0$ and the root at final time, all edges of the tree are labeled $-$.
- Quadrangulations $\leftrightarrow$ labeled trees. Causal quadr. $\leftrightarrow$ trees.

Causal triangulations/quadrangulations



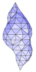
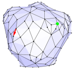


[Krikun, Yambartsev '08]

[Durhuus, Jonsson, Wheater '09]

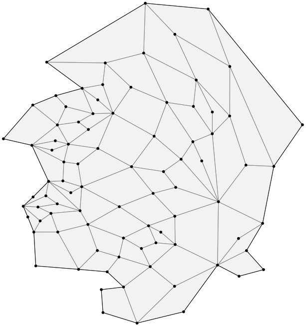
(Wednesday's talks!)

- If we take the origin at $t = 0$ and the root at final time, all edges of the tree are labeled $-$.
- Quadrangulations $\leftrightarrow$ labeled trees. Causal quadr. $\leftrightarrow$ trees.
- As a direct consequence: With N squares we can build

$\#\{$  $\}_N = C(N)$, $\#\{$  $\}_N = 2C(N)3^N$, $C(N) = \frac{1}{N+1}\binom{2N}{N}$

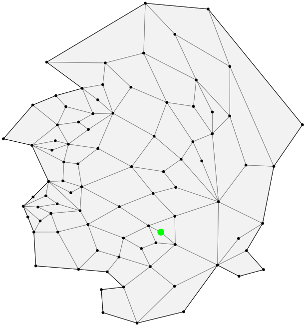
Including boundaries [Bettinelli '11]

- A quadrangulations with boundary length $2l$



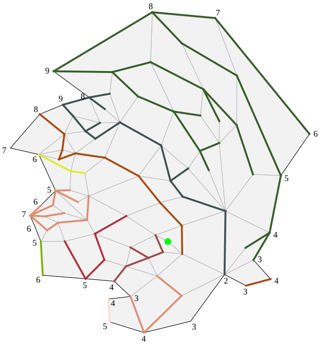
Including boundaries [Bettinelli '11]

- A quadrangulations with boundary length $2l$ and an origin.



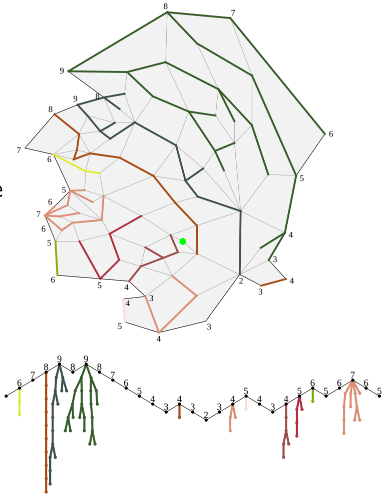
Including boundaries [Bettinelli '11]

- A quadrangulations with boundary length $2l$ and an origin.
- Applying the same prescription we obtain a forest rooted at the boundary.



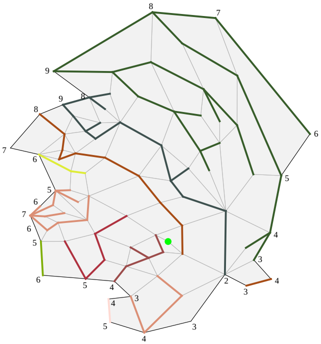
Including boundaries [Bettinelli '11]

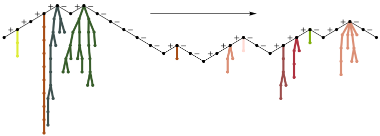
- A quadrangulations with boundary length $2l$ and an origin.
- Applying the same prescription we obtain a forest rooted at the boundary.
- The labels on the boundary arise from a (closed) random walk.



Including boundaries [Bettinelli '11]

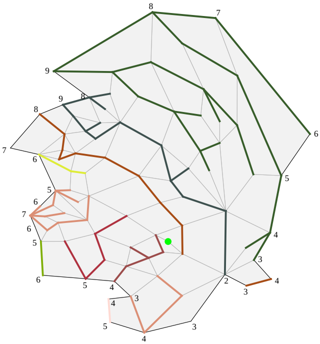
- A quadrangulations with boundary length $2l$ and an origin.
- Applying the same prescription we obtain a forest rooted at the boundary.
- The labels on the boundary arise from a (closed) random walk.
- A (possibly empty) tree grows at the end of every $+$-edge.

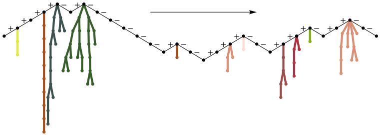




Including boundaries [Bettinelli '11]

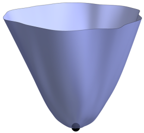
- A quadrangulations with boundary length $2l$ and an origin.
- Applying the same prescription we obtain a forest rooted at the boundary.
- The labels on the boundary arise from a (closed) random walk.
- A (possibly empty) tree grows at the end of every $+$-edge.
- There is a bijection [Bettinelli]

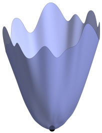




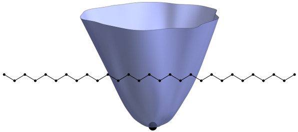
$$\left\{ \begin{array}{l} \text{Quadrangulations with origin} \\ \text{and boundary length } 2l \end{array} \right\} \leftrightarrow \{(+,-)\text{-sequences}\} \times \{\text{tree}\}^l$$

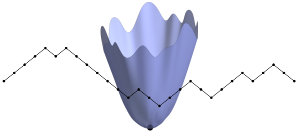
Disk amplitudes





Disk amplitudes





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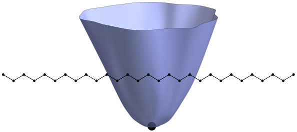
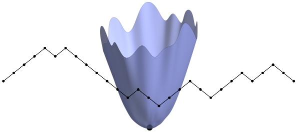


$$w(g,l) = z(g)^l$$

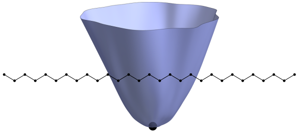
$$w(g,x) = \sum_{l=0}^{\infty} w(g,l)x^l = \frac{1}{1 - z(g)x}$$



$$w(g,l) = \binom{2l}{l} z(g)^l$$

$$w(g,x) = \frac{1}{\sqrt{1 - 4z(g)x}}$$

Disk amplitudes



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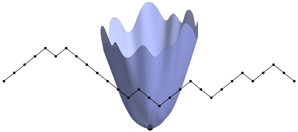

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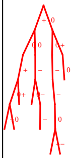
Generating function for unlabeled trees:

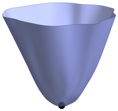
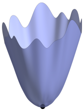
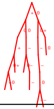
$$z(g) = \frac{1 - \sqrt{1 - 4g}}{2g}$$



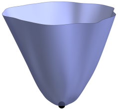
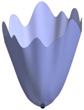
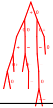
Generating function for labeled trees:

$$z(g) = \frac{1 - \sqrt{1 - 12g}}{6g}$$

Continuum limit

 $w(g,x) = \dfrac{1}{1 - zx}$	 $w(g,x) = \dfrac{1}{\sqrt{1 - 4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

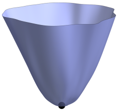
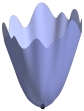
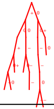
Continuum limit

 $w(g,x) = \dfrac{1}{1 - zx}$	 $w(g,x) = \dfrac{1}{\sqrt{1 - 4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1 - \Lambda\epsilon^2), \quad z(g) = z_c(1 - Z\epsilon), \quad x = x_c(1 - X\epsilon)$$

Continuum limit

 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

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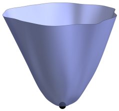
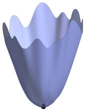
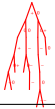
 $w(g,x) = \dfrac{1}{1-zx}$ $W_\Lambda(X) = \dfrac{1}{X+Z}$	 $w(g,x) = \dfrac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \dfrac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

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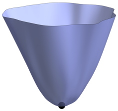
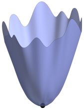
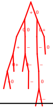
- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$

Continuum limit

 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

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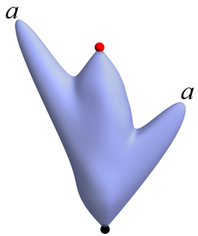
- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$
- DT disk amplitude with marked point: $W'_\Lambda(X) = \frac{1}{\sqrt{X+\sqrt{\Lambda}}}$. Integrate w.r.t. Λ to remove mark: $W_\Lambda(X) = \frac{2}{3}(X - \frac{1}{2}\sqrt{\Lambda})\sqrt{X+\sqrt{\Lambda}}$.

Generalized CDT [Ambjørn, Loll, Westra, Zohren '07]

- Assign coupling a to the local maxima of the distance function.



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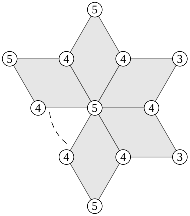

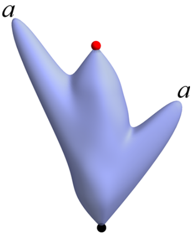

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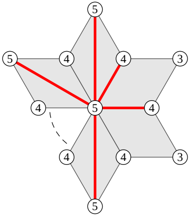

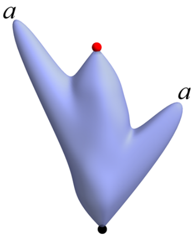

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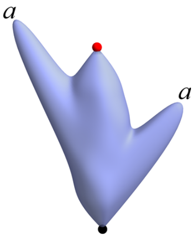

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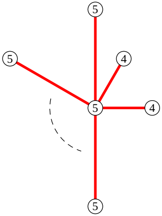

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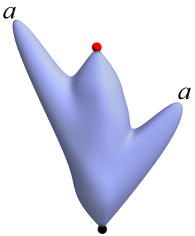

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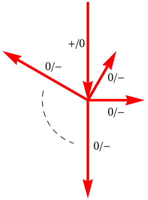

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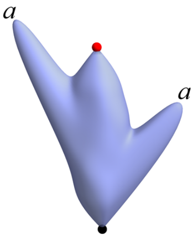

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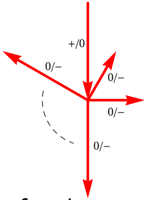
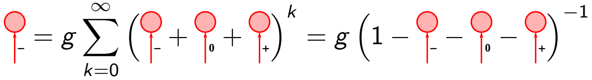


- We can write down equations for the generating functions



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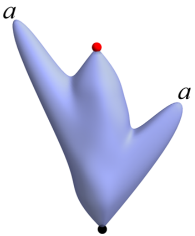

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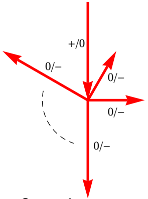

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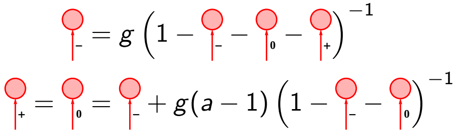
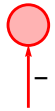
$$\overset{\circ}{\uparrow}_{-} = g\sum_{k=0}^{\infty}\left(\overset{\circ}{\uparrow}_{-}+\overset{\circ}{\uparrow}_{0}+\overset{\circ}{\uparrow}_{+}\right)^{k} = g\left(1-\overset{\circ}{\uparrow}_{-}-\overset{\circ}{\uparrow}_{0}-\overset{\circ}{\uparrow}_{+}\right)^{-1}$$

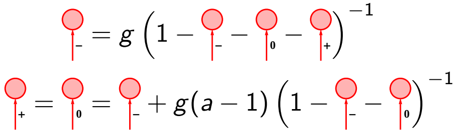
$$\overset{\circ}{\uparrow}_{+} = \overset{\circ}{\uparrow}_{0} = g\left[\sum_{k=0}^{\infty}\left(\overset{\circ}{\uparrow}_{-}+\overset{\circ}{\uparrow}_{0}+\overset{\circ}{\uparrow}_{+}\right)^{k} + (a-1)\sum_{k=0}^{\infty}\left(\overset{\circ}{\uparrow}_{-}+\overset{\circ}{\uparrow}_{0}\right)^{k}\right]$$

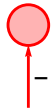
$$= \overset{\circ}{\uparrow}_{-} + g(a-1)\left(1-\overset{\circ}{\uparrow}_{-}-\overset{\circ}{\uparrow}_{0}\right)^{-1}$$



- Combine into one equation for $z_-(g,a) =$ :

$$3z_-^4 - 4z_-^3 + (1 + 2g(1-2a))z_-^2 - g^2 = 0$$



- Combine into one equation for $z_-(g,a) =$ :

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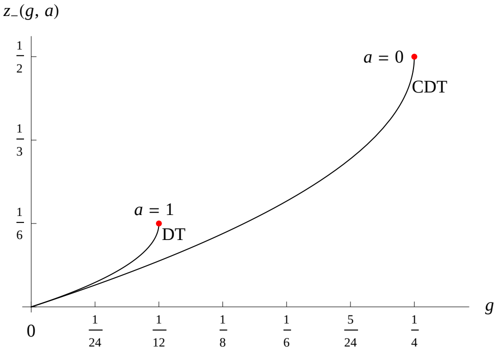
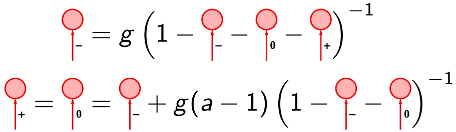
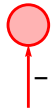




- Combine into one equation for $z_-(g, a) =$ :

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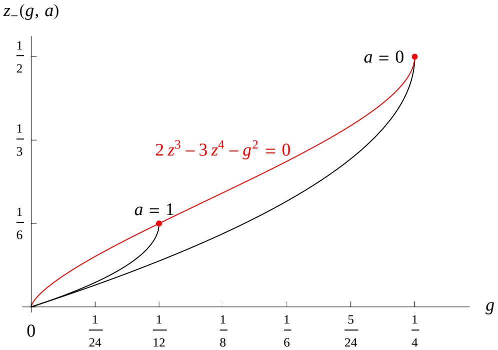

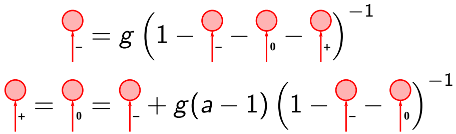

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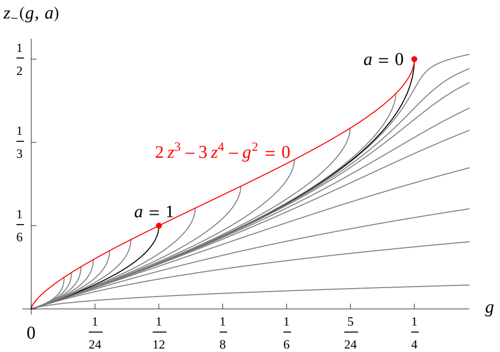
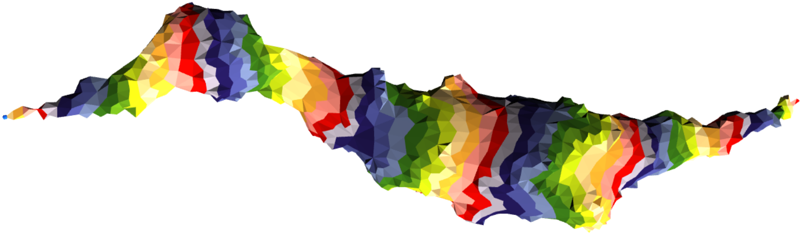
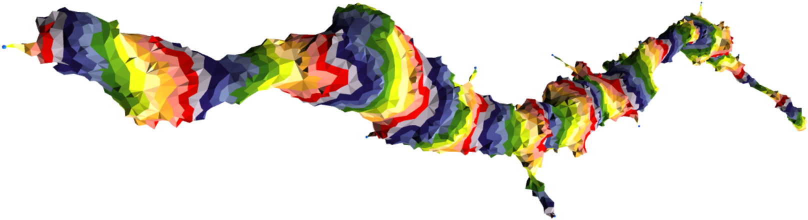


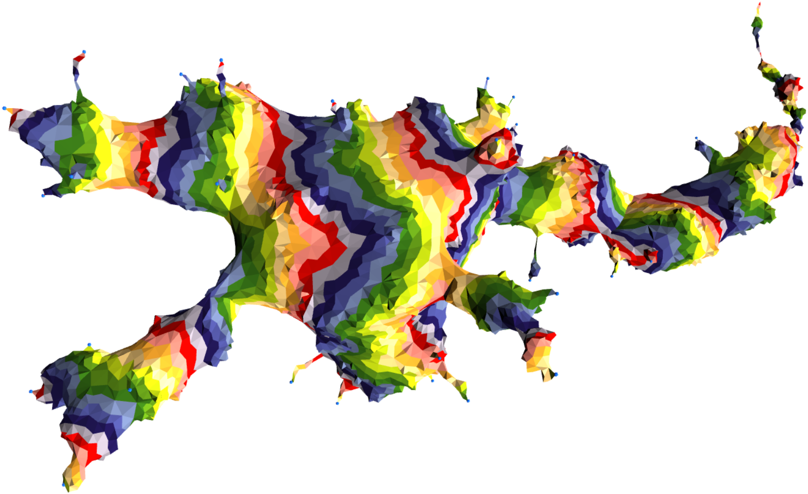
$N = 2000,\ a = 0,\ N_{max} = 0$



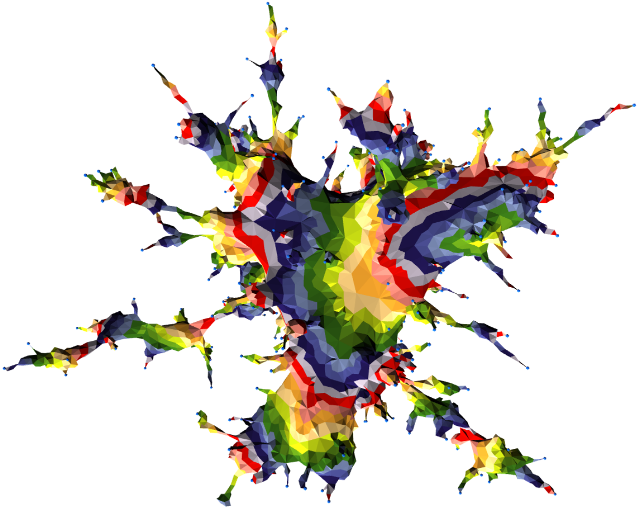
$N = 5000,\ a = 0.00007,\ N_{max} = 11$



$$N = 7000, \; a = 0.0002, \; N_{max} = 37$$

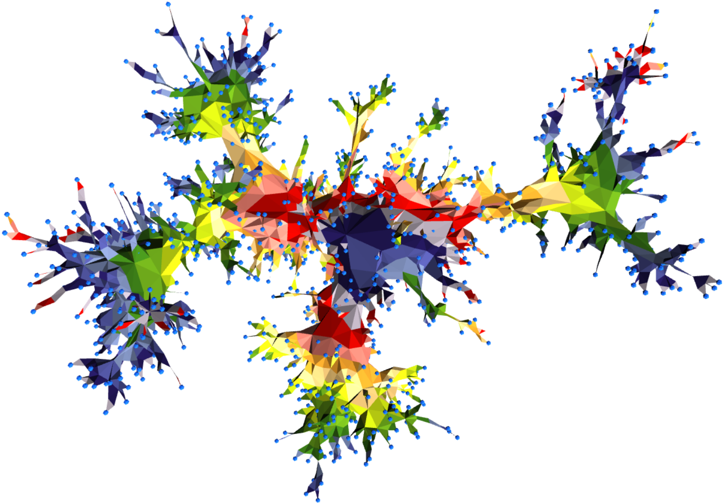


$N = 7000,\ a = 0.004,\ N_{max} = 221$



$N = 4000,\ a = 0.02,\ N_{max} = 362$

$N = 2500,\ a = 1,\ N_{max} = 1216$



- The number of local maxima $N_{max}(a)$ scales with N at the critical point,

$$\frac{\langle N_{max}(a)\rangle_N}{N} = 2\left(\frac{a}{2}\right)^{2/3} + \mathcal{O}(a), \quad \frac{\langle N_{max}(a=1)\rangle_N}{N} = 1/2$$

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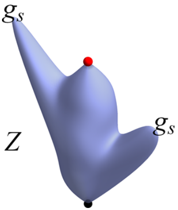


- Continuum limit $g = g_c(a)(1-\Lambda\epsilon^2)$, $z_- = z_{-,c}(1-Z\epsilon)$, $a = g_s\epsilon^3$:

$$Z^3 - \left(\Lambda + 3\left(\frac{g_s}{2}\right)^{2/3}\right) Z - g_s = 0$$

- The number of local maxima $N_{max}(a)$ scales with N at the critical point,

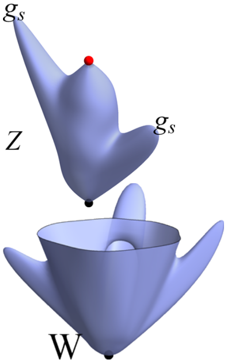
$$\frac{\langle N_{max}(a)\rangle_N}{N} = 2\left(\frac{a}{2}\right)^{2/3} + \mathcal{O}(a), \qquad \frac{\langle N_{max}(a=1)\rangle_N}{N} = 1/2$$

- Therefore, to obtain a finite continuum density of critical points one should scale $a \propto N^{-3/2}$, i.e. $a = g_s\epsilon^3$ as observed in [ALWZ '07].
- This is the only scaling leading to a continuum limit qualitatively different from DT and CDT.

- Continuum limit $g = g_c(a)(1-\Lambda\epsilon^2)$, $z_- = z_{-,c}(1-Z\epsilon)$, $a = g_s\epsilon^3$:

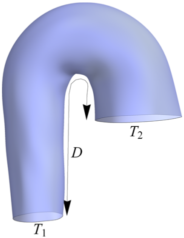
$$Z^3 - \left(\Lambda + 3\left(\frac{g_s}{2}\right)^{2/3}\right) Z - g_s = 0$$

- "Cup function" $W_\Lambda(X) = \frac{1}{X+Z}$. Agrees with [ALWZ '07].



Two loop identity in generalized CDT

- Consider surfaces with two boundaries separated by a geodesic distance D.



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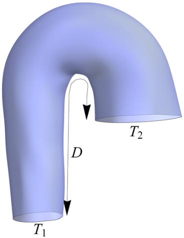
- Consider surfaces with two boundaries separated by a geodesic distance D.
- One can assign time T_1, T_2 to the boundaries ($|T_1 - T_2| \leq D$) and study a "merging" process.



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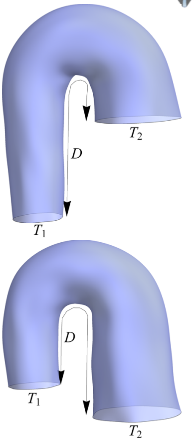

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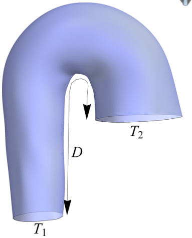

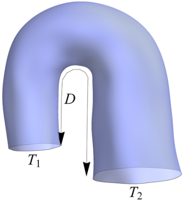

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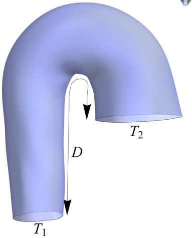

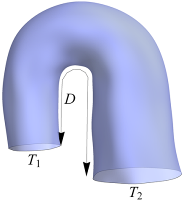

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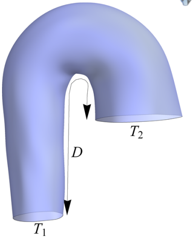

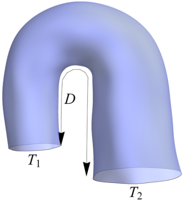

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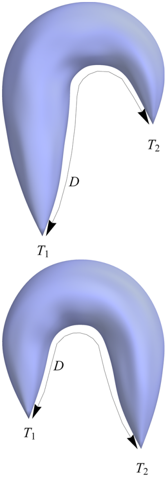
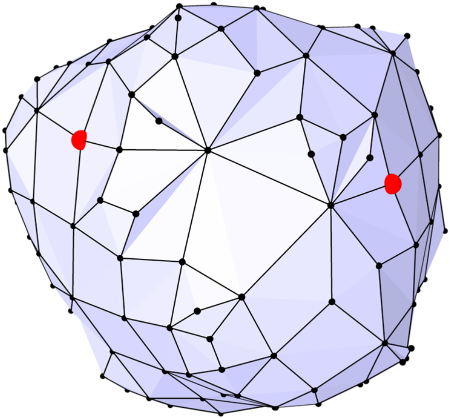
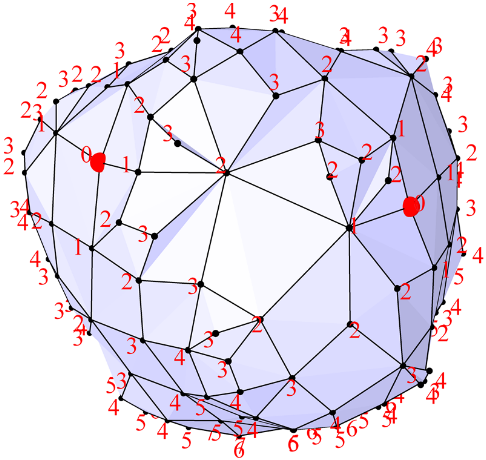
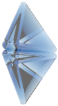
- Can we better understand this symmetry at the discrete level?
- For simplicity set the boundarylengths to zero. Straightforward generalization to finite boundaries.





$$T_1 = 0$$
$$T_2 = 0$$
$$D = 4$$





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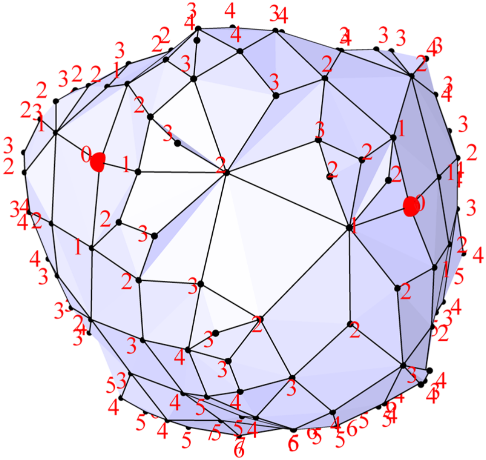

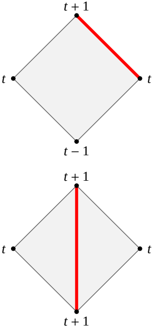

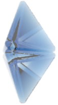

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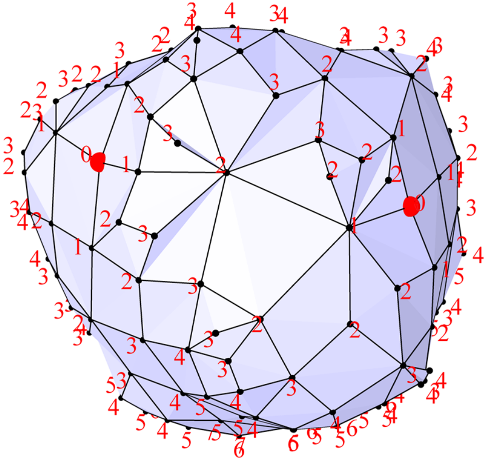

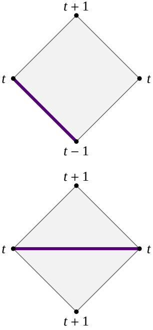

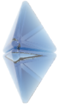

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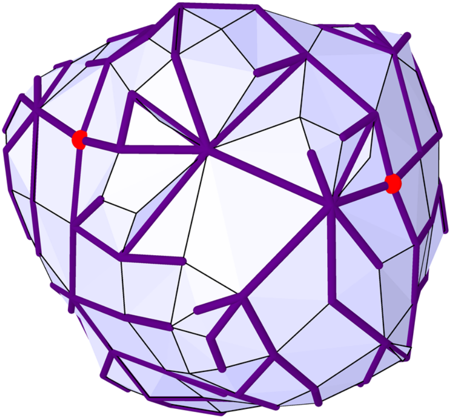

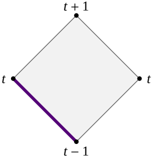

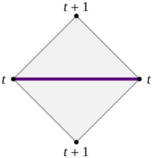

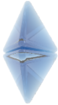

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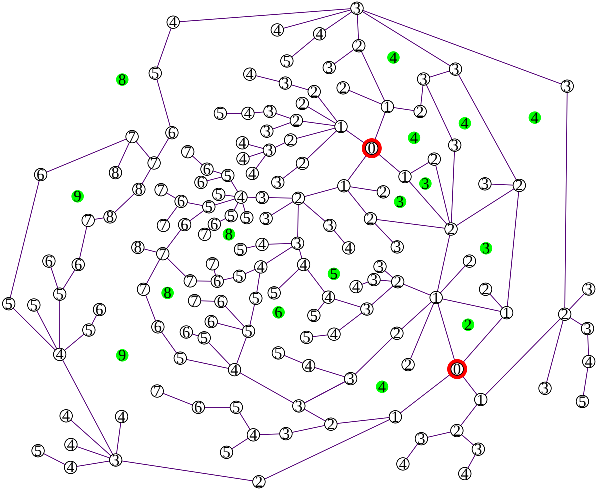

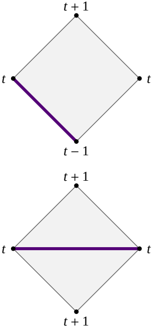

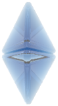

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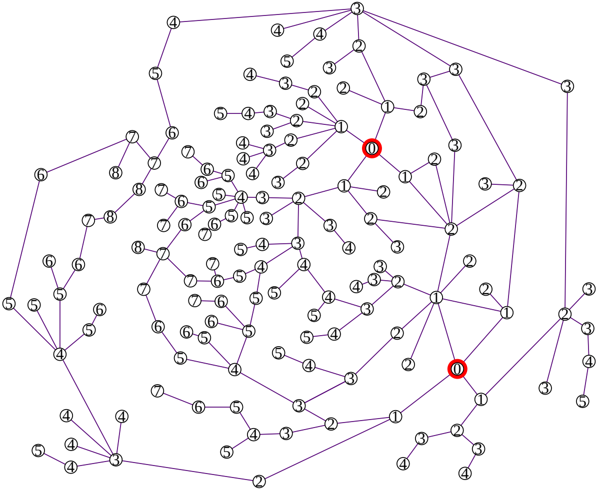

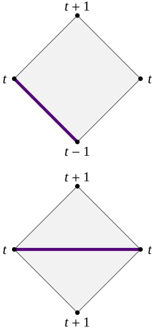

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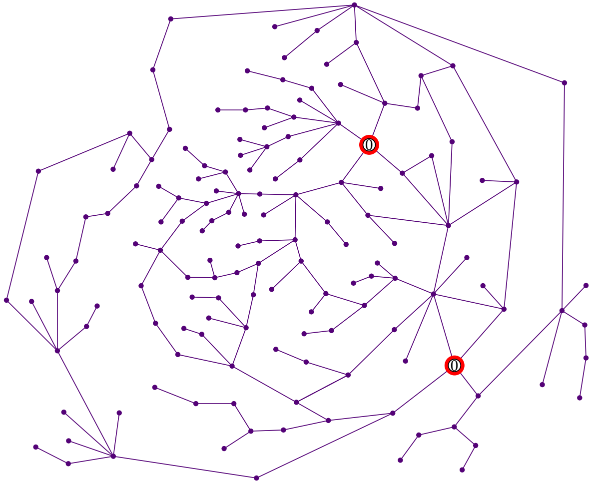
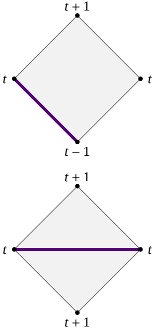
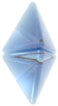
- No local minima $\Rightarrow$ the labeling is canonical!







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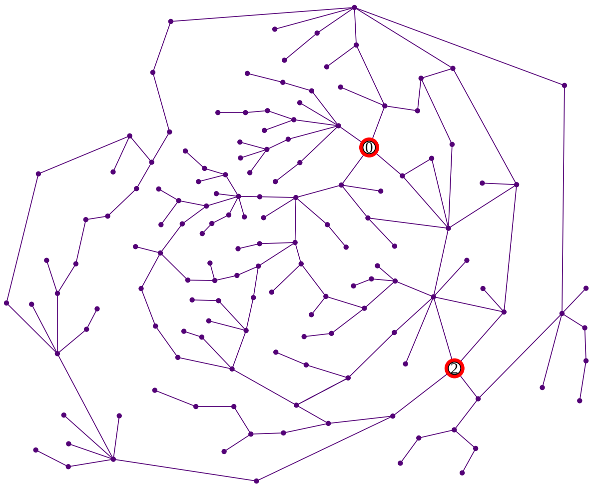

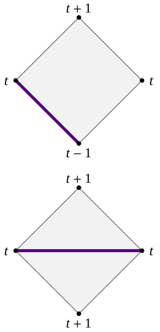

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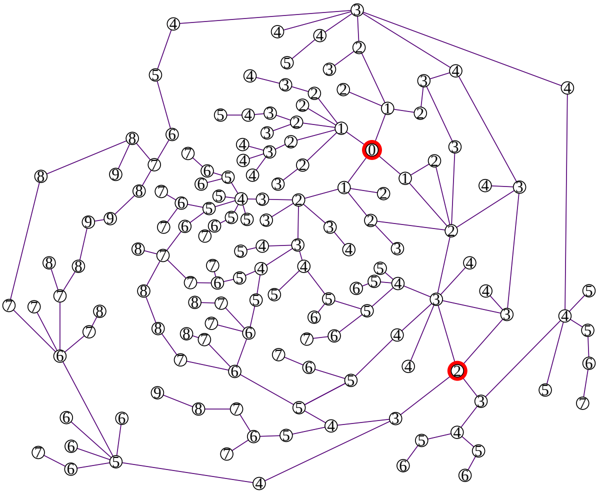

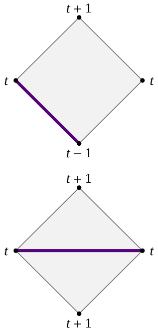

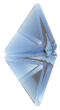

$T_1 = 0$
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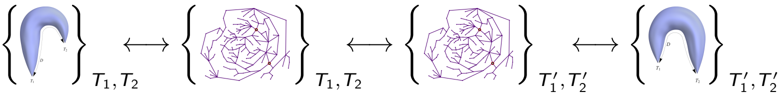
- No local minima $\Rightarrow$ the labeling is canonical!
- There exists a bijection preserving the number of local maxima:



Conclusions & Outlook

- Conlusions
  - The Cori–Vauquelin–Schaeffer bijection encodes 2d geometry in trees, which are simple objects from an analytical point of view.
  - The bijection is ideal for studying “proper-time foliations” of random surfaces.
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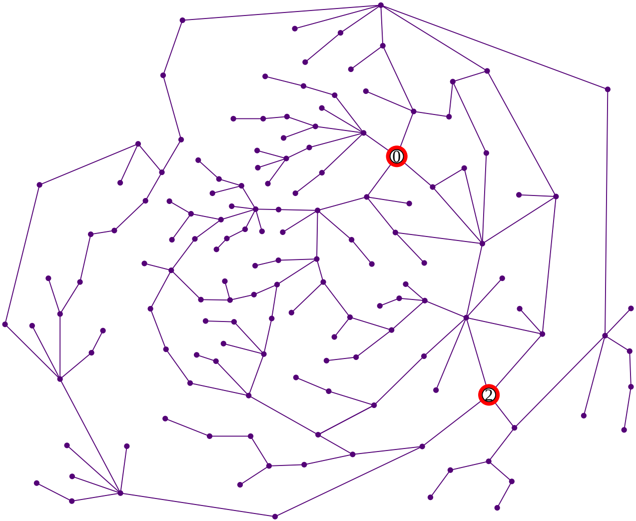
- Outlook
  - What is the exact structure of these symmetries? Related to conformal symmetry (Virasoro algebra)?
  - Straightforward extension to higher genus. Sum over topologies in non-critical string theory?

Merry Christmas!

Appendix: canonical labeling



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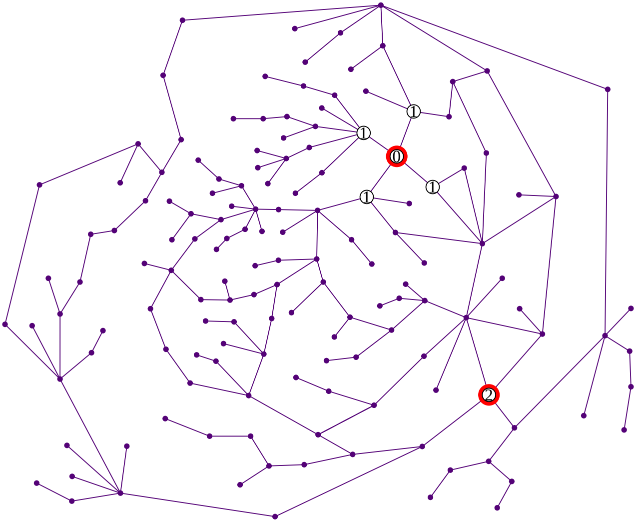

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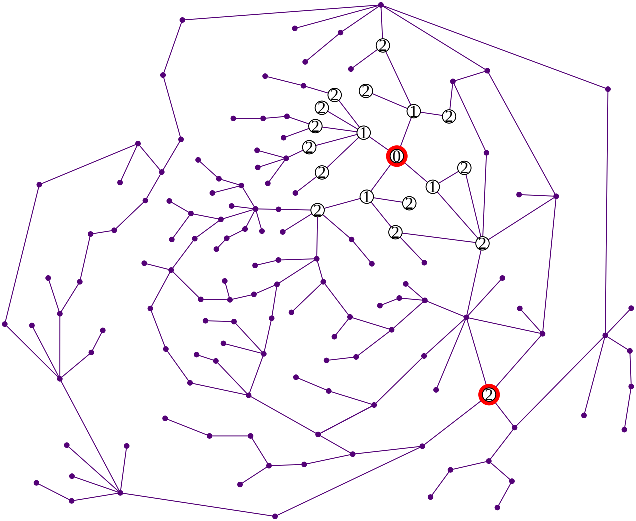

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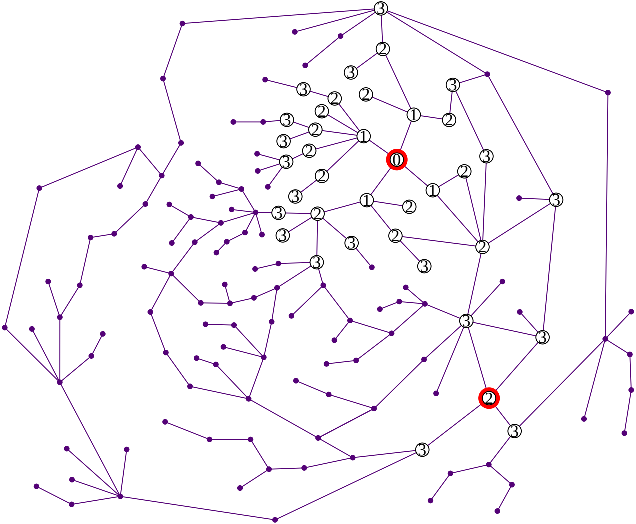

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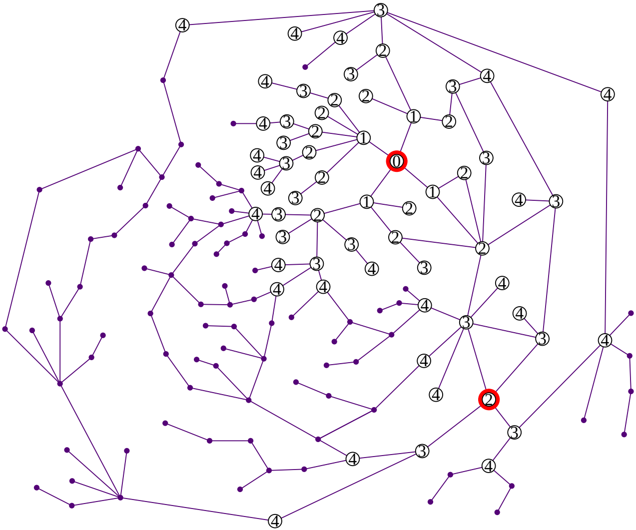

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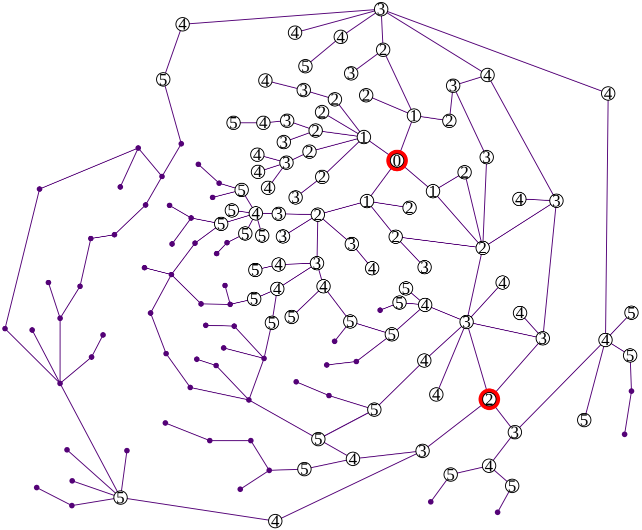

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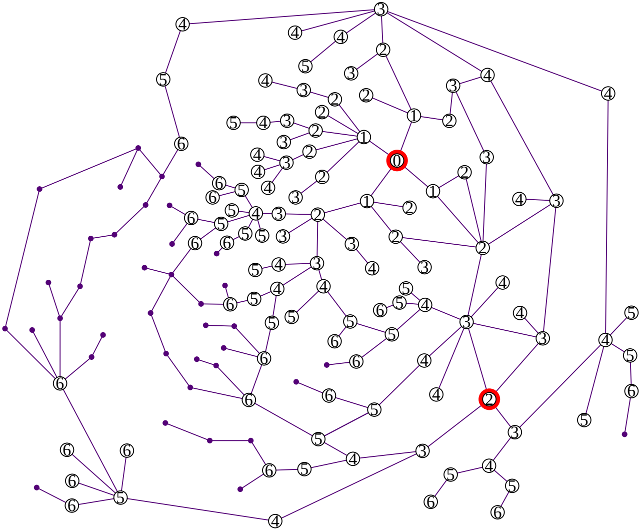

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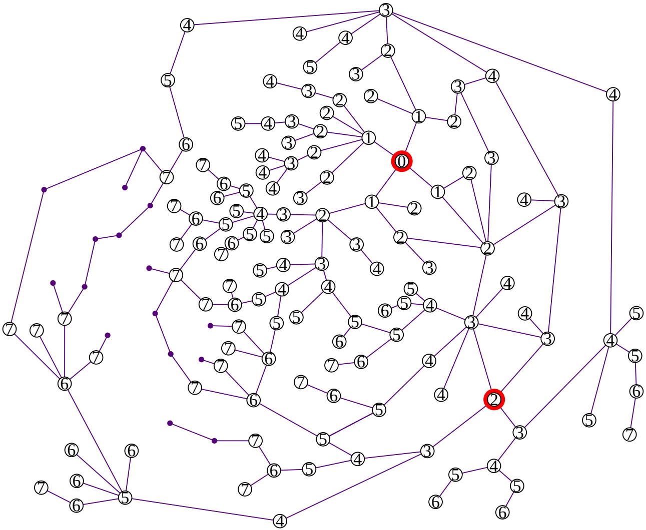

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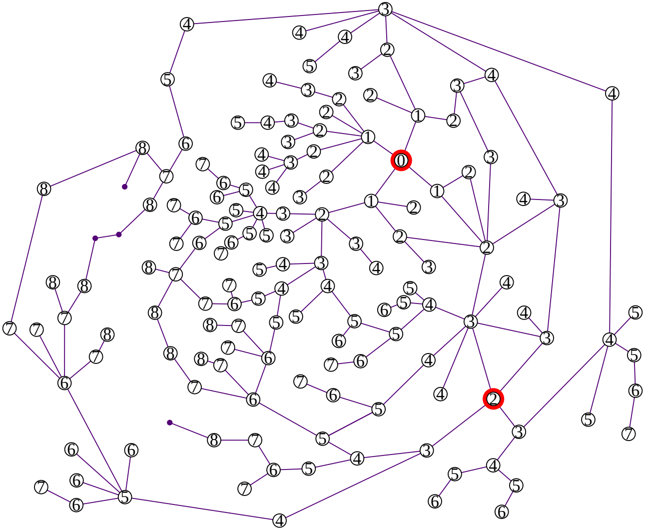

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