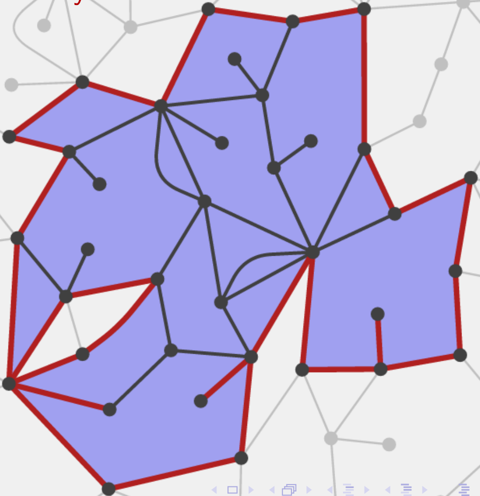
Random Planar Structures and Statistical Mechanics, Isaac Newton Institute, Cambridge, 20-04-2015

Scaling constants and the lazy peeling of infinite Boltzmann planar maps

Timothy Budd



Niels Bohr Institute
University of Copenhagen
`budd@nbi.dk`
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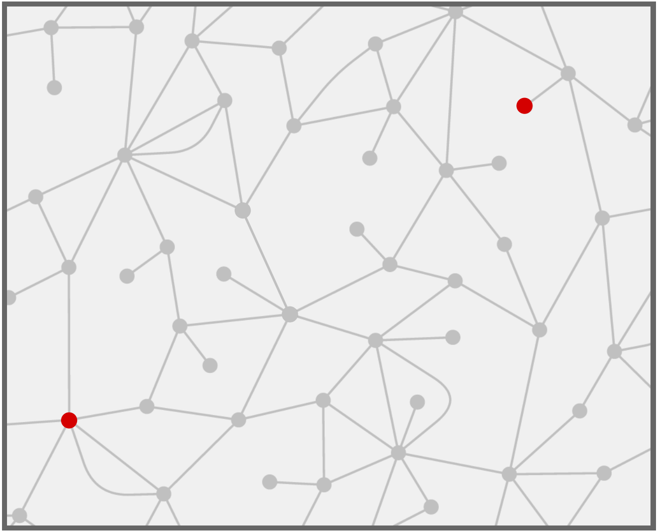
Distances on a planar map



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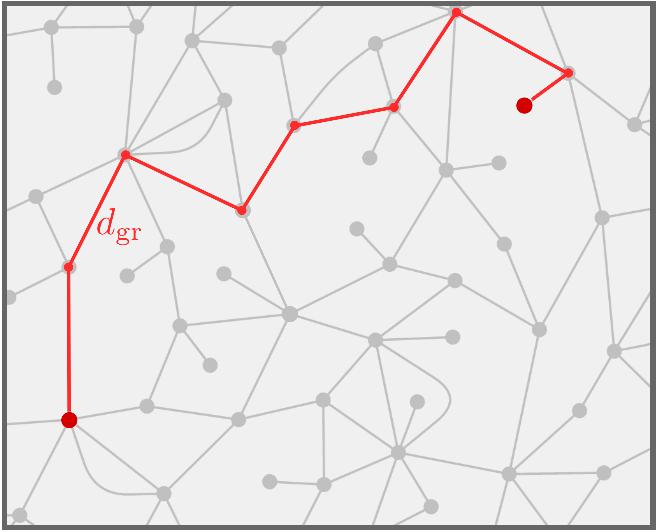

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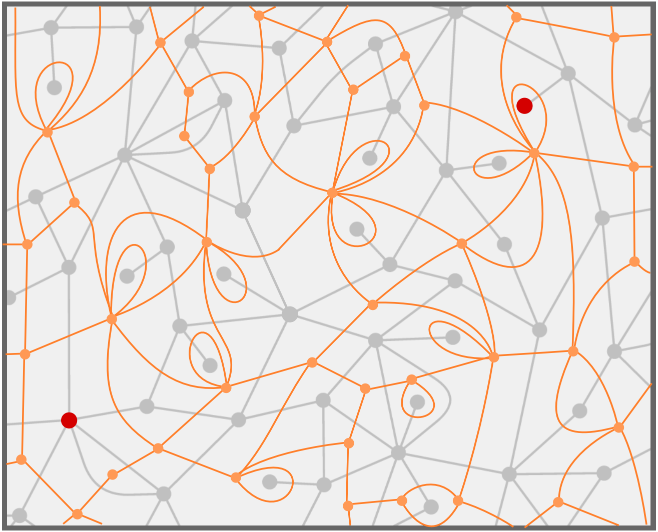

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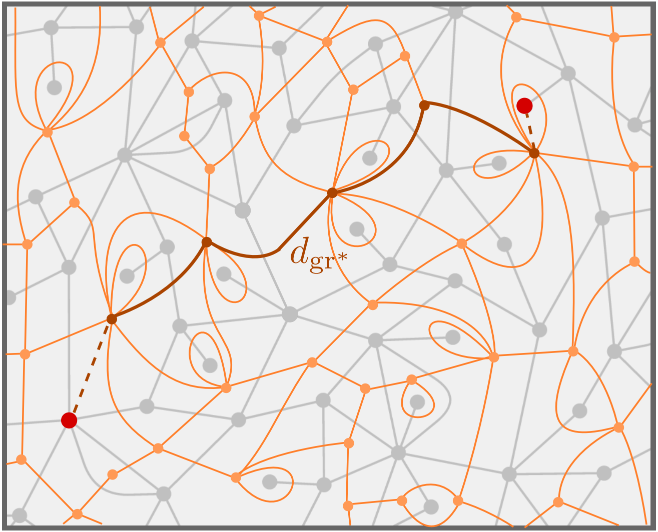

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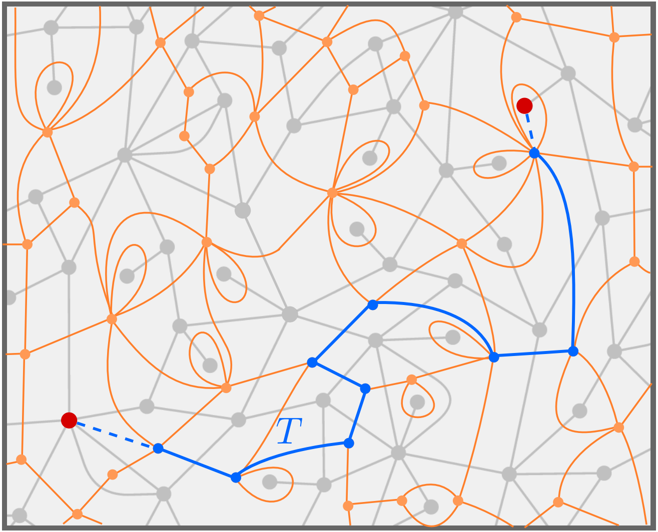

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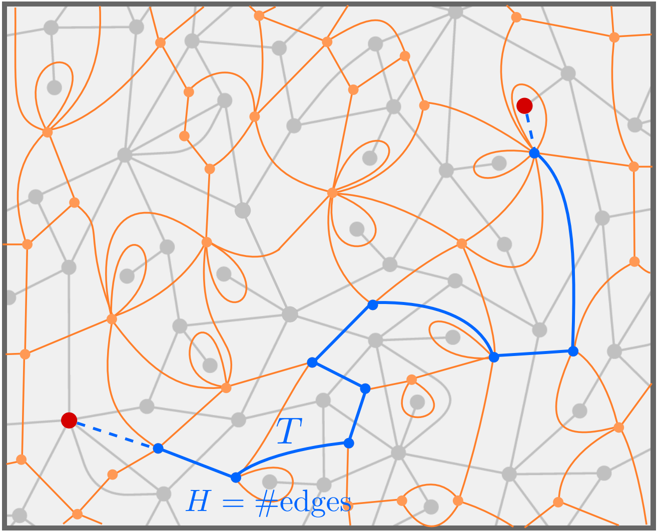

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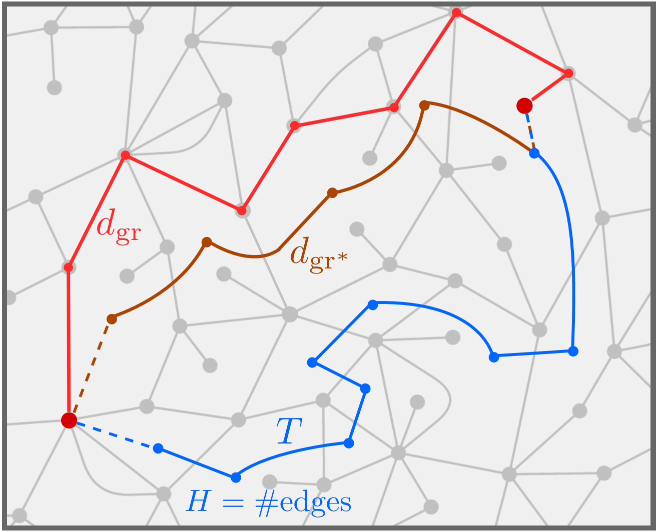

Outline

- Introduce lazy peeling of planar maps
- Description of the associated perimeter and volume processes
- Scaling limit
- Scaling constants from peeling:
  - First-passage time
  - Hop count
  - Dual graph distance
- Miermont's scaling constant for the graph distance
- Example: uniform infinite planar map.
- From lazy to simple peeling.
- Open questions.

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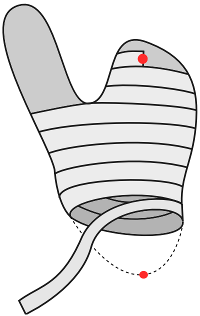

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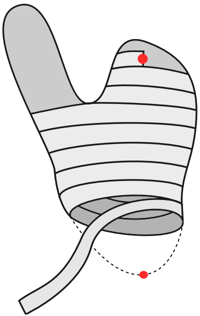

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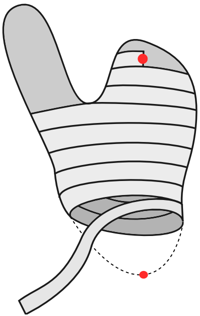

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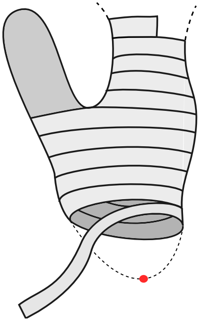

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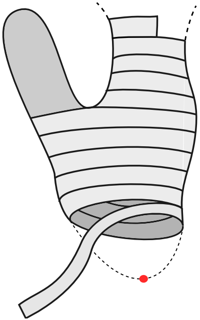

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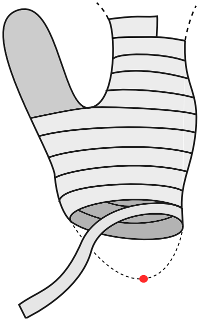
- Precise scaling limits have been obtained for the perimeter and volume of the explored region in the UIPT and UIPQ [Curien, Le Gall, '14], Le Gall's talk!



Lazy peeling of a pointed planar map

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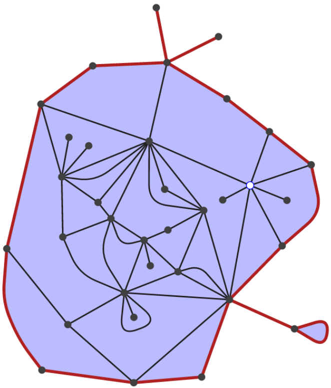

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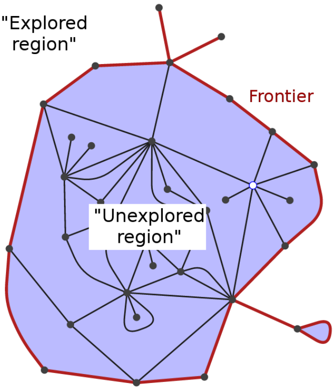

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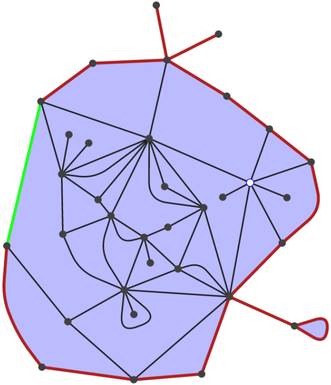

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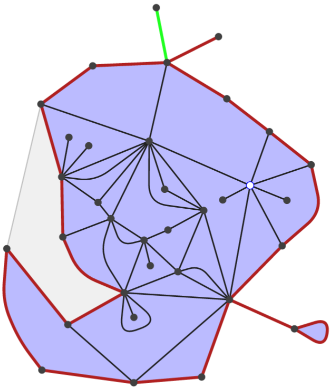

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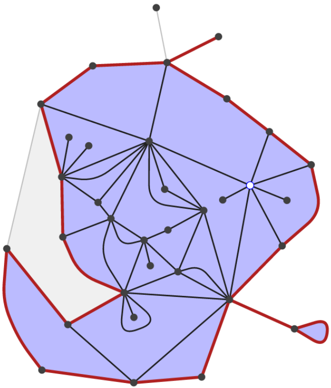

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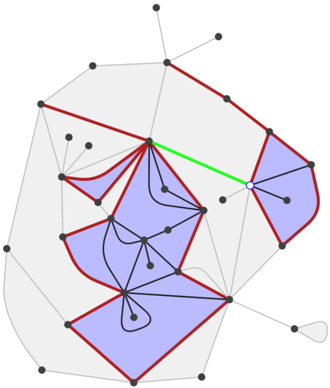

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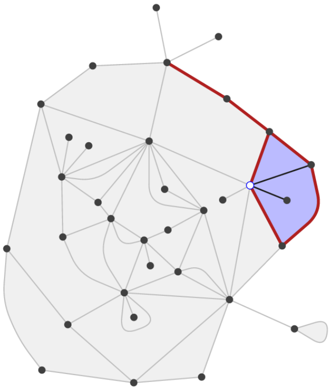

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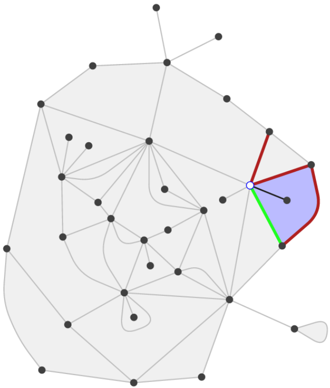

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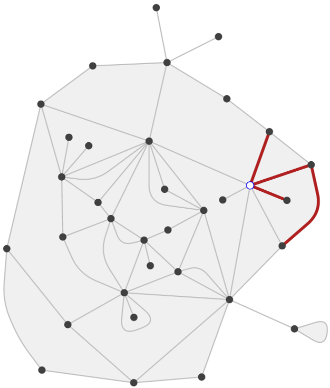

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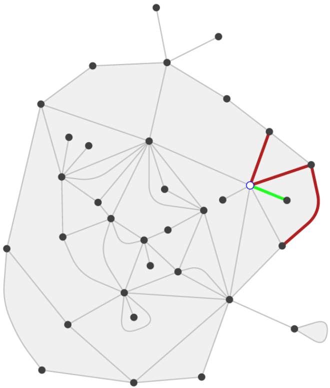

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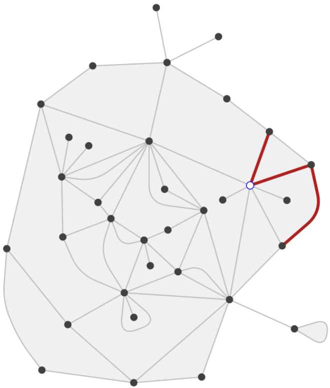

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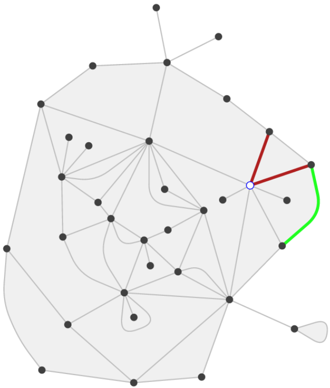

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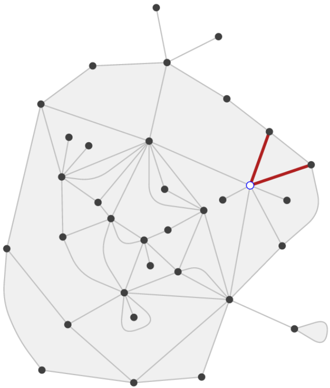

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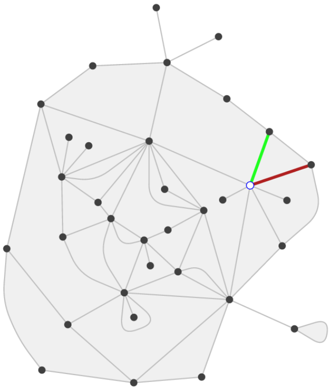

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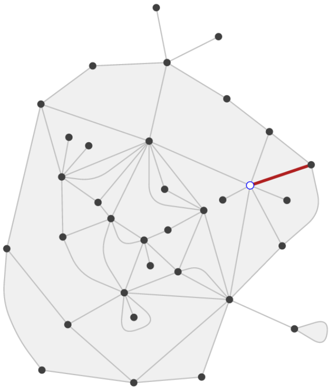

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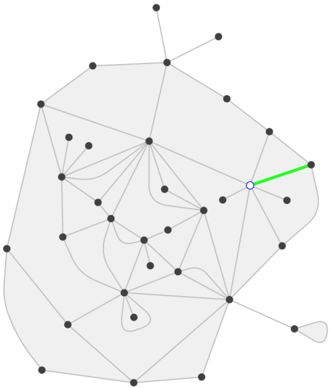

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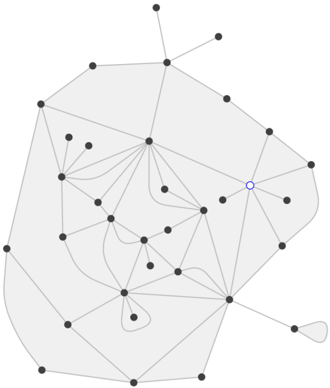

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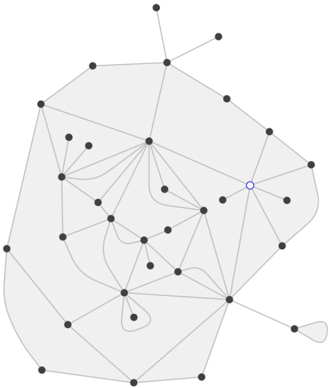
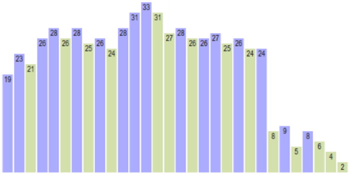
- First goal: given a random disk, what is the law of the perimeter $(l_i)_{i\geq 0}$, i.e. the length of the frontier after i steps?





Boltzmann planar maps [Miermont, '06]

- Let $\mathbf{q} = (q_k)_{k=1}^{\infty}$ be a *weight sequence* of non-negative reals, such that $q_k > 0$ for at least one $k \geq 3$.
- Call $\mathbf{q}$ *bipartite* if $q_k = 0$ for all odd k, and *non-bipartite* otherwise. For now assume $\mathbf{q}$ non-bipartite, but all I am going to say is also true in bipartite case.

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$$W^{(l)} = W^{(l)}(\mathbf{q}) := \sum_{m \in \mathcal{M}^{(l)}} \prod_{\text{non-root faces } f} q_{\deg(f)}, \tag{1}$$

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Theorem (Miermont, '06)

$$f^\bullet(x,y) := \sum_{k,k'=0}^{\infty} x^k y^{k'} \binom{2k+k'+1}{k+1}\binom{k+k'}{k} q_{2+2k+k'}, \quad f^\diamond(x,y) := \sum_{k,k'=0}^{\infty} x^k y^{k'} \binom{2k+k'}{k}\binom{k+k'}{k} q_{1+2k+k'}.$$

The sequence $\mathbf{q}$ is admissible if and only if there exist $z^+, z^\diamond > 0$ such that $f^\bullet(z^+,z^\diamond) = 1 - \frac{1}{z^+}$, $\quad f^\diamond(z^+,z^\diamond) = z^\diamond$

and the matrix $\quad \mathfrak{M}_{\mathbf{q}}(z^+,z^\diamond) := \begin{pmatrix} 0 & 0 & z^+ - 1 \\ \frac{z^+}{z^\diamond}\partial_x f^\diamond(z^+,z^\diamond) & \partial_y f^\diamond(z^+,z^\diamond) & 0 \\ \frac{(z^+)^2}{z^+-1}\partial_x f^\bullet(z^+,z^\diamond) & \frac{z^+ z^\diamond}{z^+-1}\partial_y f^\bullet(z^+,z^\diamond) & 0 \end{pmatrix}$ *has spectral radius ≤ 1.*

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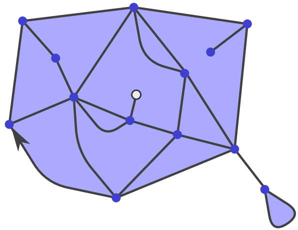
- Proof based on the Bouttier-Di Francesco-Guitter bijection between pointed planar maps and *labeled mobiles*.



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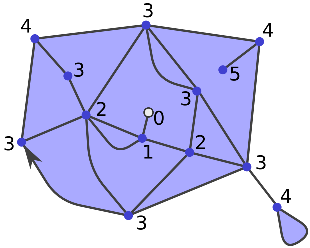

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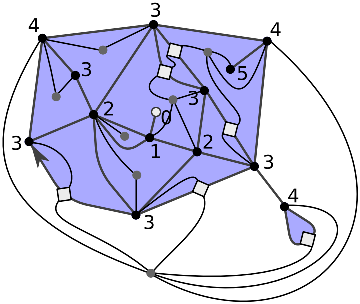

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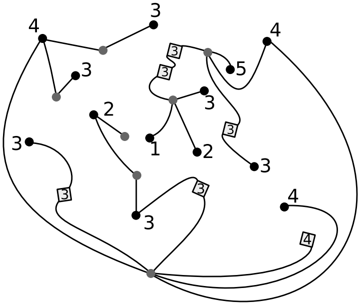

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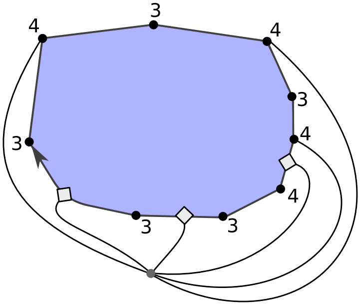

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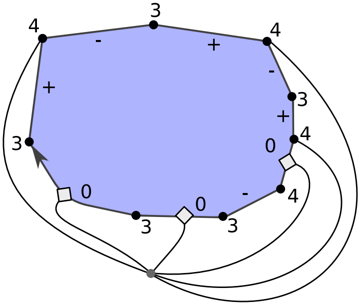

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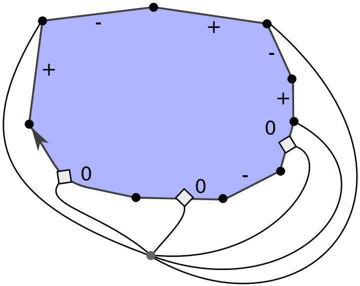

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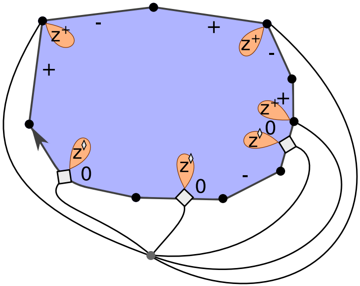

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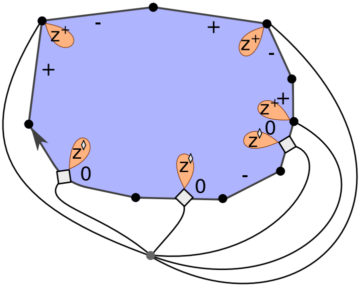
- Hence, we can express



$$W_{\bullet}^{(l)} = \sum_{k=0}^{\lfloor l/2 \rfloor} \frac{l!}{(k!)^2 (l-2k)!} (z^+)^k (z^\diamond)^{l-2k} = [z^{-l-1}] \frac{1}{\sqrt{(z - z^\diamond)^2 - 4z^+}}$$

Ingredients for peeling

- Disk function: $\mathbf{q}$ admissible iff there exist $c_\pm \in \mathbb{R}$ such that for $z > c_+ > c_-$,

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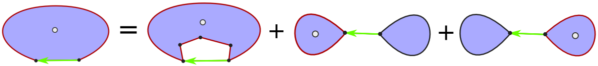

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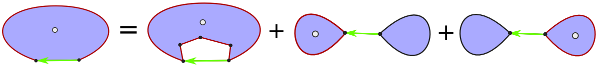

- Read off: $\mathbb{P}(l_{i+1} = l + k | l_i = l) = \frac{W_\bullet^{(l+k)}}{W_\bullet^{(l)}} \times \begin{cases} q_{k+2} & k \geq -1 \\ 2W^{(-k-2)} & k \leq -2 \end{cases}$

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  Moreover, if ν is *regular*, i.e. $\sum_{k=0}^{\infty} \nu(k) C^k < \infty$ for some $C > 1$, then this is equivalent to ν having non-positive drift, i.e. $\sum_{k=-\infty}^{\infty} k\nu(k) \leq 0$.

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- Call $\mathbf{q}$ critical if equality holds. [Miermont,'06]

Proposition

We have the following bijection between weight sequences $\mathbf{q}$ *and random walks* $(X_i)_{i\geq 0}$ *with step probabilities* ν*:*

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- Remarkable property of "lazy" peeling: the *h-function* $h_r^{(0)}$ hardly depends on $\mathbf{q}$! In particular it is the same for all bipartite $\mathbf{q}$, i.e. $r = 1$: $h_1^{(0)}(k) = 2^{-k}\binom{k}{k/2}$ for even $k \geq 0$ and 0 otherwise.

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- Linear map: $\nu(-k) = \sum_{l=1}^{\infty} \mathcal{R}_r(k,l)\nu(l) \quad (k \geq 1)$
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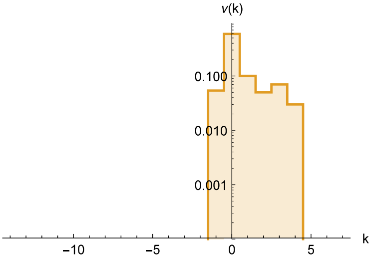

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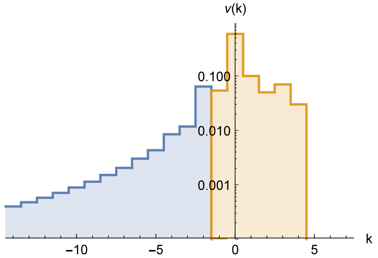

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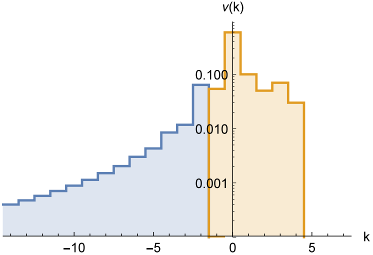

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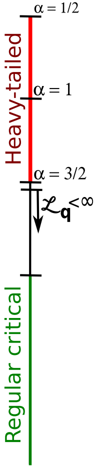

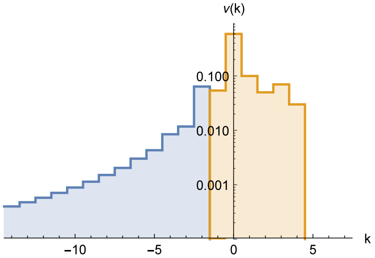

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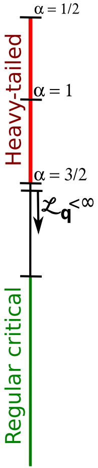

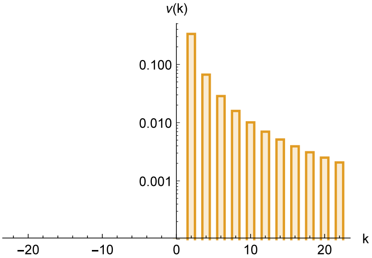

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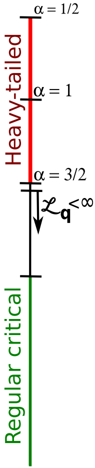

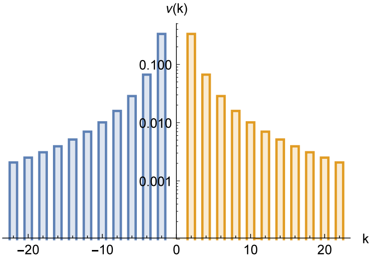

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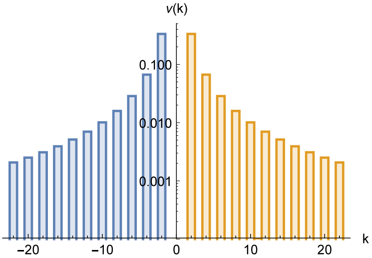
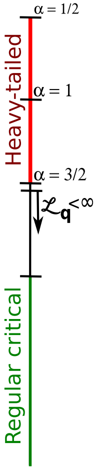
$$\nu(k) = 1/(k^2 - 1) \text{ for even } k \neq 0, \text{ otherwise } \nu(k) = 0$$





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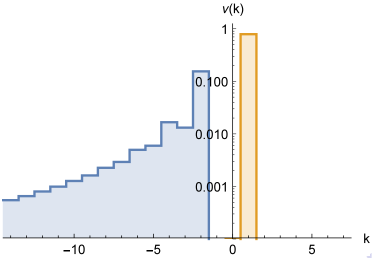

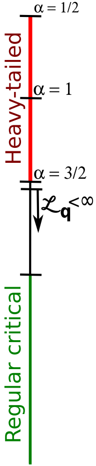

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- If $\mathcal{L}_{\mathbf{q}} < \infty$ the characteristic function of ν satisfies

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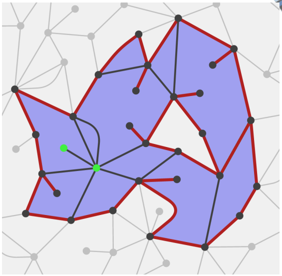

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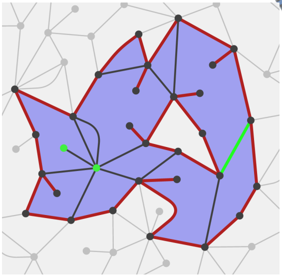

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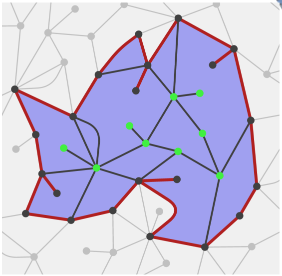

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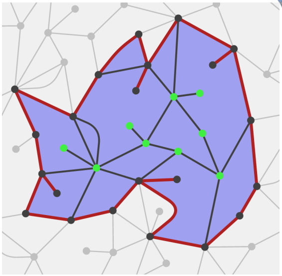

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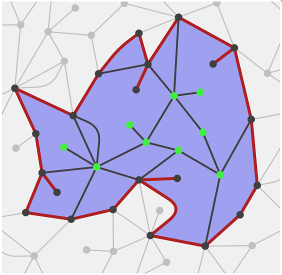

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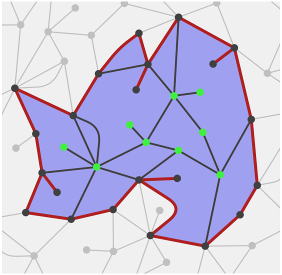
$$\mathbb{E}(V_{i+1} - V_i | l_i - l_{i+1} - 2 = l \geq 0) = \frac{W_\bullet^{(l)}}{W^{(l)}}$$

$$= \frac{h_r^{(0)}(l)\nu(-2)}{\nu(-l-2)} \sim \frac{8}{3c_+^2(1+r)\mathcal{L}_\mathbf{q}} l^2$$



- Checking the details of the proof of Curien and Le Gall one gets (see Le Gall's talk for definition of process $Z(t)$):

Theorem (Direct consequence of [Curien, Le Gall, '14])

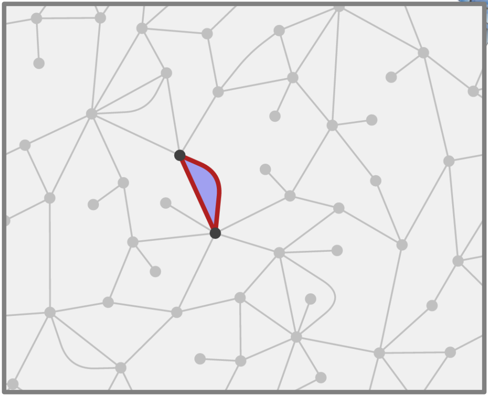
The perimeter $(l_i)_{i\geq 0}$ and volume $(V_i)_{i\geq 0}$ of a peeling of a regular critical $\mathbf{q}$*-IBPM converge jointly in distribution in the sense of Skorokhod to*

$$\left(\frac{l_{\lfloor nt\rfloor}}{\mathbf{p}_\mathbf{q}^\ell n^{2/3}}, \frac{V_{\lfloor nt\rfloor}}{\mathbf{v}_\mathbf{q}^\ell n^{4/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(\mathrm{d})} (S_{3/2}^+(t), Z(t))_{t\geq 0}$$

$$\mathbf{p}_\mathbf{q}^\ell = (\sqrt{1+r}\mathcal{L}_\mathbf{q})^{2/3}$$

$$\mathbf{v}_\mathbf{q}^\ell = \frac{8}{3c_+^2}\left(\frac{\mathcal{L}_\mathbf{q}}{1+r}\right)^{1/3}$$

First-passage percolation



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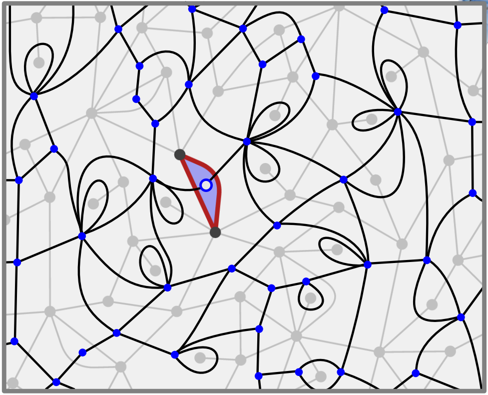
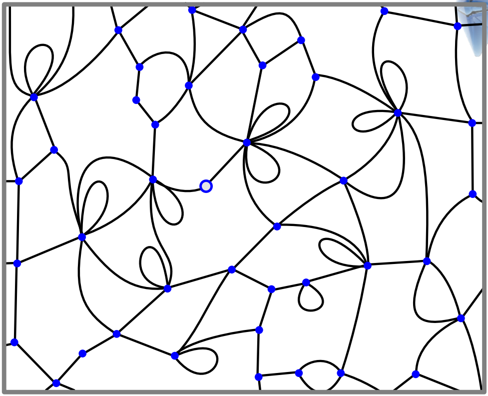


First-passage percolation

- Assign random exp(1)-lengths to dual edges.



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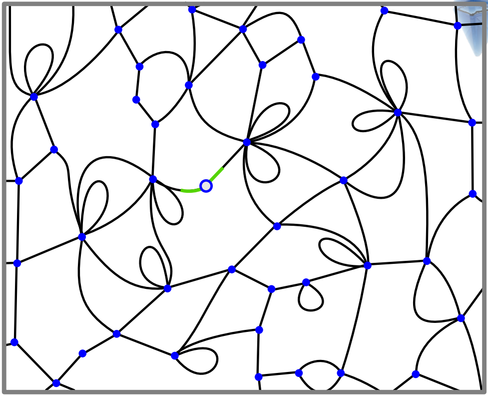

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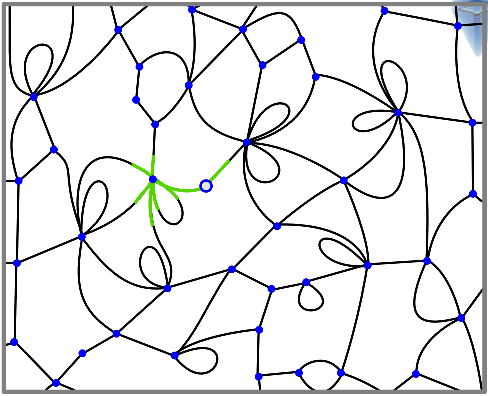

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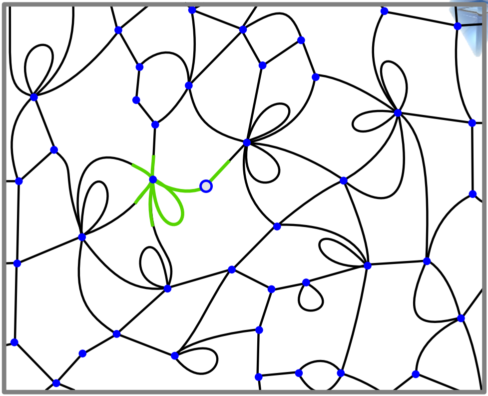

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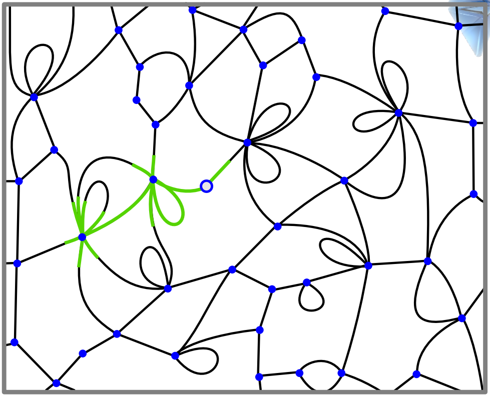

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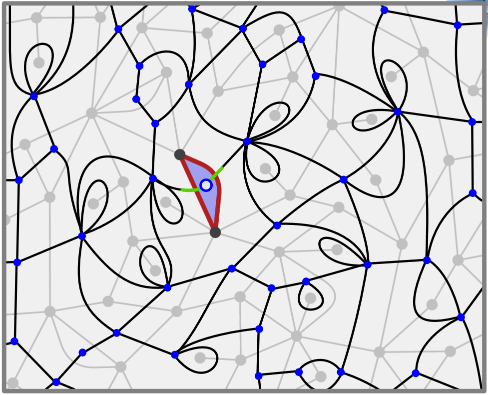

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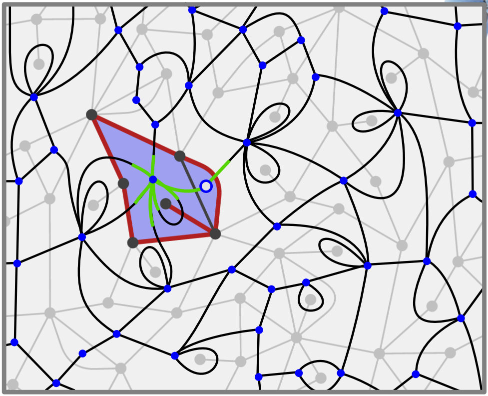

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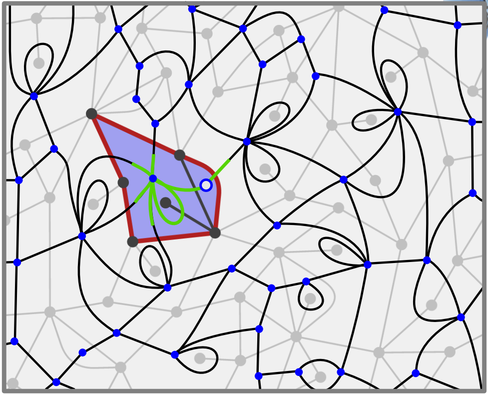

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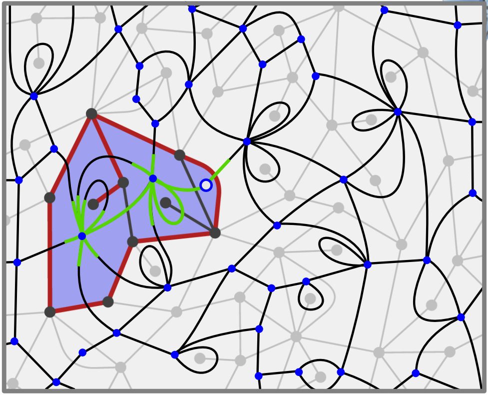

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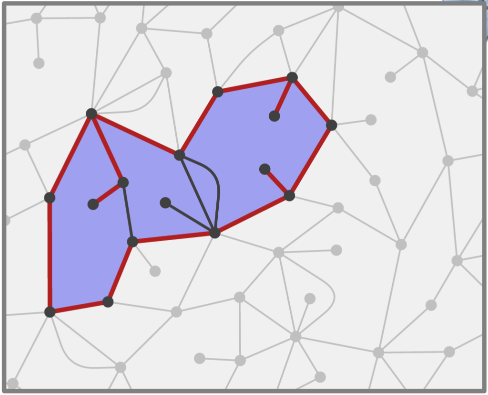

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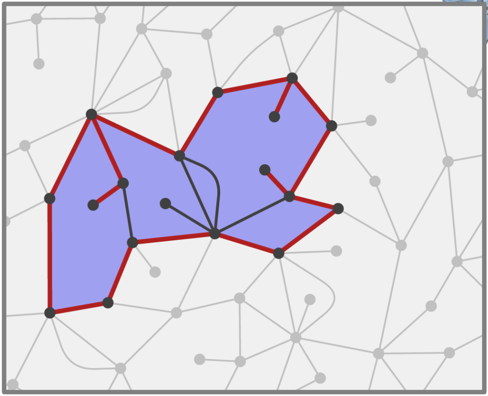

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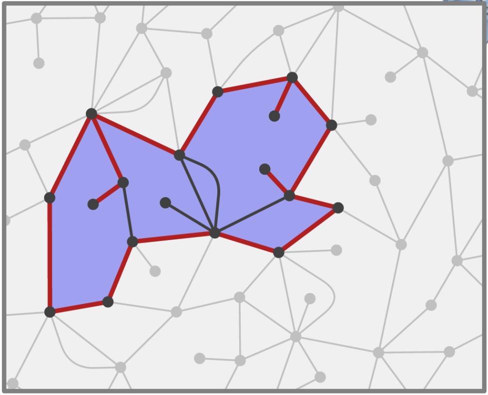

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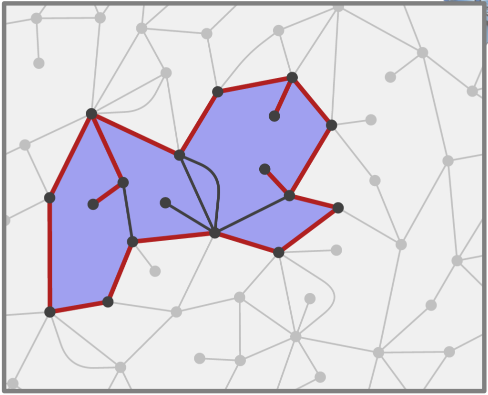

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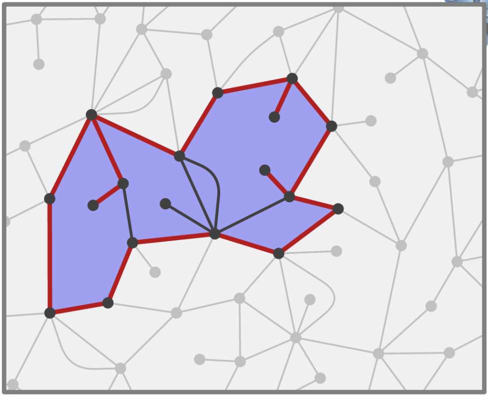

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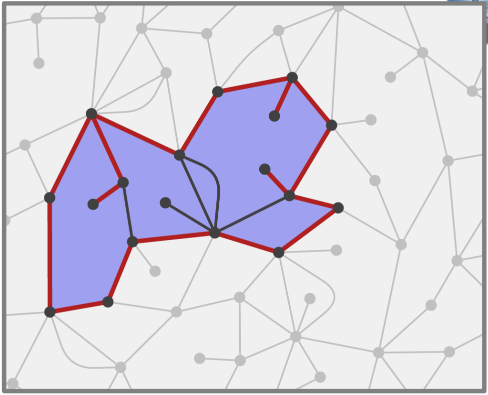

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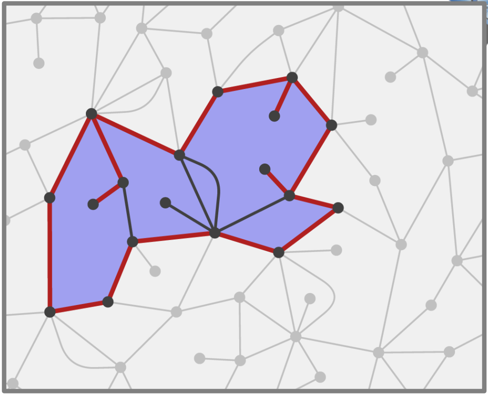

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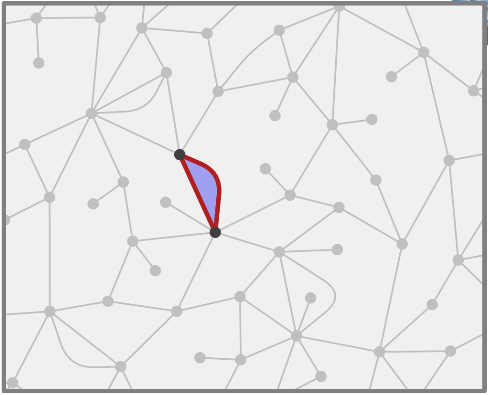
- Following [Curien, Le Gall, '14], this suggests that:

$$\left(\frac{l_{\lfloor nt \rfloor}}{\mathbf{p}_{\mathbf{q}}^{\ell} n^{2/3}}, \frac{T_{\lfloor nt \rfloor}}{(\mathbf{p}_{\mathbf{q}}^{\ell})^{-1} n^{1/3}} \right)_{t\geq 0} \xrightarrow[n\to\infty]{(\mathrm{d})} \left(S^{+}_{3/2}(t), \int_0^t \frac{\mathrm{d}t'}{S^{+}_{3/2}(t')} \right)_{t\geq 0}$$

Hop count

- Let γ_n be the shortest-time path to the edge explored at n'th step.



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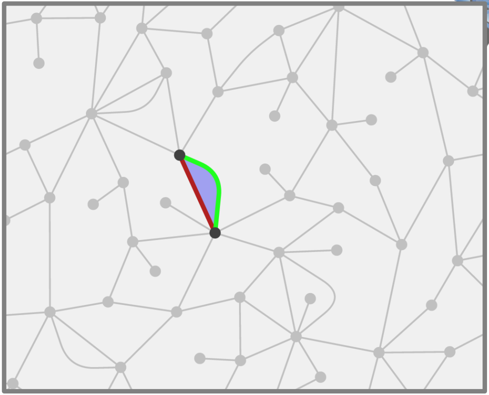

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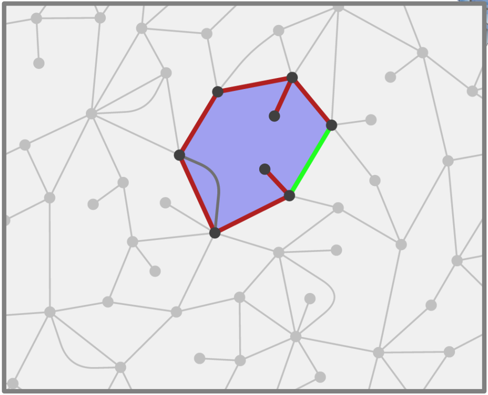

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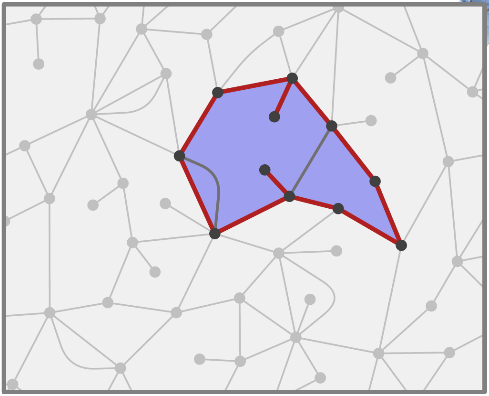

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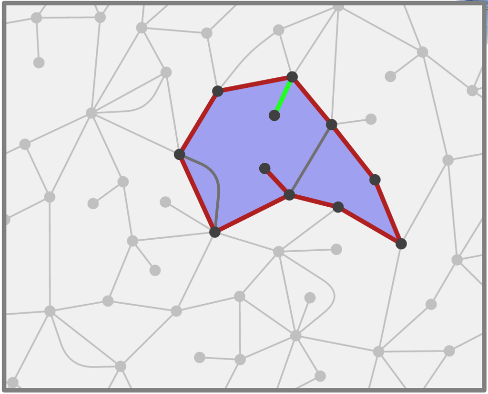

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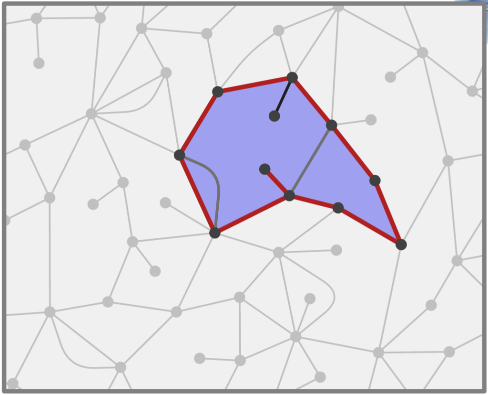

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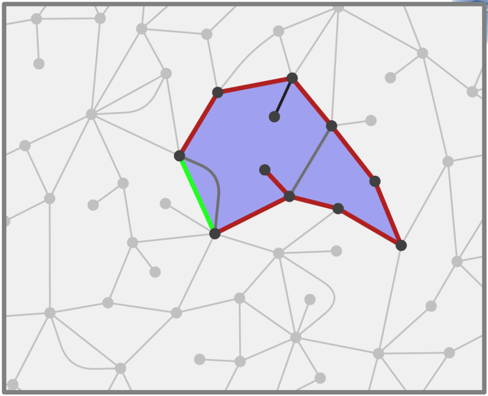

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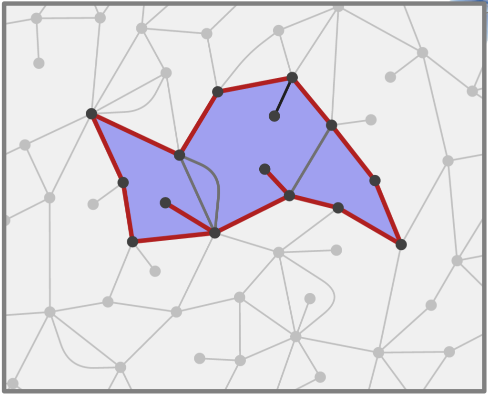

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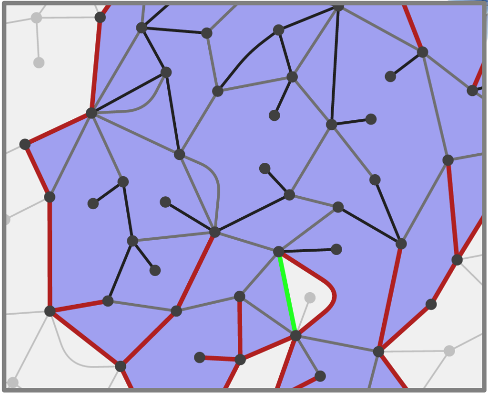

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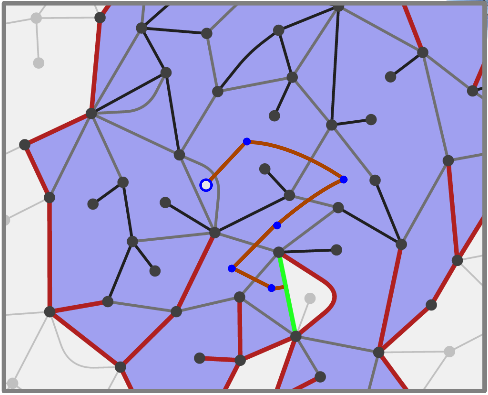

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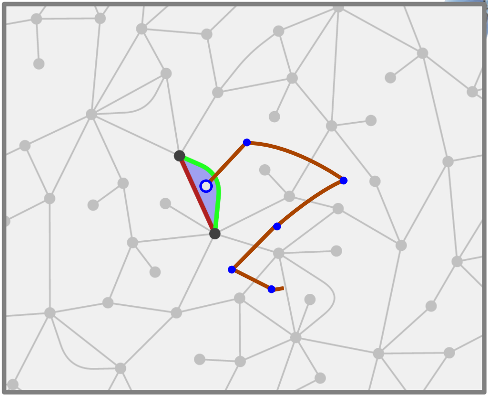

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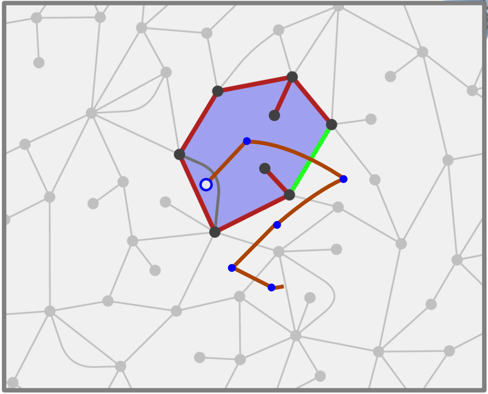

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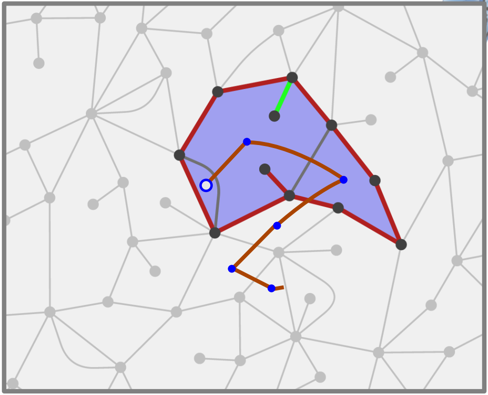

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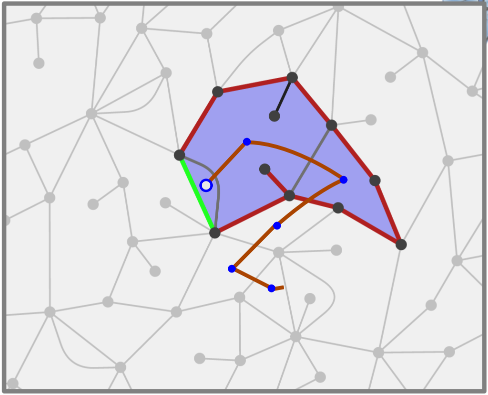

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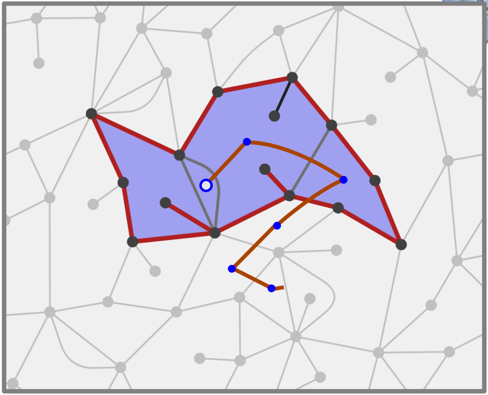

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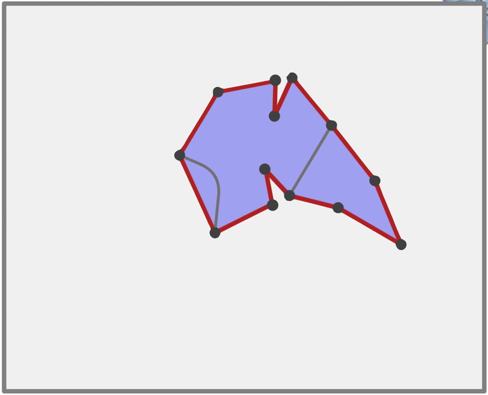

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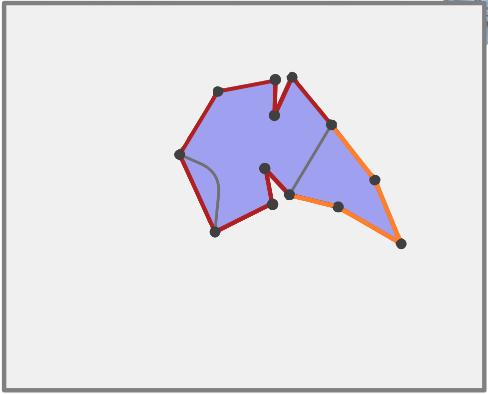

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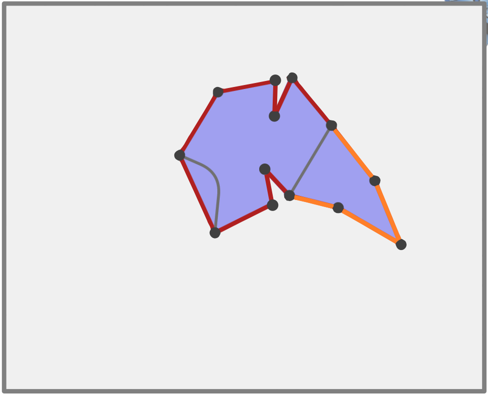

- Law of $H_i(n)$ independent of n (as long as $n > i$). Let us define the *hop count* process $(H_i)_{i\geq 0}$ as the large n limit. Then $(l_i, T_i, H_i)_{i\geq 0}$ is a Markov process with $H_i = \sum_{j=1}^{i} \mathfrak{b}_j, \mathfrak{b}_j \in \{0,1\}, \mathbb{P}(\mathfrak{b}_i = 1) = P_i$.

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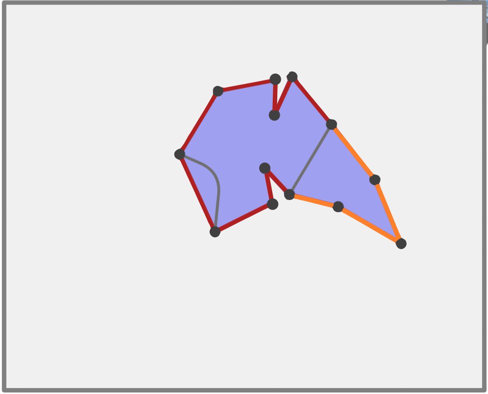

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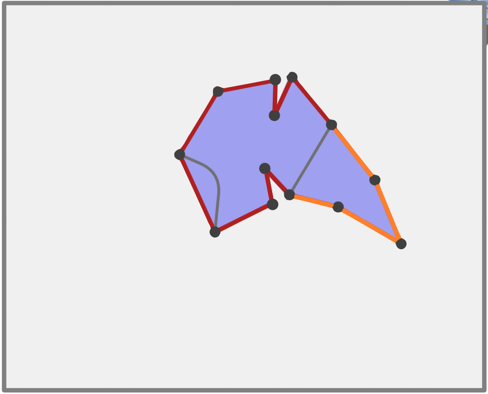

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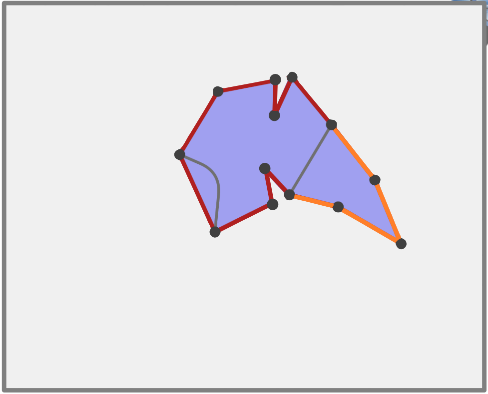
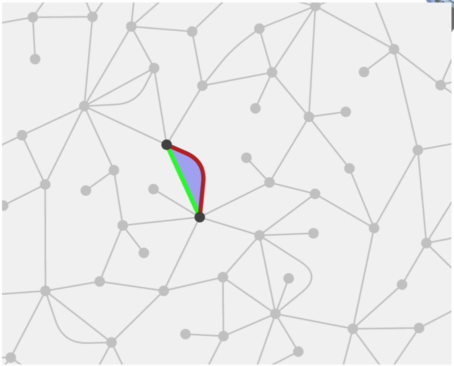


- Law of $H_i(n)$ independent of n (as long as $n > i$). Let us define the *hop count* process $(H_i)_{i\geq 0}$ as the large n limit. Then $(l_i, T_i, H_i)_{i\geq 0}$ is a Markov process with $H_i = \sum_{j=1}^{i} \mathfrak{b}_j, \mathfrak{b}_j \in \{0,1\}, \mathbb{P}(\mathfrak{b}_i = 1) = P_i$.
- For regular critical $\mathbf{q}$ we have

$$\mathbb{E}(H_{i+1}-H_i|l_i) = \sum_{k=0}^{\infty} \nu(k) \frac{k+1}{k+l_i} \frac{h_r^{(1)}(k+l_i)}{h_r^{(1)}(l_i)} = \overbrace{\sum_{k=0}^{\infty}(k+1)\nu(k)}^{\mathcal{H}_{\mathfrak{q}} \approx \lim_{i\to\infty} H_i/T_i} \mathbb{E}(T_{i+1}-T_i|l_i) + \mathcal{O}(l_i^{-1})$$

Dual graph distance

- Choose peel edge deterministically: breadth first exploration.



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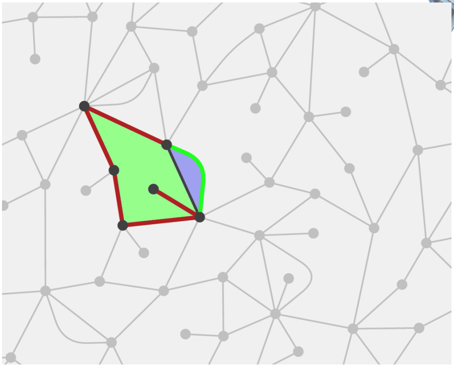

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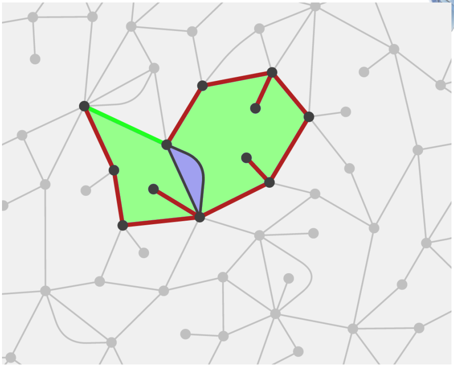

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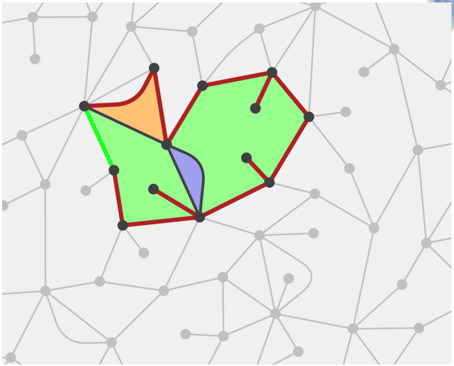

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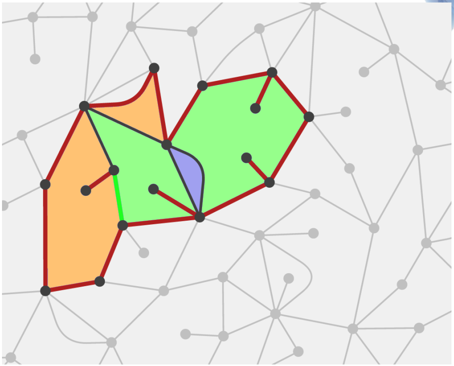

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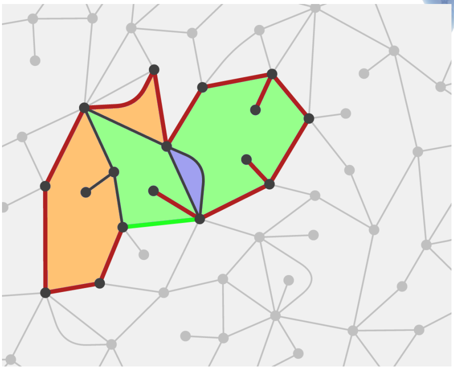

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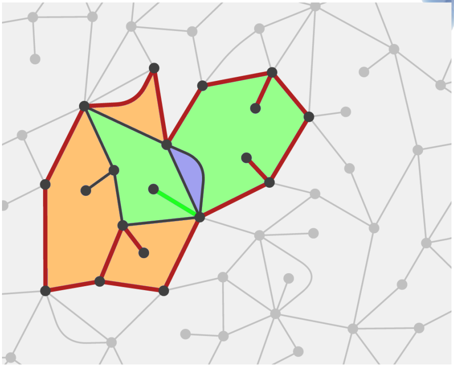

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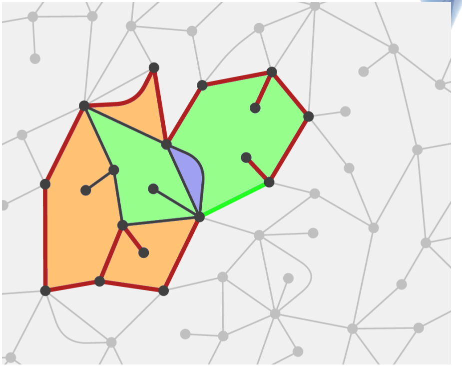

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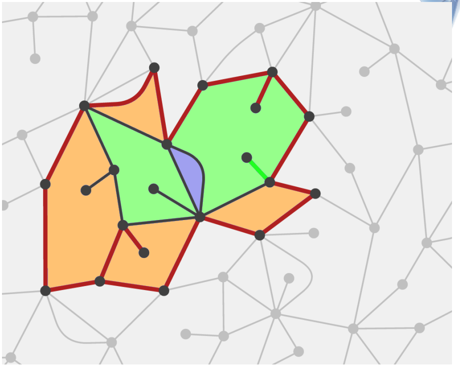

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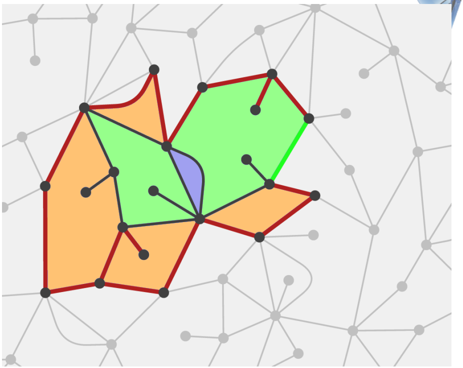

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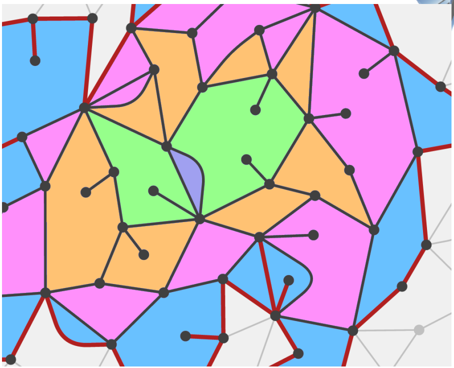

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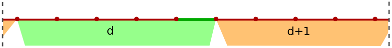

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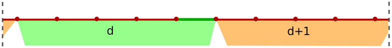

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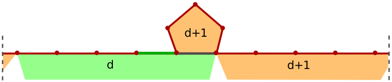

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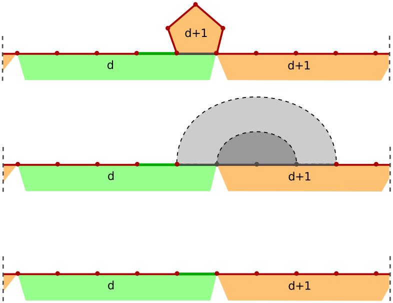

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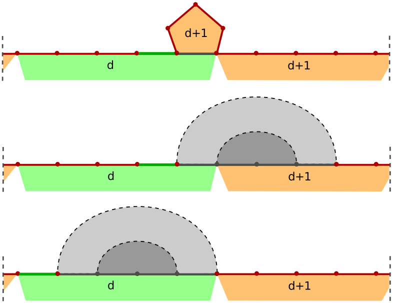

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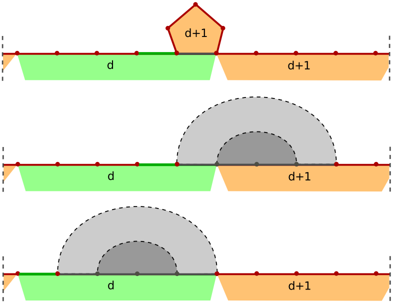

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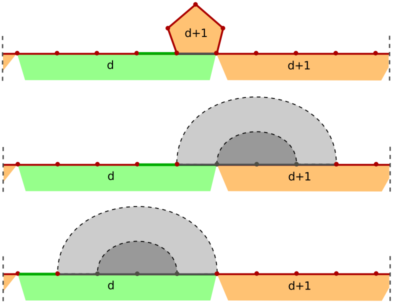

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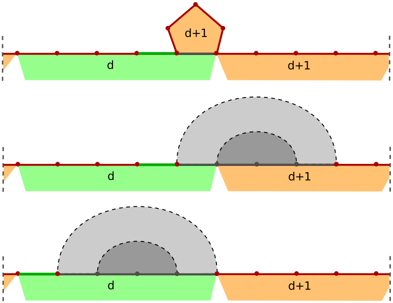

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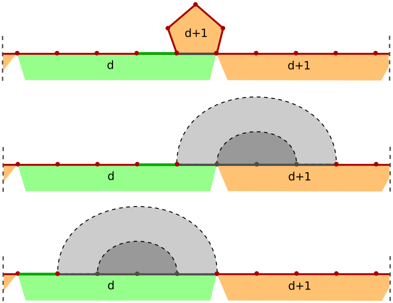

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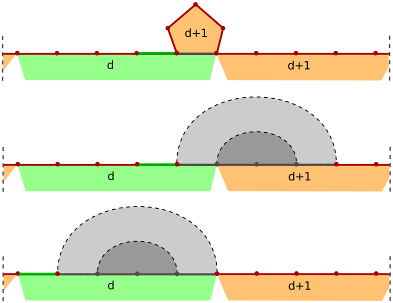

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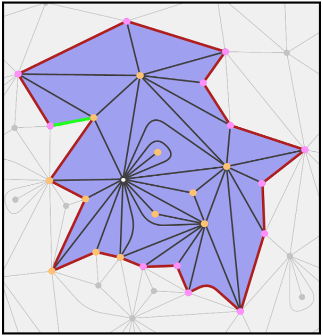
  - This suggests the asymptotic relation: $d_{\mathrm{gr}^*} \approx \frac{1}{2}(1+\mathcal{H}_{\mathbf{q}})\tilde{T} \approx \frac{1}{2}(1+\mathcal{H}_{\mathbf{q}})T \approx \frac{1}{2}(T+H)$

Graph distance

- Can adapt peeling process to graph distance: take peel edge to be closest frontier edge. In case of UIPT and UIPQ distances on boundary behave nicely (see Le Gall's talk).



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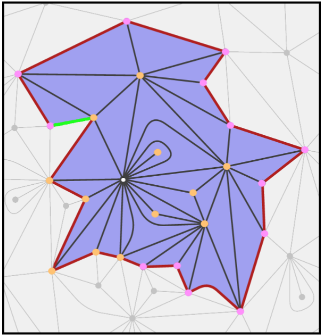

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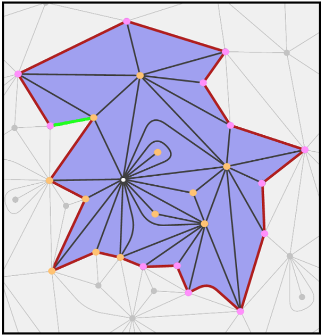

$$\left(\frac{(\mathbf{h}_\triangle)^2}{\mathbf{p}_\triangle k^2}|\partial\overline{B_{\lfloor kt\rfloor}}(m_\infty)|, \frac{(\mathbf{h}_\triangle)^4}{\mathbf{v}_\triangle k^4}|\overline{B_{\lfloor kt\rfloor}}(m_\infty)|\right)_{t\geq 0} \xrightarrow[k\to\infty]{\text{(d)}} (\mathcal{L}(t), \mathcal{M}(t))_{t\geq 0}$$

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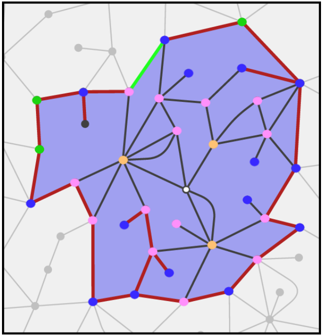

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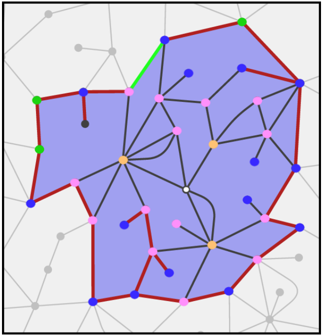

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- Convergence implies $\lim_{k\to\infty}\frac{\mathbb{E}|\overline{B_{\lfloor kt\rfloor}}(m_\infty)|}{k^4} = \frac{\mathbf{v}^\ell_\mathbf{q}}{(\mathbf{h}^\ell_\mathbf{q})^4}\mathbb{E}\mathcal{M}(t) = \frac{\mathbf{v}^\ell_\mathbf{q}}{(\mathbf{h}^\ell_\mathbf{q})^4}\frac{3}{64}t^4$.

- Luckily there is a different route:

Theorem (Miermont, '06)

If $\mathbf{q}$ *is regular critical and* m_n *is a random (rooted)* $\mathbf{q}$*-Boltzmann planar map conditioned to have* n *vertices and* v_1, v_2 *are random vertices, then there exists a* $\mathcal{C}_{\mathbf{q}} > 0$ *and a* $\mathbf{q}$*-independent random variable* d_∞ *s.t.*

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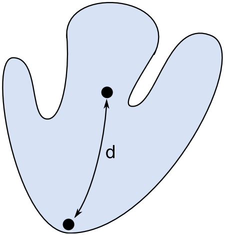
- Miermont also outline an algorithm to compute $\mathcal{C}_{\mathbf{q}}$. With some work:

Proposition

$$\mathcal{C}_{\mathbf{q}} = \left(\frac{2}{9} (z^+)^{3/2} \left(\partial_y + \sqrt{x}\partial_x\right)^2 f^\diamond(x,y)\Big|_{\substack{x=z^+ \\ y=z^\diamond}} \right)^{1/4} = \left(\frac{c_+^2}{96} (1+r)^3 \mathcal{L}_{\mathbf{q}} \right)^{1/4}.$$

- Let $|B_d(m)|$ be the number of vertices at distance $\leq d$ from the root vertex.



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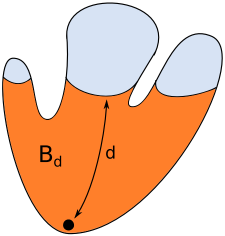

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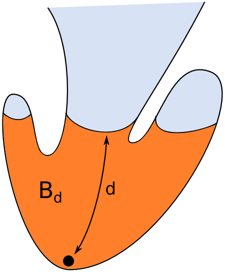

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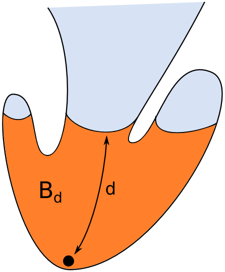

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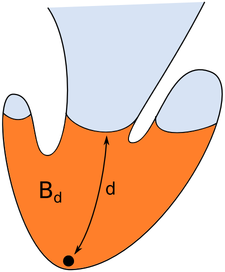

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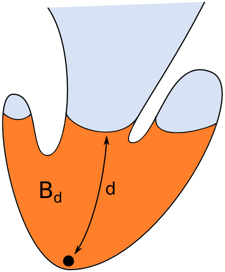

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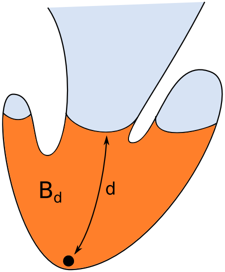
\end{tikzcd}$$



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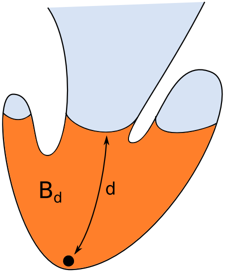

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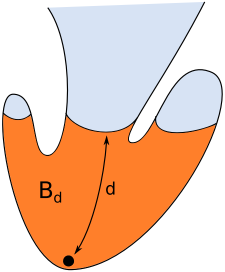

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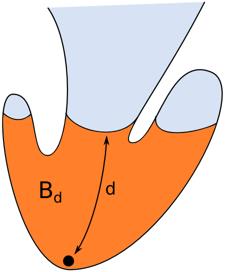

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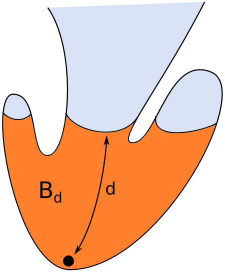

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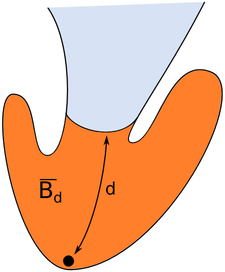

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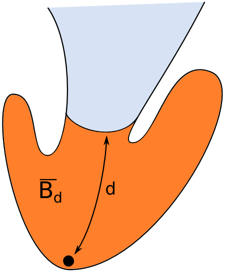

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- Then we finally get: $\frac{\mathbf{v}_{\mathbf{q}}^\ell}{(\mathbf{h}_{\mathbf{q}}^\ell)^4} = \frac{64}{3}\frac{7}{2}\frac{2}{21}\frac{1}{\mathcal{C}_{\mathbf{q}}^4}$, i.e. $\mathbf{h}_{\mathbf{q}}^\ell = \sqrt{\frac{3}{8}}(\mathbf{v}_{\mathbf{q}}^\ell)^{1/4}\mathcal{C}_{\mathbf{q}}$.

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Conjectures (in search of mathematicians!)

$$\mathbf{p}^\ell_\mathbf{q} = (\sqrt{1+r}\mathcal{L}_\mathbf{q})^{2/3}, \quad \mathbf{v}^\ell_\mathbf{q} = \frac{8}{3c_+^2}\left(\frac{\mathcal{L}_\mathbf{q}}{1+r}\right)^{1/3}, \quad \mathbf{h}^\ell_\mathbf{q} = \frac{1}{4}(1+r)^{2/3}\mathcal{L}_\mathbf{q}^{1/3}$$

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Let $(L_d)_{d\geq 0}$ be the length of the frontier when all vertices at distance d are discovered in a lazy peeling adapted to the distance of a *bipartite* regular $\mathbf{q}$-IBPM. Then $(n^{-2}L_{\lfloor nt\rfloor})_{t\geq 0}$ converges in distribution to a process independent of $\mathbf{q}$ (namely $4\mathcal{L}(t)$, see Le Gall's talk).

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Let v be a random vertex at distance d_{gr} from the root in a regular $\mathbf{q}$-IBPM, then we have the following limits in probability as $d_{\mathrm{gr}} \to \infty$ for its first-passage time T, hop count H, and dual graph distance d_{gr^*}:

$$\frac{H}{T} \to \mathcal{H}_\mathbf{q}, \quad \frac{d_{\mathrm{gr}^*}}{T} \to \frac{1+\mathcal{H}_\mathbf{q}}{2}, \quad \frac{d_{\mathrm{gr}}}{T} \to \mathbf{p}^\ell_\mathbf{q}\mathbf{h}^\ell_\mathbf{q} = \tfrac{1}{4}(1+r)\mathcal{L}_\mathbf{q}.$$

Example: Uniform infinite planar map (bivariate)

- Take $\mathbf{q}$ to be a geometric sequence. Then necessarily $\nu(k) = \alpha\sigma^k$ is a geometric sequence as well for $k \geq -1$ and $0 < \sigma < 1$, $\alpha > 0$.

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- Notice UIPM is $\sigma = \frac{5}{6}$, $\mathcal{H}_q = 3$, and duality: $\frac{\mathcal{H}_{\mathbf{q}}-1}{2} \leftrightarrow \frac{2}{\mathcal{H}_{\mathbf{q}}-1}$.

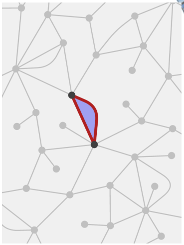
More examples

	r	c_+	$\mathcal{L}_\mathbf{q}$	$\mathcal{C}_\mathbf{q}^4$	$\rho_\mathbf{q}$
Triangulations	$2\sqrt{3}-2$	$\sqrt{6+4\sqrt{3}}$	$\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)$	$1/3$	$1+\sqrt{3}$
Quadrangulations	1	$\sqrt{8}$	$4/3$	$8/9$	3
Pentangulations	$0.70878\ldots$	$2.6098\ldots$	$2.1704\ldots$	$0.7683\ldots$	$3.3207\ldots$
$2p$-angulations	1	$\sqrt{\frac{4p}{p-1}}$	$\frac{4}{3}(p-1)$	$\frac{4}{9}p$	$\frac{p-1}{4^{-p}p\binom{2p}{p}-\frac{1}{2}}-1$
Uniform planar maps	$3/5$	$5/\sqrt{3}$	5	$16/9$	5
Uniform planar maps (biv.)	$\frac{\mathcal{H}^2-3}{\mathcal{H}^2+1}$	$\frac{(\mathcal{H}-1)^{3/2}\sqrt{\mathcal{H}+3}}{2(\mathcal{H}^2+3)}$	$\frac{1}{2}(\mathcal{H}^2+1)$	$\frac{(\mathcal{H}+1)^3}{6(\mathcal{H}+1)}$	$\frac{\mathcal{H}^2+1}{\mathcal{H}-1}$

	$\frac{\text{vertices}}{\text{faces}}$	$H/T=\mathcal{H}_\mathbf{q}$	T/d_{gr}	$d_{\mathrm{gr}^*}/d_{\mathrm{gr}}$	
Triangulations	$1/2$	$1+\frac{1}{\sqrt{3}}$	$2\sqrt{3}$	$1+2\sqrt{3}$	
Quadrangulations	1	2	$3/2$	$9/4$	
Pentangulations	$3/2$	$2.3608\ldots$	$1.0785\ldots$	$1.8123\ldots$	
$2p$-angulations	$p-1$	$\frac{2p-1}{p\binom{2p}{p}}2^{2p-1}$	$\frac{3}{2(p-1)}$	$\frac{3}{4}\left(\frac{1}{p-1}+\frac{2^{2p-2}}{p\binom{2p-2}{p}}\right)$	
Uniform planar maps	1	3	$1/2$	1	
Uniform planar maps (biv.)	$\frac{(\mathcal{H}+3)(\mathcal{H}-1)^2}{8\mathcal{H}}$	$\mathcal{H}$	$\frac{4}{\mathcal{H}^2-1}$	$\frac{2}{\mathcal{H}-1}$	

From lazy to simple peeling

- During simple peeling the frontier is kept simple, i.e. it does not touch itself.



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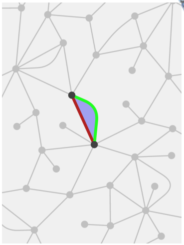

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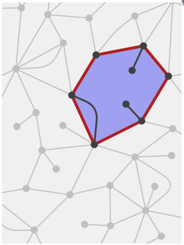

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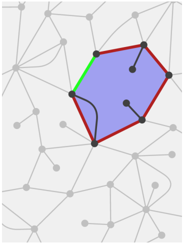

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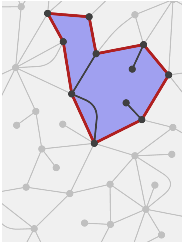

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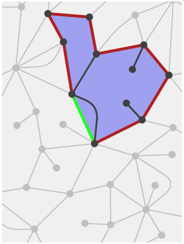

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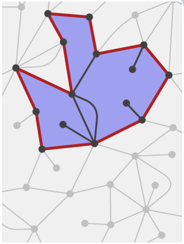

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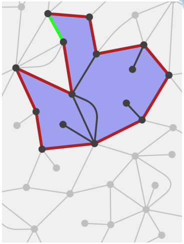

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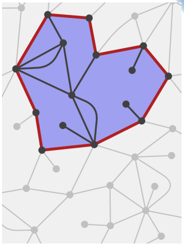

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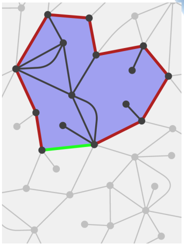

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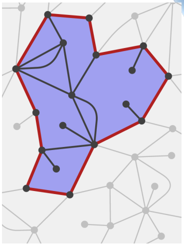

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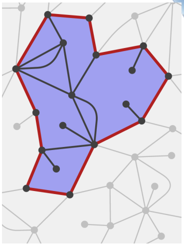

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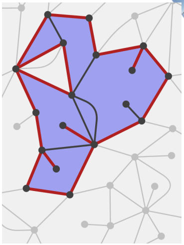

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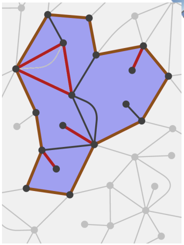

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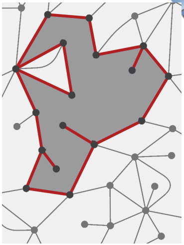

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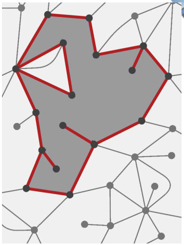

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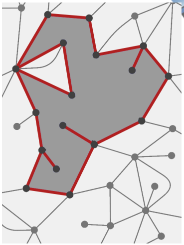

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- Do any of the geometric constructions still make sense in the heavy-tailed case ($\alpha \in [\frac{1}{2}, \frac{3}{2}]$)? Le Gall and Miermont have shown that w.r.t. the graph distance such finite maps converge to a metric space with Hausdorff dimension $2\alpha + 1$. The metric space w.r.t. dual graph distance is quite different. Does it have a limit?

Thanks for your attention!