Introduction to Subatomic Physics:

Exercises

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Exercises for Chapter 1

1.1: Natural units

Most computations in this course are carried out in *natural units*. However, often it will be desirable to convert the results of such calculations to more standard units. We consider two (important) examples here.

a) Decay widths (total or partial) are given in terms of $[E]$, and are often given in GeV. Use *e.g.* the relation between a particle's average lifetime and its total decay width to compute the average lifetime that a decay width of 1 GeV corresponds to.

b) The total cross sections of scattering processes are given in $[E]^{-2}$, and in practice are often expressed in GeV^{-2}. Compute the conversion factor used to convert from GeV^{-2} to barns (remember that $1\text{b} \equiv 10^{-28}\text{m}^2$).

Hint: these computation require the values of $\hbar$ and c in SI units, as well as the magnitude of the electron's charge:

$$\begin{aligned} \hbar &= 1.055 \cdot 10^{-34}\ \text{J s} \\ c &= 2.9979 \cdot 10^{8}\ \text{m s}^{-1} \\ e &= 1.601 \cdot 10^{-19}\ \text{C}. \end{aligned}$$

1.2: High-energy collisions

a) There are two basic operating modes for collision processes. In *fixed target* mode, still often used, a beam of particles collides with a target at rest. Here, show that the centre-of-mass energy is given (for $E_{\text{beam}} \gg m$) by

$$E_{\text{CM}} \approx \sqrt{2E_{\text{beam}} m_{\text{target}}}$$

The most common use of *collider* mode is to accelerate particles and their antiparticles to the same energy in the same beam pipe, but in the opposite direction (why would this be a natural thing to do, bending particles into a circular orbit with the help of a magnetic field?). Show that in this case $E_{\rm CM} = 2E_{\rm beam}$. Also give the more general formula for unequal masses and energies of the two particles involved, while still assuming that the collisions are *head on*, *i.e.*, the particles move in exactly opposite directions. Work this out for the case of the HERA electron-proton accelerator ($E_{\rm p} = 920$ GeV, $E_{\rm e} = 27.5$ GeV) which was in operation until 2007. (Note that in the HERA case, the electron and proton beams were accelerated in different beam pipes, and a small *crossing angle* was therefore inevitable. But this was a small effect, and we neglect it here.)

b) A standard (but still rather general) scattering process to consider is the process

$$A + B \to C + D,$$

where all particles involved may *a priori* have different masses. In the context of this sort of process, often the so-called *Mandelstam variables* are defined:

$$\begin{aligned} s &\equiv (p_A + p_B)^2 \\ t &\equiv (p_A - p_C)^2 \\ u &\equiv (p_A - p_D)^2 \end{aligned}$$

Prove the identity

$$s + t + u = (m_A^2 + m_B^2 + m_C^2 + m_D^2).$$

Show also that s is generally related to the centre-of-mass energy of collision processes by $s = E_{\rm CM}^2$. (It is for this reason that the symbol $\sqrt{s}$ is often used as an alternative for $E_{\rm CM}$.)

(Note that t and u are not very precisely defined, for an interchange of particles C and D is possible without loss of generality. But in many important processes, at least one of the particles retains its identity, and in such cases the definition can indeed be made unambiguous by representing this particle by A and C in the initial and final states, respectively.)

Finally, show that in a $2 \to 2$ body scattering process of identical particles considered in the CM frame, the Mandelstam variables take the form

$$\begin{aligned} s &= 4(|\vec{p}|^2 + m^2) \\ t &= -2|\vec{p}|^2(1 - \cos\theta) \\ u &= -2|\vec{p}|^2(1 + \cos\theta). \end{aligned}$$

Here, $\vec{p}$ and θ are the momentum and scattering angle of either of the incoming particles.

1.3: Discovery of the charged kaon

The charged kaon was discovered in 1943, simply by the fact that its mass ($m_{\rm K} = 494$ MeV) differed significantly from that of all particles known up to that time. This discovery made use of fairly simple kinematics: one cosmic ray event showed a charged particle imparting a high energy to an atomic electron. The momentum of the incoming particle (which was hardly deflected) and the electron could be measured, as well as the angle θ between the directions of the outgoing electron and the incident particle.

Show that simple two-body collision kinematics can be used to estimate the mass M of the incident particle as

$$M = p\left(\frac{E+m}{E-m}\cos^2\theta - 1\right)^{1/2},$$

where p and E are the momentum of the incident particle and the energy of the outgoing electron, respectively, and m_{e} is simply the electron mass. (Hint: we are considering a *heavy* particle, so m_{e} can be neglected compared to M.)

1.4: Particle decays

a) Consider the decay process $A \to B + C$. Show that the energy of particle B in the rest frame of the decaying particle is given by

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}.$$

What is the *velocity* of particle B?

b) One of the processes that were studied in a fair amount of detail at the LEP collider was the production of τ leptons, $\mathrm{e}^+\mathrm{e}^- \to \tau^+\tau^-$. The precise nature of the τ lepton will be covered in more detail later in this course; for now, it suffices to know that it has a mass $m_\tau = 1.777$ GeV, and an average lifetime $\tau_\tau \approx 0.3$ ps (with apologies for the re-use of symbols!).

This lifetime was barely long enough to measure with good precision (this was done by extrapolating the trajectories of longer-lived decay products back to the decay point of the τ lepton). Estimate the average distance traveled by these τ leptons before decaying, given that (in the phase relevant for this process) LEP operated at $E_{\mathrm{CM}} = 92$ GeV.

1.5: Compton scattering

Consider the Compton scattering process $\gamma + \mathrm{e}^- \to \gamma + \mathrm{e}^-$. Show that this elastic scattering process, in the case when the initial-state electron is at rest, leads to the following relation between the wave lengths of the incident and scattered light λ and λ' and the angle θ over which the light is scattered:

$$\lambda' = \lambda + \lambda_{\mathrm{c}}(1 - \cos\theta).$$

Here, the *Compton wavelength* λ_{c} is given by

$$\lambda_{\mathrm{c}} = \frac{2\pi}{m_{\mathrm{e}}}.$$

1.6: The cosmic ray muon flux

The Earth's atmosphere is constantly being bombarded by protons (and likely, heavier nuclei as well) of high energies. The ensuing *cosmic ray showers* typically give rise to the production of a significant number of muons. Despite the short lifetime of the muon ($\tau_\mu \approx 2.2\mu$s), a fair fraction of them can be observed at ground level. Assuming that the muons are produced at a hight of 10 km, and that they travel to Earth vertically, compute the probability that they reach ground level as a function of their energy. (The muon's mass is given by $m_\mu = 105.6$ MeV.)

1.7: Discovery of the antiproton

The antiproton (denoted as $\bar{\mathrm{p}}$) was discovered (in 1955) by the reaction

$$\mathrm{p}+\mathrm{p} \to \mathrm{p}+\mathrm{p}+\mathrm{p}+\bar{\mathrm{p}},$$

by shooting a beam of protons at protons that are at rest.

a) Give the minimum *centre-of-mass energy* required for this process to take place.

b) Show that the corresponding minimum proton beam energy in the *lab* frame must be $E_{\mathrm{min}} = 7m_{\mathrm{p}}$.

1.8: Rutherford scattering as a classical process

In addition to the scattering of point particles off hard spheres, there is another situation that is addressed in a fairly straightforward manner: the *classical* (and non-relativistic) scattering of two point charges off each other.

We consider this problem in the *Rutherford scattering* approximation, in which one of the particles is sufficiently heavy that it can be considered as remaining at rest throughout the scattering process. From the $1/r$ form of the potential due to this particle, it can be shown that the angle θ over which the incident particle is scattered is related to the impact parameter b by

$$b = \frac{q_1 q_2}{E} \cot(\theta/2),$$

where E is the kinetic energy of the incident particle.

a) Compute the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$.

b) Show that the total cross section is *infinite*. Can you think of a reason why this would be so?

Exercises for Chapter 2

2.1: Minimal substitution

While it is not a complete *proof*, the validity of the principle of *minimal substitution* can at least be made somewhat plausible in the context of (non-relativistic) classical electrodynamics.

a) Starting from a Lagrangian

$$L = \frac{m\vec{v}^2}{2} - q(\Phi - \vec{v}\cdot\vec{A}),$$

derive the Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi,$$

where $\vec{p}$ represents the *canonical* momentum. Also show that the explicit relation between $\vec{p}$ and the physical momentum $m\vec{v}$ is given by

$$\vec{p} = m\vec{v} + q\vec{A}.$$

b) Use Hamilton's equations to derive the Lorentz force.
Hint: use the expression for the full time dependence of $\vec{A}$:

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}.$$

2.2: The QED Lagrangian

a) Consider the Lagrangian

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

and show that application of the Euler-Lagrange equations again leads to the Dirac equation for ψ (as well as its conjugate equation for $\bar{\psi}$).
Hint: for the eight independent degrees of freedom (before application of the equations of motion), rather than taking *e.g.* the real and imaginary parts of ψ it is again easier to take the components of ψ and $\bar{\psi}$.

b) Consider the Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma}$$

and, applying the Euler-Lagrange equations, show that it leads to the Maxwell equations describing the free photon field.
Hint: here, take each component A_ν as an independent degree of freedom.

c) Finally, take the phase symmetry relevant to QED. Use the covariant derivative to construct the Lagrangian describing the interaction between the electron and the photon. Add the kinetic term describing the photon and discussed in the previous item. Use the Noether theorem (applied to ψ and $\bar{\psi}$ only) to construct the conserved current. Apply the Euler-Lagrange equations to the various fields involved. For the equation of motion of the photon field in particular, show that this procedure yields again the Maxwell equations but this time with a non-zero source term.

2.3: Møller scattering

Consider the *scalar* Møller scattering process $e^- + e^- \to e^- + e^-$, as covered during the lectures.

a) Compute also the matrix element corresponding to the "second" Feynman diagram.

b) Show that in the CM frame their sum $\mathscr{M}$ satisfies

$$\mathscr{M} = 2e^2 \left(\frac{2m^2/|\vec{p}|^2 + 3 + \cos^2\theta}{-\sin^2\theta} \right),$$

where θ is the scattering angle. To do this, argue that these two diagrams involve the Mandelstam variables t and u that we encountered in exercise 1.2. Given that we are indeed working in the CM frame, the corresponding expressions for t and u can be directly. (In the following, such diagrams will be referred to occasionally as t-channel or u-channel diagrams – even if in this case, with all identical particles, it is not really possible to label the particles unambiguously.)

c) Use the above to compute the differential cross section in the CM frame, using the following general rule for $2 \to 2$ body scattering processes:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} \mathscr{S} |\mathscr{M}|^2. \tag{2.1}$$

Here, $\mathscr{S}$ is a "spin factor" (which is simply unity for our spin-0 scattering process), $\mathscr{M}$ is the total matrix element, s is the Mandelstam variable encountered in exercise 1, and $\vec{p}_i$ and $\vec{p}_f$ are the momenta of the initial- and final-state particles (again in the CM frame).

Note also that it is useful to write the result in terms of the *fine structure constant* α, which in our system of natural units is defined as

$$\alpha \equiv \frac{e^2}{4\pi}.$$

When computing a perturbative series, this is conveniently done in terms of powers of $\alpha \approx 1/137$.

2.4: Bhabha scattering

Consider the Bhabha scattering process $e^+ + e^- \to e^+ + e^-$ in the CM frame.

a) Draw the two lowest-order Feynman diagrams relevant to this process.

b) Sticking to the language of exercise 1.2, demonstrate that of the two diagrams involved, one is an s-channel diagram and one a t-channel one. Merely from considering the propagator, can you argue which of the two will be dominant for high energies?

c) Use Eqn. 2.1 to show that the total Bhabha scattering cross section is *infinite*. This may appear strange, but it is in fact due to the fact that the electromagnetic interaction is of infinite range.
Hint: to show this, it is *not* necessary to do the complete calculation – that is optional. Instead, it suffices to consider the behaviour for small scattering angles θ.

d) Draw also a higher-order Feynman diagram corresponding to this process. What power of α is involved, relative to that of the lowest-order process?

2.5: Other QED processes

a) Draw the two Feynman diagrams that describe the Compton scattering ($e^- + \gamma \to e^- + \gamma$) process in lowest order. Also indicate whether the Mandelstam variable occurring in each diagram's propagator is s, t, or u.

b) Draw the lowest order Feynman diagrams describing the process $e^+e^- \to \gamma\gamma$. This process is relevant both for decay of positronium (an e^+e^- bound state) and for high-energy e^+e^- scattering.

2.6: Scattering of spin-1/2 electrons and muons

We consider the scattering process $e^- + \mu^- \to e^- + \mu^-$. For the purpose of this exercise, the muon can be regarded essentially like an electron, except that it is a *different* particle. This reduces the complexity of the subsequent calculations while still exhibiting most of the interesting features. $e^- + e^- \to e^- + e^-$, which an earlier exercise covered already. However, we make one important change by considering "proper" spin-1/2 electrons rather than the spin-0 substitutes encountered earlier.

a) Draw the (single) Feynman diagram that at lowest order is relevant for this process. Label the four-momenta of the incoming (outgoing) electron and muon as p_1 and p_2 (p_3 and p_4), respectively, and also label the spin states accordingly.

b) Show that the matrix element for this process is given by

$$-i\mathscr{M} = \bar{u}^{(s_3)}(p_3)(ie\gamma^\mu)u^{(s_1)}(p_1)\frac{-ig_{\mu\nu}}{t}\bar{u}^{(s_4)}(p_4)(ie\gamma^\nu)u^{(s_2)}(p_2),$$

where t is one of the Mandelstam variables.
Note: in the Feynman rules as discussed in the lectures, we've been a bit cavalier about the placement of the spinor functions $u^{(s_i)}(p_i)$ and $\bar{u}^{(s_i)}(p_i)$. To make this more precise, every connected fermion line is "started" by one u (or v) and ended by a $\bar{u}$ (or $\bar{v}$). It is easily verified that this construction always results in a "simple" complex number for each such line.

c) Prove the relation

$$(\bar{u}^{(s_i)}(p_i)\gamma^\mu u^{(s_j)}(p_j))^\dagger = \bar{u}^{(s_j)}(p_j)\gamma^\mu u^{(s_i)}(p_i)$$

and use this to write the *squared* matrix element as

$$|\mathscr{M}|^2 = \frac{e^4}{t^2}\bar{u}_3\gamma^\mu u_1\bar{u}_1\gamma^\nu u_3\bar{u}_4\gamma_\mu u_2\bar{u}_2\gamma_\nu u_4$$

(where in the last equation we've written $u_i \equiv u^{(s_i)}(p_i)$ etc.).

d) The above is the expression for one individual combination of electron and muon spins $s_1 \ldots s_4$. But often we start from unpolarized particles in the initial state and we don't measure particle spins in the final states. In this case, we need to *sum* over all spin states in the final state and *average* over all spin states in the initial state. Argue why we should be treating initial and final states differently in this respect, and that for our specific case (scattering spin-1/2 particles off each other) this results in

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathscr{M}(s_1,s_2,s_3,s_4)|^2,$$

where the dependence of $\mathscr{M}$ on the particle spins is now made explicit.

To compute the spin sums, we now make use of the specific properties of the u_i functions. It turns out that their normalisation factor N, which was left unspecified in the lecturs, is $N = \sqrt{E+m}$. With this normalisation, it can be shown that

$$(u^{(r)})^\dagger(p)u^{(s)}(p) = 2E\delta_{rs}, \quad \sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p}+m, \quad \sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = \not{p}-m.$$

e) Use the above to eliminate two of the four spin sums and write

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathscr{M}(s_1,s_2,s_3,s_4)|^2 = \frac{e^4}{4t^2}\sum_{s_3,s_4}\bar{u}_3\gamma^\mu(\not{p}_1+m)\gamma^\nu u_3\bar{u}_4\gamma^\mu(\not{p}_2+M)\gamma^\nu u_4,$$

where m and M denote the electron and muon mass, respectively.

To make further progress, it is instructive to consider *e.g.* the term $\bar{u}_3\gamma^\mu(\not{p}_1+m)\gamma^\nu u_3$ in more detail, and in particular to write it in component form:

$$\begin{aligned}
\bar{u}_3\gamma^\mu(\not{p}_1+m)\gamma^\nu u_3 &= \sum_{a,b}(\bar{u}_3)_a(\gamma^\mu(\not{p}_1+m)\gamma^\nu)_{ab}(u_3)_b \\
&= \sum_{a,b}(u_3)_b(\bar{u}_3)_a(\gamma^\mu(\not{p}_1+m)\gamma^\nu)_{ab} \\
&= \sum_{a,b}(u_3\bar{u}_3)_{ba}(\gamma^\mu(\not{p}_1+m)\gamma^\nu)_{ab} \\
&= \mathrm{Tr}(u_3\bar{u}_3\gamma^\mu(\not{p}_1+m)\gamma^\nu).
\end{aligned}$$

In addition we will use (without proof) a number of *trace theorems*:

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} & \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}) \\
\mathrm{Tr}(\not{a}\not{b}) &= 4a\cdot b & \mathrm{Tr}(\not{a}\gamma^\mu\not{b}\gamma^\nu) &= 4(a^\mu b^\nu+a^\nu b^\mu-a\cdot b g^{\mu\nu})
\end{aligned}$$

as well as the fact that the trace of any *odd* number of gamma matrices vanishes.

f) Show that the result of part (e) now becomes

$$\frac{e^4}{4t^2}L^{(e)}_{\mu\nu}L^{\mu\nu,(\mu)},$$

and give the expression for $L^{(e)}_{\mu\nu}$ (the expression for $L^{\mu\nu,(\mu)}$ follows simply from this). This is often called the *lepton tensor* for scattering of unpolarized leptons.

g) Show that the result reduces to

$$\frac{8e^4}{t^2}\left((p_3\cdot p_1)(p_1\cdot p_4)+(p_1\cdot p_2)(p_3\cdot p_4)-(p_2\cdot p_4)m^2-(p_1\cdot p_3)M^2+2m^2M^2\right).$$

The above is as far as we can get in all generality. However, there is an interesting limit case that can be studied. We have $M\gg m$, so that in the scattering of a non-relativistic electron (let us denote its momentum by $\vec{p}$) off a muon at rest, the recoil of the muon can be neglected.

h) Show that in this case, $t=-4|\vec{p}|^2\sin^2(\theta/2)$, where θ is the electron's scattering angle, and that finally the differential cross section, given again by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}\cdot\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathscr{M}(s_1,s_2,s_3,s_4)|^2$$

becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\left(\frac{\alpha}{4|\vec{p}|^2\sin^2(\theta/2)}\right)^2(m^2+|\vec{p}|^2\cos^2(\theta/2)).$$

Note that this uses again the relation $e^2=4\pi\alpha$.

This last expression is that for *Mott scattering*: it is essentially that for Rutherford scattering, apart from the last term, which arises from the fact that the particles involved have spin 1/2 rather than 0.

Exercises for Chapter 3

3.1: Isospin symmetry and transition amplitudes

The purpose of this exercise is to investigate in a bit of detail what is meant by the statement that "the physics is invariant under rotations in isospin space". Our starting point is the statement that rotations in isospin space represent *unitary* transformations, and that this implies that the isospin operator commutes with the Hamiltonian describing the strong interaction,

$$[\vec{I}, H] = 0,$$

and hence with the terms in the Hamiltonian describing the strong interaction.

a) Show that the fact that the (operator representing the) strong interaction commutes with $\vec{I}^2$ means that no transitions occur between states of different total isospin I.
(*Hint: this proof makes use of the commutator* $[\vec{I}^2, H]$.)

b) Similarly, show that no transitions occur between states of different I_3.

c) Show that strong interaction transition amplitudes depend *only* on the total isospin I, and not on I_3.
(*Hint: use the raising and lowering operators* $I_\pm = I_1 \pm iI_2$, *and their normalization*

$$I_\pm |I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)}\,|I, I_3 \pm 1\rangle,$$

again in analogy to the case of ordinary angular momentum operators.)

In conclusion, the consequences of isospin symmetry for transition amplitudes are *exactly* analogous to those of ordinary rotational invariance. (This last case is covered extensively *e.g.* in Chapter 4 of the book "Modern Quantum Mechanics" by J.J. Sakurai, or in the book "Introduction to Quantum Mechanics" by D. Griffiths.)

3.2: Isospin symmetry and deuteron production cross sections

Another example of the use of isospin symmetry is to consider the reactions

$$\begin{aligned} \mathrm{p} + \mathrm{p} &\rightarrow \mathrm{d} + \pi^+ \\ \mathrm{p} + \mathrm{n} &\rightarrow \mathrm{d} + \pi^0 \\ \mathrm{n} + \mathrm{n} &\rightarrow \mathrm{d} + \pi^-. \end{aligned}$$

Calculate the expected cross section ratios for these processes, accounting for the fact that the deuteron has no *nn* or *pp* bound state equivalents.
Hint: Clebsch-Gordan coefficients can be found, *e.g.*, on page 168 of the book "Introduction to Quantum Mechanics" by Griffiths, or on the Web as
`http://pdg.lbl.gov/2010/reviews/rpp2010-rev-clebsch-gordan-coefs.pdf`.
The latter version is also reproduced in one of the Appendices of the lecture notes.

3.3: Partial wave analysis
Note: this exercise relies heavily on the formalism as developed in the course "Kwantummechanica 2", see Chapters 1 and 6 of the lecture notes of this course, at `http://www.hef.ru.nl/~beenakker/dictaat_QM2.pdf`. *An alternative source of information is given by Chapters 4 and 11 of Griffiths' book "Introduction to Quantum Mechanics".*

Here, we use the partial-wave formalism developed for spinless particles and extend it slightly to deal with particles with spin. In particular, we consider the case of elastic pion-nucleon scattering (*e.g.*, $\pi^+\mathrm{p} \to \pi^+\mathrm{p}$), and use this formalism to demonstrate that the spin of the $\Delta(1232)$ baryon (the number in brackets is its mass in MeV) is $3/2$.

a) Show that the momentum in the CM frame is sufficiently low that only $L = 0$ and $L = 1$ partial waves have to be considered. (Hint: approximate the effective range of the interaction region by the *Compton wavelength* of the pion, $1/m_\pi$.)

We need only consider one proton spin S_z state, where we take the z axis to be the direction of the incoming particles. For the sake of definiteness, assume that $S_z = 1/2$ (due to parity conservation, the result for $S_z = -1/2$ can be obtained by merely changing $\theta \to \pi - \theta$). The relevant $|L, J, J_z\rangle$ states therefore become $|0, \frac{1}{2}, \frac{1}{2}\rangle$, $|1, \frac{1}{2}, \frac{1}{2}\rangle$, and $|1, \frac{3}{2}, \frac{1}{2}\rangle$. The spinless formula for $f(k,\theta)$ is therefore modified to

$$f(k,\theta) \sim f_0(k)|0, \frac{1}{2}, \frac{1}{2}\rangle + f_1^{(-)}(k)|1, \frac{1}{2}, \frac{1}{2}\rangle + f_1^{(+)}(k)|1, \frac{3}{2}, \frac{1}{2}\rangle.$$

b) Use the relevant Clebsch-Gordan coefficients to write this in terms of $|L, m\rangle|S, S_z\rangle$ states.

We will now assume that we are dealing either with a $J = 1/2$ or a $J = 3/2$ resonance. To find the angular distributions, we operate on the resulting $f(k,\theta)$ with the ket $\langle\vec{x}|$, and consider the result in terms of spherical coordinates. In addition, it is to be kept in mind that while states with different L can overlap (after all, the pion and nucleon don't remain as a bound state), states with different S_z are orthonormal (since S_z is – in principle – still *measurable*).

c) Use the expressions for the relevant Y_L^m to show that the angular distribution for a $J = 3/2$ resonance becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim 1 + 3\cos^2\theta.$$

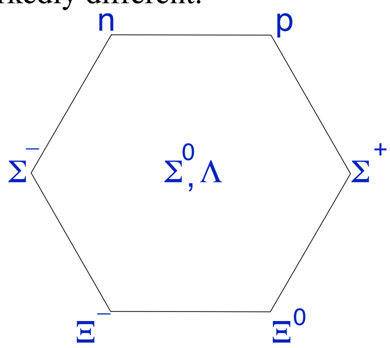
(this assumes that we don't actually measure the spin of the outgoing proton). What would the angular distribution for a $J = 1/2$ resonance look like?

This angular distribution was fairly easily measured in data, and confirmed the $J = 3/2$ nature of the $\Delta(1232)$ resonance.

3.4: The baryon octet revisited

From the case of the baryon decuplet, we learnt that only the *total* wavefunction $\psi = \psi(\text{flavour}) \cdot \psi(\text{spin}) \cdot \psi(\text{colour}) \cdot \psi(\text{space})$ for baryons is completely antisymmetric under the exchange of two particles. The introduction of colour, and the $l = 0$ nature of the octet (as well as the decuplet) imply that $\psi(\text{flavour}) \cdot \psi(\text{spin})$ must be completely symmetric. The purpose of this exercise is to show how this can be constructed for the octet, and at the same time learn why, unlike the case of the mesons, the $J = 1/2$ and $J = 3/2$ multiplets are markedly different.

For completeness, the octet is reproduced here. The key to constructing the appropriate wavefunctions is to start from *partially antisymmetric* building blocks. We first consider cases where an antisymmetrization procedure has been applied to quarks (1) and (2). For the flavour part of the wavefunction, this means that the proton can be represented as



$$|\text{p}\rangle_{(12)} = \frac{1}{\sqrt{2}} (|udu\rangle - |duu\rangle)$$

Such simple forms are appropriate for all baryons at the edges of the hexagon. The only ones for which a more complicated form has to be found are the Λ and the Σ^0 baryons. In the case of the Σ^0, the appropriate form can be obtained by applying the (isospin) lowering operator to the Σ^+; the Λ wavefunction is orthogonal to that:

$$\begin{aligned} |\Sigma^0\rangle_{(12)} &= \frac{1}{2}(|sud\rangle - |usd\rangle + |sdu\rangle - |dsu\rangle) \\ |\Lambda\rangle_{(12)} &= \frac{1}{\sqrt{12}}(2|uds\rangle - 2|dus\rangle + |usd\rangle - |sud\rangle - |sdu\rangle + |dsu\rangle) \end{aligned}$$

a) Write down similarly the spin wavefunction corresponding to $s_z = +1/2$ upon antisymmetrization under the exchange of quarks (1) and (2), *i.e.*, $\psi(\text{spin})_{(12)}$.

A completely symmetric wavefunction can be built as

$$\psi = \frac{\sqrt{2}}{3}\left(\psi_{(12)}(\text{flavour}) \cdot \psi_{(12)}(\text{spin}) + \psi_{(23)}(\text{flavour}) \cdot \psi_{(23)}(\text{spin}) + \psi_{(31)}(\text{flavour}) \cdot \psi_{(31)}(\text{spin})\right).$$

b) Use this to construct the complete (spin and flavour) wavefunction for a spin-up proton. Verify that the normalization factor above is appropriate.

c) Show that this construction implies that no equivalent of the Δ^{++}, Δ^-, or Ω^- exists in the $J = 1/2$ multiplet.

3.5: The Callan-Gross relation

As seen in the lectures, in the context of the *parton model* the differential cross section for deep-inelastic scattering is given by

$$\frac{\alpha^2}{q^4}\sum_i Q_i^2 f_i(x) L_{\mu\nu} K^{\mu\nu} ,$$

at least up to an overall normalization constant. Here, the lepton tensor $L_{\mu\nu}$ is given by

$$L_{\mu\nu} \equiv 2\left(p_{1\mu}p_{3\nu} + p_{1\nu}p_{3\mu} + g_{\mu\nu}(m_\mathrm{e}^2 - p_1\cdot p_3)\right),$$

and a similar expression (with the initial- and final-state four-momenta of the struck quark used instead of those of the electron) for $K^{\mu\nu}$.

a) Show that the expression for $K^{\mu\nu}$ becomes

$$K^{\mu\nu} = 2x\left(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p\cdot q) + 2x p^\mu p^\nu\right).$$

b) Contract this with the lepton tensor, and consider the result in the rest frame of the proton. Use the result to argue the validity of the Callan-Gross relation

$$F_2(x) = 2xF_1(x).$$

(*Hints: terms of order m_e^2 can be neglected. Use covariant notation as long as possible. Show that $p_1\cdot q = -p_3\cdot q = \frac{1}{2}q^2$, again up to terms of order m_e^2.*)

(I'm not calling this a complete *proof* in view of the lack of a complete and explicit cross-section calculation – but it's close.)

3.6: The Gottfried sum rule

Deep-inelastic scattering experiments have been carried out also using as a target the neutron rather than the proton: while the fact that the neutron is uncharged overall implies an absence of electromagnetic couplings at low energies, the fact that it *does* contain charged constituents means that the formalism developed for electron-proton scattering carries over directly to this case. The study of this process can be combined with that of the proton to help determine *individual* parton density functions (PDFs). In particular, the Gottfried sum rule provides some information about the PDFs of *anti-quarks* in the proton.

Neglecting the (weak) Q^2 dependence, as well as the presence of other (heavier) quarks, we can write the proton F_2 structure function as

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{1}{9}(\mathrm{d}^\mathrm{p}(x) + \bar{\mathrm{d}}^\mathrm{p}(x)) + \frac{4}{9}(\mathrm{u}^\mathrm{p}(x) + \bar{\mathrm{u}}^\mathrm{p}(x)) + \frac{1}{9}(\mathrm{s}^\mathrm{p}(x) + \bar{\mathrm{s}}^\mathrm{p}(x))\right),$$

where the superscript denotes that it is the PDFs for the proton that are relevant here. An identical formula holds for the neutron, but with the PDFs for the neutron instead. Now we can exploit the fact that the proton and the neutron are isospin partners, and assume

$$\begin{aligned} \mathrm{d}^{\mathrm{n}}(x) &= \mathrm{u}^{\mathrm{p}}(x) \\ \mathrm{u}^{\mathrm{n}}(x) &= \mathrm{d}^{\mathrm{p}}(x) \\ \mathrm{s}^{\mathrm{n}}(x) &= \mathrm{s}^{\mathrm{p}}(x) \end{aligned}$$

and likewise for the antiquark PDFs.

Use the above to derive the Gottfried sum rule,

$$\int_0^1 \left(F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x)\right) \frac{\mathrm{d}x}{x} = \frac{1}{3} + \frac{2}{3} \int_0^1 \left(\bar{\mathrm{u}}^{\mathrm{p}} - \bar{\mathrm{d}}^{\mathrm{p}}\right) \mathrm{d}x.$$

Hint: write the quark PDFs as the sum of a "valence" contribution (which determines the flavour of the hadron) and a "sea" contribution (resulting from the higher-order corrections shown in Fig. 3.16 of the lecture notes), and assume that the sea contribution is the same for a quark as for its anti-quark.

3.7: Colour counting

Why does the measurement of structure functions in deep-inelastic scattering not yield direct information on the number of quark colours (unlike the measurement of the cross section for the process $e^+e^- \rightarrow$ hadrons)?

Exercises for Chapter 4

4.1: Bhabha scattering at the Z resonance

a) Draw all four diagrams that are responsible for the *Bhabha scattering* process $e^+e^- \to e^+e^-$, when also the electron's weak interactions are accounted for.

b) Show that at $\sqrt{s} \approx M_Z$, the s-channel contributions are dominated by the Z boson.
Hint: the Z-boson propagator looks exactly like that for the W boson, but with M_W and Γ_W replaced with M_Z and Γ_Z, respectively.

c) Show that nevertheless, at sufficiently small scattering angles, the differential cross section is dominated by photon exchange.

This last fact was instrumental in the use of small-angle Bhabha scattering as a means to monitor the luminosity of the collider: its near-independence of the Z boson details makes it a suitable reference for the study of the Z boson.

4.2: Neutrino-electron scattering

a) Draw the Feynman diagram(s) for the scattering process $\nu_\mu + e^- \to \nu_\mu + e^-$.

b) Do the same for the process $\nu_e + e^- \to \nu_e + e^-$.

4.3: Fermion spin and helicity

The quantum mechanical spin operator $\vec{S}$ needs to be adapted to our new four-component spinors (bi-spinors); the appropriate definition is

$$\vec{S} = \frac{1}{2}\vec{\Sigma} \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},$$

where the σ^i are again the Pauli matrices.

a) Consider the Hamiltonian describing a free spin-1/2 particle,

$$H = \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m.$$

Use the basic commutation relation $[r_i, p_j] = i\delta_{ij}$ to show that the orbital angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ does not commute with H, and more specifically that

$$[H, \vec{L}] = -i\gamma^0 \vec{\gamma} \times \vec{p}.$$

Also show that the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ *does* commute with H.
Hint: it may be helpful to carry out the computations in component form rather than retaining vector notation.

This result shows once more (the explicit solution of the Dirac equation is another instance) that in a relativistic setting, one cannot simply "decouple" spin from other degrees of freedom!

b) There is one exception to the above situation, obtained by choosing as the quantisation axis the direction of movement $\hat{p} \equiv \vec{p}/|\vec{p}|$ of the particle and by considering the *helicity* operator

$$\lambda = \vec{S} \cdot \hat{p}.$$

Show that indeed $[H, \lambda] = 0$. (*Hint:* you may make use of the result obtained in item (a).)

This result implies that fermion helicity can be used as a proper quantum number.

4.4: Helicity and chirality

a) Show that in the *Björken and Drell convention* discussed in Sect. 2.3.1 of the lecture notes, the quantity $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ takes the form

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

b) Consider the bispinor solutions to the Dirac equation and show that for massless particles,

$$\gamma^5 \psi(p) = 2\lambda\, \psi(p),$$

with the helicity operator λ defined in exercise 4.3.

c) Using the above, show that the *chirality* operators $P_{\mathrm{L,R}} = (1 \mp \gamma^5)/2$ for massless particles indeed project onto left- and right-handed states, respectively. (It is furthermore easy to show that $P^2_{\mathrm{L,R}} = P_{\mathrm{L,R}}$, $P_{\mathrm{L}} P_{\mathrm{R}} = P_{\mathrm{R}} P_{\mathrm{L}} = 0$, which completes the proof of $P_{\mathrm{L,R}}$ being orthogonal projection operators.)

The physical significance of the difference between helicity and chirality in the case of *massive* particles can be clarified by considering the *Hamiltonian* that can be constructed for a free spin-1/2 particle:

$$H = \gamma^0(\vec{\gamma}\cdot\vec{p}+m).$$

d) Show that γ^5 does not commute with this Hamiltonian if $m \neq 0$. As was shown in exercise 4.3, the helicity operator *does*; however, helicity in this case is not a Lorentz-invariant quantity, as a Lorentz transformation can be applied to a frame in which the direction of the momentum is reversed, while the spin is not.

e) Prove the identity discussed in the lectures

$$\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L = \overline{\psi}\psi.$$

4.5: Interactions of the Higgs boson

a) Show that the coupling of the Higgs boson to fermions is proportional to the mass of those fermions.
Hint: from the material covered in the lectures, it is possible to show this for the charged leptons only. But the same procedure works for the down-type quarks as well.

b) However, for the up-type quarks this procedure does not work. Instead, an auxiliary field $\Phi^c \equiv i\sigma_2\Phi^*$ is introduced (which transforms under $SU(2)$ in the same way as Φ itself; what is the hypercharge of this auxiliary doublet?). Construct the term in the Lagrangian that after symmetry breaking gives rise to the masses of up-type quarks.

c) From the terms present in the Lagrangian after symmetry breaking, argue which Higgs boson interactions with the W and Z bosons will exist. Also indicate which Higgs boson *self-couplings* (terms h^n, with $n > 2$) exist.

d) From the parabolic behaviour of the Higgs potential $V(\Phi)$ after symmetry breaking, show that without access to the self-coupling constants (which haven't been determined as yet – the Higgs boson is yet to be discovered!) it isn't possible to derive an estimate of the Higgs boson mass m_H using existing information.

e) For the cases $m_H = 125$ GeV and $m_H = 160$ GeV, argue what will be the decay mode with the highest branching fraction.

f) The Higgs boson was searched for extensively at the LEP collider, in which electron-positron collisions occurred with centre-of-mass energies up to 210 GeV. Draw a Feynman diagram in which the Higgs boson is produced together with a Z boson, in these collisions. From the non-observation of the Higgs boson in this production mode, what constraint on m_H would you expect?

g) Two of the decay modes in which the Higgs boson was discovered in 2012 are to a pair of Z or a pair of W bosons. How can this happen, given that $m_H \approx 125$ GeV, *i.e.*, less than twice the W- or Z-boson mass?

4.6: Selection rules

For semileptonic decays of *strange* hadrons which change the strangeness quantum number, we have the so-called selection rule

$$\Delta S = \Delta Q = \pm 1,$$

where ΔQ represents the change in charge of the hadronic system. Using the underlying quark picture, show the validity of this selection rule.

4.7: Hadron decays in the spectator model

Draw the diagrams responsible (in the context of the spectator model) for the decays

a) $\mathrm{K}^- \to \pi^0 \mathrm{e}^- \bar{\nu}_\mathrm{e}$.

b) $\Lambda \to \mathrm{p}\,\pi^-$ (hint: there are *two* relevant diagrams for this decay.)

4.8: Decay modes of the W boson

Consider the possible decay modes of the W boson (it doesn't matter here whether we consider the W^+ or W^- boson: the only relevant assumption is that it is produced *on-shell*, *i.e.*, its four-momentum q^μ satisfies $q^2 = M_\mathrm{W}^2$. This is the case *e.g.* at LEP2: $\mathrm{e}^+\mathrm{e}^-$ scattering at $E_\mathrm{CM} > 2M_\mathrm{W}$), leptonic and hadronic. What fractions of the W boson decays will be to final states involving $\mathrm{e}^\pm$, $\mu^\pm$, $\tau^\pm$, and hadrons (and why)?

4.9: Top quark production at hadron colliders

We consider here in a bit of detail the physics of top quarks as relevant in hadron collider experiments. This is appropriate *e.g.* for the case of the Tevatron $\mathrm{p\bar{p}}$ collider (where the top quark was discovered in 1995, on the basis of its decay characteristics), but also for the LHC pp collider.

a) How many charged leptons and/or jets may be observed in the decays of $\mathrm{t\bar{t}}$ pairs? Classify the resulting final states accordingly.

b) Draw also a (tree level, *i.e.*, without loops) Feynman diagram leading to the production of a single top quark. (Hint: this can occur only through a weak process.)

4.10: Explicit solution of the two-state neutral kaon system

We extend slightly the Schrödinger equation describing the two-state neutral kaon system to include also kaon *decays*:

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\psi = (M - i\Gamma/2)\psi,$$

where M and Γ are hermitian 2×2 matrices; on general grounds (CPT invariance, for the initiated), they satisfy $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$. The purpose of this exercise is to derive the explicit solution to this Schrödinger equation seen as an eigenvalue problem.

a) Show that the eigenvalues λ of the "Hamiltonian" $\lambda = M_{11} - i\Gamma_{11}/2 + m - i\gamma/2$ (*i.e.*, we separate explicitly the real and imaginary parts and look in detail only at the effect of the off-diagonal terms) satisfy

$$\begin{aligned} m^2 - \frac{1}{4}\gamma^2 &= |M_{12}|^2 - \frac{1}{4}|\Gamma_{12}|^2, \\ m\gamma &= \Re(M_{12}\Gamma^*_{12}). \end{aligned}$$

b) Using the above equations, show that the two eigenstates can be written as

$$p|\mathrm{K}^0\rangle \pm q|\overline{\mathrm{K}^0}\rangle,$$

and derive the ratio q/p of the corresponding coefficients.

c) Show that the eigenstates are *not* in general orthogonal. Why is that?

4.11: B-meson mixing

The phenomenon of B^0-$\bar{\mathrm{B}}^0$ mixing has been studied extensively both at $\mathrm{e}^+\mathrm{e}^-$ colliders and at hadron colliders (as it yields information about the CKM matrix elements V_{td} and V_{ts} in the case of $\mathrm{B}^0_{\mathrm{d}}$-$\bar{\mathrm{B}}^0_{\mathrm{d}}$ mixing and $\mathrm{B}^0_{\mathrm{s}}$-$\bar{\mathrm{B}}^0_{\mathrm{s}}$ mixing, respectively). Using state-of-the-art detectors, it is now possible to observe the oscillation signal proper. However, it is also possible to determine the oscillation frequency Δm indirectly (Δm is the difference between the two mass eigenstates), and that method has proven useful in the case of the $\mathrm{B}^0_{\mathrm{d}}$. This indirect determination is the subject of this exercise.

a) Show that the *time-integrated* fraction of neutral B mesons produced as a B^0 meson and decaying as a $\overline{\mathrm{B}^0}$ meson χ is given by

$$\chi = \frac{1}{2}\frac{x^2}{1+x^2} \quad \text{with} \quad x \equiv \Delta m/\Gamma,$$

and where Γ is the (unique) lifetime of the B mesons (*i.e.*, we ignore any lifetime difference). Argue that this method was *not* useful in the case of the $\mathrm{B}^0_{\mathrm{s}}$ meson, for which $x \approx 22$ (in this case, the oscillation signal was demonstrated only a few years ago), while it was useful for the $\mathrm{B}^0_{\mathrm{d}}$ meson, for which $x \approx 0.8$.

(Hint: start from the corresponding formula for the kaon system. Realizing that the number of decays occurring at each time t is proportional to the remaining (undecayed) number of B mesons, integrate the decay distribution.)

b) Assume that the production mechanism of B-mesons is such that always a pair of B^0 and $\overline{B^0}$ mesons is produced, and that we are considering their production far above threshold (so that the two mesons are produced *incoherently*, *i.e.*, one particular meson out of the two was produced as a B^0; this is in contrast to the situation at so-called B-factories, operating at $\sqrt{s} = m(\Upsilon(4S))$, leading to a coherent production of B^0 and $\overline{B^0}$). Consider the case where both mesons decay semileptonically (at the quark level, this corresponds to the transition $b \to c\ell\bar{\nu}_\ell$ or its charge conjugate; or instead of the charm quark the final state could contain an up quark). Show how a consideration of the number of same-sign and opposite-sign lepton events (the numbers of events in which the two leptons have the same or opposite charge) can be used to estimate χ.

4.12: Quantum mechanics of neutrino oscillations

There are a few aspects concerning neutrino oscillations that require a proper quantum mechanical treatment, and that have not been addressed in the lecture notes.

The first of these is related to the somewhat counter-intuitive fact that the different mass eigenstates propagate coherently over macroscopic (and even very large) distances. One might wonder whether it is not possible (in principle) to *determine* the mass of the neutrino simply from the measured kinematics of the decay producing the neutrino (or otherwise, from the charged current interaction leading to a charged lepton in the final state). If this were possible, it should not be possible to observe an oscillation pattern.

a) Using the normal dispersion relation

$$m_\nu^2 = E_\nu^2 - \mathbf{p}_\nu^2,$$

and the Heisenberg uncertainty relation $\Delta p \Delta x > 1$, show that *if* the neutrino four-momentum (and hence its squared mass) can be determined sufficiently well to distinguish between different squared-mass states (say $\Delta m_{12}^2 \equiv m_1^2 - m_2^2$), the corresponding uncertainty in the neutrino position measurement (of either its creation or its destruction) becomes larger than the oscillation length describing the oscillation pattern for these masses. (Once this becomes the case, it will become impossible to observe any oscillation pattern anymore.)

The other issue is related to *decoherence*, which results from the different velocities at which the different mass eigenstates propagate. The coherence between the two eigenstates can be said to be lost when the *distance* between the different eigenstates becomes larger than the length of the propagating wave packet (which can be taken to be the uncertainty in the position at which the neutrino was created).

b) Assuming a position uncertainty of the order of a nm (*i.e.* somewhat larger than the inter-atomic spacing in a solid), show that for the typical neutrino momenta (ranging from about 1 MeV to 100 GeV), show that coherence is lost only after *very* many oscillation lengths. (Under any practical circumstances, the oscillation pattern will have long disappeared due to the finite momentum spread.)

It is useful to contrast this with the equivalent case in the quark sector: there, it would be appropriate to consider the d and s quarks. Because quarks cannot propagate freely, it is the charged pion ($m_\pi = 140$ MeV) and the charged kaon ($m_\mathrm{K} = 494$ MeV) that are to be taken as the propagating mass eigenstates instead. Show that decoherence is reached practically instantaneously.