Foundations of gravitational waves and black hole perturbation theory 2020/2021 (NWI-NM125)

Problem sheet #5: Linearized gravity and the Regge-Wheeler equation

Tutorial on Thursday 3 March 2022, 13:30 - 15:15

Exercise 5.1: Linearized gravity on maximally symmetric backgrounds

There are three maximally symmetric spacetimes that distinguish themselves by the value of the cosmological constant Λ: Minkowski ($\Lambda = 0$), de Sitter ($\Lambda > 0$) and anti-de Sitter ($\Lambda < 0$).

a) Warm up: Show that the equation for the linearized Riemann tensor in terms of the linearized perturbed metric on a Minkowski background reduces to

$$\frac{d}{d\lambda} R_{abc}{}^{d}(\lambda)\Big|_{\lambda=0} = -\nabla_c \nabla_{[a} \gamma_{b]}{}^{d} + \nabla^{d} \nabla_{[a} \gamma_{b]c} \tag{1.1}$$

b) What equation does the linearized metric perturbation on a (anti-)de Sitter spacetime satisfy?

c) Show how the implementation of the Lorenz gauge takes you from Eq. (6.21) to Eq. (6.26) in the Lecture notes.

d) Show that the linearized Ricci scalar on *any* maximally symmetric background is gauge invariant, that is, it does not change when transforming $\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} + 2\nabla_{(a}\xi_{b)}$ for any ξ_a. Hint: Determine the background Ricci tensor and use that the commutator of two covariant derivatives is proportional to the Riemann tensor.

Exercise 5.2: Gauge transformations and invariance of Regge-Wheeler quantities

Understand gauge transformations in the context of Schwarzschild spacetimes.

a) Derive the gauge transformations for the even-parity sector in Eq. (7.13). (Recall that ∇_p is the covariant derivative operator compatible with the two-dimensional subspace spanned by (t, r), much like D_A is the covariant derivative operator compatible with S_{AB}.)

b) Do the same for the odd-parity sector in Eq. (7.14).

c) Show explicitly that the odd-parity quantities $\tilde{h}_0$ and $\tilde{h}_1$ in Eq. (7.17) are indeed gauge-invariant.

Exercise 5.3: Parity odd master function

The Regge-Wheeler master equation for the parity odd case.

a) Write the Regge-Wheeler equation in terms of Ψ_{odd} in retarded null coordinates (u, r, θ, ϕ).

b) Show that the relation between the Regge-Wheeler function and the Cunningham-Price-Moncrief function in Eq. (7.25) in the Lecture notes is true.

Additional practice

Exercise 5.4: The efficiency of spherical harmonics

Become familiar with some key properties of spherical harmonics.

a) Under a parity transformation P, the angular coordinates change as

$$\theta \longrightarrow \pi - \theta \tag{4.1a}$$
$$\phi \longrightarrow \pi + \phi \tag{4.1b}$$

so that $PY_{\ell m}(\theta, \phi) = Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi)$. We therefore call the scalar spherical harmonics parity even (if it had transformed as $(-1)^{\ell+1}$, it would have been parity odd). Show how the vector and tensor harmonics transform under a parity transformation.

b) Show that the following identities for the vector harmonics hold:

$$S^{BC} D_A D_B Y^{\ell m}_C = -\ell(\ell+1) Y^{\ell m}_A \tag{4.2a}$$
$$S^{BC} D_B D_C Y^{\ell m}_A = [1 - \ell(\ell+1)]\, Y^{\ell m}_A \tag{4.2b}$$
$$S^{BC} D_A D_B X^{\ell m}_C = 0 \tag{4.2c}$$
$$S^{AC} D_A D_B X^{\ell m}_C = X^{\ell m}_B \tag{4.2d}$$
$$S^{BC} D_B D_C X^{\ell m}_A = [1 - \ell(\ell+1)]\, X^{\ell m}_A \tag{4.2e}$$

and for the tensor harmonics

$$S^{CD} D_A D_B (Y^{\ell m} S_{CD}) = 2Y^{\ell m}_{AB} - \ell(\ell+1) Y^{\ell m} S_{AB} \tag{4.3a}$$
$$S^{BC} D_C Y^{\ell m}_{AB} = \left[1 - \frac{\ell(\ell+1)}{2}\right] Y^{\ell m}_A \tag{4.3b}$$
$$S^{BC} D_C X^{\ell m}_{AB} = \left[1 - \frac{\ell(\ell+1)}{2}\right] X^{\ell m}_A \,. \tag{4.3c}$$

Exercise 5.5: Quasi-normal modes

Quasi-normal modes are the normal modes of a black hole.

a) Decompose Ψ_{odd} in terms of a Fourier expansion

$$\Psi_{\text{odd}}(t, r, \theta, \phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\ell, m} Q_{\ell m}(\omega, r) e^{-i\omega t} Y_{\ell m}(\theta, \phi) \,. \tag{5.1}$$

Show that the Regge-Wheeler equation can be written as a one-dimensional Schrödinger equation of the form:

$$\left[-\frac{d^2}{dr_*^2} + V_\ell(r)\right] Q_{\ell m} = -\omega^2 Q_{\ell m} \,. \tag{5.2}$$

where r_* is the tortoise coordinate, which runs from $-\infty$ to ∞ while r runs from $2M$ to ∞. You should also give the explicit form of V_ℓ.

b) Plot the potential V_l for various values of l and argue that there do not exist any bound states.

Extra background information: To solve this differential equation, one needs to impose boundary conditions. Similar to say the differential equation of a string, the imposition of (certain) boundary conditions selects some discrete values of ω and the resulting solutions describe the normal modes of the system. Like the normal modes of a string, ω is also discrete in the black hole setting. However, in contrast to the normal modes of a string, the normal modes of a black hole have both a real and imaginary part. To highlight this, they are called *quasi*-normal modes. These are observable by gravitational wave observatories just after the merger when the binary system settles down to a single black hole.