Foundations of gravitational waves and black hole perturbation theory 2020/2021
(NWI-NM125)

Problem sheet #2: Asymptotic flatness

Tutorial on Thursday 10 February 2022, 13:30 - 15:15

This tutorial is graded. You will only need to hand in Exercise 2.1 and 2.2 (the other two exercises are additional practice). Please hand in your tutorial *before* the start of the next tutorial on Thursday 17 February at 13:30. Tutorials received after the deadline will be marked as a 1. If you are not able to hand in your solutions in person, you can email a scan or photo of your answers to Patricia (patricia.ribesmetidieri@ru.nl).

Exercise 2.1: Asymptotic structure with a cosmological constant

How the inclusion of a cosmological constant changes the boundary of spacetime.

a) If g_{ab} satisfies Einstein's equation with cosmological constant Λ

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G \, T_{ab} \,, \tag{1.1}$$

what equation does the conformally rescaled metric $\tilde{g}_{ab}$ satisfy? Use the results from tutorial 1 for $\tilde{R}_{ab}$ and $\tilde{R}$. It is convenient to rewrite $\nabla_a \ln \Omega$ as $\Omega^{-1} \nabla_a \Omega$. Note that this is a long calculation. Explain why the conformally rescaled metric is sometimes called the "unphysical metric".

b) Define $n_a := \tilde{\nabla}_a \Omega$, multiply the equation you obtained in part (a) by Ω^2 and take the limit $\Omega \to 0$ (assume that $\lim_{\Omega \to 0} \Omega^2 T_{ab} = 0$). What is the norm of the co-vector field n_a in this limit? Indicate whether the norm of n_a is time-like, null-like or space-like for the three different scenarios: (1) a negative cosmological constant $\Lambda < 0$, (2) a vanishing cosmological constant $\Lambda = 0$ and (3) a positive cosmological constant $\Lambda > 0$.

Exercise 2.2: Conformal diagram of Minkowski spacetime

Understand the conformal diagram of Minkowski spacetime and the location of its different infinities.

The Minkowski metric in standard (t, r, θ, ϕ) coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \tag{2.1}$$

where $-\infty < t < \infty$, $0 \leq r < \infty$ and (θ, ϕ) with their standard range on the two-sphere. To make a conformal diagram we need to ensure that (1) null rays are at $\pm 45°$ with respect to the horizontal axis and (2) the coordinate range is finite. (We will suppress the angular part of the metric in the diagram and think of each point as a two-sphere.) In order to achieve the first condition, we switch to null coordinates

$$u = t - r \qquad \text{and} \qquad v = t + r \,, \tag{2.2}$$

so that now lines of constant u and v are null geodesics. To implement the second, we perform an additional coordinate transformation to

$$U = \arctan u \qquad \text{and} \qquad V = \arctan v \,, \tag{2.3}$$

where now the range of U and V is finite:

$$-\frac{\pi}{2} < U < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < V < \frac{\pi}{2} \qquad \text{and} \qquad U \le V < \frac{\pi}{2}. \tag{2.4}$$

a) Show that in these coordinates the metric takes the form

$$\mathrm{d}s^2 = \frac{1}{4\cos^2 U\, \cos^2 V}\left[-4\mathrm{d}U\mathrm{d}V + \sin^2(V-U)\,\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]. \tag{2.5}$$

b) Perform a conformal transformation to $\mathrm{d}\tilde{s}^2 = \Omega^2\mathrm{d}s^2$ with $\Omega = 2\cos U\,\cos V$. To interpret the result, write the metric after performing one last coordinate transformation to T, R coordinates:

$$T = U + V \qquad \text{and} \qquad R = V - U. \tag{2.6}$$

Show that the range of these coordinates is finite, specifically, $0 \le R < \pi$ and $|T| + R < \pi$.

c) Complete the table below indicating the location of the different infinities in various coordinate systems.

	U, V	T, R
$\mathcal{I}^+$		
$\mathcal{I}^-$	$U = -\frac{\pi}{2},\ -\frac{\pi}{2} < V < \frac{\pi}{2}$	$T = -\pi + R,\ 0 < R < \pi$
i^+		$T = \pi,\ R = 0$
i^-		
i^0		$T = 0,\ R = \pi$

d) *Extra:* Null rays are always at 45° in a conformal diagram. How do lines with $r = r_0$ (with r_0 some finite, non-zero constant) look? Answer this question by expressing $r = r(T, R)$ and draw this in a conformal diagram. Feel free to use Mathematica (or any other software) to help you answer this question.

Additional practice

Exercise 2.3: FLRW spacetimes are not asymptotically flat

Understand that not all physically interesting spacetimes are asymptoticaly flat and become more familiar with conformal freedom.

a) Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes are cosmological spacetimes that are homogeneous and isotropic. The stress-energy tensor of these spacetimes is modeled by a perfect fluid. Using a symmetry argument, explain why these spacetimes are not asymptotically flat (i.e., which of the conditions is broken?).

b) While FLRW spacetimes are not asymptotically flat, a subclass of these solutions (the ones whose expansion is decelerating) can be studied using similar techniques. A key difference is the definition of the normal to $\mathcal{I}$:

$$n_a := \Omega^{-s}\tilde{\nabla}_a\Omega\,, \tag{3.1}$$

where $0 \le s < 1$ encodes information about the equation of state that relates the energy-density and pressure of the fluid. The conformally completed metric is still defined through $\tilde{g}_{ab} = \Omega^2 g_{ab}$ with g_{ab} the physical metric. How do $\tilde{g}_{ab}$, $\tilde{g}^{ab}$, n_a and n^a change under a conformal rescaling $\Omega \longrightarrow \Omega' = \omega\Omega$? Check that your answer reduces to Eq. (2.14) in the Lecture notes when $s = 0$.

c) Show that under a conformal rescaling the divergence in the new frame is related to that in the original frame through

$$\nabla'_a n'^a \hat{=} \omega^{-1-s}\tilde{\nabla}_a n^a + (4-2s)\omega^{-2-s}\mathcal{L}_n\omega\,. \tag{3.2}$$

If one is not in a conformal divergence free frame, what should ω solve in order for $\nabla'_a n'^a \hat{=} 0$?

Exercise 2.4: Fall-off stress energy tensor

Show that a point charge on Minkowski spacetime satisfies the demand on the stress-energy tensor in the definition of asymptotic flatness.

In this exercise, we will work with electromagnetic fields on a Minkowski background. Recall that the electromagnetic field $F_{ab} := 2\nabla_{[a}A_{b]}$ can be decomposed into an electric E_a and magnetic field B_a. This decomposition is observer dependent. For an observer with ξ^a tangent to his/her wordline (and ξ^a normalized, i.e. $\xi^a\xi_a = -1$), this decomposition is

$$F_{ab} = 2\xi_{[a}E_{b]} - \epsilon_{abcd}\xi^c B^d \,. \tag{4.1}$$

a) Using the above result, show that this is consistent with the standard definition of the electric and magnetic field

$$E_a := F_{ab}\xi^b \tag{4.2}$$

$$B_a := \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}\xi^b \,. \tag{4.3}$$

You likely will need to use that

$$\epsilon^{a_1\ldots a_j a_{j+1}\ldots a_n}\epsilon_{a_1\ldots a_j b_{j+1}\ldots b_n} = (-1)^s\,(n-j)!\,j!\,\delta^{[a_{j+1}}_{b_{j+1}}\ldots\delta^{a_n]}_{b_n} \tag{4.4}$$

where $s = 0$ if the volume form has Riemannian signature and $s = 1$ if it has Lorentzian signature.

b) The electromagnetic potential for a point charge in Minkowski spacetime in Lorenz gauge is

$$A_a = -\frac{q}{r}\nabla_a t \,. \tag{4.5}$$

Calculate the electromagnetic field tensor F_{ab}, as well as the electric and magnetic field for a static observer. Is this result what you expected?

c) The stress-energy tensor for electromagnetism is

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{cd}F^{cd} \,. \tag{4.6}$$

Calculate the stress-energy tensor for the point charge in (t, r, θ, ϕ) coordinates. Make sure to check that the stress-energy tensor you obtain is traceless.

d) Take the limit to $\mathcal{I}$ by introducing $u = t - r$ and $\Omega = \frac{1}{r}$. Check that this stress-energy tensor satisfies the fall-off required in the definition of asymptotically flat spacetimes, i.e. $\Omega^{-2}T_{ab}$ has a limit to $\mathcal{I}$.

e) Based on the analysis for a static point charge, what would the limit of T_{ab} to $\mathcal{I}$ be for higher multipole moments? Your answer does not require any calculations.