Foundations of gravitational waves and black hole perturbation theory 2020/2021 (NWI-NM125)

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Contents

1 Overview

This course is divided into two parts. In the first part, we will study many different spacetimes at the same time and extract general features from this large class. Of course, we are not interested in some random selection of spacetimes: we would like to study physically interesting spacetimes such as those with gravitational radiation generated by some compact source (for instance, a black hole binary system or a rotating neutron star). While the spacetime near these compact objects is strongly curved and the details to describe these spacetimes are very complicated, as you move far away from the strong gravity regime, the spacetime curvature dies off and the spacetime starts to look like flat spacetime. Hence, these spacetimes capture the idea of an isolated system and should be asymptotically flat. We will learn how to rigorously define this intuitive concept of asymptotically flat spacetimes, what their asymptotic symmetries are and how to describe gravitational radiation. That this is possible is fortunate for us as theoretical physicists and is in no way a demand on the theory, as so nicely put in words by Geroch:

> "After all, a given physical theory — and general relativity in particular — has no need whatever of any notion of "isolated system": The theory is, at least in principle, as viable, as self-contained, as predictive without such a notion with it. A definition is sought, rather, for its convenience, for we choose to understand the Universe through analysis of smaller, simpler systems, one at a time. There are no "correct" or "incorrect" definitions, only more or less useful ones." [1, page 4]

In the second part, we will specialize to studying perturbations on a black hole background. Black holes were once thought to be merely mathematical solutions unrelated to our Universe. Nowadays they play a central role in our understanding of the Universe. The Event Horizon Telescope recently observed light emitted from the accretion disk of the black hole at the center of the galaxy M87. This black hole is a supermassive black hole with a mass about $4 \times 10^8 M_\odot$. The LIGO-Virgo collaboration has observed binaries comprised of stellar-mass black holes in the range $1 - 100 M_\odot$ by observing the gravitational radiation emitted as the two black holes orbit each other and finally merge. These recent observations add to the growing body of observations of black holes by other means such as X-ray observations and precise measurements of stellar orbits near black holes. In these lecture notes, we will first learn about gravitational perturbations on the background of a non-rotating black hole. Next, we will learn about the Geroch-Penrose-Held formalism to also study perturbations off a rotating black hole spacetime.

Conventions: In these notes, I will use geometrical units with $c = G_\mathrm{N} = 1$ where c is the speed of light and G_N is the gravitational constant, use abstract index notation unless otherwise indicated (see [2]), restrict myself to four-dimensional spacetimes and use the mostly plus convention $(- + + +)$. For all other conventions, I usually follow those in [2].

Disclaimer: These lecture notes may still contain typos (a special thanks goes out to Joost Remie, Luuk Venbrux and Luka Stam for catching many of them already). If you find any typos, mistakes or if certain parts are unclear, please send me an email (bbonga@science.ru.nl).

2 Asymptotically flat spacetimes

Asymptotically flat spacetimes are, roughly speaking, spacetimes whose metric approaches a Minkowski metric in the limit as one moves away from the source. Analytic examples are the Schwarzschild, Kerr-Newman and Vaidya spacetimes. Asymptotic flatness has many applications in gravitational science ranging all the way from numerical relativity (for instance, 'Characteristic Cauchy Extraction' in current spectral codes) to mathematical relativity (such as in the positive mass theorems).

Making the intuitive idea of becoming "flat" far away precise is delicate. How quickly should the metric "approach a Minkowski metric"? If the metric becomes flat too slowly, then the asymptotic behavior is not sufficiently close to that of Minkowski space and one does not gain anything by moving far away. If, on the other hand, the approach to flatness is too fast, then the asymptotic metric is so much like that of Minkowski spacetime that one rules out interesting spacetimes. For instance, you could imagine that in some appropriate coordinate system you demand that the metric should have no $1/r$ terms and thereby you would rule out systems with non-zero mass (such as the Schwarzschild metric). Definition of asymptotic flatness represent a compromise between these two effects. In some ways, this definition is like specifying boundary conditions general enough to describe a large class of spacetimes and rigid enough to recover an interesting structure that allows you define concepts such as energy and angular momentum. To some degree, this is as much an art as science. Perfectly reasonable scientists can disagree on what such boundary conditions should be. Fortunately, most scientists agree on the definition of asymptotic flatness. [1]

There are two ways to study asymptotically flat spacetimes: (1) in the spirit of general relativity using differential geometry, or (2) introducing coordinates. Historically, the second approach was developped first by Bondi, Sachs and others and later on this was 'geometrized' by Penrose and others. We will follow the opposite route in these lecture notes: we will first learn about the geometric approach and later on learn how this translates to the coordinates of Bondi and Sachs.

[1] Although, there are some recent exceptions advocating for more relaxed boundary conditions [3, 4, 5, 6]; but see [7] for a criticism of that approach.

Background knowledge

The following concepts are used in the remainder of this section. Most of these concepts are also introduced in text and tutorials, but if you would like some more background information, I can recommend:

- **Conformal transformations.** For more background on these transformations, see tutorial 1 and [2, App. D].
- **Hypersurfaces.** If you want to brush up your background on hypersurfaces in general relativity, I can recommend [8, first part App. D].
- **Conformal diagrams.** You will learn more about these diagrams in Sec. 2.1 and in tutorial 2. For a detailed introduction to this topic, I can recommend Tong's lecture notes [9, Ch. 4], Blau's lecture notes [10, Ch. 28] and/or App. H in Carroll's book [8].
- **Expansion, shear and twist.** These concepts describe how a geodesic congruence changes. [A geodesic congruence is a collection of geodesics in an (open) region of spacetime such that every point in that region belongs to exactly one geodesic.] Simply put, for any geodesic congruence with v^a as a tangent vector: $\nabla_a v^a$ is its expansion, $\nabla_{(a}v_{b)} - \frac{1}{n}\nabla_c v^c g_{ab}$ is its shear and $\nabla_{[a}v_{b]}$ is its twist. For a quick review, see [8, App. F] (note: Carroll calls twist 'rotation') or [2, first part of Ch. 9.2].
- **Lie derivatives.** For all practical purposes, you only need to know that

$$\begin{aligned}\mathcal{L}_v T^{abc\ldots}{}_{def\ldots} = v^m \nabla_m T^{abc\ldots}{}_{def\ldots} - T^{mbc\ldots}{}_{def\ldots}\nabla_m v^a - T^{amc\ldots}{}_{def\ldots}\nabla_m v^b \\ - T^{abm\ldots}{}_{def\ldots}\nabla_m v^c - \ldots + T^{abc\ldots}{}_{mef\ldots}\nabla_d v^m + T^{abc\ldots}{}_{dmf\ldots}\nabla_e v^m \\ + T^{abc\ldots}{}_{dem\ldots}\nabla_f v^m + \ldots\end{aligned} \tag{2.1}$$

  where ∇_a can be any derivative operator and does not necessarily have to be the covariant derivative operator. For instance, it sometimes is very convenient to use the partial derivative instead. If you would like a more thorough understanding of Lie derivatives, see [8, App. B].
- **Pullback.** Pullbacks are about maps between different manifolds. In Sec. 2.3, we use them to make concepts like the induced metric more precise. You do not need to be an expert on pullbacks (and its related concept pushforwards), but if you like to know more, see [8, App. A] or [9, Ch. 2]

2.1 Canonical example: Minkowski spacetime

The canonical example of an asymptotically flat spacetime is of course flat spacetime itself. Therefore, we will study this first and highlight some important features. From the conformal diagram in Fig. 1, it is immediately clear that there are many ways to take a limit to infinity:

- One can fix the time coordinate t and send $r \to \infty$ to reach *spatial infinity* i^0;
- One can fix the radial coordinate r and send $t \to \pm\infty$ to reach *future (past) time-like infinity* $i^\pm$;
- One can fix the null coordinate $u := t - r$ ($v := t + r$) and send $r \to \infty$ to reach *future (past) null infinity* $\mathcal{I}^\pm$ (pronounced as "scri-plus" and "scri-minus").

Since we are interested in studying gravitational radiation, which like electromagnetic and other types of radiation propagates along null geodesics (within the geometric optics approximation), the last limit is the most relevant to take; $\mathcal{I}^+$ is the natural arena where all forms of radiation find its final resting place. Writing the Minkowski metric in null coordinates

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{2.2a}$$

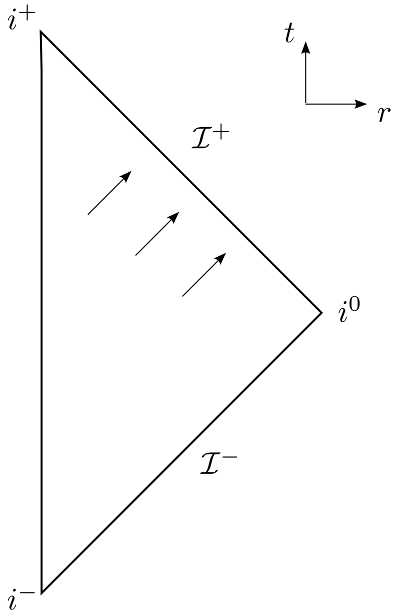

Figure 1: Conformal diagram of Minkowski spacetime. Each point represents a two-sphere with the exception of the line $r = 0$, which is one-dimensional. Radiation travels along null geodesics, which are always at a 45° angle with respect to the time-axis.

$$= -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) , \tag{2.2b}$$

we notice that if we take the limit $r \to \infty$, the metric components on the two-sphere diverge. But the situation is even worse: as we will see, also the differential $\mathrm{d}r$ diverges in this limit. You may wonder if this is an artifact of the coordinates or whether the metric as a tensor truly diverges. There are two methods to assess this: (1) contract the tensor with a complete set of smooth vector fields and assess the resulting scalars, or (2) introduce coordinates that are smooth in the limit and evaluate all the components of the tensor with respect to that smooth coordinate system. We will follow the second approach here. Introduce $\Omega = \frac{1}{r}$ so that $\Omega \to 0$ as $r \to \infty$ and $\mathrm{d}\Omega$ is a perfectly well-defined differential, also at $\Omega = 0$. Going back to $\mathrm{d}r$ and writing it in terms of Ω

$$\mathrm{d}r = \mathrm{d}(\Omega^{-1}) = -\Omega^{-2}\mathrm{d}\Omega , \tag{2.3}$$

we see that $\mathrm{d}r$ diverges as Ω^{-2} in the limit $r \to \infty$. In other words, the limit to infinity of the physical metric is not well-defined, or even better phrased: the physical metric at $r = \infty$ is *not* defined. So how should we take the limit to infinity? A clever mathematical technique to overcome this is to "bring infinity in to a finite place, and represents it by additional points attached to space-time". How can we achieve this? By a conformal transformation! Thus, we introduce an unphysical metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ for which the limit $\Omega \to 0$ is well-defined. The conformally rescaled Minkowski metric is

$$\tilde{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \Omega^2 g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Omega^2\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}\Omega + \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) , \tag{2.4}$$

which in the limit to infinity simply becomes

$$\lim_{\Omega\to 0} \tilde{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = 2\mathrm{d}u\mathrm{d}\Omega + \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 . \tag{2.5}$$

There are two interesting features of the metric at $\mathcal{I}$: (1) the metric is degenerate at $\mathcal{I}$, i.e., it has signature $(0 + +)$ (indicating that $\mathcal{I}$ is a null surface), (2) the surfaces of constant u and Ω are round two-spheres so that $\mathcal{I}$ is topologically $\mathbb{R} \times \mathbb{S}^2$. We will see that all asymptotically flat spacetimes share these two properties with Minkowski spacetime.

We were strictly speaking not allowed to take the above limit $r \to \infty$ (or equivalently, $\Omega \to 0$) as the coordinate range for Minkowski space is $0 \leq r < \infty$, in other words '$r = \infty$' is not part

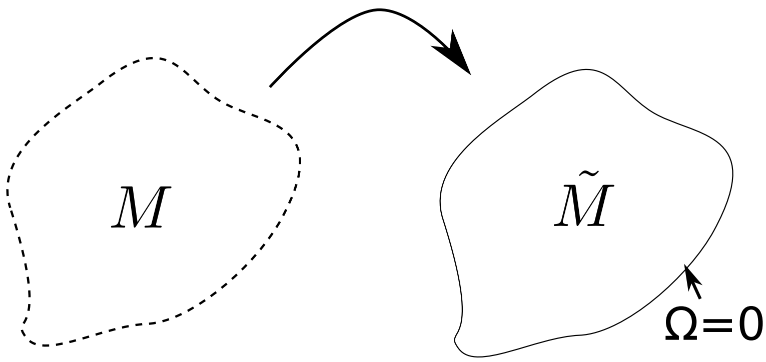

Figure 2: Visualization of a conformally completing a spacetime: the boundary $\Omega = 0$ is not part of the manifold M, but is part of the conformally completed manifold $\tilde{M}$.

of the spacetime (not so surprisingly, as the metric diverges there). In conformally rescaling the spacetime, we have "secretly" added the boundary $\Omega = 0$ to the original spacetime M. Therefore, we are in fact working with a larger manifold $\tilde{M} = M \cup \mathcal{I}$, known as the conformally completed spacetime. This procedure is illustrated in Fig. 2. Adding the boundary $\mathcal{I}$ to the manifold M is very powerful as it allows us to continue to use local differential geometry — a key tool in general relativity.

Remark 2.1. In the above, we focused on future null infinity $\mathcal{I}^+$. One can reach $\mathcal{I}^-$ by replacing u by v and then taking the limit $r \to \infty$ keeping v fixed. For most isolated systems we are interested in modeling, a reasonable assumption is to implement a 'no incoming radiation' condition, which makes $\mathcal{I}^-$ a somewhat boring surface. It is on future null infinity where one can track the dynamics of the system, and we therefore mostly focus on $\mathcal{I}^+$. However, all results for $\mathcal{I}^+$ also apply to $\mathcal{I}^-$ (up to some minor modifications). Occasionally, we refer to $\mathcal{I}$ in which case one can either read $\mathcal{I}^+$ or $\mathcal{I}^-$.

2.2 Geometric definition

Asymptotically flat spacetimes describes a large class of spacetimes that are modeled after Minkowski spacetime. The exact definition is as follows.

Definition 2.1. A space-time (M, g_{ab}) is *asymptotically flat* if there exists a manifold $\tilde{M}$ with boundary $\mathcal{I}$ equipped with a Lorentzian metric $\tilde{g}_{ab}$ and a diffeomorphism from M onto $(\tilde{M} \setminus \mathcal{I})$ such that:

(i) there exists a smooth function Ω on $\tilde{M}$ such that $\tilde{g}_{ab} = \Omega^2 g_{ab}$ on M; $\Omega = 0$ on $\mathcal{I}$; and $n_a := \tilde{\nabla}_a \Omega$ is nowhere vanishing on $\mathcal{I}$;

(ii) $\mathcal{I}$ is topologically $\mathbb{R} \times \mathbb{S}^2$;

(iii) g_{ab} satisfies Einstein's equation, i.e., $R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G\, T_{ab}$ where $\Omega^{-2}\, T_{ab}$ has a smooth limit to $\mathcal{I}$.[2]

The first condition ensures that $(\tilde{M}, \tilde{g}_{ab})$ is a conformal completion of (M, g_{ab}) in which the boundary $\mathcal{I}$ is at infinity with respect to the physical metric g_{ab} (by demanding that $\Omega = 0$ on $\mathcal{I}$). Note that we used the same general method for attaching a boundary to the physical spacetime as we did in the case of Minkowski spacetime in Sec. 2.1. The condition that $\tilde{\nabla}_a \Omega$ is nowhere vanishing on $\mathcal{I}$ ensures that we can use the conformal factor on $\mathcal{I}$ to perform Taylor expansions of physical fields and capture their decay as these fields approach $\mathcal{I}$. In terms of the physical spacetimes, the

[2] In these lecture notes, I assume that all fields are smooth. These conditions can be relaxed, and weaker differentiable structures have been studied in the literature.

condition $\tilde{\nabla}_a \Omega \neq 0$ ensures that Ω has the same leading order behavior as in Minkowski spacetime, that is, the conformal factor Ω falls off like $1/r$.

The second condition requires null infinity $\mathcal{I}$ to have the same topology as the conformal completion of Minkowski spacetime. This topological restriction is — quite surprisingly — essential for constructing a useful framework at null infinity. Without this condition, the notion of asymptotic symmetries (which we will discuss in Sec. 3) would be drastically different and one cannot introduce basic notions such as the energy flux radiated by gravitational waves.

Condition (i) and (ii) are — while motivated by physics — essentially pure math: you could apply this to any manifold with a metric, there is no physics involved. This is where point (iii) comes into play: first of all, we demand that the physical metric satisfies Einstein's equation. Moreover, to ensure that this class of spacetimes is asymptotically flat and describes isolated systems with physically reasonable properties, the curvature should decay near $\mathcal{I}$. Alternatively, matter fields should decay near $\mathcal{I}$. This is encoded in the requirement that $\Omega^{-2}\, T_{ab}$ has a smooth limit to $\mathcal{I}$. The precise power of Ω ensures that the total energy-momentum of the system is well-defined at any instant of retarded time along $\mathcal{I}$. Standard matter fields such as Maxwell fields satisfy this condition and one therefore does not expect this to rule out any physically interesting spacetimes.

We can strengthen the above definition by also demanding that $\mathcal{I}$ has the "right size" so that the spacetime is not just asymptotically flat, but asymptotically Minkowskian.

Definition 2.2. A spacetime (M, g_{ab}) is *asymptotically Minkowskian* if it is asymptotically flat and if the boundary $\mathcal{I}$ satisfies the following condition:

(iv) If Ω is chosen such that $\tilde{\nabla}_a n^a$ vanishes at $\mathcal{I}$, the vector field n^a is complete on $\mathcal{I}$.

We will see in the next section that if a spacetime is asymptotically flat, one can always choose Ω such that $\tilde{\nabla}_a n^a$ vanishes on $\mathcal{I}$, so that the first part of the sentence in condition (iv) is not a restriction. The second part of the sentence *is* and it states that $\mathcal{I}$ is complete in the $\mathbb{R}$ direction.[3] This condition does not play an essential role in most of gravitational wave theory, however, its validity is essential for certain black hole physics results as well as in the study of scattering from $\mathcal{I}^-$ to $\mathcal{I}^+$.

At this point, it is good to take a break and ask ourselves: why are these two definitions reasonable? There are three lines of independent support. First of all, this definition includes many analytic examples of interest such as rotating black holes described by the Kerr solution. Secondly, it is shown that the above definitions are linearization stable meaning that an asymptotically flat spacetime with small fluctuations generated by some compact source is also asymptotically flat [11]. Third, mathematicians have provided evidence that the class of asymptotically flat spacetimes is large and includes many interesting physical scenarios. Finally, this definition is useful in the sense that it has enough structure to allow for — among other concepts — a definition of energy-momentum radiated.

2.2.1 Implications Einstein's equation

Note that nowhere in the definition did we demand that $\mathcal{I}$ is a null surface. This is not input, but a direct consequence of Einstein's equation. In particular, if the physical metric g_{ab} satisfies Einstein's equation, then the conformally rescaled metric $\tilde{g}_{ab}$ (also known as the unphysical metric) satisfies

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} + 2\Omega^{-1}\left(\tilde{\nabla}_a n_b - \tilde{g}_{ab}\tilde{\nabla}^c n_c\right) + 3\Omega^{-2}\tilde{g}_{ab}n^c n_c = 8\pi T_{ab}\ , \tag{2.6}$$

[3] You can check completeness of a vector field tangent to geodesics by looking at its affine parameter: if the affine parameter is unbounded so that it ranges from $-\infty$ to ∞, the vector field is complete. If the affine parameter has a limited range, the vector field is not complete. (As we will see, n^a is tangent to null geodesics on $\mathcal{I}$. If we call its affine parameter u, then it satisfies $n^a \partial_a u = 1$ and the range of u determines whether n^a is complete or not.)

where as before $n_a := \tilde{\nabla}_a \Omega$ and $n^a = \tilde{g}^{ab} n_b$. Since the conformally completed metric is well-defined and smooth on $\mathcal{I}$, so are its curvature tensors. This means, for instance, that $\Omega \tilde{R}_{ab}$ evaluated on $\mathcal{I}$ vanishes. Multiplying the above equation with Ω^2 and taking the limit to $\mathcal{I}$, we therefore find that

$$3\tilde{g}_{ab} n^c n_c \mathrel{\hat{=}} 8\pi \Omega^2 T_{ab} \ , \tag{2.7}$$

where $\hat{=}$ denotes equality at $\mathcal{I}$. From condition (iii) in Def. 2.1, we see that $\Omega^{-1} T_{ab} \mathrel{\hat{=}} 0$ so also $\Omega^2 T_{ab} \mathrel{\hat{=}} 0$. Applying this to the above result, we conclude that the norm of n^a vanishes on $\mathcal{I}$. In other words, n^a is a null vector on $\mathcal{I}$ and $\mathcal{I}$ is a null 3-manifold (as its normal vector n^a is null and nowhere vanishing in $\mathcal{I}$).

Next, we can apply this trick of multiplying with a positive power of Ω and taking the limit to $\mathcal{I}$ once more to show that n^a is not just null but also hypersurface orthogonal, shear-free and geodetic on $\mathcal{I}$. In order to do so, we first contract Eq. (2.6) with $\tilde{g}^{ab}$, to find

$$\underbrace{\tilde{R} - 2\tilde{R}}_{=-\tilde{R}} + 2\Omega^{-1} \underbrace{\left(\tilde{\nabla}_a n^a - 4\tilde{\nabla}_a n^a \right)}_{-3\tilde{\nabla}_a n^a} + 12\Omega^{-2} n^a n_a = 8\pi \tilde{g}^{ab} T_{ab} \ . \tag{2.8}$$

Multiplying by Ω and evaluating the resulting expression on $\mathcal{I}$ yields

$$\lim_{\Omega \to 0} \Omega^{-1} n^a n_a \mathrel{\hat{=}} \frac{1}{2} \tilde{\nabla}_a n^a \ , \tag{2.9}$$

where the left-hand side is well-defined because $n^a n_a \mathrel{\hat{=}} 0$ (in other words, $n^a n_a = \mathcal{O}(\Omega)$). Now multiplying Eq. (2.6) with Ω and taking the limit to null infinity again, we find after using Eq. (2.9) that

$$\tilde{\nabla}_a n_b \mathrel{\hat{=}} \frac{1}{4} \tilde{\nabla}_c n^c \tilde{g}_{ab} \ . \tag{2.10}$$

From this expression, we find that

$$n_{[a} \tilde{\nabla}_b n_{c]} \mathrel{\hat{=}} 0 \tag{2.11}$$

$$\tilde{\nabla}_{(a} n_{b)} - \frac{1}{4} \tilde{\nabla}_c n^c \tilde{g}_{ab} \mathrel{\hat{=}} 0 \tag{2.12}$$

$$n^a \tilde{\nabla}_a n^b \mathrel{\hat{=}} f n^b \quad \text{for some } f \tag{2.13}$$

where the first line states that n_a is hypersurface orthogonal, the second that n_a is shear-free and the third that n^a is geodetic. These properties of n^a reveal that null infinity is not any random surface, but has a lot of structure.

2.2.2 Conformal freedom

You may have wondered about the special role of Ω and how unique it is. As it turns out: not at all! The only important properties that Ω needs to satisfy are (1) that it vanishes on $\mathcal{I}$, and (2) that its gradient is non-vanishing on $\mathcal{I}$. For instance, in the conformal completion of Minkowski spacetime, instead of $\Omega = 1/r$, we could also have chosen $\Omega = 3/r$ or more general $\Omega = f(u, \theta, \phi)/r$ as long as the function f does not have any zeroes. Therefore, no physical result should depend on this choice. This is an important check one has to perform at the end of your calculations.

Now of course — as is often the case in physics — some choices of Ω will make calculations easier than others (just as using spherical coordinates whenever you have spherical symmetry is typically a good idea, while Cartesian coordinates in this case unnecessarily complicate your calculations). Let us first investigate how the physical fields transform under a conformal rescaling: $\Omega \to \Omega' = \omega \Omega$, where ω is any arbitrary function that is nowhere vanishing on $\mathcal{I}$:

$$\tilde{g}_{ab} \longrightarrow g'_{ab} = \omega^2 \tilde{g}_{ab} \tag{2.14a}$$

$$\tilde{g}^{ab} \longrightarrow g'^{ab} = \omega^{-2}\tilde{g}^{ab} \tag{2.14b}$$

$$n_a \longrightarrow n'_a = \nabla'_a(\omega\Omega) = \omega n_a + \Omega\tilde{\nabla}_a\omega \tag{2.14c}$$

$$n^a \longrightarrow n'^a = g'^{ab}n'_b = \omega^{-1}n^a + \Omega\,\omega^{-2}\tilde{g}^{ab}\tilde{\nabla}_b\omega\ . \tag{2.14d}$$

A popular choice is to use this conformal freedom to impose

$$\tilde{\nabla}_a n^a \mathrel{\hat{=}} 0\ . \tag{2.15}$$

It is known as a *(conformal) divergence free frame* and this choice is made only to make intermediate calculations simpler. No physical result depends on this choice! The reason for its convenience is that the derivative of n^a at null infinity is completely determined by its divergence (see Eq. (2.10)) and thus in a divergence free frame $\tilde{\nabla}_a n_b \mathrel{\hat{=}} 0$ (i.e., $\mathcal{L}_n g_{ab} \mathrel{\hat{=}} 0$). In other words, the extrinsic curvature of $\mathcal{I}$ vanishes on $\mathcal{I}$ in a divergence free frame. Moreover, from Eq. (2.9), it is also clear that in such a frame, the normal vector n_a is not just null at $\mathcal{I}$ but also a bit off of $\mathcal{I}$ because now $n^a n_a = \mathcal{O}(\Omega^2)$ instead of $\mathcal{O}(\Omega)$.

If one is not in a conformal divergence free frame, one can always transform to one by choosing ω appropriately. In particular, we first note that

$$\nabla'_a n'^a = g'^{ab}\nabla'_a n'_b = \omega^{-2}\tilde{g}^{ab}\left(\tilde{\nabla}_a n'_b - \tilde{C}^c_{ab}n'_c\right) \tag{2.16}$$

$$= \omega^{-2}\tilde{g}^{ab}\tilde{\nabla}_a n'_b + 2\omega^{-3}\tilde{g}^{cd}\tilde{\nabla}_d\omega\, n'_c \tag{2.17}$$

$$\begin{aligned} &= \omega^{-1}\tilde{g}^{ab}\tilde{\nabla}_a n_b + \omega^{-2}\tilde{g}^{ab}\tilde{\nabla}_a\omega\, n_b + \omega^{-2}\tilde{g}^{ab}\tilde{\nabla}_a\left(\Omega\tilde{\nabla}_b\omega\right) \\ &\quad + 2\omega^{-2}\tilde{g}^{ab}n_a\tilde{\nabla}_b\omega + 2\omega^{-3}\Omega\tilde{g}^{ab}\tilde{\nabla}_a\omega\tilde{\nabla}_b\omega \end{aligned} \tag{2.18}$$

$$= \omega^{-1}\tilde{\nabla}_a n^a + 4\omega^{-2}n^a\tilde{\nabla}_a\omega + \Omega\left[\omega^{-2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\omega + 2\omega^{-3}\tilde{g}^{ab}\tilde{\nabla}_a\omega\tilde{\nabla}_b\omega\right] \tag{2.19}$$

where in going from the first to the second line, we used that

$$\begin{aligned} \tilde{C}^c_{ab} = \omega^{-1}\tilde{g}^{cd}\left(\tilde{g}_{ad}\tilde{\nabla}_b\omega + \tilde{g}_{bd}\tilde{\nabla}_a\omega - \tilde{g}_{ab}\tilde{\nabla}_d\omega\right) \implies \tilde{g}^{ab}\tilde{C}^c_{ab} &= \omega^{-1}\tilde{g}^{cd}\left(\tilde{\nabla}_d\omega + \tilde{\nabla}_d\omega - 4\tilde{\nabla}_d\omega\right) \\ &= -2\omega^{-1}\tilde{g}^{cd}\tilde{\nabla}_d\omega \end{aligned} \tag{2.20}$$

and in going from the second to the third line we used the transformation law in Eq. (2.14c). So if we want to impose on $\mathcal{I}$ that $\nabla'_a n'^a \mathrel{\hat{=}} 0$, we should choose ω such that

$$\omega^{-1}\tilde{\nabla}_a n^a + 4\omega^{-2}n^a\tilde{\nabla}_a\omega \mathrel{\hat{=}} 0 \quad \implies \quad \mathcal{L}_n \ln\omega \mathrel{\hat{=}} -\frac{1}{4}\tilde{\nabla}_a n^a\ . \tag{2.21}$$

Since this is a simple ordinary differential equation along each integral curve of n^a for ω, there always exists a solution (at least locally). Thus, one can always choose to work in a conformal divergence free frame.

Does this choice fix all the conformal freedom near $\mathcal{I}$? No. There is *residual* conformal freedom. In particular, once we are in a divergence free frame, one can still change the conformal factor by say $\tilde{\omega}$ as long as $\mathcal{L}_n\tilde{\omega} \mathrel{\hat{=}} 0$ so that the conformal transformation does not take you out of the conformal divergence free frame.

2.3 Universal structure

Let us recap what we have accomplished so far. Using the mathematical technique of conformally completing a spacetime (M, g_{ab}), we have learned about the geometric definition for asymptotically flat spacetimes. From the few conditions in Def. 2.1, we derived that:

- $\mathcal{I}$ is topologically $\mathbb{R}\times\mathbb{S}^2$ (see condition (ii) in Def. 2.1);
- $\mathcal{I}$ is a null surface (Eq. (2.7));

- $\mathcal{I}$ is ruled by null geodesics (because n^a is geodetic, see Eq. (2.13));
- the normal to $\mathcal{I}$, n_a, is hypersurface orthogonal and shear-free (Eqs. (2.11) and (2.12)), moreover, the conformal factor can be chosen such that it also has a vanishing expansion (Eq. (2.15)).

The last statement is equivalent to saying that $\tilde{\nabla}_a n_b \mathrel{\hat{=}} 0$, because its anti-symmetric, traceless-symmetric and trace part all vanish. There is even more structure on $\mathcal{I}$. The conformally completed metric $\tilde{g}_{ab}$ induces a metric on $\mathcal{I}$ through its pullback (that is, the operation that maps fields on M to $\mathcal{I}$). In particular, the induced metric is $\tilde{q}_{ab} = \underleftarrow{\tilde{g}_{ab}}$ with the underarrow denoting the pullback.[4] Since $\mathcal{I}$ is a null surface, $\tilde{q}_{ab}$ has signature $(0++)$ and is degenerate (i.e., it does not have a unique inverse).[5] Moreover, $\mathcal{L}_n \tilde{q}_{ab} \mathrel{\hat{=}} 0$ in a conformal divergence free frame, because $\mathcal{L}_n \tilde{g}_{ab} \mathrel{\hat{=}} 0$ in such a frame. That means that infinitesimal area elements remain unchanged in size as they are parallel transported along n^a. (In fact, they also remain unchanged in shape and form due to the vanishing twist and shear of n^a, so that infinitesimal area elements are invariant along $\mathcal{I}$ when $\mathcal{L}_n \tilde{q}_{ab} \mathrel{\hat{=}} 0$.)

In summary, *all* asymptotically flat spacetimes have a future null infinity with the above properties. The key *intrinsic* fields on $\mathcal{I}$ are $(\tilde{q}_{ab}, n^a)$. Due to the freedom in the choice of the conformal factor $\Omega \mapsto \omega\Omega$, these fields are determined by the physical spacetime only up to equivalence under the conformal transformations $(\tilde{q}_{ab}, n^a) \mapsto (\omega^2 \tilde{q}_{ab}, \omega^{-1} n^a)$. Stated differently, the *universal structure* common to all spacetimes satisfying 2.1 is given by:

1. a smooth manifold $\mathcal{I} \cong \mathbb{R} \times \mathbb{S}^2$,
2. an equivalence class of pairs $(\tilde{q}_{ab}, n^a)$ on $\mathcal{I}$ where n^a is a vector field and $\tilde{q}_{ab}$ is a (degenerate) metric with $\tilde{q}_{ab} n^b = 0$ and $\mathcal{L}_n \tilde{q}_{ab} = 0$, and,
3. any two members of the equivalence class are related by the map $(\tilde{q}_{ab}, n^a) \mapsto (\omega^2 q_{ab}, \omega^{-1} n^a)$for some $\omega > 0$ satisfying $\mathcal{L}_n \omega = 0$.

The above universal structure is only true in a conformal divergence free frame (for instance, the condition $\mathcal{L}_n \omega = 0$ is a result of this choice), but since one can always make such a choice, this is not a genuine restriction. Physically different spacetimes are distinguishable only in the "next-order" structure such as the derivative operator on $\mathcal{I}$ induced by the derivative operator compatible with $\tilde{g}_{ab}$ on $\tilde{M}$ and the curvature tensors.

From the universal structure, it is clear that the induced metric $\tilde{q}_{ab}$ on $\mathcal{I}$ for any asymptotically flat spacetime is the same as that of Minkowski spacetime (up to the freedom in one's choice of conformal factor). The same is true for the null vectors n^a ruling $\mathcal{I}$. This implies that if the spacetime is asymptotically flat, one can always find coordinates such that:

$$\tilde{q}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrel{\hat{=}} 2\mathrm{d}u\mathrm{d}\Omega + \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \tag{2.22a}$$

$$n^\mu \partial_\mu \mathrel{\hat{=}} \frac{\partial}{\partial u} \, . \tag{2.22b}$$

This is exactly the induced metric and null normal of $\mathcal{I}$ of Minkowski spacetime we discussed in Sec. 2.1. Gravitational radiation and other physical fields show up at the next order structure. It is important to stress that nowhere in this construction did we introduce a split of the background and "gravitational waves" (as opposed to studying linearized gravity, where this is the first step!). A split occurs naturally at null infinity: where the universal structure is like a background, and the first-order structure contains gravitational radiation. This is an absolutely beautiful result and allows us to study gravitational radiation fully non-linearly.

[4]Note that $\underleftarrow{n_a} \mathrel{\hat{=}} 0$, since $n_a = \tilde{\nabla}_a \Omega$ and $\Omega \mathrel{\hat{=}} 0$, but $\underleftarrow{n^a}$ is not zero on $\mathcal{I}$. Moreover, the pullback commutes with exterior differentiation and taking Lie derivates.

[5]Degeneracy of the induced metric $\tilde{q}_{ab}$ also follows from the fact that $0 \mathrel{\hat{=}} \underleftarrow{n_a} \mathrel{\hat{=}} \underleftarrow{\tilde{g}_{ab} n^b} \mathrel{\hat{=}} \tilde{q}_{ab} \underleftarrow{n^b}$.

Remark 2.2. Nowadays with the indirect and direct observational evidence of gravitational waves, it is difficult to imagine that the existence of gravitational waves was debated for several decades. One year after the publication of Einstein field equation in 1915, Einstein himself proposed the existence of gravitational waves. Although the idea of gravitational waves was already explored by others including Lagrange and Poincaré, Einstein's 1916 paper provided a relativistic description by linearizing the field equations off a Minkowski background. However, not everyone was convinced that these waves exist in full general relativity, beyond the linear approximation. Some argued that these waves were merely a mathematial artifact of linearizing an inherently non-linear theory. The coordinate freedom inherent to general relativity further complicated the arguments: what convincing appeared as "wavy" in one coordinate system could appear stationary in another. This debate was finally resolved by theoretical work on asymptotically flat spacetimes (and a few years later, the Hulse-Taylor binary pulsar provided (indirect) observational evidence). For a more complete historic account, I recommend 'Traveling at the speed of thought: Einstein and the quest for gravitational waves' by D. Kennefick [12].

3 Asymptotic symmetries

Before we explore the notion of asymptotic symmetries, we will briefly review the role of exact symmetries in general relativity and the symmetries of Minkowski spacetime.

Background knowledge

- **Killing vector fields.** These describe symmetries in general relativity and their defining equation is Eq. (3.1). If you would like to review these, see [8, Ch. 3.8].
- **Poincaré algebra and group.** These describe the symmetries of Minkowski spacetime and will be reviewed in Sec. 3.2 and tutorial 3. You may find it useful to first refresh your memory about these symmetries as they will likely be presented in a slightly different way than you have seen them before.
- **Lie bracket.** There is lots to say about Lie brackets, but for this course it suffices to know that $[v,w]^a = \mathcal{L}_v w^a$, which using the expression for the Lie derivative in Eq. (2.1) can also be written as $[v,w]^a = v^b\nabla_b w^a - w^b\nabla_b v^a$ or $[v,w]^a = v^b\partial_b w^a - w^b\partial_b v^a$ or with any other convenient derivative operator.

3.1 Symmetries

Symmetries play an essential role in all of fundamental physics; general relativity is no exception. In general relativity, one is typically interested in the symmetries of the metric, known as *isometries*. Intuitively, isometries provide a motion in space-time under which all the physics is invariant. The most important isometries are those generated by Killing vector fields K^a, which are vector fields satisfying:

$$\mathcal{L}_K g_{ab} = 0 \quad \iff \quad \nabla_{(a}K_{b)} = 0\ . \tag{3.1}$$

A general space-time of course has no Killing vectors, because most realistic systems have no symmetries. Killing vectors are important, not because they are common in actual systems, but because they allow us to do "something" where otherwise we might not be able to do anything.

In particular, Killing vectors give rise to conserved quantities. These come in two varieties: conservation of quantities along geodesics known as "constants of motion" and those associated with the stress-energy tensor. The first type is extremely useful when studying geodesics. In particular, if u^a is a tangent vector of some geodesic γ so that $u^a\nabla_a u^b = 0$, then u^aK_a is constant along γ. Put differently, if K^a is a Killing vector field, then $u^a\nabla_a(u^bK_b) = 0$. This is easily proven:

$$u^a\nabla_a(u^bK_b) = u^a\nabla_a u^b\ K_b + u^a u^b\nabla_a K_b \tag{3.2}$$

$$= u^a u^b \left(\nabla_{(a} K_{b)} + \nabla_{[a} K_{b]}\right) = 0 \,, \tag{3.3}$$

where in going from the first to the second line, we used that u^a is a tangent vector to a geodesic and that any rank-two tensor can be decomposed into its symmetric and anti-symmetric part. For the final equality, we used the Killing equation in Eq. (3.1) and that $u^a u^b$ is symmetric upon interchanging $a \leftrightarrow b$. The second type of conserved quantity is associated with T_{ab}. Specifically, we first note that T_{ab} contracted with a Killing vector field is divergence-free:

$$\nabla_a (T^{ab} K_b) = \underbrace{\nabla_a T^{ab}}_{=0} \; K_b + T^{ab} \; \underbrace{\nabla_a K_b}_{=\nabla_{(a} K_{b)}} \; = 0 \tag{3.4}$$

where the first term vanishes due to conservation of the stress-energy tensor and the second because T^{ab} is a symmetric tensor and K^a a Killing vector field. Since $\nabla_a (T^{ab} K_b) = 0$, so is its integral over some four-dimensional volume $\int \nabla_a (T^{ab} K_b) \; d^4 V = 0$. If the stress-energy T_{ab} has compact support, i.e., it is non-zero only within some finite region, by Stokes' theorem, we find that

$$0 = \int \nabla_a (T^{ab} K_b) \; d^4 V = \int_\Sigma T^{ab} K_b d\Sigma_a - \int_{\Sigma'} T^{ab} K_b d\Sigma_a \tag{3.5}$$

where Σ, Σ' are three-dimensional hypersurfaces (see Fig. 3). Consequently, the value of $\int_\Sigma T^{ab} K_b d\Sigma_a$ is independent of the hypersurface Σ and in that sense "conserved". (If T_{ab} is not of compact support, or if the surface Σ and Σ' are such that they do not extend beyond the region of compact support of T_{ab} then the "sides" of the region also need to be taken into account in Eq. (3.5) and $\int_\Sigma T^{ab} K_b d\Sigma_a$ will generically no longer be independent of the surface Σ.)

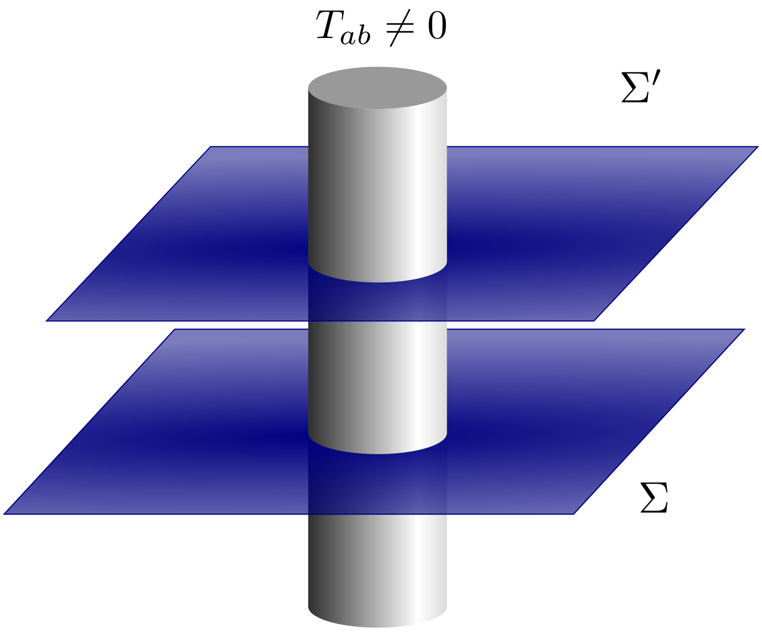

Figure 3: The gray region denotes a four-dimensional worldtube in which T_{ab} is non-zero, elsewhere T_{ab} vanishes. Σ and Σ' represent three-dimensional hypersurfaces.

By analogy with the Killing vector fields of Minkowski spacetime, if a Killing vector is time-like, then the conserved quantity $u^a K_a$ is called the energy of the particle flowing along the geodesic and the conserved quantity associated with T_{ab} is called the energy of the region with non-zero T_{ab}. Similarly, conserved quantities associated with Killing vectors generating spatial translations are called (components of the) momentum, and the conserved quantities associated with Killing vectors generating rotations are called (components of the) angular momentum. The conserved quantities associated with boosts are typically less useful and have no special name.

3.2 Symmetries of Minkowski spacetime

The symmetry group of Minkowski spacetime is the Poincaré group. The action of the generators of this group is represented by the ten Killing vector fields of Minkowski spacetime, which can be decomposed as:

$$K_a = \underline{F}_{ab} x^b + \underline{K}_a \tag{3.6}$$

where $\underline{F}_{ab}$ is a constant antisymmetric tensor field (so $\nabla_c \underline{F}_{ab} = 0$), $\underline{K}_a$ is a constant vector field ($\nabla_a \underline{K}_b = 0$) and x^a is the position vector relative to some origin O of Minkowski spacetime (so that $\nabla_a x^b = \delta_a{}^b$). The four translations of Minkowski spacetime are generated by the constant part in K_a, that is, $\underline{K}_a$. When $\underline{K}_a$ is time-like, the Killing vector field generates time translations; when it is space-like, it generates spatial translations. The part $\underline{F}_{ab} x^b$ generates Lorentz boosts and rotations. What is called a boost and what a rotation, depends on one's choice of a unit time-like vector t^a with respect to the origin O:

- The "electric" part of $\underline{F}_{ab} t^b$ generates Lorentz boosts in the direction of $\underline{F}_{ab} t^b$;
- The "magnetic part of $\underline{F}_{ab} t^b$, which is given by $\frac{1}{2}\epsilon_a{}^{bcd} \underline{F}_{bc} t_d$, generates spatial rotations.[6]

Of course, if you change the (arbitrary) choice of origin, then the decomposition of the Killing vector field in Eq. (3.6) also changes. Specifically, if your new choice of origin O' is shifted by c^a relative to the original choice O, then the position vectors x^a and x'^a relative to the origins O and O' are related by $x'^a = x^a - c^a$ (see Fig. 4). Substituting this into Eq. (3.6), we find that the Killing vector field is now decomposed as $K_a = \underline{F}_{ab} x'^b + \underline{K}'_a$ with the constant part given by $\underline{K}'_a = \underline{K}_a + \underline{F}_{ab} c^b$.

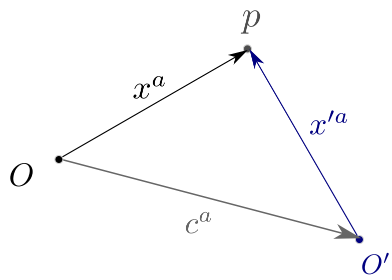

Figure 4: Shift in origin from O to O' by c^a with the associated relabelling of the point p.

An important property of Killing vector fields is that if K_1^a and K_2^a are both Killing vectors, so is its commutator/Lie bracket $[K_1, K_2]^a$. In mathematical notation, this statement translates to

$$K_3^a := [K_1, K_2]^a = K_1^b \nabla_b K_2^a - K_2^b \nabla_b K_1^a \quad \text{satisfies} \quad \mathcal{L}_{K_3} g_{ab} = 0 \tag{3.7}$$

if $\mathcal{L}_{K_1} g_{ab} = 0$ and $\mathcal{L}_{K_2} g_{ab} = 0$. Thus, Killing vectors of any spacetime have the structure of a Lie algebra.[7] It is natural to ask if there is a special structure to the algebra of the Killing vector fields of Minkowski spacetime, which is called the Poincaré algebra $\mathfrak{iso}(1,3)$. As you may expect, there is. Let us first consider the Lie bracket of a translation Killing vector $\underline{K}_a$ with another translation Killing vector $\underline{K}'_a$:

$$[\underline{K}, \underline{K}']^a = \underline{K}^b \nabla_b \underline{K}'^a - \underline{K}'^b \nabla_b \underline{K}^a = 0\,. \tag{3.8}$$

Since this commutator vanishes, translations form an abelian Lie subalgebra denoted by $\mathbb{R}^{1,3} \subset \mathfrak{iso}(1,3)$. Next, let us consider the Lie bracket of a translation Killing vector field and a Lorentz

[6] Of course, $\underline{F}_{ab}$ has nothing to do with electromagnetism, and the name giving is only based on the mathematical similarities with the Maxwell tensor F_{ab} and its decomposition into the electric and magnetic field.

[7] Recall that a Lie algebra is a vector space with a linear, closed, antisymmetric bracket operation called a Lie bracket, subject to the Jacobi identity.

transformation $L^a = \underline{F}^a{}_b x^b$:

$$[\underline{K}, L]^a = \underline{K}^b \nabla_b (\underline{F}^a{}_c x^c) - \underline{F}^b{}_c x^c \nabla_b \underline{K}^a = \underline{F}^a{}_b \underline{K}^b \ . \tag{3.9}$$

The result is a constant vector, so another translation Killing vector field. Therefore, the abelian subalgebra $\mathbb{R}^{1,3}$ is an *invariant* subalgebra also known as a Lie ideal. Since $\mathbb{R}^{1,3}$ is a Lie ideal, the quotient algebra $\mathfrak{iso}(1,3)/\mathbb{R}^{1,3}$ is also a Lie algebra. This quotient algebra is exactly the Lorentz algebra $\mathfrak{so}(1,3)$. Note, however, that there is no unique Lorentz algebra because a Killing vector in Minkowski spacetime defines an origin-independent $\underline{F}_{ab}$, but no origin-independent $\underline{K}_a$. As a result, there exist as many Lorentz subalgebras as there are origins (i.e., infinitely many!). Consequently, the Poincaré algebra cannot be written as a direct product of $\mathbb{R}^{1,3}$ and $\mathfrak{so}(1,3)$, but is a semi-direct product:

$$\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \rtimes \mathbb{R}^{1,3} \ . \tag{3.10}$$

The asymptotic symmetries of asymptotically flat spacetimes will be distinct from the Poincaré algebra, but will share a similar structure.

Remark 3.1. Given a physical system with conserved stress-energy tensor T_{ab} in Minkowski spacetime, the ten Poincaré generators K^a define ten conserved quantities, namely P_a the 4-momentum, and M_{ab} the relativistic angular momentum:

$$P_a \underline{K}^a + M_{ab} \underline{F}^{ab} := \int_\Sigma T_{ab} K^b \, d\Sigma^a \, , \tag{3.11}$$

where the integral is performed on a Cauchy surface Σ of Minkowski spacetime.[8]

3.3 Asymptotic symmetry algebra

Just as symmetries represented by Killing vector fields are the motions under which the metric is invariant, so are asymptotic symmetries of asymptotically flat spacetimes, roughly speaking, the "asymptotic motions under which the asymptotic structure is invariant". More precisely, the *asymptotic symmetry algebra* is the algebra of infinitesimal diffeomorphisms of $\mathcal{I}$ which preserves this universal structure. We focus only on the asymptotic symmetry *algebra* rather than the asymptotic symmetry *group* as this requires the strengthened definition of 2.2.

Concretely, the asymptotic symmetry algebra consists of all smooth vector fields ξ^a on $\mathcal{I}$ that map one pair $(\tilde{q}_{ab}, n^a)$ to another equivalent pair (q'_{ab}, n'^a) within the universal structure. First of all, note that the vector fields need to be smooth, because $(\tilde{q}_{ab}, n^a)$ are. Now let us consider an arbitrary vector field ξ^a that maps $\tilde{q}_{ab}$ to a conformally related $\tilde{q}_{ab}$ and similarly for n^a:

$$\mathcal{L}_\xi \tilde{q}_{ab} = \kappa \, \tilde{q}_{ab} \tag{3.13a}$$

$$\mathcal{L}_\xi n^a = \lambda \, n^a \tag{3.13b}$$

where κ and λ are *a priori* arbitrary functions on $\mathcal{I}$.[9] Are there any restrictions on κ and λ for this vector field to preserve the universal structure? Yes, in order to preserve that $\mathcal{L}_n \tilde{q}_{ab} = 0$ also

[8]Under a change of origin from $O \to O'$, we have

$$P_a \to P_a \tag{3.12}$$

while M_{ab} changes. In fact, three of the six components of M_{ab} – corresponding to the three boosts in the rest frame of the system – can be transformed away by an judicious choice of origin. The non-trivial information encoded in M_{ab} are only three values, conveniently collected in the angular momentum 3-vector $\vec{J}^a$. Thus, although there are ten conserved quantities, all the physical information is encoded entirely in the 4-momentum P_a and the angular momentum 3-vector $\vec{J}^a$.

[9]Since the universal structure is intrinsic to $\mathcal{I}$ all equal signs in this section refer to equalities on $\mathcal{I}$. To avoid notational clutter, I have omitted the hat on top of the equations as it is clear from the context that these equalities refer to $\mathcal{I}$ only.

after the transformation, we immediately conclude that $\mathcal{L}_n\kappa = 0$ because $\mathcal{L}_n(\kappa\,\tilde{q}_{ab}) = \mathcal{L}_n\kappa\,\tilde{q}_{ab} + \kappa\,\mathcal{L}_n\tilde{q}_{ab} = \mathcal{L}_n\kappa\,\tilde{q}_{ab}$. Second, we obtain an additional constraint by considering how some other equivalence pair in this class, say $q'_{ab} = \omega^2\tilde{q}_{ab}$ and $n'^a = \omega^{-1}n^a$, transforms under a diffeomorphism by the same ξ^a. A short computation shows that

$$\begin{aligned}\mathcal{L}_\xi q'_{ab} &= \mathcal{L}_\xi\left(\omega^2\tilde{q}_{ab}\right) = \omega^2\kappa\tilde{q}_{ab} + 2\omega\mathcal{L}_\xi\omega\,\tilde{q}_{ab}\\ &= \left(\kappa + 2\omega^{-1}\mathcal{L}_\xi\omega\right)q'_{ab} \qquad (3.14a)\\ \mathcal{L}_\xi n'^a &= \mathcal{L}_\xi(\omega^{-1}n^a) = \omega^{-1}\lambda n^a - \omega^{-2}\mathcal{L}_\xi\omega\,n^a\\ &= \left(\lambda - \omega^{-1}\mathcal{L}_\xi\omega\right)n'^a\,. \qquad (3.14b)\end{aligned}$$

If we take $\kappa = 2\alpha_{(\xi)}$ and $\lambda = -\alpha_{(\xi)}$, then we see that

$$\mathcal{L}_\xi\tilde{q}_{ab} = 2\alpha_{(\xi)}\tilde{q}_{ab} \qquad \text{and} \qquad \mathcal{L}_\xi n^a = -\,\alpha_{(\xi)}n^a\,, \qquad (3.15)$$

and also

$$\mathcal{L}_\xi q'_{ab} = 2\alpha'_{(\xi)}q'_{ab} \qquad \text{and} \qquad \mathcal{L}_\xi n'^a = -\,\alpha'_{(\xi)}n'^a\,, \qquad (3.16)$$

where $\alpha_{(\xi)}$ is any function depending on the vector field ξ^a on $\mathcal{I}$ such that $\mathcal{L}_n\alpha_{(\xi)} = 0$ (and similarly for $\alpha'_{(\xi)}$). Hence, the vector fields ξ^a satisfying Eq. (3.15) preserve the universal structure and are therefore asymptotic symmetry vector fields. Moreover, if ξ^a and ζ^a both satisfy the conditions in Eq. (3.15), so does $[\xi,\zeta]^a$. Thus, these vector fields form a Lie algebra which we denote by $\mathfrak{b}$ for Bondi-Metzner-Sachs (BMS) algebra.

3.3.1 Supertranslations

Let us explore the structure of this Lie algebra. Since $\mathcal{I}$ is ruled by its null normal n^a, let us first consider vector fields of the form $\xi^a = fn^a$. In particular, such a vector field transforms the induced metric as

$$\mathcal{L}_{fn}\tilde{q}_{ab} = fn^c\partial_c\tilde{q}_{ab} + 2\tilde{q}_{c(a}\partial_{b)}(fn^c) = f\underbrace{\mathcal{L}_n\tilde{q}_{ab}}_{=0} + 2\underbrace{q_{c(a}n^c}_{=0}\partial_{b)}f = 0 \qquad (3.17)$$

and the normal vector as

$$\mathcal{L}_{fn}n^a = fn^b\partial_b n^a - n^b\partial_b(fn^a) = f\underbrace{\mathcal{L}_n n^a}_{=0} - \mathcal{L}_n f\,n^a\,. \qquad (3.18)$$

From the transformation of the induced metric we conclude that $\alpha_{(fn)} = 0$, and thus $\mathcal{L}_{fn}n^a$ should vanish as well. Hence,vector fields of the form $\xi^a = fn^a$ satisfy Eq. (3.15) if and only if

$$\alpha_{(fn)} = 0 \quad \text{and} \quad \mathcal{L}_n f = 0\,. \qquad (3.19)$$

Since $\mathcal{L}_n f = 0$, f does not change as you move along $\mathcal{I}$ and thus is only a function on the space of generators of $\mathcal{I}$, which is topologically $\mathbb{S}^2$. Put plainly, if you introduce spherical coordinates on $\mathbb{S}^2$, then f is an arbitrary function of θ and ϕ.

Now let us consider the Lie bracket of two such vector fields, $\xi^a = fn^a$ and $\zeta^a = hn^a$ (with of course $\mathcal{L}_n f = 0$ and $\mathcal{L}_n h = 0$):

$$[\xi,\zeta]^a = fn^b\partial_b(hn^a) - hn^n\partial_b(fn^a) = (f\mathcal{L}_n h - h\mathcal{L}_n f)\,n^a\,. \qquad (3.20)$$

The resulting vector field is again of the form "some function" $\times\, n^a$, with in this case that function being zero as $\mathcal{L}_n f$ and $\mathcal{L}_n h$ both vanish. Consequently, these vector fields form an infinite-dimensional abelian subalgebra $\mathfrak{s} \subset \mathfrak{b}$. This algebra is infinite-dimensional because of the functional freedom in multiplying n^a. The vector fields that make up this algebra are called *supertranslations*. Where does this name come from? Let us go back to the translation Killing vector fields of

Minkowski spacetime, which also form an abelian Lie subalgebra (but now the subalgebra is four-dimensional instead of infinite-dimensional). In Cartesian coordinates, these translational vector fields are simply $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. If we write these vector fields in terms of the null coordinates in Sec. 2.1 and take the limit to $\mathcal{I}$, we find that

$$\frac{\partial}{\partial t} \mathrel{\widehat{=}} \frac{\partial}{\partial u} = n^a \partial_a \tag{3.21a}$$

$$\frac{\partial}{\partial x} \mathrel{\widehat{=}} -\sin\theta\cos\phi \frac{\partial}{\partial u} = -\sin\theta\cos\phi\; n^a\partial_a \tag{3.21b}$$

$$\frac{\partial}{\partial y} \mathrel{\widehat{=}} -\sin\theta\sin\phi \frac{\partial}{\partial u} = -\sin\theta\sin\phi\; n^a\partial_a \tag{3.21c}$$

$$\frac{\partial}{\partial z} \mathrel{\widehat{=}} -\cos\theta \frac{\partial}{\partial u} = -\cos\theta\; n^a\partial_a\,. \tag{3.21d}$$

Thus, the time and spatial translation vector fields of Minkowski spacetime are also of the form "some function" $\times\, n^a$ on $\mathcal{I}$. However, now the function multiplying n^a cannot be any arbitrary function but has to be a linear combination of the spherical harmonics with $l = 0, 1$. Thus, supertranslations are a generalization of translations as illustrated in Fig. 5. (Note that supertranslations, despite 'super' in the name, have nothing to do with supersymmetry in particle physics.) The presence of these supertranslations is also the reason why the asymptotic symmetry algebra of asymptotically flat spacetimes is not the Poincaré algebra, but the larger BMS algebra.

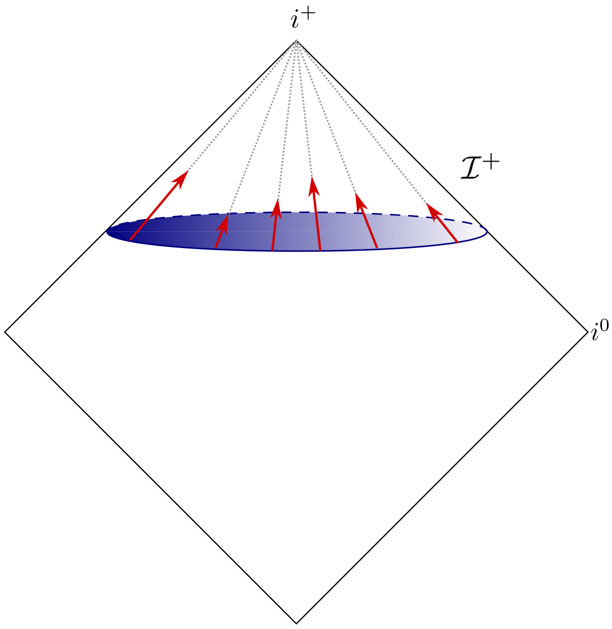

Figure 5: Supertranslations are vector fields $\xi^a = fn^a$ with $\mathcal{L}_n f = 0$ and thus can be thought of as angle-dependent translations; whereas translations are more "rigid" and f can only be a linear combination of spherical harmonics $Y_{lm}(\theta, \phi)$ with $l = 0, 1$. The arrows illustrate an example of a supertranslation at a given cross-section of $\mathcal{I}^+$. The length of the arrows for a time translation, for instance, would all have the same length.

Next, let us consider the Lie bracket of any vector field ξ^a in $\mathfrak{b}$ and a supertranslation fn^a

$$[\xi, fn]^a = f\mathcal{L}_\xi n^a + \mathcal{L}_\xi f\; n^a = \left(-\alpha_{(\xi)}\, f + \mathcal{L}_\xi f\right) n^a\,. \tag{3.22}$$

The resulting vector field has the form of a supertranslation, but before we can conclude that is in fact a supertranslation, we need to show that

$$\mathcal{L}_n \left(-\alpha_{(\xi)}\, f + \mathcal{L}_\xi f\right) \stackrel{?}{\mathrel{\widehat{=}}} 0\,. \tag{3.23}$$

Let us start by using the sum and chain rule, to rewrite the left-hand side as:

$$-\mathcal{L}_n \alpha_{(\xi)} f - \alpha_{(\xi)} \mathcal{L}_n f + \mathcal{L}_n \mathcal{L}_\xi f = \mathcal{L}_n \mathcal{L}_\xi f \;, \tag{3.24}$$

where the first term vanishes because of the conditions in Eq. (3.15) and the second because $\mathcal{L}_n f = 0$. The remaining term can be rewritten as

$$\mathcal{L}_n \mathcal{L}_\xi f = \underbrace{\mathcal{L}_{[n,\xi]} f}_{=\alpha \mathcal{L}_n f} + \mathcal{L}_\xi \mathcal{L}_n f \;, \tag{3.25}$$

where both terms vanish because $\mathcal{L}_n f = 0$ (note that for the first term we used Eq. (3.22) to conclude that $[n, \xi]^a = \alpha_{(\xi)} n^a$). Hence, the right-hand-side in Eq. (3.22) is indeed a supertranslation. Therefore, the supertranslation subalgebra $\mathfrak{s}$ is a Lie ideal in $\mathfrak{b}$; just as the translation subalgebra is a Lie ideal in the Poincaré algebra $\mathfrak{iso}(1,3)$. Therefore, we can quotient $\mathfrak{b}$ with $\mathfrak{s}$ to get a Lie algebra $\mathfrak{b}/\mathfrak{s}$.

Lorentz & translation subalgebra

You are not expected to be able to reproduce the content in the next two subsections 3.3.2 and 3.3.3, which is rather mathematical in nature. I have included these sections for completeness. Nonetheless, I hope you will get the general gist that there there are many Lorentz subalgebras in the BMS algebra and that there is a unique four-dimensional translation subalgebra.

3.3.2 Lorentz algebra

To obtain a concrete realization of this quotient $\mathfrak{b}/\mathfrak{s}$, let us first introduce the base space S of generators of $\mathcal{I}$. Throughout, we have assumed that $\mathcal{I}$ is a "nice" surface so that the integral curves generated by n^a are not closed (or nearly closed). As a result, each point on $\mathcal{I}$ belongs uniquely to one of the integral curves of n^a. The set of all maximally extended curves of n^a is a two-dimensional manifold, known as the base space S. There is a unique projection from points on $\mathcal{I}$ to S , let's denote it $\pi : \mathcal{I} \to S$. The inverse of this map ℓ is of course not unique, but there should be a map from $S \to \mathcal{I}$ such that $\pi \circ \ell$ is the identity on S. This (non-unique) inverse is called a "lift" and when we speak of a cross-section of $\mathcal{I}$, we mean the lift from S back to $\mathcal{I}$. Since $\mathcal{I}$ is topologically $\mathbb{R} \times \mathbb{S}^2$, it follows that the base space S is topologically $\mathbb{S}^2$.

Back to the quotient $\mathfrak{b}/\mathfrak{s}$. Consider two vector fields ξ^a and ξ'^a in $\mathfrak{b}$. If $\xi^a - \xi'^a = f n^a$, then they are considered equivalent in $\mathfrak{b}/\mathfrak{s}$. So the quotient is an equivalence class of elements of $\mathfrak{b}$. How can we describe $\mathfrak{b}/\mathfrak{s}$ more efficiently without having to carry all this additional information around? As it turns out, the projection to the base space is invaluable here: using the projection map, $\xi^a \in \mathfrak{b}/\mathfrak{s}$ can be identified with vector fields on the base space S. (If $\xi^a \in \mathfrak{s}$, ξ^a is mapped to zero under this projection.) The second condition for $\xi \in \mathfrak{b}$ in Eq. (3.15) now simply ensures that this map is well-defined (because the supertranslations preserve the generators of $\mathcal{I}$) and from the first condition we find that any $X^a \in \mathfrak{b}/\mathfrak{s}$ satisfies

$$\mathcal{L}_X \underline{q}_{ab} = 2\alpha_{(X)} \underline{q}_{ab} \,, \tag{3.26}$$

where $\underline{q}_{ab}$ is the positive-definite metric on the space of generators S whose lift to $\mathcal{I}$ yields q_{ab}. In other words, X^a is a conformal Killing vector field on $\mathbb{S}^2$. At this point, a remarkable fact about two-spheres helps understand what this means. Two-spheres carry a unique conformal structure: every metric on $\mathbb{S}^2$ is conformal to the unit two-sphere. This is a special property of two dimensions and does not hold in any other dimension! This is also the reason that the condition that $\mathcal{I}$ has topology $\mathbb{R} \times \mathbb{S}^2$ is included in the definition of asymptotic flatness. If null infinity were allowed to have any other topology such as $\mathbb{R}^3$ or $\mathbb{T}^3$, we could not have appealed to this special conformal structure, which buys a lot (and does not seem to exclude any spacetimes of interest). The fact that the metric on $\mathbb{S}^2$ is conformal to the unit two-sphere and X^a are conformal Killing vector fields of this metric, is equivalent to saying that the Lie algebra $\mathfrak{b}/\mathfrak{s}$ is the algebra of conformal isometries of the unit 2-sphere. A well-known fact in the study of Lie algebras is that this Lie algebra is isomorphic to the Lorentz algebra $\mathfrak{so}(1,3)$. This implies immediately that the quotient $\mathfrak{b}/\mathfrak{s}$ is the Lorentz algebra. Hence, the asymptotic symmetry algebra of asymptotically flat spacetimes is given by the semi-direct sum of supertranslations and the Lorentz algebra, i.e.,

$$\mathfrak{b} \cong \mathfrak{s} \rtimes \mathfrak{so}(1,3)\,. \tag{3.27}$$

3.3.3 Translation subalgebra

Lastly, the BMS algebra $\mathfrak{b}$ admits a unique 4-dimensional Lie ideal of translations $\mathbb{R}^{1,3}$. This is the only finite-dimensional Lie ideal in $\mathfrak{b}$ (while $\mathfrak{s}$ is also a Lie ideal in $\mathfrak{b}$, it is infinite dimensional). The easiest way to derive this is to further restrict the conformal freedom by demanding that the induced metric on $\mathcal{I}$ pulled back to the base space S is the unit two-sphere metric. This is called a "Bondi conformal frame". It is a convenient choice in many calculations, but it is difficult to check conformal invariance of the final results. Nonetheless, if we go to such a frame and introduce the standard spherical coordinates θ, ϕ for S, then translations are vector fields fn^a with $f = f(\theta, \phi)$ a linear combination of spherical harmonics with only $l = 0, 1$ (see also Eq. (3.21)). In contrast, a generic supertranslation is a linear combination of spherical harmonics with arbitrary l. To show that translations are a Lie ideal, we need to show that the Lie bracket of a translation with an arbitrary BMS vector field is again a translation. This amounts to showing that for any $\xi^a = f_1 n^a$ with f_1 comprised of spherical harmonics with $l = 0, 1$:

$$[f_1 n, f_2 n]^a = f_3 n^a \tag{3.28}$$

$$[f_1 n, X]^a = f_4 n^a\;, \tag{3.29}$$

where f_2 is an arbitrary function of θ, ϕ (so that $f_2 n^a$ represents a supertranslation) but f_3 and f_4 are also linear combinations of spherical harmonics with l restricted to $0, 1$. We will not show this explicitly here, but it is certainly true!

3.3.4 Summary

At this point, it is worthwhile to recall what the BMS algebra is good for. Killing vectors play a pivotal role in spacetimes with symmetries by providing conserved quantities with a physical interpretation, similarly, asymptotic symmetry vector fields provide 'conserved quantities' at null infinity (and their rate of change) with a physical interpretation. Quantities associated with translations are called energy-momentum, those with rotations angular momentum. In addition, there is an interesting connection between supertranslations and the gravitational memory effect (see Sec. 5). Moreover, the BMS algebra gives rise to infinitely many conservation laws from past null infinity to future null infinity in certain scattering processes, which are conjectured to play a role in the resolution of the black hole information paradox.

In summary, asymptotic symmetries are vector fields that preserve the universal structure on $\mathcal{I}$. For asymptotically flat spacetimes, these vector fields form the BMS algebra $\mathfrak{b}$, which is the semi-direct product of supertranslations and the Lorentz algebra. This structure is an infinite-dimensional generalization of the Poincaré algebra of Minkowski spacetime, which is the semi-direct product of translations and the Lorentz algebra. The BMS algebra also contains a unique four-dimensional Lie ideal $\mathfrak{t}$ representing time and spatial translations. This "enlargement" from a finite dimensional algebra $\mathfrak{iso}(1,3)$ to an infinite dimensional algebra $\mathfrak{b}$ can be intuitively understood as follows. If you have some physical spacetime which is asymptotically flat, and expand in $1/r$, you can always write it as:

$$ds^2 = \underbrace{-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}_{=\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} + \mathcal{O}(1/r)\;, \tag{3.30}$$

where the $1/r$ fall-off refers to fall-off in a Cartesian chart of the flat "background" metric $\eta_{\mu\nu}$. However, their is no unique choice of flat metric to approach. For instance, if we were to make a simple coordinate transformation to say a "rotating" frame $t = t' + f(\theta, \phi)$ with for instance $f(\theta, \phi) = \cos\theta$, then

$$\begin{aligned} \mathrm{d}t &= \mathrm{d}t' - \frac{xz}{r^3}\mathrm{d}x - \frac{xy}{r^3}\mathrm{d}y + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)\mathrm{d}z \\ &= \mathrm{d}t' - \frac{\sin\theta\cos\theta\cos\phi}{r}\mathrm{d}x - \frac{\sin\theta\cos\theta\sin\phi}{r^3}\mathrm{d}y + \frac{\sin^2\theta}{r}\mathrm{d}z \,. \end{aligned} \tag{3.31}$$

Substituting this back into Eq. (3.30), it is clear that this transformation does not change the form of the equation

$$ds^2 = \underbrace{-\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2}_{=\eta'_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} + \mathcal{O}(1/r') \,, \tag{3.32}$$

but $\eta_{\mu\nu}$ is different from $\eta'_{\mu\nu}$ and — of course — the $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r')$ parts are different. While in both cases the leading order part is a flat spacetime, these flat spacetimes are distinct. Consequently, they also do not have the same Poincaré algebra. Their translational Killing vector fields are the same, but they do not share the same Lorentz subalgebra. The vectors comprising the Poincaré algebra of $\eta_{\mu\nu}$ are asymptotic Killing vector fields of the asymptotically flat spacetime, but so are those of $\eta'_{\mu\nu}$. This is ultimately the reason for the large symmetry algebra: the BMS algebra can be thought of as the collection of the Poincaré algebras associated with all these Minkowski metrics that are related to each other by angle dependent translations. In short, even though the physical metric is the same, it can approach infinitely many Minkowski metrics on $\mathcal{I}$ with their own Poincaré algebras and the collection of all these is the BMS algebra. When there is no radiation and the spacetime is stationary, one can in fact reduce the BMS algebra to the Poincaré algebra. Thus, ultimately gravitational radiation is responsible for the existence of supertranslations.

Remark 3.2. You may have been a bit uncomfortable throughout this analysis because the asymptotic symmetries are in fact diffeomorphisms. In general relativity, a diffeomorphism (=gauge transformation) is at a different footing from an isometry, yet when discussing asymptotic symmetries it seems like we equated the two. While this was a bit obscured by using the language of universal structure and such, an alternative description of the asymptotic symmetry algebra is those diffeomorphisms that preserve the asymptotic conditions of the metric modulo those diffeomorphisms that are asymptotically identity. What makes these "boundary diffeomorphisms" special is that they alter the physical boundary data. This becomes remarkably clear by analyzing the degenerate directions of the (pre)-symplectic two-form in the covariant phase space: diffeomorphisms are generically part of the degenerate directions (so they are in the kernel), but asymptotic symmetries are not. Closely related to this point is that asymptotic symmetries give rise to well-defined and finite conserved quantities, whereas diffeomorphisms give rise to zero charges — regardless of the physical spacetime under consideration. The fact that the diffeomorphisms on the "boundary" of spacetime play a special role is analogous to the distinction between "small" and "large" gauge transformations in gauge theory.

4 Bondi-Sachs coordinates

We have defined asymptotically flat spacetimes in a geometric and, accordingly, in an inherently covariant way. For certain practical applications, it is handy to have a suitable coordinate system at one's disposal. Starting from the geometric definition, we will now construct such coordinates, for both the conformally completed unphysical spacetime and the physical spacetime. These coordinates are known as *Bondi-Sachs coordinates*. The metric of any asymptotically flat spacetimes takes a particular form in these coordinates and the metric coefficients fall off at a specified rate as one approaches $\mathcal{I}$. In fact, one can also *define* asymptotically flat spacetimes as those spacetimes for which in a Bondi-Sachs coordinates, the metric coefficients satisfy the prescribed fall-off behavior. This coordinate definition is completely equivalent to the geometric definition. In some ways, the coordinate definition is easier to understand and allows you to build intuition for asymptotically flat spacetimes. On the other hand, one should always be very careful in drawing conclusions for asymptotically flat spacetimes purely based on the coordinate definition. One could be let astray and believe that certain properties are generic, whereas they are in fact a property of the coordinates themselves and not the class of spacetimes they describe. Closely related, properties derived from the coordinates are not coordinate-invariant (by construction) and you should pay extra care to make sure that your results are covariant. Also, be aware that Bondi-Sachs coordinates are not "divinely prescribed" in any meaningful sense. One can construct other coordinates that can be similarly useful. For instance, the (closely related) conformal Gaußian coordinates near $\mathcal{I}$ are convenient in the study of asymptotic flatness in higher dimensions.

4.1 Conformal Bondi-Sachs coordinates

We first construct *conformal Bondi-Sachs* coordinates for the unphysical metric $\tilde{g}_{ab}$ in a neighborhood of $\mathcal{I}$ using an asymptotic expansion. The conformal Bondi-Sachs coordinates can then be used to setup Bondi-Sachs-type coordinates for the physical metric g_{ab}, in which $\mathcal{I}$ is located at an "infinite radial distance" in the physical spacetime. We start by choosing coordinates on $\mathcal{I}$. Let u be the parameter along the null generators n^a so that $n^a\tilde{\nabla}_a u \mathrel{\hat{=}} 1$, and let $S_u \cong \mathbb{S}^2$ be the cross-sections of $\mathcal{I}$ with $u =$ constant. On some cross-section, say S_{u_0} with $u \mathrel{\hat{=}} u_0$, we pick coordinate functions x^A and parallel transport them to other cross-sections S_u along the null generators. This means that $n^a\nabla_a x^A \mathrel{\hat{=}} 0$. The precise choice of coordinates x^A on S_{u_0} is not relevant. You can choose standard angular coordinates (θ, ϕ), or stereographic coordinates $(z, \bar{z})$ or any other coordinates you fancy.[10] Hence, (u, x^A) serve as coordinates on $\mathcal{I}$.

We also need to pick a coordinate away from $\mathcal{I}$. Ω is the natural candidate. In fact, the conditions in the definition of asymptotically flat spacetimes ensure that it is a good coordinate. Additionally, it is convenient that $\Omega = 0$ on $\mathcal{I}$ itself. Having chosen Ω as the coordinate away from $\mathcal{I}$, we can now also extend the coordinates (u, x^A) away from $\mathcal{I}$. Consider the null hypersurfaces transverse to $\mathcal{I}$ that intersect $\mathcal{I}$ in the cross-sections S_u. In a sufficiently small neighborhood of null infinity, such null hypersurfaces do not intersect each other and thus generate a null foliation. We extend the coordinate u by demanding that it is constant along these null hypersurfaces into the spacetime. Using this definition of u, we can define a co-vector $l_a := -\tilde{\nabla}_a u$, which is the future-directed null normal to these hypersurfaces. In other words, $l^a l_a = 0$ everywhere (or at least in the region in which the Bondi-Sachs coordinates are valid) and we normalize it so that at null infinity $n^a l_a \mathrel{\hat{=}} -1$. Then, we extend the angular coordinates as well by parallel transporting them along l^a, that is, $l^a\tilde{\nabla}_a x^A = 0$. This concludes the setup of the *conformal Bondi-Sachs coordinates* (u, Ω, x^A) in a neighborhood of null infinity.

As a result of the above construction, the general form of the conformally completed metric in these coordinates is

$$\mathrm{d}\tilde{s}^2 = -We^{2\beta}\mathrm{d}u^2 + 2e^{2\beta}\mathrm{d}u\mathrm{d}\Omega + h_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u) , \tag{4.1}$$

where W, β, h_{AB} and U^A are all smooth functions of (u, Ω, x^A). The inverse metric is then given by

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} 0 & e^{-2\beta} & 0 \\ & We^{-2\beta} & e^{-2\beta}U^A \\ & & h^{AB} \end{pmatrix} . \tag{4.2}$$

[10] In general, we need of course more than one coordinate patch to cover all of $S_{u_0} \cong \mathbb{S}^2$ but this subtlety will not be important.

Note that the metric coefficients $\tilde{g}_{\Omega\Omega}$ and $\tilde{g}_{\Omega A}$ are zero. This follows directly from the condition that l_a is null and that the x^A coordinates are parallel propagated along l^a. This is more easily seen for the inverse metric coefficients. In particular, $l^a l_a = \tilde{g}^{ab} l_a l_b = \tilde{g}^{ab} \tilde{\nabla}_a u \tilde{\nabla}_b u = \tilde{g}^{uu}$ so that $l^a l_a = 0$ implies that $\tilde{g}^{uu} = 0$. Additionally, take for instance $x^A = (\theta, \phi)$, then

$$l^a \tilde{\nabla}_a x^A = 0 \quad \Longrightarrow \quad l^a \tilde{\nabla}_a \theta = \tilde{g}^{ab} l_a \tilde{\nabla}_b \theta = -\tilde{g}^{u\theta} = 0 \tag{4.3}$$

$$l^a \tilde{\nabla}_a \phi = \tilde{g}^{ab} l_a \tilde{\nabla}_b \phi = -\tilde{g}^{u\phi} = 0 \, . \tag{4.4}$$

In other words, $l^a \tilde{\nabla}_a x^A = 0$ implies that $\tilde{g}^{uA} = 0$. Inverting the metric then establishes that these two conditions imply that $\tilde{g}_{\Omega\Omega} = 0 = \tilde{g}_{\Omega A}$. The metric coefficient $e^{2\beta}$ could also have been called something else altogether, such as simply β. However, in the original paper introducing these coordinates, Bondi and his collaborators choose $e^{2\beta}$ to highlight that this is a positive definite function. This has been convention ever since. It makes explicit that the determinant of the metric is negative and thus that this is a Lorentzian metric:

$$\det(\tilde{g}_{ab}) = -e^{4\beta} \det(h_{AB}) < 0 \, . \tag{4.5}$$

From the fact that $\mathcal{I}$ is smooth, it is reasonable to assume that the metric components in (4.1) have an asymptotic expansion in integer powers of Ω near $\mathcal{I}$:

$$W = W^{(0)} + W^{(1)}\Omega + W^{(2)}\Omega^2 + \mathcal{O}(\Omega^3) \tag{4.6a}$$

$$\beta = \beta^{(0)} + \beta^{(1)}\Omega + \beta^{(2)}\Omega^2 + \mathcal{O}(\Omega^3) \tag{4.6b}$$

$$U^A = U^A_{(0)} + U^A_{(1)}\Omega + U^A_{(2)}\Omega^2 + \mathcal{O}(\Omega^3) \, , \tag{4.6c}$$

with all the coefficients functions of (u, x^A), and similarly for h_{AB}. Now at this point the metric components are still rather generic, but in fact, we can further restrict this freedom. Let us first focus on h_{AB}. As we saw in Sec. 2.2.2, we can always choose the conformal factor such that n^a is divergence-free at $\mathcal{I}$. However, we also noted that this choice does not completely fix the conformal factor: there is still residual conformal freedom. In particular, consider $\Omega \to \Omega' = \omega\Omega$ and let us also expand ω in terms of Ω: $\omega = \omega^{(0)} + \omega^{(1)}\Omega + \ldots$. The divergence free condition does not (even) fix $\omega^{(0)}$ entirely. There is still freedom to change $\omega^{(0)}$ as long as $\mathcal{L}_n \omega^{(0)} \hat{=} 0$. In terms of these coordinates, that means that $\omega^{(0)}$ can be any arbitrary function of x^A but cannot depend on u. Let us pick an arbitrary cross-section of $\mathcal{I}$, say S_{u_0}, then we can use this freedom to make h_{AB} the unit two-sphere metric at $\mathcal{I}$ (recall that all metrics on $\mathbb{S}^2$ are conformally related to the unit two-sphere metric). This of course fixes $\omega^{(0)}$ completely. This choice may not sound particularly interesting at first, until we realize that in a conformal divergence free frame $\mathcal{L}_n \tilde{q}_{ab} \hat{=} 0$ and consequently $\mathcal{L}_n h_{AB} \hat{=} 0$. Thus, if $h_{AB} = S_{AB}$ for $u = u_0$ with S_{AB} denoting the unit two-sphere metric, it is also true for any other value of u. So we find that

$$h_{AB} = S_{AB} + C_{AB}\Omega + d_{AB}\Omega^2 + \mathcal{O}(\Omega^3) \, . \tag{4.7}$$

Can we use the remaining freedom in $\omega^{(1)}, \omega^{(2)}, \ldots$ to impose that also C_{AB} and d_{AB} are restricted? And what would be natural choices? First of all, note that since C_{AB} and d_{AB} are symmetric rank-two tensors, they both have three independent components, while ω being a scalar only has one independent component. Therefore, we do not have the freedom to change C_{AB} and d_{AB} into anything we would like by a conformal transformation. As it turns out, a convenient choice is to pick the conformal factor away from $\mathcal{I}$ such that the determinant of h_{AB} (=one free function) is equal to the determinant of the unit two-sphere everywhere (currently, this is only true on $\mathcal{I}$). As a result, spheres of constant u and Ω will have area 4π everywhere. This completely exhaust the freedom in the conformal factor. Calculating the determinant of h_{AB} yields:

$$\det(h_{AB}) = \det(S_{AB}) \left[1 + S^{CD}C_{CD}\Omega + \left(-\tfrac{1}{2}C^{CD}C_{CD} + S^{CD}d_{CD}\right)\Omega^2\right] + \mathcal{O}(\Omega^3) \, , \tag{4.8}$$

so that $\det(h_{AB}) = \det(S_{AB})$ requires

$$S^{AB}C_{AB} = 0\,, \qquad S^{AB}d_{AB} = \tfrac{1}{2}C^{AB}C_{AB}\,, \tag{4.9}$$

where the angular indices $A, B, \ldots$ are raised and lowered with the unit round metric S_{AB}.

Let us return to the expansions of W, β and U^A in Eq. (4.6). These can also be further simplified. Since we are in a conformal divergence free frame now, n^a is not just geodetic but in fact geodesic. The affine parameter along these null generators is u because $n^a\tilde{\nabla}_a u = -n^a l_a \,\hat{=}\, 1$. Therefore, $\tilde{g}^{u\Omega} \,\hat{=}\, 1$, which in turn implies that $e^{-2\beta} \,\hat{=}\, 1$ and thus $\beta^{(0)} = 0$. Moreover, we also have $n^a\nabla_a x^A \,\hat{=}\, 0$ so that $\tilde{g}^{\Omega A} \,\hat{=}\, 0$ and $e^{-2\beta}U^A \,\hat{=}\, 0$, which sets $U^A_{(0)} = 0$. Moreover, we have $\tilde{\nabla}_a n_b \,\hat{=}\, 0$ so that

$$\tilde{\nabla}_a n_b = \tilde{\nabla}_a\tilde{\nabla}_b\Omega = \partial_a\partial_b\Omega - \tilde{\Gamma}^c_{ab}\partial_c\Omega \,\hat{=}\, 0. \tag{4.10}$$

Looking for instance at the $\Omega\Omega$-component, we find that

$$0 \,\hat{=}\, \Gamma^{\Omega}_{\Omega\Omega} = \tilde{g}^{u\Omega}\partial_\Omega\tilde{g}_{u\Omega} = 2\partial_\Omega\beta = 2\beta^{(1)} + \mathcal{O}(\Omega) \tag{4.11}$$

and thus not only $\beta^{(0)} = 0$ but also $\beta^{(1)} = 0$. Continuing this for the other components and using Eq. (2.10) to probe one order deeper, we also find that $U^{A(1)}$ vanishes and $W^{(2)} = 1$.

To summarize, in the conformal Bondi-Sachs coordinates (u, Ω, x^A) we have the unphysical metric (4.1) with the following asymptotic expansions

$$W = \Omega^2 + \Omega^3 W^{(3)} + \mathcal{O}(\Omega^4) \tag{4.12a}$$

$$\beta = \Omega^2\beta^{(2)} + \mathcal{O}(\Omega^3) \tag{4.12b}$$

$$U^A = \Omega^2 U^A_{(2)} + \Omega^3 U^A_{(3)} + \mathcal{O}(\Omega^4) \tag{4.12c}$$

$$h_{AB} = S_{AB} + \Omega C_{AB} + \Omega^2 d_{AB} + \mathcal{O}(\Omega^3)\,, \tag{4.12d}$$

where C_{AB} is traceless and the trace of d_{AB} is specified in Eq. (4.9). The coefficients $W^{(3)}$ and $U^A_{(3)}$ are often written as

$$W^{(3)} \overset{\text{def}}{=} -2M \qquad U^A_{(3)} \overset{\text{def}}{=} N^A\,, \tag{4.13}$$

where M and N^A are referred to as the mass and angular momentum aspect, respectively, because M and N^A encode information about the mass and angular momentum of the spacetime at null infinity.[11] This interpretation is supported by explicit examples such as the Kerr-Newman and Vaidya metric, the Landau-Lifschitz approach to defining balance laws and put on a firm ground by an analysis of the covariant phase space of asymptotically flat spacetimes.

4.2 Bondi-Sachs coordinates for the physical metric

These conformal Bondi-Sachs coordinates for the unphysical spacetime constructed above can be used to obtain asymptotic coordinates for the physical metric. In the conformal Bondi-Sachs coordinates, the physical metric is

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= \Omega^{-2}\tilde{g}_{\mu\nu}dx^\mu dx^\nu \\ &= -\Omega^{-2}e^{2\beta}W\,du^2 + 2\Omega^{-2}e^{2\beta}\,dud\Omega + \Omega^{-2}h_{AB}\left(dx^A - U^A\,du\right)\left(dx^B - U^B\,du\right)\,. \end{aligned} \tag{4.14}$$

Surfaces of constant u are outgoing null surfaces for both the unphysical and the physical spacetime, since conformal transformations do not change the properties of null geodesics. To put this metric in a more familiar form, we define a radial coordinate r in the physical spacetime so that null

[11] Be aware: conventions differ on the precise definition of the angular-momentum aspect, and some authors shift N^A by terms proportional to C_{AB} and its derivatives, and/or multiply it by a numerical factor.

infinity is approached as the radial coordinate goes to infinity along the null surfaces of constant u. The natural candidate for this radial coordinate is $r = \Omega^{-1}$:

$$g_{\mu\nu}dx^\mu dx^\nu = -e^{2\beta}\frac{V}{r}\, du^2 - 2e^{2\beta}\, dudr + r^2 h_{AB}\left(dx^A - U^A\, du\right)\left(dx^B - U^B\, du\right) , \tag{4.15}$$

where — to make contact with the standard notation for asymptotically flat spacetimes — we have introduced $V := r^3 W$. The expansions of the metric coefficients in Ω in Eq.(4.12) naturally translate to expansions in terms of $1/r$. Moreover, if we impose Einstein's equations order by order in $1/r$, we find relations between some of the coefficients. In particular, we find that

$$\beta = -\frac{1}{32}C^{AB}C_{AB}\frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{4.16}$$

$$\frac{V}{r} = 1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{4.17}$$

$$U^A = -\frac{D_B C^{AB}}{2r^2} + \frac{N^A}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{4.18}$$

$$h_{AB} = S_{AB} + \frac{C_{AB}}{r} + \frac{d_{AB}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{4.19}$$

and evolution equations for M and N^A. These are:

$$\dot{M} = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_A D_B N^{AB} \tag{4.20}$$

$$\dot{N}^A = -D_B\left[\left(\frac{2}{3}M - \frac{1}{8}C_{CD}N^{CD}\right)S^{AB} + \frac{1}{3}D_C D^{[A}C^{B]C}\right] + \frac{1}{2}N^A{}_C D_B C^{BC} - \frac{1}{6}N^B{}_C D_B C^{AC} \tag{4.21}$$

with the dot denoting a partial derivative with respect to u and we have defined the Bondi News tensor $N_{AB} := \dot{C}_{AB}$. N_{AB} is called the news tensor, because it determines the energy flux of gravitational radiation and thus carries news about the presence of radiation. One can show that the news tensor is a geometrically determined tensor field independent of the choice of u-foliation. For the unphysical metric, we had only used part of Einstein's equation through the conditions on n^a. Of course, we could have implemented these equations already for the conformally completed metric, but in that case we would have to be careful to use Eq. (2.6) as it is the physical metric that satisfies Einstein's equation, not the conformally rescaled one.

Remark 4.1. Minkowski spacetime itself is of course given by $\beta = 0, V/r = 1, U^A = 0$ and $h_{AB} = S_{AB}$ and the Schwarzschild spacetime is $\beta = 0, V/r = 1 - 2M/r, U^A = 0$ and $h_{AB} = S_{AB}$.

5 Gravitational waves

Having learned about asymptotically flat spacetimes from two complementary pespectives — the geometric definition relying on conformal completions and the coordinate definition using Bondi-Sachs coordinates — we are finally ready to study gravitational radiation. In principle, we could do this again from both perspectives. However, in the interest of time, we will take the second route only and focus on the physical metric in Bondi-Sachs coordinates that we discussed in Sec. 4.2.[12]

[12] The geometric approach is rather involved due to the fact that $\tilde{q}_{ab}$ is not invertible and therefore raising indices *on* $\mathcal{I}$ is subject to a certain ambiguity. Challenges in defining a covariant derivative for all fields on $\mathcal{I}$ goes hand in hand with this observation, and derivatives are essential in defining the 'next-order' structure that contains information about radiation.

5.1 Radiation, mass and angular momentum

What can we learn from these metric expansions about gravitational radiation? First of all, note that to know the metric near $\mathcal{I}$, we need to know C_{AB}, M and N^A. Since we know the time evolution of both M and N^A, this significantly reduces the freedom of M and N^A: we can only freely determine M and N^A at some initial time u_0. In other words, we have the freedom to specify the "initial data" $M(u_0, x^A)$ and $N^A(u_0, x^A)$. The tensor C_{AB} is left undetermined by the field equations. As mentioned before, it contains two degrees of freedom, since it is traceless. These degrees of freedom are exactly the two radiative degrees of freedom of the gravitational field and can be directly related to the plus and cross polarization of gravitational waves, often described in textbooks on this topic. If C_{AB} is time independent so that $C_{AB} = C_{AB}(x^A)$ and $N_{AB} = 0$, the two-sphere metric near $\mathcal{I}$ will be distorted but there is, of course, no radiation. Thus, the real litmus test in terms of these coordinates of whether radiation is present is not $C_{AB} \neq 0$, but $N_{AB} \neq 0$.

The time-dependent Bondi mass $m(u)$ for an isolated system is defined as:

$$m(u) := \frac{1}{4\pi} \int M(u, x^A)\, d^2S \tag{5.1}$$

with d^2S denoting the volume element on the unit two-sphere. The celebrated Bondi mass-loss formula follows from Eq. (4.20). Integrating this equation over the unit 2-sphere on both sides, we obtain

$$\dot{m}(u) = -\frac{1}{32\pi} \int N_{AB} N^{AB}\, d^2S \tag{5.2}$$

where the first term on the right in Eq. (4.20) describes the flux of gravitational-wave energy, and the second term integrates to zero due to the divergence theorem. If there is no gravitational radiation, i.e. $N_{AB} = 0$, the Bondi mass is constant. If a system emits gravitational waves, then its Bondi mass must decrease since $\dot{m} < 0$.

Defining angular momentum is more tricky, however. Why? This is a result of the "supertranslation ambiguity". Let us go back to the situation in Minkowski spacetime first, to highlight the similarities and differences. Recall that the Poincaré algebra is the semi-direct product of translations with the Lorentz algebra. As a result, there is no unique Lorentz subalgebra: there is one associated to each origin. The location of the origin can be shifted by translations, so there is a 4-parameter family of Lorentz subalgebras. Since angular momentum refers to the Lorentz algebra, angular momentum in Minkowski spacetime also comes with a 4-parameter ambiguity. This ambiguity is physical and corresponds precisely to the choice of an origin about which angular momentum is defined. Now the BMS algebra is the semi-direct product of supertranslations with the Lorentz algebra. As a result, the BMS algebra has an *infinite* parameter family of Lorentz subalgebras that are related to each other by supertranslations. Therefore, the ambiguity in angular momentum is infinite-dimensional for asymptotically flat spacetimes and cannot be traced to the choice of an origin in spacetime.

However, when the spacetime is stationary during for some extended period, the supertranslation ambiguity can be removed. Because in the absence of gravitational waves, one can naturally reduce the BMS algebra to the Poincaré algebra and the supertranslations disappear. Since most systems of interest are typically assumed to be stationary in the remote past, this is not a severe restriction on the space of physically motivated spacetimes. At a practical level, this can be implemented by adopting a preferred Bondi frame. The key element is to require that C_{AB} vanishes in the remote past (instead of being merely constant, which is required by the stationarity condition: $N_{AB} = 0$). A supertranslation would turn an initially zero C_{AB} into a non-zero C_{AB}. This can be seen as follows. A supertranslation is generated by a vector field ξ^μ with components

$$\xi^\mu = \left(f, \frac{1}{2} D^A D_A f + \mathcal{O}\left(\frac{1}{r}\right), -\frac{1}{r} D^A f + \mathcal{O}\left(\frac{1}{r^2}\right) \right) \tag{5.3}$$

in the (u, r, x^A) Bondi-Sachs coordinates and $f = f(x^A)$ an arbitrary function on the two-sphere. From the transformation, $\mathcal{L}_\xi g_{\mu\nu}$, one then obtains that a supertranslation changes C_{AB} in the following way:

$$\delta_f C_{AB} = f N_{AB} - 2D_A D_B f + S_{AB} D^C D_C f \ . \tag{5.4}$$

This non-zero C_{AB} violates the requirement of the preferred Bondi frame. Thus, this eliminates the freedom to perform supertranslations and thereby makes angular momentum unambiguous.

In gravitational wave modeling, one is of course not interested in spacetimes that remain stationary. So after some finite period, there typically is radiation and the supertranslation ambiguity in defining angular momentum reappears. In the case of a binary merger, the final configuration is a rotating black hole spacetime. In other words, in those scenarios the final configuration is also stationary. Thus, in such cases, one can unambiguously define angular momentum during the early non-radiative regime and the late non-radiative regime. One should, however, be extremely careful in comparing the two, since the initial angular momentum of the binary and the final angular momentum of the black hole generically refer to distinct rotation subalgebras of the BMS algebra. Only if one has kept track of the full information in C_{AB} throughout the radiative regime can one meaningfully compare the initial and final angular momentum.

5.2 Gravitational memory effect

There is another interesting effect associated with gravitational radiation, which is known as the memory effect. Consider a system of test particles, i.e., freely falling objects that do not backreact on the spacetime, near null infinity. If a gravitational wave passes by, it will induce oscillations and the relative positions of these test particles will change. This is of course not surprising in anyway. What is interesting is that after the wave has passed and the spacetime is stationary again, the test particles will generically *not* return to their original position. In other words, the spacetime metric before is distinct from the metric after the wave has passed. Compare this to a more ordinary experience. If you were to jump into a swimming pool, the waves you created will fade and if you are back on the side of the pool, the water will return to its original, flat configuration. The water in the pool has no "memory" of the waves you created, while the spacetime metric "remembers" the passing of a burst of gravitational waves.

The relative change in the positions of the freely falling particles near $\mathcal{I}$ can be calculated exactly by integrating the geodesic deviation equation twice:

$$\ddot{X}^a = R_{bcd}{}^a X^b T^c T^d \tag{5.5}$$

where X^a is the deviation vector, T^a the tangent vector to the geodesics and an overdot indicates derivation along T^a. Let us make this more concrete. Suppose that an asymptotically flat spacetime is stationary at early and late times, so for $u < u_0$ and $u > u_1$ with $u_0 < u_1$, the Bondi news vanishes. Consider two freely falling test masses near $\mathcal{I}$ with worldlines initially tangent to $(\partial/\partial u)^a$ and initial deviation vector X_0^A. During the stationary era, the deviation vector $X^A(u)$ will simply be the parallel transport of X_0^A, but in the radiative regime it will differ. As a result, in the late time stationary era for which $u > u_1$, the deviation vector X_1^A will be different from the parallel transport of X_0^A. This difference is encoded in the *memory tensor* Δ_{AB}:

$$X_1^{(1)\,A} = \Delta^A{}_B \, X_0^{(1)\,B} \ . \tag{5.6}$$

where the superscript (1) indicates the $1/r$ part of the initial and final deviation vectors X_0^A and X_1^A. This memory tensor can be directly related to the difference in C_{AB} at early and late times:

$$\Delta_{AB} = \frac{1}{2} \, C_{AB} \Big|_{u=-\infty}^{u=\infty} \ . \tag{5.7}$$

From the above discussion about angular momentum, you may worry that this is not gauge invariant since a supertranslation can change C_{AB} (see Eq. (5.4)). While it is indeed true that at any given u, you can change C_{AB} to any value you like using a supertranslation, you can only do this for one value of u since $f(x^A)$ is u-independent. Consequently, the difference between C_{AB} at two different instances of retarded time cannot be changed by a supertranslation.[13] The memory effect is more difficult to observe in ground-based gravitational wave observatories than the gravitational wave itself, but there are predictions that this is within reach of advanced LIGO and certainly with the Einstein Telescope or other next generation observatories. In analogy with electrical circuits, the memory effect is sometimes referred to as the DC part of the signal and the gravitational wave itself as the AC part.

6 Linearized gravity

We have studied a large class of spacetimes describing solutions to Einstein's equation that are asymptotically flat. This framework is very general and allowed us to study the consequences of Einstein's equations in the full non-linear theory. We only assumed that far away from the sources describing the isolated objects of interest, the spacetime curvature died off at some appropriate rate. Near the source, the curvature could be as strong and wild as one would like. However, the framework does have the drawback that we only have access to information of the spacetime near null infinity. One may also be interested in situations in which the spacetime is only "slightly different" from some analytic solution such as the Schwarzschild or Kerr spacetime. To what does Einstein's equation reduce in such a scenario? This is exactly the realm of perturbation theory, which is used in all areas of physics. More precisely, suppose we set $g_{ab} = \bar{g}_{ab} + \gamma_{ab}$ with $\bar{g}_{ab}$ some exact solution to Einstein's equation and γ_{ab} is "small". We could then imagine expanding Einstein's equation to first order in the perturbation γ_{ab}. The resulting equations are called the linearized Einstein equation and we will derive it in this section.

It is convenient to adopt an approach in which one does not have to keep track of the orders of various terms in the perturbation expansion. Let $g_{ab}(\lambda)$ be a one-parameter family of metrics on a fixed manifold M, where the parameter λ has a range between zero and one. Let $g_{ab}(0)$ be the analytic background metric you are interested in perturbing (e.g., the Schwarzschild metric). The plan is to evaluate $d/d\lambda$ of various quantities at $\lambda = 0$. This allows us to automatically keep the appropriate terms in our perturbation expansion. Here we are only interested in the leading and linear order, but in principle this technique applies to any order.

Since Einstein's equation is a differential equation in which the covariant derivative plays a key role, let us focus on this derivative operator first. Denote the one-parameter family of covariant derivative operators compatible with the one-parameter family of metrics $g_{ab}(\lambda)$ as $\overset{\lambda}{\nabla}_a$. Fixing a λ-independent vector field k_a on M, we find that

$$\frac{d}{d\lambda}\left(\overset{\lambda}{\nabla}_a\, k_b\right)\Big|_{\lambda=0} = -C^c_{ab}k_c \tag{6.1}$$

for some tensor field C^c_{ab} which is symmetric in its covariant indices. This tensor field represents the λ-rate of change of the derivative operator evaluated at $\lambda = 0$ and in some ways is reminiscent of the Christoffel symbol relating the covariant derivative operator to, say, the partial derivative. Now the question is of course: what is C^c_{ab}? By definition, the covariant derivative operator satisfies $\overset{\lambda}{\nabla}_a\, g_{bc}(\lambda) = 0$, so that

$$0 = \frac{d}{d\lambda}\left(\overset{\lambda}{\nabla}_a\, g_{bc}(\lambda)\right)\Big|_{\lambda=0} \tag{6.2}$$

[13] If at early time, one adopts a preferred Bondi frame then the memory tensor is of course simply the value of C_{AB} at late times because C_{AB} at early times vanishes.

$$= -C^d_{ab}\, g_{cd}(0) - C^d_{ac}\, g_{bd}(0) + \overset{0}{\nabla}_a \frac{d}{d\lambda} \left(g_{bc}(\lambda)\right)|_{\lambda=0} \tag{6.3}$$

$$= -C^d_{ab}\, \bar{g}_{cd} - C^d_{ac}\, \bar{g}_{bd} + \nabla_a \gamma_{bc} \tag{6.4}$$

where in going from the first to the second line, we used Eq. (6.1), and in going from the second to the third, we defined $g_{ab}(0) = \bar{g}_{ab}$, $\overset{0}{\nabla}_a = \nabla_a$ and $\frac{d}{d\lambda}\left(g_{ab}(\lambda)\right)|_{\lambda=0} = \gamma_{ab}$. Using this in

$$0 = \frac{d}{d\lambda}\left(\overset{\lambda}{\nabla}_a g_{bc}(\lambda) - \overset{\lambda}{\nabla}_b g_{ac}(\lambda) - \overset{\lambda}{\nabla}_c g_{ab}(\lambda)\right)\Big|_{\lambda=0} \tag{6.5}$$

and solving for C^c_{ab}, we find that

$$C^d_{bc}\bar{g}_{da} = -\frac{1}{2}\left(\nabla_a\gamma_{bc} - \nabla_b\gamma_{ac} - \nabla_c\gamma_{ab}\right)\ . \tag{6.6}$$

Contracting this with $\bar{g}^{am}$, we finally obtain the desired expression

$$C^m_{bc} = \frac{1}{2}\bar{g}^{am}\left(\nabla_b\gamma_{ac} + \nabla_c\gamma_{ab} - \nabla_a\gamma_{bc}\right)\ . \tag{6.7}$$

The Riemann tensor for each value of λ is defined in the usual way from the λ-dependent derivative:

$$2\, \overset{\lambda}{\nabla}_{[a} \overset{\lambda}{\nabla}_{b]}\, k_c = R_{abc}{}^d(\lambda) k_d \tag{6.8}$$

for any co-vector field k_a. Denoting the Riemann tensor at $\lambda = 0$ by $\bar{R}_{abc}{}^d$ and taking the derivative with respect to λ, we obtain the linearized Riemann tensor:

$$\frac{d}{d\lambda} R_{abc}{}^d(\lambda)\Big|_{\lambda=0} k_d = 2\left[\frac{d}{d\lambda}\overset{\lambda}{\nabla}_{[a}\overset{\lambda}{\nabla}_{b]}\, k_c\right]\Big|_{\lambda=0} \tag{6.9}$$

$$= -2\nabla_{[a}\left(C^d_{b]c} k_d\right) - 2C^d_{[ab]}\nabla_d k_c - 2C^d_{c[a}\nabla_{b]} k_d \tag{6.10}$$

$$= -2\nabla_{[a} C^d_{b]c} k_d - 2\nabla_{[a} k_{|d|} C^d_{b]c} - 2C^d_{c[a}\nabla_{b]} k_d \tag{6.11}$$

$$= -2\nabla_{[a} C^d_{b]c} k_d \tag{6.12}$$

where in going from the first to the second line, we used Eq. (6.1), in going from the second to the third we used that $C^c_{[ab]} = 0$ and expanded the derivatives and in the final step we canceled the last two terms. Hence, we obtain

$$\frac{d}{d\lambda} R_{abc}{}^d(\lambda)\Big|_{\lambda=0} = -2\nabla_{[a} C^d_{b]c}\ , \tag{6.13}$$

since k_a is arbitrary. Substituting the expression for C^c_{ab} in terms of the background and perturbed metric in Eq. (6.7), we thus find

$$\frac{d}{d\lambda} R_{abc}{}^d(\lambda)\Big|_{\lambda=0} = -\nabla_{[a}\nabla_{b]}\gamma^d{}_c - \nabla_{[a}\nabla_{|c|}\gamma_{b]}{}^d + \nabla_{[a}\nabla^d\gamma_{b]c} \tag{6.14}$$

where we raised and lowered indices with the background metric $\bar{g}_{ab}$. This convention generically does not lead to any confusions, but be careful with minus signs and inverse metrics:

$$\frac{d}{d\lambda} g^{ab}(\lambda)\Big|_{\lambda=0} = -\gamma^{ab} \tag{6.15}$$

and not $+\gamma^{ab}$ so that

$$0 \overset{?}{=} \frac{d}{d\lambda}\delta^a_b\Big|_{\lambda=0} = \frac{d}{d\lambda}\left(g^{ac}g_{cb}\right)\Big|_{\lambda=0} = \frac{d}{d\lambda}g^{ac}\Big|_{\lambda=0}\bar{g}_{cb} + \bar{g}^{ac}\frac{d}{d\lambda}g_{cb}\Big|_{\lambda=0}$$

$$= -\gamma^{ac}\bar{g}_{cb} + \bar{g}^{ac}\gamma_{cb} = -\gamma^a_{\ b} + \gamma^a_{\ b} \overset{!}{=} 0 \,. \tag{6.16}$$

To determine Einstein's equation, we also need to know the linearized Ricci tensor and scalar:

$$\begin{aligned}
\frac{d}{d\lambda}\, R_{ac}(\lambda)|_{\lambda=0} &= \frac{d}{d\lambda}\left(\delta^b_{\ d}\, R_{abc}{}^d(\lambda)\right)\Big|_{\lambda=0} \\
&= \cancel{-\frac{1}{2}\nabla_a\nabla_b\gamma^b_{\ c}} + \frac{1}{2}\nabla_b\nabla_a\gamma^b{}_c - \frac{1}{2}\nabla_a\nabla_c\gamma_b{}^b + \frac{1}{2}\nabla_b\nabla_c\gamma_a{}^b \cancel{+\frac{1}{2}\nabla_a\nabla^b\gamma_{bc}} - \frac{1}{2}\nabla_b\nabla^b\gamma_{ac} \\
&= \nabla_b\nabla_{(a}\gamma^b{}_{c)} - \frac{1}{2}\nabla_a\nabla_c\gamma_b{}^b - \frac{1}{2}\nabla_b\nabla^b\gamma_{ac} \,.
\end{aligned} \tag{6.17}$$

The linearized Ricci scalar is

$$\frac{d}{d\lambda}\, R(\lambda)|_{\lambda=0} = \frac{d}{d\lambda}\, (g^{ac}R_{ac})|_{\lambda=0} = -\gamma^{ac}\bar{R}_{ac} + \bar{g}^{ac}\frac{d}{d\lambda}\, R_{ac}(\lambda)|_{\lambda=0} \tag{6.18}$$

$$= -\gamma^{ac}\bar{R}_{ac} + \nabla_a\nabla_c\gamma^{ac} - \nabla_a\nabla^a\gamma^c_{\ c} \,, \tag{6.19}$$

where we denoted $R_{ab}(\lambda = 0) = \bar{R}_{ab}$.

Now we suppose our family of metrics represents a family of solutions to Einstein's equation, so we also have a family of stress-energy tensors $T_{ab}(\lambda)$. At the background level, the form of Einstein's equation remains the same but now with bars on top of all the tensors (with $\bar{T}_{ab} := T_{ab}(\lambda = 0)$). At the linearized level, using the results for the Ricci tensor and scalar in Eq. (6.17)-(6.19), we obtain

$$\Box\gamma_{ac} + \nabla_a\nabla_c\gamma - 2\nabla_b\nabla_{(a}\gamma^b_{\ c)} + \gamma_{ac}\bar{R} + \bar{g}_{ac}\left(\nabla_b\nabla_d\gamma^{bd} - \Box\gamma - \gamma^{bd}\bar{R}_{bd}\right) = -16\pi\frac{d}{d\lambda}\, T_{ac}(\lambda)|_{\lambda=0} \,, \tag{6.20}$$

where the d'Alembertian operator $\Box$ is simply $\nabla^a\nabla_a$ and $\gamma = \gamma^a_{\ a} = \bar{g}^{ab}\gamma_{ab}$.

This equation simplifies on a Minkowski background for which all background curvature tensors vanish:

$$\Box\gamma_{ac} + \nabla_a\nabla_c\gamma - 2\nabla_b\nabla_{(a}\gamma^b_{\ c)} + \eta_{ac}\left(\nabla_b\nabla_d\gamma^{bd} - \Box\gamma\right) = -16\pi\frac{d}{d\lambda}\, T_{ac}(\lambda)|_{\lambda=0} \,. \tag{6.21}$$

Most likely, you have seen this equation before in an even more simplified form. This can be achieved by a gauge transformation, for which the linearized perturbation changes as follows

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} + 2\nabla_{(a}\xi_{b)} \,. \tag{6.22}$$

Gauge transformations change the linearized metric without changing the linearized curvature on Minkowski spacetime; just like gauge transformations in electromagnetism change the vector potential A_a but do not change the electric or magnetic field. By choosing ξ_a cleverly, you can make calculations significantly simpler. In particular, note that under a gauge transformation

$$\nabla_c\left(\gamma'^{ac} - \frac{1}{2}\gamma'\eta^{ac}\right) = \nabla_c\left(\gamma^{ac} - \frac{1}{2}\gamma\eta^{ac}\right) + \nabla_c\left(\nabla^a\xi^c + \nabla^c\xi^a - \nabla_d\xi^d\eta^{ac}\right) \tag{6.23}$$

$$= \nabla_c\left(\gamma^{ac} - \frac{1}{2}\gamma\eta^{ac}\right) + \nabla_c\nabla^c\xi^a \,, \tag{6.24}$$

where in going from the first to the second line, the third and fifth term canceled each other. So you can always choose ξ^a such that

$$\nabla_c\left(\gamma'^{ac} - \frac{1}{2}\gamma'\eta^{ac}\right) = 0 \,. \tag{6.25}$$

This is known as the Lorenz gauge (occasionally also refered to as the de Donder gauge), and the linearized equations further reduce to

$$\Box\left(\gamma_{ac} - \frac{1}{2}\gamma\eta_{ac}\right) = -16\pi\frac{d}{d\lambda}\, T_{ac}(\lambda)|_{\lambda=0} \,. \tag{6.26}$$

In the above, we made a convenient gauge choice to simplify the equations. Depending on the background spacetime, other gauge choices might be more suitable as we will see in the next section on linearized perturbations off the Schwarzschild spacetime.

Remark 6.1. While it is not true that every gauge invariant quantity is observable, gauge invariance is a *necessary* condition for quantities to be observable. Gauge invariance is a subtle subject in general relativity and even more so at the linearized level (this was one of the reasons for the decades long debate on the physical reality of gravitational waves). There is, however, a very powerful but somewhat underappreciated lemma known as the Stewart-Walker lemma that clarifies many points. It states the following:

> A linear perturbation of any tensor field $T^{c\ldots d}_{a\ldots b}$ is gauge invariant if and only if its corresponding background quantity is a constant tensor field, i.e., a linear combination of delta functions, or identically zero.

As a corollary, this states that the linearized Ricci tensor is only gauge invariant when the background metric has a vanishing Ricci tensor or is constant. For instance, the linearized Ricci tensor is gauge invariant off a de Sitter background but it is not off a FLRW background. The linearized Weyl tensor is gauge invariant for both de Sitter and FLRW spacetimes, since the Weyl tensor vanishes for both background spacetimes.

7 Perturbations off the Schwarzschild spacetime

No realistic physical system involving a black hole is exactly described by the Schwarzschild or Kerr spacetime. The Schwarzschild solution models an isolated, unchanging black hole far removed from other influences. Instead, the astrophysical black holes in our Universe are the sites of the most energetic events known to occur (and most certainly are rotating!). Nonetheless, whenever the influence of external processes near black holes are small compared to the curvature produced by the black hole, we can linearize the Einstein field equations around the exact black hole solution. Moreover, when rotation is small, we can work with the simpler Schwarzschild spacetime instead of the Kerr spacetime.

7.1 Spherical harmonics

Before delving into the relevant equations, let us first review some facts about spherical harmonics. These harmonics come in three different types: scalar, vector and tensor spherical harmonics. Here, the classification of a spherical harmonic into scalar, vector or tensor relies on the unit two-sphere S_{AB}. Concretely, the scalar harmonics are just the usual spherical-harmonic functions $Y_{\ell m}(x^A)$ satisfying the eigenvalue equation

$$S^{AB} D_A D_B Y_{\ell m} = -\ell\,(\ell+1)\,Y_{\ell m} \tag{7.1}$$

with $x^A = (\theta, \phi)$ and D_A the covariant derivative compatible with S_{AB}. The vector harmonics come in two types: even-parity Y_A (also known as electric type) and odd-parity X_A (also known as magnetic type), which are related to the scalar harmonics through the covariant derivative operator D_A in the following way:

$$Y_A^{\ell m} := D_A Y_{\ell m} \tag{7.2}$$

$$X_A^{\ell m} := \epsilon_A{}^B D_B Y_{\ell m}\,. \tag{7.3}$$

These even- and odd-parity harmonics are orthogonal in the sense that $\int d^2S\ \bar{Y}^A_{\ell m} X_A^{\ell' m'} = 0$. The tensor harmonics also come in the same two types:

$$Y_{AB}^{\ell m} := D_{(A} Y_{B)}^{\ell m} - \frac{1}{2} S_{AB} D_C Y^C_{\ell m} \tag{7.4}$$

$$X^{\ell m}_{AB} := \epsilon_{(A}{}^{C} D_{B)} Y^{\ell m}_{C} \, . \tag{7.5}$$

These operators are traceless, i.e., $S^{AB} Y^{\ell m}_{AB} = 0 = S^{AB} X^{\ell m}_{AB}$ and orthogonal in the same sense as the vector harmonics are. The beauty of these decompositions is that the spherical symmetry of the background spacetime prevents modes with different parity from mixing. As a result, perturbations for each parity can be derived separately.

7.2 Set-up and strategy

To simplify the calculations, we will assume that the matter distribution generating gravitational radiation is confined to a bounded volume (as is the case for compact binaries) and that our domain of interest is outside this volume. This allows us to set the stress-energy tensor to zero and solve the vacuum Einstein equations. Of course, when you want to infer properties of the source by studying the waves generated by the source, you will need to link the waves with their source. Nonetheless, there are occasions when the vacuum equations are all you need. The equations with a source are very similar; just with more terms representing the matter pieces.

A popular gauge in studying perturbations on a Schwarzschild background is the Regge-Wheeler gauge. This gauge is named after Regge and Wheeler, who introduced this gauge to study stability of the Schwarzschild solution. It eliminates four components of the metric, thereby reducing the number of independent components from ten to six. This may be puzzling at first as – even in this gauge – Einstein's equations yield ten differential equations. Thus, a priori, the system appears overdetermined. Fortunately, the (contracted) Bianchi identities come to the rescue. In vacuum, these reduce to the statement that $\nabla^a R_{ab} = 0$. These four additional relations provide constraints on the ten differential equations obtained by linearizing Einstein's equations, thereby reducing the number of independent equations to exactly six.

In this section, we will derive the equations in the Regge-Wheeler gauge, but the final results will be gauge invariant. At this point, you should be on your guard. Occasionally, people will call results that are completely gauge fixed also gauge invariant and you may suspect that this is what is meant by "the final results are gauge invariant". This is *not* the case. Here, we mean truly gauge invariant and not gauge fixed.[14] How is this possible? This is analogous to the construction of Bardeen variables in cosmological perturbation theory. The procedure is surprisingly simple: you investigate how different quantities transform under gauge transformations and then you take clever combinations of these quantities so that the gauge transformation of the resulting composed quantity all cancel each other, and thus is gauge invariant. With this knowledge in hand, you perform your calculation in the simpler gauge (like the Regge-Wheeler gauge in the case of Schwarzschild) and at the end replace the quantities by their gauge invariant counterparts. Of course, there is a caveat. The final quantities need to be gauge invariant themselves, if they are not, it of course does not help whether you replace objects with their gauge invariant counterparts. Fortunately, by the Stewart-Walker lemma, the linearized Einstein's equations are gauge invariant in vacuum. Hence, we can apply this procedure to the case of perturbations on a Schwarzschild background.

Having set the stage, let us start by splitting the metric in the background Schwarzschild metric and perturbations, where the Schwarzschild line element in standard (t, r, θ, ϕ) coordinates is given by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \, , \tag{7.6}$$

where $f(r) = 1 - 2M/r$. We will decompose the linear perturbations using the spherical harmonics:

[14] If expressions are completely gauge fixed, they are of course gauge invariant under the allowed residual gauge transformation – of which there are none! This is likely the origin of this – in my opinion – abuse of language.

$$\gamma_{pq} = \sum_{\ell,m} f_{pq}^{\ell m}\,(t,r)\,Y_{\ell m}, \tag{7.7a}$$

$$\gamma_{pA} = \sum_{\ell,m} j_p^{\ell m}\,(t,r)\,Y_A^{\ell m} + h_p^{\ell m}\,(t,r)\,X_A^{\ell m}, \tag{7.7b}$$

$$\gamma_{AB} = \sum_{\ell,m} r^2\,K^{\ell m}(t,r)Y_{\ell m}\,S_{AB} + r^2\,G^{\ell m}\,(t,r)\,Y_{AB}^{\ell m} + h_2^{\ell m}\,(t,r)\,X_{AB}^{\ell m}, \tag{7.7c}$$

where p, q are indices referring to the (t, r) components only. The sum over ℓ is restricted to $\ell \geq 2$ (and m ranges from $-\ell$ to ℓ). The $\ell = 0$ and $\ell = 1$ multipoles are non-radiative and require special treatment. We will not treat these here. It turns out to be convenient to also give the (t, r) components their own name. In particular, standard notation is to introduce

$$f_{pq}^{\ell m} = \begin{pmatrix} f(r)H_0^{\ell m}(t,r) & H_1^{\ell m}(t,r) \\ H_1^{\ell m}(t,r) & \frac{1}{f(r)}H_2^{\ell m}(t,r) \end{pmatrix} \tag{7.8}$$

and for the even-parity modes of γ_{pA}

$$j_p^{\ell m} = \begin{pmatrix} j_0^{\ell m}(t,r) \\ j_1^{\ell m}(t,r) \end{pmatrix} \tag{7.9}$$

with similar notation for the odd-parity modes

$$h_p^{\ell m} = \begin{pmatrix} h_0^{\ell m}(t,r) \\ h_1^{\ell m}(t,r) \end{pmatrix}\,. \tag{7.10}$$

Hence, there are seven even-parity modes $(H_0, H_1, H_2, j_0, j_1, K, G)$ and three odd-parity modes (h_0, h_1, h_2). Under a gauge transformation generated by some arbitrary vector field ξ^a, the perturbation changes as follows (see Eq. (6.22)):

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} + 2\nabla_{(a}\xi_{b)} \tag{7.11}$$

and as a results, the different components transform accordingly. If we decompose ξ^a into spherical harmonics as well, we can make this more concrete. The four components of ξ^a are split into three even-parity modes (a_0, a_1, b) and one odd-parity mode (c):

$$\xi_p = \sum_{\ell,m} a_p^{\ell m}(t,r)Y_{\ell m} \tag{7.12a}$$

$$\xi_A = \sum_{\ell,m} b^{\ell m}(t,r)Y_A^{\ell m} + c^{\ell m}(t,r)X_A^{\ell m}\,. \tag{7.12b}$$

Under a gauge transformation the even-parity sector transforms as

$$f_{pq} \longrightarrow f'_{pq} = f_{pq} + 2\nabla_{(p}a_{q)} \tag{7.13a}$$

$$j_p \longrightarrow j'_p = j_p + a_p + \nabla_p b - \frac{2}{r}\nabla_p r\, b \tag{7.13b}$$

$$K \longrightarrow K' = K - \frac{\ell(\ell+1)}{r^2}b + \frac{2}{r}\left(1 - \frac{2M}{r}\right)a_1 \tag{7.13c}$$

$$G \longrightarrow G' = G + \frac{2}{r^2}b \tag{7.13d}$$

where ∇_p refers to the covariant derivative of the two-dimensional subspace spanned by (t, r) (so this derivative is compatible with the line element $ds^2 = -f dt^2 + f^{-1}dr^2$). We have suppressed the labels referring to the spherical harmonics to avoid notational clutter, but they are still present (just invisible!). It is clear that the even- and odd-parity modes do not mix in the above expression: only even-parity modes appear, as expected. Similarly, gauge transformations for the odd parity sector can only be changed by $c(t,r)$:

$$h_0 \longrightarrow h_0' = h_0 + \partial_t c \tag{7.14a}$$
$$h_1 \longrightarrow h_1' = h_1 + \partial_r c - \frac{2}{r}c \tag{7.14b}$$
$$h_2 \longrightarrow h_2' = h_2 + 2c\,. \tag{7.14c}$$

From these explicit gauge transformations, we learn two things. First, there are two even-parity gauge invariant quantities

$$\tilde{f}_{pq} := f_{pq} - 2\nabla_{(p}j_{q)} + 2r\nabla_{(p}r\nabla_{q)}G + r^2\nabla_p\nabla_q G \tag{7.15}$$
$$\tilde{K} := K + \frac{1}{2}\ell(\ell+1)G - \frac{2}{r}j_1 + r\,\partial_r G \tag{7.16}$$

and two odd-parity gauge invariant quantities

$$\tilde{h}_0 = h_0 - \frac{1}{2}\partial_t h_2 \tag{7.17a}$$
$$\tilde{h}_1 = h_1 - \frac{1}{2}\partial_r h_2 + \frac{1}{r}h_2\,. \tag{7.17b}$$

Second, one can always choose a and b such that in the even-parity sector $j_p = 0 = G$. This is exactly the Regge-Wheeler gauge and in this gauge we see that the Regge-Wheeler quantities are simply equal to the gauge-invariant quantities: $\tilde{f}_{pq} = f_{pq}$ and $\tilde{K} = K$. Similarly for the odd-parity sector, where we can always choose c such that $h_2 = 0$. For this choice, we also find that the gauge invariant quantities are simply equal to the metric components themselves $\tilde{h}_0 = h_0$ and $\tilde{h}_1 = h_1$. This is the reason why in the final equations for the linearized Einstein's equations you are allowed to "upgrade" f_{pq}, K, h_0 and h_1 in the Regge-Wheeler gauge to their gauge-invariant counterparts $\tilde{f}_{pq}, \tilde{K}, \tilde{h}_0$ and $\tilde{h}_1$.

7.3 Odd-parity equations

We will discuss the magnetic-parity sector in detail here as the equations are significantly simpler. The methods for the even-parity sector are similar, but simply more messy. In vacuum, Einstein's equations simply reduce to $R_{ab} = 0$ so that we will only need the linearized Ricci tensor. The non-zero components are:

$$\frac{d}{d\lambda}\,R_{tA}(\lambda)|_{\lambda=0} = \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(-\partial_r^2 h_0 + \left[\partial_r + \frac{2}{r}\right]\partial_t h_1\right) + \left(\frac{l(l+1)}{r^2} - \frac{4M}{r^3}\right)h_0\right]X_A \tag{7.18a}$$
$$\frac{d}{d\lambda}\,R_{rA}(\lambda)|_{\lambda=0} = \frac{1}{2}\left[-\frac{1}{1-\frac{2M}{r}}\left[\partial_r - \frac{2}{r}\right]\partial_t h_0 + \frac{(l-1)(l+2)}{r^2}h_1 + \frac{1}{1-\frac{2M}{r}}\partial_t^2 h_1\right]X_A \tag{7.18b}$$
$$\frac{d}{d\lambda}\,R_{AB}(\lambda)|_{\lambda=0} = \left[-\frac{1}{1-\frac{2M}{r}}\partial_t h_0 + \partial_r\left[\left(1-\frac{2M}{r}\right)h_1\right]\right]X_{AB}\,, \tag{7.18c}$$

where the first equation is a consequence of the other two. Using the third equation (7.18c) to replace $\partial_t h_0$ in the second (7.18b), we find that

$$-\partial_t^2 h_1 + \left(\partial_r - \frac{2}{r}\right)\left[\left(1-\frac{2M}{r}\right)\partial_r\left(\left(1-\frac{2M}{r}\right)h_1\right)\right] - \left(1-\frac{2M}{r}\right)\frac{(l-1)(l+2)}{r^2}h_1 = 0\,. \tag{7.19}$$

Introducing the *Regge-Wheeler function*

$$\Psi_{\mathrm{RW}} := \frac{1-\frac{2M}{r}}{r}h_1\,, \tag{7.20}$$

the above equation can be rewritten as

$$\underbrace{\left[-\frac{1}{1-\frac{2M}{r}}\partial_t^2+\left(1-\frac{2M}{r}\right)\partial_r^2+\frac{2M}{r^2}\partial_r\right]}_{=\Box}\Psi_{\rm RW}-\underbrace{\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right]}_{:=V_{\rm odd}}\Psi_{\rm RW}=0\,, \tag{7.21}$$

where here $\Box$ refers to the two-dimensional d'Alembertian operator on the subspace spanned by (t, r). This wave equation with the above potential is known as the Regge-Wheeler equation and plays a key role in black hole perturbation theory. Solving this single equation for $\Psi_{\rm RW}$ determines the odd-parity metric coefficients: h_1 is obtained by inverting Eq. (7.20), and $\partial_t h_0$ is determined by taking the appropriate derivatives of h_1 (see Eq. (7.18c)). The metric function h_2 is pure gauge and therefore not relevant.

The Regge-Wheeler equation is gauge invariant: it takes the same form in any gauge. This follows directly from the fact that $\Psi_{\rm RW}$ can be made gauge invariant by "upgrading" h_1 to $\tilde{h}_1$. This simple procedure is one of the reasons why the Regge-Wheeler gauge is so frequently used. In the presence of a non-zero perturbed stress-energy tensor, the right hand side of the Regge-Wheeler equation contains the source terms.

Of course, other coordinates can be used for the background as well, such as the retarded or advanced null coordinates:

$$ds^2=-f(r)du^2-2dudr+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right) \tag{7.22}$$

$$=-f(r)dv^2+2dudr+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\,, \tag{7.23}$$

where $u=t-r-2M\ln\left(\frac{r}{2M}-1\right)$ and $v=t+r+2M\ln\left(\frac{r}{2M}-1\right)$. The Regge-Wheeler takes the same form as in Eq. (7.21) but with the two-dimensional d'Alembertian operator expressed in the appropriate coordinates.

It may appear strange at first that gravitational radiation is encoded in $\tilde{h}_0$ and $\tilde{h}_1$, which are the *tt*- and *tr*-components of the perturbed metric instead of the angular components. However, this is a consequence of the coordinates used here. If we transform these results to Bondi-Sachs coordinates and calculate the time-integral of the Bondi news tensor, we find that

$$C_{AB}(u,\theta,\phi)=\Psi_{\rm odd}(u,r=\infty)X_{AB}(\theta,\phi)\,, \tag{7.24}$$

where $\Psi_{\rm odd}$ is the Cunningham-Price-Moncrief function evaluated at $r=\infty$. This function is a close cousin to the Regge-Wheeler function. In particular, up to an overall factor of a half, the Regge-Wheeler function is the (retarded) time derivative of the *Cunningham-Price-Moncrief function* in vacuum:

$$\Psi_{\rm RW}=\frac{1}{2}\partial_t\Psi_{\rm odd}\,, \tag{7.25}$$

or, equivalently, $\Psi_{\rm RW}=\frac{1}{2}\partial_u\Psi_{\rm odd}$. Since the time derivative commutes with the d'Alembertian above, the Cunningham-Price-Moncrief function satisfies exactly the same equation as the Regge-Wheeler function. This fact combined with the simple relation between $\Psi_{\rm odd}$ and the Bondi news tensor is the reason why one nowadays typically regards the Cunningham-Price-Moncrief function as the fundamental odd-parity master function instead of the Regge-Wheeler function. Of course, the Bondi news tensor could be expressed in terms of the u-integral of the Regge-Wheeler function, but this additional integration is often inconvenient. One can also show using Eq. (7.18b) that the Cunningham-Price-Moncrief function $\Psi_{\rm odd}$ can directly be expressed in terms of the gauge invariant metric components

$$\Psi_{\rm odd}=\frac{2r}{(l-1)(l+2)}\left[\left(\partial_r-\frac{2}{r}\right)\tilde{h}_0-\partial_t\tilde{h}_1\right]\,. \tag{7.26}$$

In the presence of sources, the situation is slightly more complex: the relation between Cunningham-Price-Moncrief function and Regge-Wheeler function also includes a term involving the rA-component of the perturbed stress-energy tensor. Moreover, Ψ_{RW} and Ψ_{odd} do not satisfy the exact same equation in that case because the source terms will be different for both. Nonetheless, also in the presence of sources, the Regge-Wheeler equation for the Cunningham-Price-Moncrief function provides the fundamental odd-parity master equation.

7.4 Even-parity perturbations

The manipulations for the even-parity sector are very long and tedious. Therefore, we shall simply give the final result. The final even-parity master equation is expressed in terms of the Zerilli-Moncrief function Ψ_{even}:

$$\Psi_{\mathrm{even}} := \frac{2r}{l(l+1)} \left[\tilde{K} + \frac{2}{\lambda} \left(\left(1 - \frac{2M}{r} \right)^2 \tilde{f}_{rr} - r \left(1 - \frac{2M}{r} \right) \partial_r \tilde{K} \right) \right] , \tag{7.27}$$

where $\lambda := \mu + \frac{6M}{r}$ and $\mu := (l-1)(l+2)$. This function satisfies the Zerilli equation, which in vacuum is given by

$$(\Box - V_{\mathrm{even}}) \Psi_{\mathrm{even}} = 0 \tag{7.28}$$

with

$$V_{\mathrm{even}} := \frac{1}{\lambda^2} \left[\mu^2 \left(\frac{\mu+2}{r^2} + \frac{6M}{r^3} \right) + \frac{36M^2}{r^4} \left(\mu + \frac{2M}{r} \right) \right] . \tag{7.29}$$

The Zerilli-Moncrief function contributes to the Bondi news tensor in a similar way as the Cunningham-Price-Moncrief function

$$C_{AB}(u, \theta, \phi) = \Psi_{\mathrm{even}}(u, r = \infty) Y_{AB}(\theta, \phi) . \tag{7.30}$$

Be careful: if you compare these results with the literature, you may find slightly different equations as there are many different conventions for what physicists call “the” Zerilli-Moncrief function. I find this convention particularly attractive given its direct link to the Bondi news tensor.

How to solve for the odd- and even-parity master equations is a different ball game all together. There are different (numerical) techniques, but the details of these are beyond the scope of this course.

8 Spin coefficient formalism

When studying metric perturbations off Kerr, the resulting wave equation is not separable. In other words, the solution to the resulting wave equation cannot be written as $\Psi = f_1(t, r) f_2(\theta, \phi)$. This makes studying the solutions significantly more complicated. Fortunately, there is an alternative approach to studying perturbations off Kerr: instead of using metric variables, we will use spin coefficients. Teukolsky showed that the wave equation is separable in the spin coefficient formalism, as we will see in the next chapter. First, we will need to learn the basics of this powerful formalism.

8.1 Tetrads versus coordinates

When working with a specific spacetime (or some n-parameter family of spacetimes with n integer), it is often practical to introduce coordinates. However, an alternative is to use an orthonormal basis and express everything in terms of this basis. To be more concrete, consider for instance a spherically symmetric vacuum solution:

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) . \tag{8.1}$$

Any tensor equation can be expressed in terms of a coordinate basis ∂_a

$$\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right\} \tag{8.2}$$

or in terms of an orthonomal basis e_μ[15]

$$\left\{\frac{1}{\sqrt{f}}\frac{\partial}{\partial t}, \sqrt{f}\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right\} =: \{e_0, e_1, e_2, e_3\} \tag{8.3}$$

with e_0 simply the name for the first basis element, and similarly for the other elements (so that $e_0^t = 1/\sqrt{f}, e_0^r = 0$, etc.). To illustrate this, take the time translation Killing vector field T. This can be equally well expressed in terms of the coordinate and tetrad bases

$$T = T^a \partial_a = \frac{\partial}{\partial t} = T^\mu e_\mu = \sqrt{f} e_0 \tag{8.4}$$

with the non-zero component in the coordinate basis $T^t = 1$ and in the tetrad basis $T^0 = \sqrt{f}$. Of course, neither basis is unique. In terms of the coordinates, we are always allowed to perform a coordinate transformation from $x^a \to x'^a$. As a result, the components of a vector also change accordingly

$$V^a(x) \longrightarrow V'^a(x') = \frac{\partial x'^a}{\partial x^b} V^b(x) \tag{8.5}$$

The orthonormal basis is also not unique, but should remain orthonormal at every point in spacetime, that is,

$$g_{ab}(x) e^a_{\ \mu}(x) e^b_{\ \nu}(x) = \eta_{\mu\nu} \ . \tag{8.6}$$

The transformations that preserve this condition are exactly the Lorentz transformations:

$$e^a_{\ \mu}(x) \longrightarrow e'^a_{\ \mu}(x) = \Lambda_\mu^{\ \nu} e^a_{\ \nu}(x) \tag{8.7}$$

because

$$g_{ab}(x) e'^a_{\ \mu}(x) e'^b_{\ \nu}(x) \stackrel{?}{=} \eta_{\mu\nu} \tag{8.8}$$

$$= g_{ab}\, \Lambda_\mu^{\ \kappa} e^a_{\ \kappa}\, \Lambda_\nu^{\ \lambda} e^a_{\ \lambda} \tag{8.9}$$

$$= \eta_{\kappa\lambda} \Lambda_\mu^{\ \kappa} \Lambda_\nu^{\ \lambda} \tag{8.10}$$

$$\stackrel{!}{=} \eta_{\mu\nu} \tag{8.11}$$

where in going from the first to the second line we substituted the transformation law in Eq. (8.7), in going to the third line we used Eq. (8.6) and finally we used that Lorentz transformations leave the Minkowski metric invariant. The picture to have in mind is that once you have picked an orthonormal basis, to ensure that this basis remains orthonormal, you can only rotate or boost it. This is different from a coordinate transformation which changes the coordinates x^a to *any* arbitrary x'^a, while the Lorentz transformations only change the orientation of the tetrad in spacetime. Because of the normalization condition in Eq. (8.6), tetrads are sometimes also referred to as the square root of the metric. At a practical level, one can also think of $e^a_{\ \mu}$ as a square matrix. The inverse of $e^a_{\ \mu}$ also exists and satisfies

$$e^a_{\ \mu}\, e^\mu_{\ b} = \delta^a_b \qquad \Longleftrightarrow \qquad e^a_{\ \mu}\, e^\nu_{\ a} = \delta^\mu_\nu \ . \tag{8.12}$$

This is very convenient, because this allows us to transform the internal "Lorentz" indices $\mu, \nu, \ldots$ to "spacetime" indices $a, b, \ldots$ and vice versa. For instance, we have

$$V^a = e^a_{\ \mu} V^\mu \qquad \Longleftrightarrow \qquad V^\mu = e_a^{\ \mu}\, V^a \ . \tag{8.13}$$

[15]This also goes by the name of tetrad, or vielbein and in the case of four spacetime dimensions a vierbein.

At this point, all is good and well. This formalism becomes really interesting when we start to differentiate objects (or if you are interested in describing fermions, which can only be coupled to the spacetime geometry through tetrads). Recall that the covariant derivative can be related to a partial derivative as follows

$$\nabla_a T^b{}_c = \partial_a T^b{}_c + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d \tag{8.14}$$

where if the covariant derivative is compatible with the metric ($\nabla_a g_{bc} = 0$), then the Christoffel symbols are symmetric in the lower indices $\Gamma^a_{bc} = \Gamma^a_{cb}$. How about derivatives of objects with Lorentz indices? Taking their derivative yields

$$\nabla_a T^\mu{}_\nu = \partial_a T^\mu{}_\nu + \omega_a{}^\mu{}_\lambda T^\lambda{}_\nu - \omega_a{}^\lambda{}_\nu T^\mu{}_\lambda \tag{8.15}$$

where $\omega_a{}^\mu{}_\lambda$ is called the Ricci rotation coefficient, gauge connection or spin connection.[16] The role of this spin connection is to ensure that $X_a{}^\mu{}_\nu := \nabla_a T^\mu_\nu$ has the correct transformation properties, that is,

$$X_a{}^\mu{}_\nu(x) \longrightarrow X'_a{}^\mu{}_\nu(x') = \Lambda^\mu_\lambda \Lambda_\nu{}^\kappa \frac{\partial x^b}{\partial x'^a} X_b{}^\lambda{}_\kappa(x) \ . \tag{8.16}$$

Thus, the role of the spin connection is similar to that of the Christoffel connection which ensures that tensors remain tensors after covariant differentiation. Now if the covariant derivative is metric compatible, this also implies a symmetry property for the spin connection:

$$\omega_{a\mu\nu} = -\omega_{a\nu\mu} \ . \tag{8.17}$$

This can in fact be derived once you know that the spin connection can also be written as

$$\omega_{a\mu\nu} = e_\mu{}^b \nabla_a e_{\nu\, b} \ . \tag{8.18}$$

Instead of symmetric like the Christoffel symbols, the spin connection is anti-symmetric in its last two indices. Counting the number of independent components, this implies that in n spacetime dimensions

$$\Gamma^a_{bc} : n \times \frac{n(n+1)}{2} \tag{8.19}$$

$$\omega_{a\mu\nu} : n \times \frac{n(n-1)}{2} \tag{8.20}$$

which for $n = 4$ reduces to 40 independent components for the Christoffel symbols and only 24 for the spin connection. If you are not yet impressed by the efficiency of the spin connection, you should be once you realize that all differential geometry (so the Riemann tensor, Ricci tensor, etc.) can be expressed in terms of spin connections without the need of Christoffel symbols. Therefore, in principle, the calculations using spin coefficients are a lot more efficient (once you get the hang of it). The Bianchi identities and Einstein's equations can also be written entirely in terms of the spin connection and this formulation forms the basis of the Geroch-Held-Penrose formalism.

8.2 GHP formalism

Newman and Penrose used a formulation of Einstein's equation in terms of the spin connections together with a clever choice of tetrad [13]. In particular, since their goal was to study gravitational radiation, they introduced a *null* tetrad. The Newman-Penrose (NP) formalism was a decade later superseded by the even more efficient and more explicitly covariant formalism of Geroch, Held and Penrose (GHP) [14]. Therefore, we will use the GHP formalism here. Unfortunately, many results in the literature still use the older NP formalism. So be aware! At first, both formalisms may seem like a lot of names without much content and it does take some time to get used to it. However, in studying perturbations off Kerr, it has become an invaluable tool.

[16] Some authors seem to distinguish between $\omega_{a\mu\nu}$ versus $\omega_{\lambda\mu\nu}$ when naming these objects; however, there does not seem to be a universal naming convention and I will use the different terms interchangeably.

We start by introducing a (complex) null tetrad $\{e^a_\mu\} = \{l^a, n^a, m^a, \bar{m}^a\}$ (the bar denotes complex conjugation) with normalization

$$l^a n_a = -1, \quad m^a \bar{m}_a = 1, \tag{8.21}$$

and with all other inner products vanishing.[17] In terms of the tetrad vectors, the metric can be written as

$$g_{ab} = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}. \tag{8.22}$$

For instance, in Minkowski spacetime such a null tetrad is:

$$l^a\partial_a = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\right) \tag{8.23a}$$

$$n^a\partial_a = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right) \tag{8.23b}$$

$$m^a\partial_a = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right) \tag{8.23c}$$

with $\bar{m}^a$ the complex conjugate of m^a. Of course, just as with any tetrad, we can perform a Lorentz transformation and preserve the orthogonality and normalization conditions. Let us assume for now that we have a spacetime in which l^a and n^a are special and we should not rotate/boost these directions. Then the 6-parameter Lorentz freedom is reduced to two parameters.[18] This remaining Lorentz freedom leaves the direction of l^a and n^a unchanged but rescales them by $\tilde{\alpha}$, and rotates m^a and $\bar{m}^a$ in the $(m, \bar{m})$-plane by some angle $\tilde{\beta}$:

$$l^a \longrightarrow \tilde{\alpha}\, l^a \qquad \text{and} \qquad n^a \longrightarrow \tilde{\alpha}^{-1} n^a \tag{8.24}$$

and

$$m^a \longrightarrow e^{i\tilde{\beta}} m^a \qquad \text{and} \qquad \bar{m}^a \longrightarrow e^{-i\tilde{\beta}} \bar{m}^a\ , \tag{8.25}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are both real.[19] These two transformations can be combined into one by introducing $\lambda^2 = \tilde{\alpha} e^{i\tilde{\beta}}$:

$$\eta \longrightarrow \lambda^p \bar{\lambda}^q \eta\ , \tag{8.26}$$

where η can be any tensorial object. For instance, the transformation in Eq. (8.24) is a transformation with $p = q = 1$ for l^a and $p = q = -1$ for n^a; the transformation in Eq. (8.25) is a transformation with $p = -q = 1$ for m^a and $p = -q = -1$ for $\bar{m}^a$. The metric g_{ab} is not altered by a Lorentz transformation of the tetrad, so it has $p = q = 0$. Similarly, the Riemann tensor, Ricci tensor and Weyl tensor have $p = q = 0$. If you contract for instance the Weyl tensor with four tetrad vectors to obtain a scalar, the resulting p and q will be determined by the vectors you contract it with. To illustrate this, $C_{abcd}n^a l^b n^c m^d$ has $p = 0 - 1 + 1 - 1 + 1 = 0$ and $q = 0 - 1 + 1 - 1 - 1 = -2$. All important objects in the GHP formalism have a well-defined type $\{p, q\}$. From p and q, you can determine the spin-weight $s = (p - q)/2$ and boost-weight $b = (p + q)/2$. Only objects of the same type can be added together, just as only objects with the same units can be added: things like 3 kilogram plus 2 meters make as little sense as adding an object with boost-weight 2 to an object with boost-weight 0. Multiplication of two objects with $\{p_1, q_1\}$ and $\{p_2, q_2\}$ gives an object with $\{p_1 + p_2, q_1 + q_2\}$. This is incredibly helpful in checking your (intermediate) calculations.

[17] Also this formalism is – miserably – plagued with different sign conventions (for the metric signature, the normalization of the tetrad and even in the definition of the Riemann tensor $R_{abc}{}^d v_d = 2\nabla_{[a}\nabla_{b]}v_c$ versus $-2\nabla_{[a}\nabla_{b]}v_c$). Consider yourself warned!

[18] If "rotations" leaving l^a unchanged are allowed, then $l^a \to l^a, m^a \to m^a + a\, l^a, \bar{m}^a \to \bar{m}^a + \bar{a}\, l^a, n^a \to n^a + \bar{a}\, m^a + a\, \bar{m}^a + a\bar{a}\, l^a$. If "rotations" leaving n^a unchanged are allowed, then $n^a \to n^a, m^a \to m^a + b\, l^a, \bar{m}^a \to \bar{m}^a + \bar{b}\, l^a, l^a \to l^a + \bar{b}\, m^a + b\, \bar{m}^a + b\bar{b}\, n^a$. The complex parameters a and b are the remaining four parameters of the six-parameter Lorentz group.

[19] In the literature, this is sometimes called a type III rotation.

Part of the efficiency of this formalism lies with the fact that it is using complex scalars to combine two real scalars into one. This reduces the number of relevant equations, since if you know that $A = B$, you also known $\bar{A} = \bar{B}$. The plot thickens as there are additional discrete symmetries inherent to the GHP formalism. Specifically, there are three discrete transformations corresponding to simultaneous interchange of the tetrad vectors:

1. complex conjugation $\bar{}$: $m^a \leftrightarrow \bar{m}^a$ which also interchanges $\{p, q\} \to \{q, p\}$;
2. priming $'$: $l^a \leftrightarrow n^a$ and $m^a \leftrightarrow \bar{m}^a$, with correspondingly $\{p, q\} \to \{-p, -q\}$;
3. starring $*$ (also known as the Sachs symmetry operation): $l^a \to m^a$, $n^a \to -\bar{m}^a$, $m^a \to -l^a$, $\bar{m}^a \to n^a$, with correspondingly $\{p, q\} \to \{p, -q\}$.

These additional discrete tranformations further reduce the number of required equations significantly.

In order to write the Bianchi identities and Einstein's equation, we need to know the Riemann tensor, which can be completely expressed in terms of the spin connection. Recall that there are 24 independent components of the spin connection, which can be repackaged into 12 complex scalars. These are known as the spin coefficients. Of these, the eight with well-defined GHP type are

$$\kappa = -l^a m^b \nabla_a l_b, \qquad \text{with} \quad \{3, 1\} \tag{8.27}$$
$$\sigma = -m^a m^b \nabla_a l_b, \qquad \text{with} \quad \{3, -1\} \tag{8.28}$$
$$\rho = -\bar{m}^a m^b \nabla_a l_b, \qquad \text{with} \quad \{1, 1\} \tag{8.29}$$
$$\tau = -n^a m^b \nabla_a l_b, \qquad \text{with} \quad \{1, -1\} \tag{8.30}$$

along with their primed variants, κ', σ', ρ' and τ'. Let us verify that κ indeed has $\{p, q\} = \{3, 1\}$:

$$\begin{aligned}\kappa = -l^a m^b \nabla_a l_b \longrightarrow -(\lambda\bar{\lambda} l^a)(\lambda\bar{\lambda}^{-1} m^b)\nabla_a(\lambda\bar{\lambda} l_b) &= -\lambda^3\bar{\lambda} l^a m^b \nabla_a l_b - \lambda^2 l^a \nabla_a \left(\lambda\bar{\lambda}\right) \underbrace{m^b l_b}_{=0} \\ &= \lambda^3\bar{\lambda}\kappa \, . \end{aligned} \tag{8.31}$$

The remaining spin coefficients are

$$\beta = \frac{1}{2}(m^a \bar{m}^b \nabla_a m_b - m^a n^b \nabla_a l_b), \qquad \epsilon = \frac{1}{2}(l^a \bar{m}^b \nabla_a m_b - l^a n^b \nabla_a l_b), \tag{8.32}$$

along with their primed variants, β' and ϵ'. These spin coefficients have *no* well-defined GHP type, but never appear explicitly in covariant equations because they always combine with certain directional derivative operators. For instance, if we perform a GHP transformation of β we find:

$$\beta \longrightarrow \frac{1}{2}\left(m^a \bar{m}^b \nabla_a(\lambda\bar{\lambda}^{-1} m_b) - \bar{\lambda}^{-2} m^a n^b \nabla_a(\lambda\bar{\lambda} l_b)\right) \tag{8.33}$$
$$= \lambda\bar{\lambda}^{-1}\frac{1}{2}(m^a \bar{m}^b \nabla_a m_b - m^a n^b \nabla_a l_b) + \frac{1}{2} m^a \nabla_a \left(\lambda\bar{\lambda}^{-1}\right) \underbrace{\bar{m}^b m_b}_{=1} - \frac{1}{2}\bar{\lambda}^{-2} m^a \nabla_a \left(\lambda\bar{\lambda}\right) \underbrace{n^b l_b}_{=-1} \tag{8.34}$$
$$= \lambda\bar{\lambda}^{-1}\beta + \frac{1}{2}\left[m^a \nabla_a \left(\lambda\bar{\lambda}^{-1}\right) + \bar{\lambda}^{-2} m^a \nabla_a \left(\lambda\bar{\lambda}\right)\right] \tag{8.35}$$
$$= \lambda\bar{\lambda}^{-1}\beta + \bar{\lambda}^{-1} m^a \nabla_a \lambda \, , \tag{8.36}$$

so that the second term prohibits a well-defined $\{p, q\}$ for β. In order to see how they combine with directional derivatives, let us first define the four independent directional derivatives

$$D = l^a \nabla_a \qquad \text{and} \qquad \delta = m^a \nabla_a \, , \tag{8.37}$$

where by priming/complex conjugation we also have

$$D' = n^a \nabla_a \qquad \text{and} \qquad \delta' = \bar{\delta} = \bar{m}^a \nabla_a \, . \tag{8.38}$$

Acting with any of these derivative operators on a scalar of type $\{p, q\}$ generically does not produce a scalar with a well-defined GHP type, but the following combinations do:

$$þ\,\eta = [D - p\,\epsilon - q\,\bar{\epsilon}]\,\eta \tag{8.39}$$

$$ð\,\eta = \left[\delta - p\beta + q\bar{\beta}'\right]\eta \tag{8.40}$$

where η is a scalar of type $\{p, q\}$. The operator þ is pronounced as "thorn" and ð as "e(d)th". Their primed versions are simply:

$$þ'\,\eta = [D' + p\,\epsilon' + q\,\bar{\epsilon}']\,\eta \tag{8.41}$$

$$ð'\,\eta = \left[\delta' + p\beta' - q\bar{\beta}\right]\eta \tag{8.42}$$

(also note the change in sign in front of p and q). Let us check that if η has GHP type $\{p, q\}$, then $ð\,\eta$ also has a well-defined GHP type:

$$ð\,\eta \longrightarrow \left[\lambda\bar{\lambda}^{-1}\delta - p\lambda\bar{\lambda}^{-1}\beta - p\bar{\lambda}^{-1}m^a\nabla_a\lambda + q\lambda\bar{\lambda}^{-1}\bar{\beta}' - q\lambda\bar{\lambda}^{-2}m^a\nabla_a\bar{\lambda}\right](\lambda^p\bar{\lambda}^q\eta) \tag{8.43}$$

$$= \lambda^{p+1}\bar{\lambda}^{q-1}\,ð\,\eta + \eta\underbrace{\left[\underbrace{\lambda\bar{\lambda}^{-1}\delta(\lambda^p\bar{\lambda}^q)}_{=p\lambda^p\bar{\lambda}^{q-1}\delta\lambda + q\lambda^{p+1}\bar{\lambda}^{q-2}\delta\bar{\lambda}} \quad -p\lambda^p\bar{\lambda}^{q-1}\delta\lambda - q\lambda^{p+1}\bar{\lambda}^{q-2}\delta\bar{\lambda}\right]}_{=0}, \tag{8.44}$$

where I used the transformation of β in Eq. (8.36) and the fact that $\bar{\beta}'$ transforms as

$$\bar{\beta}' \longrightarrow \lambda\bar{\lambda}^{-1}\bar{\beta}' - \lambda\bar{\lambda}^{-2}m^a\nabla_a\bar{\lambda}. \tag{8.45}$$

Hence, operating with ð on a scalar of type $\{p, q\}$ produces another scalar with type $\{p+1, q-1\}$. In general, the action of a GHP derivative causes the type to change by an amount $\{p, q\} \to \{p + u, q + v\}$ where $\{u, v\}$ for each of the operators is given by:

$$þ : \{1, 1\} \qquad \text{and} \qquad þ' : \{-1, -1\} \tag{8.46}$$

$$ð : \{1, -1\} \qquad \text{and} \qquad ð' : \{-1, 1\} \, . \tag{8.47}$$

Note that as a result of acting with þ on some $\{p, q\}$ object, it raises the boost-weight by one and does not alter the spin-weight. Conversely, þ′ lowers the boost-weight by one and also does not alter the spin-weight. Hence, þ and þ′ are sometimes also referred to as boost raising and lowering operators, respectively. Similarly, ð and ð′ can be interpreted as spin raising and lowering operators, respectively.

In vacuum spacetimes, the only non-zero components of the Riemann tensor are given by the tetrad components of the Weyl tensor, which can be represented by five complex Weyl scalars,

$$\Psi_0 = C_{abcd}l^a m^b l^c m^d \tag{8.48a}$$

$$\Psi_1 = C_{abcd}l^a n^b l^c m^d \tag{8.48b}$$

$$\Psi_2 = C_{abcd}l^a m^b \bar{m}^c n^d = \frac{1}{2}C_{abcd}\left(l^a n^b l^c n^d - l^a n^b m^c \bar{m}^d\right) \tag{8.48c}$$

$$\Psi_3 = C_{abcd}l^a n^b \bar{m}^c n^d \tag{8.48d}$$

$$\Psi_4 = C_{abcd}n^a \bar{m}^b n^c \bar{m}^d \tag{8.48e}$$

with types inherited from the tetrad vectors that appear in their definition,

$$\Psi_0 : \{4,0\}, \quad \Psi_1 : \{2,0\}, \quad \Psi_2 : \{0,0\}, \quad \Psi_3 : \{-2,0\}, \quad \Psi_4 : \{-4,0\}.$$

With these definitions in hand, we can finally list the complete set of Einstein vacuum field equations:

$$ð\rho - ð'\sigma = (\rho - \bar{\rho})\,\tau + (\bar{\rho}' - \rho')\,\kappa - \Psi_1 \tag{8.49a}$$
$$þ\,\rho - ð'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau' \tag{8.49b}$$
$$þ\,\sigma - ð\kappa = (\rho + \bar{\rho})\,\sigma - (\tau + \bar{\tau}')\,\kappa + \Psi_0 \tag{8.49c}$$
$$þ\,\rho' - ð\tau' = \rho'\bar{\rho} + \sigma\sigma' - \tau'\bar{\tau}' - \kappa\kappa' - \Psi_2 \tag{8.49d}$$

together with their primed and starred version (note that Eqs. (8.49c) and (8.49d) are star-invariant). The vacuum Bianchi identities are given by

$$þ\,\Psi_1 - ð'\Psi_0 = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 \tag{8.50a}$$
$$þ\,\Psi_2 - ð'\Psi_1 = \sigma'\Psi_0 - 2\tau'\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 \tag{8.50b}$$

To make this set complete, we of course also need to consider their primed and starred versions. Finally, the Einstein field equations and Bianchi identities are supplemented by the commutation relations for any quantity η of type $\{p, q\}$:

$$[þ, þ']\,\eta = \Big((\bar{\tau} - \tau')\,ð + (\tau - \bar{\tau}')\,ð' - p\,(\kappa\kappa' - \tau\tau' + \Psi_2) - q\left(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2\right)\Big)\,\eta \tag{8.51a}$$
$$[þ, ð]\,\eta = (-\bar{\tau}'\,þ - \kappa\,þ' + \bar{\rho}ð + \sigma ð' - p\,(\rho'\kappa - \tau'\sigma + \Psi_1) - q\,(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}'))\,\eta\,. \tag{8.51b}$$

The commutation relation of say $ð$ and $ð'$ can be obtained by considering the star of the first line. Similarly, any other commutation relation can be obtained from the above two either by starring or priming the above results. The equations in Eqs. (8.49)-(8.51) form the basis for the analysis of perturbations off the Kerr spacetime.

9 Perturbations off the Kerr spacetime

The spacetime of a spinning black hole is described by the Kerr metric. In Boyer-Lindquist coordinates, its line-element is

$$ds^2 = -\left[1 - \frac{2Mr}{\Sigma}\right] dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \left[\Delta + \frac{2Mr(r^2+a^2)}{\Sigma}\right]\sin^2\theta\,d\phi^2, \tag{9.1}$$

where M denotes the mass of the blackhole, a is its angular momentum per unit mass, $\Sigma := r^2 + a^2\cos^2\theta$ and $\Delta := r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$ with $r_\pm := M \pm \sqrt{M^2 - a^2}$ the locations of the inner and outer horizons (r_+ is the event horizon).

It is no coincidence that the GHP formalism restricted itself to Lorentz transformations that leave the direction of l^a and n^a unchanged. The Kerr spacetime is a special solution due to its symmetries: in particular, it belongs to the algebraically special solutions called type-D spacetimes. A generic spacetime has four independent null vectors k^a that satisfy

$$k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0\,. \tag{9.2}$$

For algebraically special solutions some of the null vectors coincide (i.e. are degenerate). A spacetime is algebraically special if two of these null vectors coincide at every point $k_1^a = k_2^a = l^a$, in which case $C_{abc[d}l_{e]}l^b l^c = 0$. This also goes by the name Petrov type II. A subclass of this class are Petrov type D spacetimes. For any type D spacetime, there is another pair of coinciding null vectors ($k_3^a = k_4^a = n^a$). In other words, there are only two instead of four non-degenerate principle null directions and thus they select two "special" null directions. It is natural to align two of the "legs" of the complex null tetrad with these two principle null directions. Specifically, let $l^a = k_1^a = k_a^2$ align with the outward null direction and $n^a = k_3^a = k_4^a$ with the inward null direction. Of course, as with the GHP formalism, there is still freedom to rescale l^a and n^a and change the orientation of the remaining two null vectors m^a and $\bar{m}^a$. As a result, there are various tetrads being used in the literature such as the Kinnersley tetrad, the Hartle-Hawking tetrad and Carter's canonical tetrad. All of these are related by the freedom discussed in Eq. (8.24) and Eq. (8.25). In Teukolsky's orginal derivation, he used the Kinnersley tetrad. In Boyer-Lindquist coordinates (t, r, θ, ϕ), it takes the following form:

$$l^a \partial_a = \frac{1}{\Delta}\left(r^2 + a^2, \Delta, 0, a\right) \tag{9.3a}$$

$$n^a \partial_a = \frac{1}{2\Sigma}\left(r^2 + a^2, -\Delta, 0, a\right) \tag{9.3b}$$

$$m^a \partial_a = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right). \tag{9.3c}$$

We will not use the explicit form of this tetrad until the end of this section.

One of the reasons for why the spin coefficient formalism is such a fruitful formalism in studying perturbations off the Kerr spacetime is that many of the spin coefficients of the Kerr spacetime vanish. In particular, for any type D spacetime, the Goldberg-Sachs theorem implies that four of the spin coefficients vanish

$$\kappa = \kappa' = \sigma = \sigma' = 0, \tag{9.4}$$

which is equivalent to stating that l^a and n^a are geodesic and shear-free. In addition, four out of the five Weyl scalars vanish

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0. \tag{9.5}$$

This is regardless of the details of the null tetrad you picked, provided that l^a and n^a are aligned with the principle null directions.

Given all these facts, we will derive the equations that govern how gravitational waves propagate on a Kerr spacetime with no sources (just as we obtained the Regge-Wheeler and Zerilli equation for the case with the perturbed stress-energy tensor set to zero). This equation is known as the Teukolsky equation. In his original paper, Teukolsky uses the older Newman-Penrose notation but the key ideas are the same. We will use linear perturbation theory, so we will expand any GHP quantity as a "background" quantity associated to the Kerr spacetime and a perturbation. For instance, we have:

$$\Psi_1 = \Psi_1^{(0)} + \Psi_1^{(1)} = \Psi_1^{(1)} \tag{9.6}$$

$$\Psi_2 = \Psi_2^{(0)} + \Psi_2^{(1)} \tag{9.7}$$

$$\kappa = \kappa^{(0)} + \kappa^{(1)} = \kappa^{(1)} \tag{9.8}$$

$$\eth = \eth^{(0)} + \eth^{(1)}, \tag{9.9}$$

where in the above lines we used that some of the background quantities are zero for the Kerr spacetime. However, for the first part of the derivation, we will not use this split and work completely generically. The goal is to derive a "wavelike" equation for $\Psi_0^{(1)}$.

Let's start with the Bianchi equation in Eq. 8.50a and act with $\eth$ on it to obtain:

$$\eth \,\text{þ}\, \Psi_1 - \eth\eth'\Psi_0 = -\eth(\tau'\Psi_0) + 4\eth(\rho\Psi_1) - 3\eth(\kappa\Psi_2). \tag{9.10}$$

Taking the star transform of this equation, we find

$$-\text{þ}\,\eth\Psi_1+\text{þ}\,\text{þ}'\,\Psi_0=\text{þ}(\rho'\Psi_0)-4\,\text{þ}(\tau\Psi_1)+3\,\text{þ}(\sigma\Psi_2)\,. \tag{9.11}$$

Now simply adding these two equations, we obtain a second order differential equation for Ψ_0 coupled to Ψ_1 and Ψ_2 with first order derivatives:

$$\begin{aligned}&(\text{þ}\,\text{þ}'-\eth\eth')\,\Psi_0-\rho'\,\text{þ}\,\Psi_0+\tau'\eth\Psi_0-4\rho\eth\Psi_1+4\tau\,\text{þ}\,\Psi_1+3\kappa\eth\Psi_2-3\sigma\,\text{þ}\,\Psi_2\\&+(\eth\,\text{þ}-\text{þ}\,\eth)\,\Psi_1+\Psi_0\,(\eth\tau'-\text{þ}\,\rho')-4\Psi_1\,(\eth\rho-\text{þ}\,\tau)-3\Psi_2\,(\text{þ}\,\sigma-\eth\kappa)=0\,.\end{aligned} \tag{9.12}$$

This equation can be simplified using that the first term on the second line is simply the commutator $[\eth,\text{þ}]$ on a $\{2,0\}$ quantity, the term multiplying Ψ_0 on the second line is the left hand side of Eq. (8.49d) and the term multiplying Ψ_2 is the left hand side of Eq. (8.49c). The term multiplying Ψ_1 can be rewritten using Eq. (8.49a) and its star:

$$-\text{þ}\,\tau+\text{þ}'\,\kappa=(\bar{\tau}'-\tau)\,\rho+(\tau'-\bar{\tau})\,\sigma-\Psi_1\,. \tag{9.13}$$

Substituting all these results, Eq. (9.12) becomes

$$\begin{aligned}&(\text{þ}\,\text{þ}'-\eth\eth'-\rho'\,\text{þ}+\tau'\eth-\rho'\bar{\rho}-\sigma\sigma'+\tau'\bar{\tau}'+\kappa\kappa'+\Psi_2)\,\Psi_0\\&+\left[(4\tau+\bar{\tau}')\,\text{þ}+\kappa\,\text{þ}'-(4\rho+\bar{\rho})\eth-\sigma\eth'+4\,(\text{þ}'\,\kappa-\eth'\sigma)\right.\\&\left.+2\,(\rho'\kappa-\tau'\sigma+\Psi_1)+4\,(-\tau'\sigma-\bar{\tau}'\rho+\bar{\tau}\sigma+\bar{\rho}\tau-\bar{\rho}'\kappa+\rho\kappa+2\Psi_1)\right]\Psi_1\\&+3\,(\kappa\eth-\sigma\,\text{þ}-(\rho+\bar{\rho})\,\sigma+(\tau+\bar{\tau}')\,\kappa-\Psi_0)\,\Psi_2=0\,.\end{aligned} \tag{9.14}$$

At this point, it is convenient to start splitting all the quantities into background and perturbed quantities. At linear order, since the background values for Ψ_0 and Ψ_1 are zero, the terms multiplying Ψ_0 and Ψ_1 all have to be background quantities. For the terms multiplying Ψ_2, this is different as $\Psi_2^{(0)}\neq 0$. Let us focus on those terms first and extract the linear part (ignoring the Ψ_0 term which we will include with the last term on the first line):

$$\begin{aligned}3\,(\kappa\eth-\sigma\,\text{þ}-(\rho+\bar{\rho})\,\sigma+(\tau+\bar{\tau}')\,\kappa)\,\Psi_2\overset{\text{lin}}{=}3\,\Big(&\kappa^{(1)}\eth^{(0)}-\sigma^{(1)}\,\text{þ}^{(0)}-\left(\rho^{(0)}+\bar{\rho}^{(0)}\right)\sigma^{(1)}\\&+\left(\tau^{(0)}+\bar{\tau}'^{(0)}\right)\kappa^{(1)}\Big)\,\Psi_2^{(0)}\end{aligned} \tag{9.15}$$

where I used that $\kappa^{(0)}=0=\sigma^{(0)}$. Using Eq. (8.50b) and its star for the background, i.e.,

$$\text{þ}^{(0)}\,\Psi_2^{(0)}=3\rho^{(0)}\Psi_2^{(0)} \tag{9.16}$$

$$\eth^{(0)}\Psi_2^{(0)}=3\tau^{(0)}\Psi_2^{(0)}\,, \tag{9.17}$$

the linear part simplifies to

$$3\,(\kappa\eth-\sigma\,\text{þ}-(\rho+\bar{\rho})\,\sigma+(\tau+\bar{\tau}')\,\kappa)\,\Psi_2\overset{\text{lin}}{=}3\left(\left(4\tau^{(0)}+\bar{\tau}'^{(0)}\right)\kappa^{(1)}-\left(4\rho^{(0)}+\bar{\rho}^{(0)}\right)\sigma^{(1)}\right)\Psi_2^{(0)}\,. \tag{9.18}$$

Hence, the linear part of Eq. (9.14) is simply

$$\begin{aligned}&(\text{þ}\,\text{þ}'-\eth\eth'-\rho'\,\text{þ}+\tau'\eth-\rho'\bar{\rho}+\tau'\bar{\tau}'-2\Psi_2)^{(0)}\,\Psi_0^{(1)}\\&+\left[(4\tau+\bar{\tau}')\,\text{þ}-(4\rho+\bar{\rho})\eth+4\,(-\rho\bar{\tau}'+\bar{\rho}\tau)\right]^{(0)}\Psi_1^{(1)}\\&+3\left[\left(4\tau^{(0)}+\bar{\tau}'^{(0)}\right)\kappa^{(1)}-\left(4\rho^{(0)}+\bar{\rho}^{(0)}\right)\sigma^{(1)}\right]\Psi_2^{(0)}=0\,,\end{aligned} \tag{9.19}$$

where many terms dropped out because their background quantities are zero. Recall that the goal is to obtain a differential equation for $\Psi_0^{(1)}$. We will do this by removing the derivatives on $\Psi_1^{(1)}$ using Eq. (8.50a) and its star:

$$\eth\Psi_1-\text{þ}'\,\Psi_0=-\rho'\Psi_0+4\tau\Psi_1-3\sigma\Psi_2\,. \tag{9.20}$$

This yields

$$
\begin{aligned}
&\left(\text{þ}\,\text{þ}' - \eth\eth' - \rho'\,\text{þ} - (4\rho+\bar\rho)\,\text{þ}' + \tau'\eth + (4\tau+\bar\tau')\eth' - \rho'\bar\rho + \tau'\bar\tau' - (4\tau+\bar\tau')\tau' + (4\rho+\bar\rho)\rho' - 2\Psi_2\right)^{(0)}\Psi_0^{(1)} \\
&+ \left[4\left(-\rho\bar\tau' + \bar\rho\tau\right) + (4\tau+\bar\tau')4\rho - (4\rho+\bar\rho)4\tau\right]^{(0)}\Psi_1^{(1)} = 0\,, \qquad (9.21)
\end{aligned}
$$

where all the terms proportional to Ψ_2 canceled out. Simply cleaning this expression up, this finally reduces to the famous Teukolsky equation

$$
\left(\text{þ}\,\text{þ}' - \eth\eth' - \rho'\,\text{þ} - (4\rho+\bar\rho)\,\text{þ}' + \tau'\eth + (4\tau+\bar\tau')\eth' + 4\rho\rho' - 4\tau\tau' - 2\Psi_2\right)^{(0)}\Psi_0^{(1)} = 0\,. \qquad (9.22)
$$

By priming this equation, we obtain a similar equation for $\Psi_4^{(0)}$. The first two terms are like the d'Alembertian operator in Kerr and the remaining terms play the role of a "potential"; this is somewhat analogous to the Regge-Wheeler and Zerilli master equations. It turns out that electromagnetic and massless scalar perturbations on a Kerr background satisfy a very similar equation (only the value of some coefficients are slightly changed). This can be compactly expressed by introducing the "spin weight" parameter s: $s = 0$ for scalar perturbations, $s = \pm 1$ for electromagnetic perturbations (with $s = +1$ for ϕ_0 and $s = -1$ for ϕ_2) and $s = \pm 2$ for gravitational perturbations (with $s = +2$ for $\Psi_0^{(1)}$ and $s = -2$ for $\Psi_4^{(1)}$). In these notes, we will work with $s = 2$ only.

In this form, the Teukolsky equation is rather abstract. To get a more concrete understanding of this equation, let us use the Kinnersley tetrad in Eq. (9.3) and express the spin-coefficients in Boyer-Lindquist coordinates

$$\rho^{(0)} = -\frac{1}{r - ia\cos\theta} \qquad (9.23a)$$

$$\rho'^{(0)} = \frac{\Delta}{2\Sigma(r - ia\cos\theta)} \qquad (9.23b)$$

$$\tau^{(0)} = -\frac{ia\sin\theta}{\sqrt{2}\Sigma} \qquad (9.23c)$$

$$\tau'^{(0)} = -\frac{ia\sin\theta}{\sqrt{2}(r^2 - 2i\,ar\cos\theta - a^2\cos^2\theta)} \qquad (9.23d)$$

$$\beta^{(0)} = \frac{\cot\theta}{2\sqrt{2}(r + ia\cos\theta)} \qquad (9.23e)$$

$$\beta'^{(0)} = \frac{\cot\theta}{2\sqrt{2}(r - ia\cos\theta)} - \frac{ia\sin\theta}{\sqrt{2}(r^2 - 2i\,ar\cos\theta - a^2\cos^2\theta)} \qquad (9.23f)$$

$$\epsilon^{(0)} = 0 \qquad (9.23g)$$

$$\epsilon'^{(0)} = \frac{\Delta}{2\Sigma(r - ia\cos\theta)} - \frac{r - M}{2\Sigma} \qquad (9.23h)$$

as well as the non-vanishing background Weyl scalar

$$\Psi_2^{(0)} = -\frac{M}{(r - ia\cos\theta)^3}\,. \qquad (9.24)$$

Inserting this into the wave equation, together with the expressions for the directional derivative operators yields

$$
\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2}\Psi_0^{(1)} + \frac{4Mar}{\Delta}\frac{\partial^2}{\partial t\partial\phi}\Psi_0^{(1)} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial^2\phi}\Psi_0^{(1)} \\
&- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\Psi_0^{(1)}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\frac{\partial}{\partial\phi}\Psi_0^{(1)} \\
&- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial}{\partial t}\Psi_0^{(1)} + \left[s^2\cot^2\theta - s\right]\Psi_0^{(1)} = 0\,. \qquad (9.25)
\end{aligned}
$$

This is a marvelous result as in this form, the equation is separable. Meaning that if we make the following ansatz

$$\Psi_0^{(1)} = \int_{-\infty}^{\infty} d\omega \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} R_{\ell m\omega}(r)\, S_{\ell m}(\theta,\phi;a\omega) e^{-i\omega t} \tag{9.26}$$

and substitute this back into the above equation, we find two decoupled equations

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \frac{K^2 - 2si(r-M)K}{\Delta} + 4si\omega r - \lambda_{\ell m}\right] R_{\ell m\omega} = 0 \tag{9.27}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + a^2\omega^2\cos^2\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} - 2a\omega s\cos\theta + s + A\right] S_{\ell m} = 0\,, \tag{9.28}$$

where $K := \left(r^2 + a^2\right)\omega - am$, the separation constant $A := \lambda_{\ell m} + 2am\omega - a^2\omega^2$ and the eigenvalue $\lambda_{\ell m}$ depends on the value of $a\omega$. The first equation is an ordinary differential equation in the radial direction. The second equation is — if you impose regularity at the poles $\theta = 0$ and $\theta = \pi$ — an eigenvalue problem for the separation constant A. The solutions are spin-weighted spheroidal harmonics. These functions are a generalization of spherical harmonics and excellent numerical recipes exist to calculate them to arbitrary precision. In general, the spin-weighted spheroidal harmonics are rather complicated. However, like the standard spherical harmonics, the dependence of the spin-weighted spheroidal harmonics on ϕ is simple:

$$S_{\ell m}(\theta,\phi;a\omega) = S_{\ell m}(\theta,0;a\omega)e^{im\phi}\,. \tag{9.29}$$

Hence, the challenging part in solving for gravitational perturbations propagating on a Kerr background is to solve for the radial differential equation. There are various (numerical) methods that tackle this problem and in some special cases analytic solutions exist. Of course, in the presence of a non-zero stress-energy tensor, the equation will have a source term on the right hand side and the problem will be more complicated.

The trivial solution to the Teukolsky equation, $\Psi_0^{(1)} = 0$, only changes the mass and the angular momentum of the spacetime but is otherwise non-dynamical (assuming that the solution is well-behaved). As a result, one can show that the *complete* gravitational metric perturbation can be recovered from $\Psi_0^{(1)}$ up to infinitesimal changes in the mass and angular momentum of the Kerr background. There is even an 'algorithm' to do exactly this, which goes by the name metric reconstruction and was pioneered by Chrzanowski.

Of course, the Teukolsky equation can also be used to solve for perturbations on Schwarzschild spacetime: simply set $a = 0$ in the above equations. The resulting equation is called the Bardeen-Press equation. This equation is not identical to the even or odd master equation derived in Sec. 7, since the Bardeen-Press equation is a differential equation for $\Psi_0^{(1)}$ instead of the gauge invariant combinations of the metric perturbations Ψ_{even} and Ψ_{odd}. Nonetheless, the two equations contain of course the same information and an explicit map between $\Psi_0^{(1)}$ and Ψ_{even} & Ψ_{odd} exists.

The Teukolsky equation has many applications in gravitational science: from the study of (linear) stability of Kerr black holes to super radiant phenomena, from quasi-normal modes to floating orbits. I hope your interest is piqued and you are now ready to delve into these fascinating topics yourself!

A Derivative operators

In this appendix, I collect some key properties of various derivative operators. Derivative operators are highly non-unique. Three types, however, are particularly frequently used and their key properties are collected in Table A.

	Covariant derivative	Lie derivative	Exterior derivative
Generalization of	partial/ordinary derivative	directional derivative	gradient
Applies to	any tensor field	any tensor field	p-forms (i.e., anti-symmetric $(0,p)$-tensors)
Map	(k,l) tensor $\longrightarrow$ $(k,l+1)$ tensor	(k,l) tensor $\longrightarrow$ (k,l) tensor	p-form $\longrightarrow$ $p+1$-form
Requires	nothing	any contravariant vector field ξ^a	nothing
Concomitant	no	yes	yes
Index notation	$\nabla_a T^{b_1\ldots b_k}{}_{c_1\ldots c_l}$	Eq. (2.1)	$(d\omega)_{ab_1\ldots b_p} = (p+1)\nabla_{[a}\omega_{b_1\ldots b_p]}$
Comments	non-unique, but popular choice is a torsion-free covariant derivative compatible with the metric ($\nabla_a g_{bc} = 0$)	beautiful connection with Lie brackets through $\mathcal{L}_\xi v^a = [\xi, v]^a$	satisfies a modified Leibniz rule $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^p\omega\wedge d\eta$ for any $p-$form ω and q-form η

Table 1: Collection of some key properties of covariant, Lie and exterior derivatives. Note: a concomitant is a type of derivative operator that is independent of the choice of derivative operator.

Some extra words about the covariant derivative. Any covariant derivative operator ∇ on a manifold M is a map which takes a smooth[20] tensor field of type (k,l) to a smooth tensor field of type $(k,l+1)$ and satisfies

1. Linearity:

$$\nabla_c\left(\alpha A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta B^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) = \alpha\,\nabla_c A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta\,\nabla_c B^{a_1\ldots a_k}{}_{b_1\ldots b_l} \tag{A.1}$$

   for any $(k,l)-$tensor fields $A^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ and $B^{a_1\ldots a_k}{}_{b_1\ldots b_l}$, and constants $\alpha,\beta\in\mathbb{R}$.

2. Leibniz rule:

$$\nabla_c\left(A^{a_1\ldots a_k}{}_{b_1\ldots b_l} B^{c_1\ldots c_{k'}}{}_{d_1\ldots b_{l'}}\right) = \left(\nabla_c A^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) B^{c_1\ldots c_{k'}}{}_{d_1\ldots b_{l'}} + A^{a_1\ldots a_k}{}_{b_1\ldots b_l}\nabla_c B^{c_1\ldots c_{k'}}{}_{d_1\ldots b_{l'}} \tag{A.2}$$

   for any $(k,l)-$tensor field $A^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ and $(k',l')-$tensor fields $B^{c_1\ldots c_{k'}}{}_{d_1\ldots b_{l'}}$.

3. Commutativity with contraction (which is essentially equivalent to the statement that $\nabla_a\delta^b{}_c = 0$).

[20] Or at least differentiable.

4. On scalar fields, the covariant derivative reduces to the standard ordinary derivative:

$$\nabla_a f = \partial_a f \ . \tag{A.3}$$

   Put differently, the covariant derivative is consistent with the notion of tangent vectors as directional derivatives on scalar fields: $v(f) = v^a \nabla_a f$ for any $f \in \mathcal{F}(M)$ and v^a a contravariant vector field.

In general relativity, we also add the condition that the covariant derivative has to be torsion free:

5 Torsion free:

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f \tag{A.4}$$

   for all $f \in \mathcal{F}(M)$.

In other theories of gravity, this condition could be relaxed (for instance, in teleparallel gravity).

B Further Reading

In preparing these lectures notes, I have relied on the following resources:

- Sec. 2 and 3 are based on the review article by Geroch [1], Ch. 11 in [2] and Ch. 35-38 in [15].
- Sec. 4 is based largely on [16], which in turn is based on the original papers by Bondi, Sachs and collaborators [17, 18] as well as the review by Mädler and Winicour [19].
- The discussion in Sec. 5 about the angular momentum ambiguity is nicely explained in a recent paper [20]. For the discussion on the memory effect, I relied on the summary in the introduction of [21].
- For Sec. 6, I have closely followed Ch. 39 in the excellent notes by Geroch [15]. For the part regarding the Stewart-Walker lemma, I relied on Sec. 1.6 in [22].
- Sec. 7 is based on the classic paper by Martel and Poisson [23] and Martel's PhD thesis [24]. Note, however, that Martel and Poisson have an additional minus sign in their definition of X_A and as a result some equations are slightly different.
- Sec. 8.1 is a *very* succinct version of the discussion about tetrads and vielbiens in [2] and [8]. Sec. 8.2 is based on the orginal paper by Geroch, Held and Penrose [14].
- The groundwork for Sec. 9 is performed in the seminal work by Teukolsky [25]. The equations in this paper, however, rely on the older less convenient Newman-Penrose formalism. For the presentation in terms of the GHP formalism [14], I relied on [26].

If you would like more background information on some of the "standard" concepts in general relativity, I recommend the excellent textbooks by Robert Wald [2] and Sean Carroll [8].

For more information on any of the following topics, I recommend the above resources in addition to:

- For a proof of the fact that translations form a Lie ideal in $\mathfrak{b}$ (which was mentioned but not provided in Sec. 3.3.3), set $s = 0$ in Appendix A of [27].
- In Sec. 5, I sketched the key elements to go to a canonical Bondi frame in the absence of radiation, but for the details see Sec. II D of [28] (which relies on earlier work by van der Burg and Bondi [29] and Newman and Penrose [30]).

- If you are interested in studying asymptotic flatness in higher dimensions, you want to learn about conformal Gaussian coordinates, which are closely related to Bondi-Sachs coordinates and are convenient in the study of asymptotic flatness in higher dimensions [31, 32].
- A succinct review of the Teukolsky equation and its relation to the Regge-Wheeler and Zerilli equation can be found in [33]. (This review also discusses an interesting applications of the Teukolsky equation: the self-force problem in general relativity.)
- If you are interested in metric reconstruction, the original paper by Chrzanowski [34] shows how to obtain the metric perturbation from $\Psi_0^{(1)}$ (or $\Psi_4^{(1)}$) in the ingoing/outgoing radiation gauge (up to infinitesimal transformations in the mass and angular momentum of the Kerr black hole). Recently, some alternative reconstruction schemes have been developed that allow for metric reconstruction in other gauges (see [35, 36]).

Personally, I have also learned a tremendous amount from discussions with my PhD supervisor Abhay Ashtekar and collaborators Alex Grant, Aruna Kesevan, Eric Poisson, Kartik Prabhu and Huan Yang as well as with colleagues at conferences. Some of the explanations in these notes may originally be theirs. I strongly recommend you also deepen your learning by discussing these topics with others!

If you miss any important references and would like to see these included, feel free to let me know.

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