Foundations of gravitational waves and black hole perturbation theory 2020/2021 (NWI-NM125)

Problem sheet #7: Perturbations on Kerr

Tutorial on Thursday 17 March 2022, 13:30 - 15:15

Exercise 7.1: Teukolsky equation in different forms

There are many equivalent ways of writing the Teukolsky equation.

a) An alternative way of writing the Teukolsky equation is:

$$\mathscr{O}\Psi_0^{(1)} = 0 \tag{1.1}$$

with

$$\mathscr{O} = (\text{þ} - 4\rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - 4\tau - \bar{\tau}')(\eth' - \tau') - 3\Psi_2^{(0)} \, . \tag{1.2}$$

Show that this is equivalent to Eq. (9.22) in the Lecture notes.

b) Given the Teukolsky equation for $\Psi_0^{(1)}$, what equation does $\Psi_4^{(1)}$ satisfy?[1] Make sure to write the derivative operators in the same order as those in the equation for $\Psi_0^{(1)}$ (i.e. $\text{þ}\,\text{þ}' - \eth\eth'$).

c) Introduce $y_{\ell m} = \Delta^{s/2}(r^2+a^2)^{1/2} R_{\ell m}$ and the coordinate r_* through $\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta}$. Show that $y_{\ell m}$ satisfies

$$\left[\frac{d^2}{dr_*^2} + \frac{1}{(r^2+a^2)^2}\Big(K^2 - 2is(r-M)K + \Delta\left(4ir\omega s - \lambda\right)\Big)\right] y_{\ell m} = 0 \, . \tag{1.3}$$

d) Show that in the limit $r \to \infty$, the differential equation for $y_{\ell m}$ becomes

$$\left[\frac{d^2}{dr_*^2} + \omega^2 + \frac{2i\omega s}{r}\right] y_{\ell m} \approx 0 \, , \tag{1.4}$$

so that $y_{\ell m} \approx r^{\pm s} e^{\mp i\omega r_*}$ are asymptotically solutions. In terms of the original variable $R_{\ell m}$, these asymptotic solutions are

$$R_{\ell m} \approx \frac{1}{r} e^{-i\omega r_*} \qquad \text{and} \qquad R_{\ell m} \approx \frac{1}{r^{2s+1}} e^{i\omega r_*} \, . \tag{1.5}$$

e) *Extra:* Show that near the event horizon $r \to r_+$, the radial equation for Y becomes

$$\left[\frac{d^2}{dr_*^2} + k^2 - 2isk\frac{r_+ - M}{2Mr_+} - s^2\frac{(r_+-M)^2}{(2Mr_+)^2}\right] y_{\ell m} \approx 0 \, , \tag{1.6}$$

where $k = \omega - m\omega_+$ and $\omega_+ = \frac{a}{2Mr_+}$ with solutions $y_{\ell m} \approx e^{\pm i\left(k - is\frac{r_+ - M}{2Mr_+}\right) r_*}$. Hint: Use a Taylor series in $r = r_+$. You may use Mathematica or some alternative algebraic manipulation software for this subquestion.

[1] The vacuum Teukolsky equation for $\Psi_4^{(1)}$ that typically appears in the literature is $\mathscr{O}(\zeta^4\Psi_4^{(1)}) = 0$ instead of $\mathscr{O}'\Psi_4^{(1)} = 0$, where ζ appears in the Killing tensor for Kerr and in Boyer-Lindquist coordinates is given by $\zeta = r - ia\cos\theta$. This alternative form is based on the identity: $\mathscr{O}'\Psi_4^{(1)} = \zeta^{-4}\mathscr{O}\zeta^4\Psi_0^{(1)}$.