Foundations of gravitational waves and black hole perturbation theory 2020/2021
(NWI-NM125)

Problem sheet #6: GHP formalism

Tutorial on Thursday 10 March 2022, 13:30 - 15:15

This tutorial is graded. To get full credits, you only need to hand in two completed exercises (you are free to choose which ones).

Hand in the tutorial *before* the start of the tutorial on Thursday 17 March at 13:30.Tutorials received after the deadline will be marked as a 1. If you are not able to hand in your solutions in person, you can email a scan or photo of your answers to Patricia (patricia.ribesmetidieri@ru.nl).

Exercise 6.1: GHP gymnastics

Getting more familiar with the properties of the GHP quantities and manipulating them.

a) Proof the symmetry property of the spin coefficient in Eq. (8.17) in the Lecture notes.

b) Show explicitly that $\text{þ}\,\eta$ with η some GHP quantity with weight $\{p, q\}$, is a quantity with GHP weight $\{p+1, q+1\}$. Hint: See the discussion about $\eth\eta$ in the Lecture notes (in particular, the part between Eq. (8.33) and Eq. (8.45)).

c) In the GHP formalism, the direction of l^a and n^a is fixed. However, imagine we had the freedom to rotate around l^a and n^a:

$$\text{I:}\quad l^a \to l^a, m^a \to m^a + a\,l^a, \bar{m}^a \to \bar{m}^a + \bar{a}\,l^a, n^a \to n^a + \bar{a}\,m^a + a\,\bar{m}^a + a\bar{a}\,l^a \tag{1.1}$$

$$\text{II:}\quad n^a \to n^a, m^a \to m^a + b\,l^a, \bar{m}^a \to \bar{m}^a + \bar{b}\,l^a, l^a \to l^a + \bar{b}\,m^a + b\,\bar{m}^a + b\bar{b}\,n^a\ , \tag{1.2}$$

where a and b are complex parameters. The former are called class I rotations and the latter class II. Show explicitly for the class I "rotations" that they do not change the normalization of the tetrad: that is, all vector remain null and the only non-vanishing inner products are $n \cdot l = -1$ and $m \cdot \bar{m} = 1$. These are the remaining four parameters of the six-parameter Lorentz group.

d) *Extra:* Show that the Weyl scalars Ψ_0, Ψ_1, Ψ_2 change in the following way under a type I rotation:

$$\Psi_0 \longrightarrow \Psi_0 \tag{1.3a}$$

$$\Psi_1 \longrightarrow \Psi_1 + \bar{a}\Psi_0 \tag{1.3b}$$

$$\Psi_2 \longrightarrow \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0 \tag{1.3c}$$

Hint: To find the result for Ψ_2, first show that $C_{abcd}l^a m^b \bar{m}^c m^d = C_{abcd}l^a m^b l^c n^d$ (rewrite $m^b\bar{m}^c$, use Eq. (8.22) in the Lecture notes and the tracelessness of the Weyl tensor).

Exercise 6.2: Priming starring and conjugating

Understand how a general GHP quantity with weight $\{p, q\}$ changes by a combined transformation of starring, priming or conjugating.

a) Complex conjugation and priming commute with each other, this is not true for starring. Moreover, while taking the complex conjugate of the same object twice returns the original object, this is not always true for priming or starring the same object twice. In particular, the following identities hold

$$(\eta^*)^* = (-1)^q \eta \tag{2.1}$$
$$(\eta')^* = (-1)^q (\eta^*)' \tag{2.2}$$
$$(\bar{\eta^*}) = (-i)^{p+q} (\bar{\eta}')^* \tag{2.3}$$
$$(\eta')' = (-1)^{p+1} \eta \ , \tag{2.4}$$

where η has weight $\{p, q\}$. Apply these identities to the four null vectors in the tetrad and verify that they hold.

b) Show that

$$þ^* = ð \qquad þ'^* = -ð' \qquad ð^* = -þ \qquad ð'^* = þ' \ .$$

c) What is the star of any of the five Weyl scalars? And what is the star of the five Weyl scalars after first haven taken its complex conjugate? In other words, what is Ψ_0^* and $\bar{\Psi}_0^*$ (and similarly for the four other Weyl scalars)? Your answer should be another Weyl scalar or its conjugate. Hint: Use the hint in Ex. 6.1d.

d) *Extra:* Show that the Einstein vacuum equations in Eqs. (8.48c) and (8.48d) are indeed star-invariant.

Exercise 6.3: GHP for Maxwell fields and the famous peeling theorem

The GHP formalism is also convenient to study electromagnetic fields.

The GHP formalism can also be very convenient to study electromagnetic radiation on curved spacetimes. The key quantity in electromagnetism is the Maxwell tensor F_{ab}. Since it has six independent components (given that it is anti-symmetric), these can be conveniently repackaged into three complex scalars:

$$\phi_0 = -F_{ab} l^a m^b \tag{3.1a}$$
$$\phi_1 = -\frac{1}{2} F_{ab} \left(l^a n^b - m^a \bar{m}^b \right) \tag{3.1b}$$
$$\phi_2 = F_{ab} n^a \bar{m}^b \ . \tag{3.1c}$$

a) What are the $\{p, q\}$ weights of the electromagnetic scalars ϕ_0, ϕ_1 and ϕ_2.

b) The eight real Maxwell equations in vacuum ($\nabla_a F^{ab} = 0$ and $\nabla_{[a} F_{bc]} = 0$) can be recast into four complex equations. One of those equations is

$$þ \phi_1 - ð' \phi_0 = -\tau' \phi_0 + 2\rho \phi_1 - \kappa \phi_2 \ . \tag{3.2}$$

Obtain the three other equations from this one. This nicely illustrates the efficiency of the GHP formalism! Hint: Consider starring, priming and a combination of the two.

Maxwell's equations without external sources are conformally invariant, meaning that F_{ab} on (M, g_{ab}) and $\tilde{F}_{ab} = F_{ab}$ on $(\tilde{M}, \tilde{g}_{ab} = \Omega^2 g_{ab})$ satisfy:

$$g^{ab} \nabla_a F_{bc} = 0 \quad \text{and} \quad \nabla_{[a} F_{bc]} = 0 \qquad \text{versus} \qquad \tilde{g}^{ab} \tilde{\nabla}_a \tilde{F}_{bc} = 0 \quad \text{and} \quad \tilde{\nabla}_{[a} \tilde{F}_{bc]} = 0 \ . \tag{3.3}$$

Since $\mathcal{I}$ is simply a sub-manifold of the conformally completed spacetime, $\tilde{F}_{ab}$ is smooth on $\mathcal{I}$.

c) Using the null tetrad for Minkowski spacetime in Eq. (8.23) in the Lecture notes, first show that $\tilde{l}^a$, $\tilde{n}^a$, $\tilde{m}^a$ and its complex conjugate have smooth, non-vanishing limits to $\mathcal{I}$ with

$$\tilde{l}^a = \Omega^{-2} l^a \qquad \tilde{n}^a = n^a \qquad \tilde{m}^a = \Omega^{-1} m^a \ . \tag{3.4}$$

d) *Extra:* Based on the results in c), show that near $\mathcal{I}$, the Maxwell scalars decay as:

$$\phi_2 = \Omega\phi_2^{(0)} + \mathcal{O}(\Omega^2) \tag{3.5a}$$
$$\phi_1 = \Omega^2\phi_1^{(0)} + \mathcal{O}(\Omega^3) \tag{3.5b}$$
$$\phi_0 = \Omega^3\phi_0^{(0)} + \mathcal{O}(\Omega^4)\ . \tag{3.5c}$$

This hierarchy in the decay rate of the Maxwell scalars on Minkowski spacetime turns out to hold also for Maxwell fields on any asymptotically flat spacetime. Moreover, a similar hierarchy also applies to the Weyl scalars Ψ_i near null infinity of any asymptotically flat spacetimes. This goes by the name of "peeling".