Foundations of gravitational waves and black hole perturbation theory 2020/2021
(NWI-NM125)

Problem sheet #4: Bondi-Sachs coordinates and gravitational waves

Tutorial on Thursday 24 February 2022, 13:30 - 15:15

This tutorial is graded. To get full credits, you only need to hand in Ex. 4.1 (the others are additional practice material). Hand in the tutorial *before* the start of the next tutorial on Thursday 3 March at 13:30. Tutorials received after the deadline will be marked as a 1. If you are not able to hand in your solutions in person, you can email a scan or photo of your answers to Patricia (patricia.ribesmetidieri@ru.nl).

A general hint: the expression for the Lie derivative is valid for any derivative operator (including the partial derivative operator). Choosing the "right" derivative operator can sometimes greatly simplify intermediate steps.

Exercise 4.1: Asymptotic symmetries in Bondi-Sachs coordinates

Derive the asymptotic symmetry algebra using (conformal) Bondi-Sachs coordinates instead of the universal structure.

Let ξ^a be any vector field on M and let $\xi_a = g_{ab}\xi^b$. The physical metric perturbation generated by a diffeomorphism along ξ^a is $\gamma_{ab} = \mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}$. Let this vector field ξ^a extend smoothly to the conformally completed spacetime $\tilde{M}$ (which is needed to preserve the zeroth-order smooth structure of $\tilde{M}$) so that $\tilde{\xi}^a = \xi^a$ and $\tilde{\xi}_a = \tilde{g}_{ab}\xi^b = \Omega^2\xi_a$. The corresponding perturbation to the conformally completed metric is given by

$$\tilde{\gamma}_{ab} = \Omega^2\gamma_{ab}\ . \tag{1.1}$$

a) Show that Eq. (1.1) in terms of $\tilde{\xi}^a$ is

$$\tilde{\gamma}_{ab} = 2\tilde{\nabla}_{(a}\tilde{\xi}_{b)} - 2\Omega^{-1}\tilde{\nabla}_c\Omega\ \tilde{\xi}^c\tilde{g}_{ab} = 2\tilde{\nabla}_{(a}\tilde{\xi}_{b)} - 2\Omega^{-1}\, n_c\tilde{\xi}^c\tilde{g}_{ab}\ . \tag{1.2}$$

Also show that this equation implies that $\tilde{\xi}^\Omega \mathrel{\hat{=}} 0$, where $\tilde{\xi}^\Omega$ is the Ω-component of $\tilde{\xi}^a$.

b) Next, expand the components of $\tilde{\xi}^a$ in terms of Ω, e.g. $\tilde{\xi}^u = \tilde{\xi}^u_{(0)}(u,x^A) + \Omega\tilde{\xi}^u_{(1)}(u,x^A) + \mathcal{O}(\Omega^2)$, and substitute this into Eq. (1.1). It is common practice to call $\tilde{\xi}^u_{(0)}(u,x^A) = F$ and $\tilde{\xi}^A_{(0)}(u,x^A) = Y^A$. Show that

$$\tilde{\gamma}_{uu} \mathrel{\hat{=}} 0 \tag{1.3a}$$
$$\tilde{\gamma}_{u\Omega} \mathrel{\hat{=}} -\tilde{\xi}^\Omega_{(1)} + \partial_u F \tag{1.3b}$$
$$\tilde{\gamma}_{uA} \mathrel{\hat{=}} S_{AB}\partial_u Y^B \tag{1.3c}$$
$$\tilde{\gamma}_{\Omega\Omega} \mathrel{\hat{=}} 2\tilde{\xi}^u_{(1)} \tag{1.3d}$$
$$\tilde{\gamma}_{\Omega A} \mathrel{\hat{=}} D_A F + S_{AB}\tilde{\xi}^B_{(1)} \tag{1.3e}$$
$$\tilde{\gamma}_{AB} \mathrel{\hat{=}} 2D_{(A}Y_{B)} - 2S_{AB}\tilde{\xi}^\Omega_{(1)} \tag{1.3f}$$
$$\tag{1.3g}$$

and

$$\tilde{\gamma}^a_a = 2\partial_u F - 6\tilde{\xi}^\Omega_{(1)} + 2D_A Y^A \,. \tag{1.4}$$

Hint: To simplify your intermediate calculations, use early on that $\Omega = 0$ on $\mathcal{I}$.

c) Since the *leading order* structure at $\mathcal{I}$ is fixed, we require $\tilde{\gamma}_{ab} \,\hat{=}\, 0$. Using this, show that

$$2D_{(A}Y_{B)} - S_{AB}D_C Y^C = 0 \qquad \text{and} \qquad F = f + \frac{1}{2}uD_A Y^A \tag{1.5}$$

with $\partial_u f = 0 = \partial_u Y^A$ and D_A the derivative operator compatible with the unit two-sphere metric.

d) Bringing together the results from part c), we can conclude that a general BMS vector has the form $\tilde{\xi}^a = Fn^a + Y^a$. We can split this vector field into a vector field $\chi^a = fn^a$, responsible for the supertranslations, and a vector field $\zeta^a = \frac{u}{2}D_A Y^A\, n^a + Y^a$, responsible for the Lorentz transformations. Show that

$$\alpha_{(f)} = 0 \qquad \text{and} \qquad \alpha_{(Y)} = \tfrac{1}{2}D_A Y^A \,, \tag{1.6}$$

as expected.

e) Verify explicitly that these vector fields form a closed algebra since for an arbitrary BMS vector field $\tilde{\xi}^a = Fn^a + Y^a$, we find that:

$$\left[\tilde{\xi}, Y\right]^a = -\left(\mathcal{L}_Y f + u\,\mathcal{L}_Y \alpha_{(Y)}\right) n^a \in \mathfrak{b} \tag{1.7a}$$

$$\left[\tilde{\xi}, g(\theta,\phi)n\right]^a = \left(\mathcal{L}_Y g - \alpha_{(Y)} g\right) n^a \in \mathfrak{s} \tag{1.7b}$$

where the second line shows in fact that the supertranslations form a Lie ideal.

f) How does a BMS vector field in physical (u, r, θ, ϕ) coordinates look? (The answer to this subquestion should be a very short calculation.)

Additional practice

Exercise 4.2: Supertranslations

Understand how the radiation field and the Bondi mass aspect transform under a supertranslation.

a) Verify Eq. (5.4) in the Lecture notes using Eq. (5.3). Note that $\delta_f C_{AB}$ refers to $\mathcal{O}(r)$ part in $\mathcal{L}_\xi g_{AB}$.

b) Similarly, show that a supertranslation changes the Bondi mass aspect in the following manner

$$\delta_f M = f\partial_u M + \frac{1}{4}N^{AB}D_A D_B f + \frac{1}{2}D_A f D_B N^{AB} \,. \tag{2.1}$$

You will need to use that the r-component of ξ^a is given by

$$\xi^r = \frac{1}{2}D^A D_A f - \frac{1}{2r}\left(D_A f D_B C^{AB} + \frac{1}{2}C^{AB}D_A D_B f\right) + \mathcal{O}(r^{-2}) \,. \tag{2.2}$$

c) Take the u-derivative of Eq. (2.1) and show that this result is consistent with taking δ_f of the evolution equation in Eq. (4.20) in the Lecture notes.

Exercise 4.3: Memory effect

The electromagnetic analog of gravitational wave memory is the change in a detector in the asymptotic region after an electromagnetic wave has passed. Instead of a distortion in the detector, as occurs in the gravitational wave case, you will show that there is a velocity kick to the charges in the detector.

a) First, show that if the source current in Minkowski spacetime has the following decay towards null infinity

$$\rho = j_r = \frac{L(u, x^A)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{3.1a}$$

$$j_A = \mathcal{O}\left(\frac{1}{r^3}\right) \tag{3.1b}$$

then the electromagnetic field decays as

$$E_r = \frac{W^{(0)}(u, x^A)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{3.2a}$$

$$B_r = \frac{Z^{(0)}(u, x^A)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{3.2b}$$

$$E_A = E_A^{(0)}(u, x^A) + \mathcal{O}\left(\frac{1}{r}\right) \tag{3.2c}$$

$$B_A = B_A^{(0)}(u, x^A) + \mathcal{O}\left(\frac{1}{r}\right) . \tag{3.2d}$$

Hint: A possible approach is to think about the required fall-off in Cartesian coordinates and use (some of) Maxwell's equations (specifically, $\nabla_a E^a = 4\pi\rho$ and $\nabla_a B^a = 0$).

b) Using the Lorentz force

$$F_a = q\left(E_a + \epsilon_{abc} v^b B^c\right) , \tag{3.3}$$

show that the velocity kick experienced by a charged test particle (=detector) with negligible initial velocity following a trajectory of the time translation vector field $\partial/\partial t$ so that $(r, \theta, \phi) = (r_0, \theta_0, \phi_0)$, is to leading order given by

$$|\Delta\vec{v}| = \frac{q}{m r_0}\left|\int_{-\infty}^{\infty} du\; E_A^{(0)}(u, \theta_0, \phi_0)\right| . \tag{3.4}$$

c) Consider an electromagnetic wave generated by a time-varying dipole moment p_a, so that near $\mathcal{I}$ the electric field is given by

$$E_a = \frac{1}{r}\partial_u^2\left(p_a - \hat{r}^b p_b\, \hat{r}_a\right) + \mathcal{O}\left(\frac{1}{r^2}\right) , \tag{3.5}$$

where $\hat{r}^a$ is the unit vector in the radial direction. Calculate the induced velocity kick. Repeat the same calculation, but consider now the slow motion limit, so that the dipole moment is

$$p_a = \sum_k q_{(k)} v_a^{(k)} \tag{3.6}$$

where k runs over all objects with charge $q^{(k)}$ and velocity $v_a^{(k)}$.