Foundations of gravitational waves and black hole perturbation theory 2020/2021
(NWI-NM125)

Problem sheet #3: Exact and asymptotic symmetries

Tutorial on Thursday 17 February 2022, 13:30 - 15:15

Exercise 3.1: Conformal Killing vectors

A short and sweet exercise about a generalization of Killing vector fields: conformal Killing vector fields.

a) A conformal Killing vector field on (M, g_{ab}) satisfies

$$\nabla_{(a}K_{b)} = \frac{1}{n}\nabla_c K^c \, g_{ab} \,, \tag{1.1}$$

where n is the number of spacetime dimensions. Show that if K^a is a Killing vector field with respect to g_{ab}, it is a conformal Killing vector field with respect to $\tilde{g}_{ab} = \Omega^2 g_{ab}$.

b) In the lecture notes, you have seen that $g_{ab}u^a K^b$ is constant along geodesics of g_{ab} with K^a a Killing vector field of g_{ab} and u^a tangent to the geodesic. Show that $\tilde{g}_{ab}u^a C^b$ is conserved along null geodesics with C^a a conformal Killing vector field of $\tilde{g}_{ab}$.

c) We also saw that $J^a := T^{ab}K_b$ is a conserved current if K^a is a Killing vector field (i.e., $\nabla_a J^a = 0$). What condition should T^{ab} satisfy for the current to be conserved if K^a is not a Killing vector field, but a conformal Killing vector field? Can you give an example of such a stress-energy tensor?

Exercise 3.2: The structure of the BMS algebra

Some practice with Lie algebras.

a) In the lecture notes, it is stated that if ξ^a satisfies

$$\mathcal{L}_\xi q_{ab} = 2\alpha_{(\xi)} q_{ab} \qquad \text{and} \qquad \mathcal{L}_\xi n^a = -\,\alpha_{(\xi)} n^a \qquad \text{with} \qquad \mathcal{L}_n \alpha_{(\xi)} = 0 \,, \tag{2.1}$$

and ζ^a as well, then so does $[\xi, \zeta]^a$. Prove this. (Recall: $\mathcal{L}_{[\xi,\zeta]}T = \mathcal{L}_\xi \mathcal{L}_\zeta T - \mathcal{L}_\zeta \mathcal{L}_\xi T$ for any tensor field T.)

b) *Extra:* In tutorial 2, you learned that decelerating FLRW spacetimes are not asymptotically flat, but can be studied using similar techniques. To incorporate the presence of matter, the induced metric and normal vector transformed under a conformal rescaling $\Omega \to \Omega' = \omega\Omega$ on $\mathcal{I}$ as

$$q'_{ab} = \omega^2 \tilde{q}_{ab} \qquad n'^a = \omega^{-1-s} n^a \,. \tag{2.2}$$

What is the equivalent of Eq. (2.1) in this case? Hint: Read the first part of Sec. 3.3 in the Lecture notes.

Exercise 3.3: Symmetries of Minkowski spacetime

Derive the form of the ten Killing vector fields of Minkowski spacetime.

a) In flat spacetime, a Killing vector field satisfies $\partial_{(a}K_{b)} = 0$. Take an additional derivative, and two cyclic permutations and combine them in such a way to conclude that K_a should be linear in the coordinates $x^a = (t, x, y, z)$:

$$K_a = \underline{K}_a + \underline{F}_{ab}x^b \tag{3.1}$$

with $\underline{K}_a$ and $\underline{F}_{ab}$ constant, but otherwise arbitrary tensor fields. Next, substitute this result back into the Killing vector equation to conclude that $\underline{F}_{ab}$ has to be anti-symmetric.

b) Give the explicit expressions in Cartesian coordinates for the translations, boosts and rotations for a static observer in flat spacetime.

c) Using the results in b), verify that Eq. (3.8) and (3.9) in the Lecture notes hold.

d) What is the Lie bracket between a boost in the x-direction and a boost in the y-direction? How about a boost in the x-direction with a rotation around the x-axis? And a boost in the x-direction with a rotation around the y-axis?

Additional practice

Exercise 3.4: Exact symmetries become asymptotic symmetries

If a spacetime contains isometries, its Killing vector fields become tangential to $\mathcal{I}$.

Let K^a be a Killing vector on (M, g_{ab}) and extend it smoothly to the conformally completed spacetime $(\tilde{M}, \tilde{g}_{ab})$. Show that if the norm of K^a with respect to $\tilde{g}_{ab}$ is finite near $\mathcal{I}$,

i) K^a satisfies the conformal Killing equation on $\mathcal{I}$, and

ii) K^a is tangential to $\mathcal{I}$.

Hint: The calculation is short! Use the results from exercise 3.1.