Foundations of gravitational waves and black hole perturbation theory 2020/2021
(NWI-NM125)

Problem sheet #1: Conformal transformations

Tutorial on Thursday 3 February 2022, 13:30 - 15:15

This tutorial will not be graded but contains essential material of this course. You are not expected to finish all the exercises during the tutorial and likely will need some additional hours to complete all exercises. Feel free to (virtually) meet with your fellow students to work on these exercises.

Exercise 1.1 and 1.2 are essential, while exercise 1.3 and 1.4 are recommended for further practice. Be aware that exercise 1.1. takes by far the most amount of time.

A conformal transformation is a mathematical tool with numerous applications in general relativity and we will rely on it heavily in this course. It describes a local change of scale, that is, a conformal transformation re-scales distances without changing angles. In particular, let (M, g_{ab}) be a spacetime. A conformal transformation on g_{ab} consists of replacing g_{ab} by the metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ where Ω is a positive scalar field on M and is called the conformal factor. The inverse metric of the conformally rescaled metric is $\tilde{g}^{ab} = \Omega^{-2} g^{ab}$ so that $g^{ab} g_{bc} = \delta^a{}_c = \tilde{g}^{ab}\tilde{g}_{bc}$.

Exercise 1.1: Derivative operators and curvature tensors

How to relate the covariant derivative operator ∇_a and $\tilde{\nabla}_a$ to each other? And how to relate curvature tensors associated to g_{ab} and $\tilde{g}_{ab}$?

Let ∇_a and $\tilde{\nabla}_a$ be covariant derivative operators with respect to g_{ab} and $\tilde{\nabla}_{ab}$, respectively (so $\nabla_a g_{bc} = 0$ and $\tilde{\nabla}_a \tilde{g}_{bc} = 0$). Since both are derivative operators, there exists a tensor field $C^c_{ab} = C^c_{(ab)}$ on M such that

$$\tilde{\nabla}_m T^{a\ldots b}{}_{c\ldots d} = \nabla_m T^{a\ldots b}{}_{c\ldots d} + C^a_{mn} T^{n\ldots b}{}_{c\ldots d} + \ldots + C^b_{mn} T^{a\ldots n}{}_{c\ldots d} - C^n_{mc} T^{a\ldots b}{}_{n\ldots d} - \ldots - C^n_{md} T^{a\ldots b}{}_{c\ldots n} \quad (1.1)$$

for any tensor field $T^{a\ldots b}{}_{c\ldots d}$ on M. In other words, the tensor field C^c_{ab} relates the two (covariant) derivative operators.

a) Using $\tilde{\nabla}_a \tilde{g}_{bc} = 0$, show that C^c_{ab} is given by

$$C^c_{ab} = \Omega^{-1} g^{cd} \left(g_{ad} \nabla_b \Omega + g_{bd} \nabla_a \Omega - g_{ab} \nabla_d \Omega \right) . \quad (1.2)$$

Hint: Use Eq. (1.1) to write $\tilde{\nabla}_a \tilde{g}_{bc} = 0$ in terms of ∇_a and C^c_{ab}, use that $\nabla_a g_{bc} = 0$ and substitute this in $\tilde{\nabla}_a \tilde{g}_{bc} - \tilde{\nabla}_b \tilde{g}_{ca} - \tilde{\nabla}_c \tilde{g}_{ab} = 0$.

b) Using the following definition of the Riemann tensor $\tilde{R}_{abc}{}^d v_d = 2\tilde{\nabla}_{[a}\tilde{\nabla}_{b]} v_c$ for any v_a, show that

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d - 2\nabla_{[a} C^d_{b]c} + 2 C^m_{c[a} C^d_{b]m} \quad (1.3)$$

and

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d + 2 g_{[a}{}^d \nabla_{b]} \nabla_c \ln\Omega - 2 g^{de} g_{c[a} \nabla_{b]} \nabla_e \ln\Omega - 2 g_{[a}{}^d \nabla_{b]} \ln\Omega \nabla_c \ln\Omega$$
$$+ 2 g_{c[a} \nabla_{b]} \ln\Omega \, g^{de} \nabla_e \ln\Omega - 2 g_{c[a} g_{b]}{}^d g^{ef} \nabla_e \ln\Omega \nabla_f \ln\Omega . \quad (1.4)$$

c) Finally, by contracting the above equation with the metric to obtain the Ricci tensor and scalar, show that

$$\tilde{R}_{ab} = R_{ab} - (n-2)\nabla_a\nabla_b \ln\Omega - g_{ab}g^{cd}\nabla_c\nabla_d \ln\Omega + (n-2)\nabla_a \ln\Omega\nabla_b \ln\Omega \\ - (n-2)g_{ab}g^{cd}\nabla_c \ln\Omega\nabla_d \ln\Omega \tag{1.5}$$

$$\tilde{R} = \Omega^{-2}\left[R - 2(n-1)g^{ab}\nabla_a\nabla_b \ln\Omega - (n-2)(n-1)g^{ab}\nabla_a \ln\Omega\nabla_b \ln\Omega\right] , \tag{1.6}$$

where n is the spacetime dimension so that $g^{ab}g_{ab} = n$.

d) *Extra:* Show that the Weyl tensor is unchanged by a conformal transformation, i.e. $\tilde{C}_{abc}{}^d = C_{abc}{}^d$ (note the index position: this is essential!). Recall that the Weyl tensor is defined as

$$C_{abc}{}^d = R_{abc}{}^d - \frac{2}{n-2}\left(g_{c[a}R_{b]}^d - g^d_{\ [a}R_{b]c}\right) + \frac{2}{(n-1)(n-2)}Rg_{c[a}g_{b]}^d . \tag{1.7}$$

Exercise 1.2: Schwarzschild spacetime

You will show explicitly that Schwarzschild spacetime is asymptotically flat.

a) The Schwarzschild metric in (t, r, θ, ϕ) coordinates is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{2.1}$$

where $f(r) = 1 - 2M/r$. Perform a coordinate transformation to coordinates convenient to take the limit to $\mathcal{I}$ and conformally complete the spacetime.

b) Show that the Schwarzschild spacetime satisfies all the conditions in the definition of an asymptotically flat spacetime (see Def. 2.1 in the Lecture notes).

Additional practice

Exercise 1.3: Geodesics and conformal transformations

You will learn that conformal transformations do not change the causal structure of a spacetime by studying null geodesics.

a) Show that if v^a is a null vector with respect to g_{ab}, it is also a null vector with respect to the conformally rescaled metric $\tilde{g}_{ab}$.

b) In part (a), you showed that conformal transformations leave null vectors invariant; now you need to show that a vector that is null *and* geodesic with respect to g_{ab} remains null and geodesic. Specifically, show that conformal transformations leave null geodesics invariant. Recall that the geodesic equation is

$$v^b\nabla_b v^a = \alpha\, v^a , \tag{3.1}$$

where α is an arbitrary function. If the geodesics is affinely parametrized, $\alpha = 0$ (you can always reparametrize your curve so that this is true). To highlight the difference, some physicists will call the above equation with $\alpha \neq 0$ ‘geodetic’ and only when $\alpha = 0$ ‘geodesic’.

Note that the invariance of geodesics with respect to conformal transformations is not true for time- or space-like geodesics: these are changed by conformal transformations!

Exercise 1.4: Curvature in two-dimensions

Any two-dimensional metric can locally be written as a flat metric multiplied by a conformal factor.

Show that for two-dimensional spacetimes, a conformal transformation can be found such that (at least locally) the curvature of the transformed metric vanishes provided that $g^{ab}\nabla_a\nabla_b$ is invertible. Note that generically, this can only be done locally and not over the entire manifold.