RADBOUD UNIVERSITY NIJMEGEN

FACULTY OF SCIENCE

Making waves in cosmology

A STUDY OF THE EFFECTS OF COSMOLOGY ON GRAVITATIONAL RADIATION

THESIS BSC PHYSICS & ASTRONOMY

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Abstract

The Bondi-Sachs formalism has enabled the study of gravitational radiation in asymptotically flat spacetimes at the full non-linear level. Recently, this formalism was extended to include a large class of decelerating cosmological spacetimes, including decelerating Friedmann-Lemaître-Robertson-Walker spacetimes. In this thesis, we will solve Einstein's equations to obtain the metric coefficients near null infinity, and generalize the evolution equations for the mass- and angular momentum aspects. We also find Bondi-Sachs coordinates for exact FLRW spacetimes, and check whether its linear perturbations are consistent with the exact solutions near null infinity.

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1 Introduction

Gravity, as described by Einstein's theory of general relativity, is a non-linear theory. This property makes solving the field equations extremely hard, and the few known exact solutions are often highly symmetric and do not describe realistic situations. Many methods of studying gravitational effects therefore involve the use of perturbation theory, which destroys gravity's characteristic non-linearity. Moreover, the equivalence principle, which lies at the heart of general relativity as a geometric theory, guarantees that there is no canonical way to locally split spacetime into a 'background' and a 'wave' part.

Shortly after publishing his field equations, Einstein found that linear perturbations of Minkowski space solve a wave equation. It was not clear whether these solutions correspond to the weak field limit of some full non-linear solution of Einstein's field equations, however, since these perturbations contain gauge freedom. The possibility was considered that some sporadic solutions exist only in the linearized case, but not in the non-linear case where it may be transformed away with a coordinate transformation. It took nearly fifty years for the first theoretical evidence for the existence of gravitational waves to emerge through the works of Bondi and collaborators [1, 2]. They found an elegant solution which uses our intuition about gravity that as one moves further away from a source, the gravitational field becomes weaker. Concretely, they described a class of spacetimes called asymptotically flat spacetimes in which the metric approaches the Minkowski metric as one moves along light rays out to infinity. The flatness at null infinity can then be used as a canonical background on which waves can be found in the next order structure away from infinity. Using these ideas, many non-trivial results due to gravity's non-linearity can be derived, one famous example being the loss of mass.

Of course, in realistic situations no source is ever truly isolated, since as one moves away we expect to keep finding galaxies, each of which is curving spacetime. What's more, due to the non-linear nature of curvature, it is not possible to simply 'add' the curvature due to an isolated system to, for example, a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. As is often the case in general relativity, generalizing the results derived from asymptotic flatness is difficult. When Joe Weber asked at a conference in Warsaw, 1962, why flatness was chosen instead of FLRW as boundary conditions at infinity, Bergmann and Bondi responded with [3]:

> P.G. Bergmann:
> *The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.*
>
> H. Bondi:
> *[...] I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.*

In the end, neither Bondi nor any other scientist from that generation was able to treat cosmological spacetimes. Most recently there has been a surge of interest in different boundary conditions at infinity. The case $\Lambda > 0$, which has a spacelike boundary like our own universe, has gotten a lot of attention. Only very recently progress was finally made, when some of the methods used to study asymptotic flatness have been extended to include a class of decelerating cosmological spacetimes that, like asymptotically flat spacetimes, posses a null infinity [4]. This class includes decelerating FLRW spacetimes. A notable example is the matter dominated universe, which, even though our own universe is accelerating, is a great approximation both in our galactic neighborhood where matter dominates dark energy, and on very large scales where incoming waves will have spent the majority of their time in the pre-accelerated era.

The Bondi-Sachs formalism allows us to study complicated compact gravitational wave sources. In the context of cosmology, these have not been studied before, not even at the linear level, where perturbation theory has obvious applications in the study of the small anisotropy of the cosmic microwave background. The Einstein telescope is projected to be able to probe these complicated sources, so their study has never been more relevant.

This thesis aims to answer two questions. Firstly, does the metric of spacetimes with a cosmological null asymptote, upon being linearized, coincide with the solution obtained through linear perturbation theory? To this end we use decelerating FLRW spacetimes — the only known explicit example of a spacetime with a cosmological null asymptote which is not asymptotically flat — as our background. Secondly, how are mass and angular momentum radiated in this new class of spacetimes? In §3, we derive the Bondi-Sachs coordinates for exact FLRW spacetimes, and in §4 the linearized field equations on an FLRW background are derived and solved near infinity. Then, in §5, we solve the full non-linear field equations for metrics with a cosmological null asymptote and find the generalized form of the Bondi mass-loss formula.

2 Preliminaries

In this section, we briefly review a coordinate based approach to asymptotic flatness in §2.1. In §2.2, we present the metric describing spacetimes with a cosmological null asymptote, which will be used in the remainder of this work.

2.1 Bondi-Sachs coordinates

Bondi-Sachs coordinates (u, r, x^A) are constructed from a family of light cones $u = constant$. The normal to the light cones is null so that $g^{\mu\nu}(\nabla_\mu u)(\nabla_\nu u) = 0$, which implies that $g^{uu} = 0$. To each light ray with tangent vector $g^{\mu\nu}\nabla_\nu u$, we assign two angular coordinates x^A such that $g^{\mu\nu}(\nabla_\mu u)(\nabla_\nu x^A) = 0$, which implies that $g^{uA} = 0$. Typical choices for x^A are the angular coordinates (θ, ϕ) and the stereographic coordinates $(z, \bar{z})$. Lastly, we choose the coordinate r to measure distance along the light rays by requiring that the spheres of constant u and r have an area of $4\pi r^2$. This restricts the determinant of the angular part of the metric to $\det g_{AB} = r^4 \det q_{AB}$, where q_{AB} is the metric on the unit sphere. The Bondi-Sachs line element $g_{\mu\nu}dx^\mu dx^\nu$ can then be written as

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}du dr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du), \tag{2.1}$$

where $\det h_{AB} = \det q_{AB}$. Note that $g_{rr} = g_{rA} = 0$ since $g^{u\lambda}g_{\lambda\mu} = g^{ur}g_{r\mu} = \delta^u{}_\mu$. Choosing the four coordinates in this particular way has reduced the independent components of the metric to six. V, β, U^A and h_{AB} are still arbitrary functions of (u, r, x^A), and Eq. (2.1) therefore describes more than just asymptotically flat spacetimes. For example, the de Sitter space metric — which describes a spacetime with constant scalar curvature Λ everywhere — can be written in these coordinates as $V = r - \frac{\Lambda}{3}r^3$, $\beta = U^A = 0$, and $h_{AB} = q_{AB}$. To give meaning to the words asymptotic flatness we therefore provide additional restrictions on the metric coefficients by requiring that in the limit $r \to \infty$ with (u, x^A) constant, they coincide with the metric coefficients of flat spacetime:

$$\lim_{r\to\infty} ds^2 = -du^2 - 2du dr + r^2 q_{AB}dx^A dx^B. \tag{2.2}$$

Expanding the functions β, V, U^A and h_{AB} in inverse powers of r near infinity and solving Einstein's equations term by term, we then find

$$V = r - 2M + \mathcal{O}\left(\frac{1}{r}\right), \tag{2.3a}$$

$$\beta = -\frac{C^{AB}C_{AB}}{32r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{2.3b}$$

$$U^A = -\frac{D_B C^{AB}}{2r^2} + \frac{N^A}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{2.3c}$$

$$h_{AB} = q_{AB} + \frac{C_{AB}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{2.3d}$$

where D_A is the covariant derivative compatible with the metric q_{AB}. M and N^A are constants of integration and are called the mass- and angular momentum aspect, respectively, partly motivated by many examples of exact solutions in which they are related to the mass and angular momentum of the spacetime. Expanding the determinant condition in inverse powers of r provides the constraint $\det(q_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(\frac{1}{r^2})) = \det q_{AB}(1 + \frac{q^{CD}C_{CD}}{r} + \mathcal{O}(\frac{1}{r^2})) = \det q_{AB}$, i.e. C_{AB} is trace-less. Finding V, β, and U^A in terms of h_{AB} eliminates four additional independent components of the metric, and we are thus left with two independent components which, to lowest order, are contained in the trace-less C_{AB}. Their time evolution is determined by Einstein's equations. The remaining components of Einstein's equations are then either trivially satisfied, or trivially satisfied up to a constant, due to Bianchi's identities. Resolving these constants yields additional equations which are used to find the time evolution of M and N^A:

$$\partial_u M = -\frac{1}{8}N^{AB}N_{AB} + \frac{1}{4}D_A D_B N^{AB}, \tag{2.4a}$$

$$\partial_u N_A = -\frac{2}{3}D_A M + \frac{1}{8}D_A(C^{BC}N_{BC}) + \frac{1}{6}D^B D^C D_{[A}C_{B]C} - \frac{1}{6}D_B(N^{BC}C_{AC}) + \frac{1}{3}N_{AB}D_C C^{BC} - \frac{1}{3}N^{BC}D_B C_{AC}, \tag{2.4b}$$

where we defined $N_{AB} := \partial_u C_{AB}$ which is called the Bondi news tensor. This name comes from the fact that an observer at infinity would be able to determine the geometry given the news tensor, and initial data (in the form of $M(u_0, x^A)$ and $N^A(u_0, x^A)$ on some surface $u = u_0$). News therefore arives at infinity through the news tensor. The Bondi mass $m(u)$ of this spacetime is defined as the monopole moment of the mass aspect. Its evolution is given by

$$\partial_u m = -\frac{1}{32\pi}\oint N^{AB}N_{AB}d^2S \leq 0. \tag{2.5}$$

Here, integration is performed over the unit sphere. Mass is radiated if and only if there is news. Intuitively this makes sense, since a static configuration could produce a non-zero C_{AB} but should not radiate.

Note that this approach is fully non-linear and contains many results that could not have been found through (linear) perturbation theory. An example would be the loss of mass. Treating C_{AB} as a linear perturbation would make the integrant of Eq. (2.5) vanish.

To conclude this section, I would like to remark that a geometric (coordinate free) definition of asymptotic flatness exists. This definition is equivalent to the one we gave based on coordinates, but allows the use of some of the more sophisticated tools of differential geometry to analyse its properties. Refer to [5] for the details. The main idea is to construct a new spacetime in which points at infinity are added. The Bondi-Sachs metric (2.1) diverges at infinity, so a new (unphysical) metric is constructed conformal to the (physical) metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ for some suitable Ω, which can be used as a coordinate around infinity. In Bondi-Sachs coordinates, $\Omega = \frac{1}{r}$ is an example of a suitable conformal factor. Einstein's equations in terms of the unphysical metric are

$$\tilde{G}_{\mu\nu} - 8\pi T_{\mu\nu} = 2\Omega^{-1}(\tilde{g}_{\mu\nu}\tilde{\nabla}^\lambda n_\lambda - \tilde{\nabla}_\mu n_\nu) - 3\Omega^{-2}\tilde{g}_{\mu\nu}n^\lambda n_\lambda, \tag{2.6}$$

where $n_\mu := \tilde{\nabla}_\mu \Omega$. In some cases it can be advantageous to work with the unphysical metric, since it is of order Ω^0 and hence so are its curvature tensors.

2.2 Spacetimes with a cosmological null asymptote

While many interesting systems are asymptotically flat, on cosmological scales our universe is not. In a recent paper [4], the Bondi-Sachs formalism was generalised to include a large class of decelerating cosmological spacetimes which — like the asymptotically flat class — posses a null-infinity. The conformal factor Ω now includes a scale factor. Hence the Ricci curvature, and by extension the stress-energy, do not vanish at null infinity unlike the asymptotically flat case. In fact, the stress energy diverges. In particular, the diverging part of the stress-energy can be written in terms of geometric quantities as

$$8\pi T_{\mu\nu} = 2s\Omega^{2(s-1)}n_\mu n_\nu + 2s\Omega^{s-1}n_{(\mu}\tau_{\nu)} + \mathcal{O}(1), \tag{2.7}$$

where $n_\mu = \frac{1}{1-s}\nabla_\mu \Omega^{1-s}$, and $0 \leq s < 1$ is a parameter related to the degree to which the stress-energy diverges. In the case of FLRW, $s = \frac{2}{3(1+w)}$ is related to the equation of state parameter $w = \frac{p}{\rho}$ where p and ρ are the pressure and energy density of a perfect fluid that fills the spacetime. Like for asymptotically flat spacetimes a coordinate based definition exists for spacetimes with a cosmological null asymptote. The generalised Bondi-Sachs coordinates are

$$ds^2 = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}} \left[-\frac{\tilde{V}}{\tilde{r}} e^{2\beta} du^2 - 2e^{2\beta} du d\tilde{r} + \tilde{r}^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)\right]. \tag{2.8}$$

Much like how asymptotically flat spacetimes are asympotically identical to Minkowski spacetime, spacetimes with a cosmological null asymptote are in some sense asymptotically identical to FLRW spacetimes. It follows from Eq. (2.7) and Einstein's equations that the metric coefficients fall off as

$$\tilde{V} = \tilde{V}^{(-1)}\tilde{r} + \tilde{V}^{(0)} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{2.9a}$$

$$\beta = \frac{\beta^{(1)}}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right), \tag{2.9b}$$

$$U^A = \frac{U^{(1)A}}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right), \tag{2.9c}$$

$$h_{AB} = q_{AB} + \frac{C_{AB}}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right). \tag{2.9d}$$

A detailed calculation will be given in §4.

3 FLRW in Bondi-Sachs coordinates

In this section we derive the Bondi-Sachs coordinates of decelerating FLRW spacetimes. These coordinates are an extension of those found in §5.3 of [4], which are only valid near infinity. As it turns out, the metric coefficients are only algebraic in a few cases. Two notable special cases are the radiation dominated universe and the matter dominated universe, which will be presented in §3.1.

The FLRW metric is conformal to the Minkowski metric, so a natural place to start would be the FLRW metric in Bondi-Sachs-like coordinates conformal to Minkowski in Bondi-Sachs coordinates

$$ds^2 = (\mathring{r} + \mathring{u})^{\frac{2s}{1-s}}[-d\mathring{u}^2 - 2d\mathring{u}d\mathring{r} + \mathring{r}^2 q_{AB} d\mathring{x}^A d\mathring{x}^B]. \tag{3.1}$$

Spheres of constant $\mathring{u}$ and $\mathring{r}$ are $\mathring{u}$-dependent however, unlike the spheres of constant u and $\tilde{r}$ of the metric in Eq. (2.8). We find the Bondi-Sachs coordinate $\tilde{r}$ by setting $\mathring{x}^A = x^A$ and demanding that the $dx^A dx^B$ metric coefficients of Eq. (2.8) and Eq. (3.1) are equal:

$$\begin{aligned} \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}} \tilde{r}^2 q_{AB} &= (\mathring{r}A)^{\frac{2s}{1-s}} \mathring{r}^2 q_{AB} \\ \implies \tilde{r} &= (1-s)\mathring{r}A^s, \end{aligned} \tag{3.2}$$

where we defined $A := 1 + \frac{\mathring{u}}{\mathring{r}}$. Setting $u = (1-s)\mathring{u}$ for convenience, we find that A must be a solution to the equation

$$A = 1 + \frac{u}{\tilde{r}} A^s. \tag{3.3}$$

In general, no algebraic solution exists. For example, if $s = \frac{1}{5}$, $\sqrt[5]{A} \equiv x$ is a root of the quintic equation $x^5 - \frac{u}{\tilde{r}}x - 1 = 0$, which famously does not have an algebraic solution even in the case $\frac{u}{\tilde{r}} = 1$. Furthermore, from this example it is clear that multiple solutions may exist. However, using Descartes' rule of signs we can deduce that there is only one positive real solution, since the coefficients of the polynomial $x^p - \frac{u}{\tilde{r}} x^q - 1 = 0$ have signs $(+, \pm, -)$. The trick to solving Eq. (3.3) is to first find the inverse of A, which is easily found to be

$$\frac{u}{\tilde{r}} = A^{1-s} - A^{-s}. \tag{3.4}$$

Using Lagrange's inversion theorem we can find a series expansion of A. If $z = f(w)$ and $f'(c) \neq 0$ then this theorem states that around a point c where f is analytic,

$$w = c + \sum_{n=1}^{\infty} \lim_{u\to c} \frac{d^{n-1}}{dx^{n-1}} \left[\left(\frac{x-c}{f(x)-f(c)} \right)^n \right] \frac{(z-f(c))^n}{n!}. \tag{3.5}$$

We take $f(w) = w^{1-s} - w^{-s}$ around the point $w = 1$ which will get us the positive real solution since f diverges, and hence is not analytic, at $w = 0$ for $s > 0$. Using

$$\frac{x-1}{x^{1-s} - x^{-s}} = x^s, \qquad \frac{d^n}{dx^n} \frac{x^{sn}}{n!} = \binom{sn}{n} x^{n(s-1)}, \tag{3.6}$$

and including $n = 0$ we get an expression for A:

$$\begin{aligned} A(u, \tilde{r}) &= \sum_{n=0}^{\infty} \binom{sn}{n} \frac{1}{n(s-1)+1} \left(\frac{u}{\tilde{r}} \right)^n \\ &= 1 + \frac{u}{\tilde{r}} + s \left(\frac{u}{\tilde{r}} \right)^2 + \frac{1}{2} s(3s-1) \left(\frac{u}{\tilde{r}} \right)^3 + \frac{1}{3} s(2s-1)(4s-1) \left(\frac{u}{\tilde{r}} \right)^4 + \mathcal{O}\left(\frac{u}{\tilde{r}} \right)^5 \end{aligned} \tag{3.7}$$

The first few terms agree with the approximation found in [4]. A^s is then found through Eq. (3.3):

$$A^s(u, \tilde{r}) = \sum_{n=0}^{\infty} \binom{s(n+1)}{n} \frac{1}{n+1} \left(\frac{u}{\tilde{r}} \right)^n. \tag{3.8}$$

Lastly we will need A^{2s}, which can be found using the Hagen-Rothe identity [6]:

$$\sum_{k=0}^{n} \binom{x+kz}{k} \frac{x}{x+kz} \binom{y+(n-k)z}{n-k} \frac{y}{y+(n-k)z} = \binom{x+y+nz}{n} \frac{x+y}{x+y+nz}. \tag{3.9}$$

Squaring the sum Eq. (3.8) we find that the n-th coefficient of the resulting sum is of the form Eq. (3.9) with $x = y = z = s$, so that

$$\begin{aligned} A^{2s}(u, \tilde{r}) &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{s(k+1)}{k} \frac{1}{k+1} \binom{s(n-k+1)}{n-k} \frac{1}{n-k+1} \right] \left(\frac{u}{\tilde{r}} \right)^n \\ &= \sum_{n=0}^{\infty} \binom{s(n+2)}{n} \frac{2}{n+2} \left(\frac{u}{\tilde{r}} \right)^n. \end{aligned} \tag{3.10}$$

In the coordinates $(u, \tilde{r}, x^A)$ the scale factor becomes

$$(\mathring{r} A)^{\frac{2s}{1-s}} = \left(\frac{\tilde{r}^s}{1-s} \right)^{\frac{2}{1-s}} (1-s)^2 A^{2s}. \tag{3.11}$$

Finally, we compute the metric coefficients Eq. (2.8)

$$\begin{aligned}\frac{\tilde V}{\tilde r}e^{2\beta} &= (1-s)^2A^{2s}[(\partial_u\mathring{u})^2+2\partial_u\mathring{u}\partial_u\mathring{r}] = A^{2s}-2\tilde r\partial_u A^s\\ &= \sum_{n=0}^{\infty}\binom{s(n+2)}{n}\frac{2(1+n-s(n+2))}{n+2}\left(\frac{u}{\tilde r}\right)^n, \qquad (3.12a)\\ e^{2\beta} &= (1-s)^2A^{2s}\partial_u\mathring{u}\partial_{\tilde r}\mathring{r} = A^s-\tilde r\partial_{\tilde r}A^s = \tilde r\partial_u A\\ &= \sum_{n=0}^{\infty}\binom{s(n+1)}{n}\left(\frac{u}{\tilde r}\right)^n, \qquad (3.12b)\\ h_{AB} &= q_{AB}, \qquad (3.12c)\\ U^A &= 0, \qquad (3.12d)\end{aligned}$$

which concludes the coordinate transform. The FLRW metric in Bondi-Sachs coordinates is

$$ds^2 = \left(\frac{\tilde r^s}{1-s}\right)^{\frac{2}{1-s}}\left(\left[\sum_{n=0}^{\infty}-\binom{s(n+2)}{n}\frac{2(n+1-s(n+2))}{n+2}\left(\frac{u}{\tilde r}\right)^n du^2-2\binom{s(n+1)}{n}\left(\frac{u}{\tilde r}\right)^n dud\tilde r\right]+\tilde r^2q_{AB}dx^Adx^B\right). \qquad (3.13)$$

$\binom{s(n+1)}{n} < n+1$ for $0 \le s < 1$, which means that this series converges quicker than a geometric series, which converges for $|\frac{u}{\tilde r}| < 1$, and hence this metric covers $|u| < \tilde r < \infty$.

FLRW is spatially homogenous and isotropic. Isotropy is seen through spherical symmetry, since $h_{AB} = q_{AB}$, U^A vanishes, and none of the remaining metric coefficients depend on the coordinates x^A. Bondi-Sachs coordinates are not adapted to homogeneity however, which is why (3.13) looks much more complicated compared to the more widely used coordinates which are adapted to these symmetries: $ds^2 = -dt^2 + t^{2s}(dr^2 + r^2q_{AB}dx^Adx^B)$. The fact that the Ricci curvature is non-vanishing is reflected by the fact that the area of the spheres of constant u and $\tilde r$ is $\left(\frac{\tilde r^s}{1-s}\right)^{\frac{2}{1-s}}4\pi\tilde r^2$.

3.1 The radiation and matter dominated universes

For some values of s, the metric Eq. (3.13) can be simplified further. These values include $s = \frac{1}{2}$, in which case the metric describes a radiation filled universe, and $s = \frac{2}{3}$ in which case the metric describes a universe filled with dust. When deriving the series in the previous section, it was convenient to express new series in terms of derivatives and sums of old series, while avoiding multiplication as much as possible, since we could add and differentiate term by term quite easily. Here, the strategy will be to first find A, and to then compute the metric coefficients. The following identities will be useful:

$$e^{2\beta} = \frac{A^{s+1}}{(1-s)A+s}, \qquad (3.14a)$$

$$\frac{\tilde V}{\tilde r}e^{2\beta} = (A^s+1)^2-1-2e^{2\beta}. \qquad (3.14b)$$

Deriving these identities involves repeatedly using the defining equation of A (3.3). To compute A for $s = \frac{1}{2}$, we first notice that the coefficients of the series Eq. (3.8) split into two categories: for odd n, $\binom{\frac{1}{2}(n+1)}{n} = 0$ except for $n = 1$. For even n, we rewrite the coefficients using the identity $\binom{p}{q} \equiv \frac{p}{q}\binom{p-1}{q-1}$ by applying it $\frac{n}{2}$-times. The solution Eq. (3.8) then simplifies to

$$\begin{aligned}\sqrt{A} &= \frac{u}{2\tilde r}+\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{u}{2\tilde r}\right)^{2n}\\ &= \frac{u}{2\tilde r}+\sqrt{1+\left(\frac{u}{2\tilde r}\right)^2}. \qquad (3.15)\end{aligned}$$

From which we find:

$$ds^2 = \frac{\tilde{r}^2}{16}\left[-\left(\frac{\frac{u}{2\tilde{r}}\left(\frac{u}{2\tilde{r}}+\sqrt{1+\left(\frac{u}{2\tilde{r}}\right)^2}\right)^2}{\sqrt{1+\left(\frac{u}{2\tilde{r}}\right)^2}}\right)du^2 - 2\left(\frac{2\left(\frac{u}{2\tilde{r}}+\sqrt{1+\left(\frac{u}{2\tilde{r}}\right)^2}\right)^3}{1+\left(\frac{u}{2\tilde{r}}+\sqrt{1+\left(\frac{u}{2\tilde{r}}\right)^2}\right)^2}\right)du d\tilde{r} + \tilde{r}^2 q_{AB} dx^A dx^B\right]. \tag{3.16}$$

We could also have solved Eq. (3.3) directly, as it is now a simple quadratic in $\sqrt{A}$. This is how we will find A for $s = \frac{2}{3}$. The positive real solution to the cubic equation $A^3 - ((\frac{u}{\tilde{r}})^3 + 3)A^2 + 3A - 1 = 0$ is

$$A = 1 + 9\left(\frac{u}{3\tilde{r}}\right)^3 + \frac{3\left(\frac{u}{3\tilde{r}}\right)^3\left(2+9\left(\frac{u}{3\tilde{r}}\right)^3\right)}{\Delta} + 3\Delta, \tag{3.17}$$

$$\text{where}\quad \Delta := \sqrt[3]{\frac{1}{2}\left(\frac{u}{3\tilde{r}}\right)^3 + 9\left(\frac{u}{3\tilde{r}}\right)^6 + 27\left(\frac{u}{3\tilde{r}}\right)^9 + \sqrt{\frac{1}{4}\left(\frac{u}{3\tilde{r}}\right)^6 + \left(\frac{u}{3\tilde{r}}\right)^9}}.$$

We obtain the somewhat awkward metric coefficients

$$e^{2\beta} = \frac{\left(9\left(\frac{u}{3\tilde{r}}\right)^6 + \left(\frac{u}{3\tilde{r}}\right)^3(2+3\Delta)+\Delta^2\right)\left(\Delta + 3\left(9\left(\frac{u}{3\tilde{r}}\right)^6 + \Delta^2 + \left(\frac{u}{3\tilde{r}}\right)^3(2+3\Delta)\right)\right)}{\Delta^2\left(9\left(\frac{u}{3\tilde{r}}\right)^7 + \left(\frac{u}{3\tilde{r}}\right)^4(2+3\Delta) + \left(\frac{u}{3\tilde{r}}\right)\Delta(1+\Delta)\right)}, \tag{3.18a}$$

$$\frac{\tilde{V}}{\tilde{r}}e^{2\beta} = \frac{\left(9\left(\frac{u}{3\tilde{r}}\right)^6 + \left(\frac{u}{3\tilde{r}}\right)^3(2+3\Delta)+\Delta^2\right)\left(9\left(\frac{u}{3\tilde{r}}\right)^6 + 2\Delta\left(\frac{u}{3\tilde{r}}\right) + \left(\frac{u}{3\tilde{r}}\right)^3(2+3\Delta)+\Delta^2\right)}{\Delta^2\left(\frac{u}{3\tilde{r}}\right)^2} - 2e^{2\beta}. \tag{3.18b}$$

4 The linear perturbations of FLRW

In this section we solve the linearized field equations for perturbations on an FLRW background in Bondi-Sachs coordinates Eq. (3.13). While these linear equations are much simpler than Einstein's full non-linear field equations, they are still difficult to solve. In §4.1 we therefore start by identifying and eliminating excess degrees of freedom. In §4.2 we use separation of variables to split and simplify them further. Finally, we find the linear perturbations of FLRW near infinity in §4.3.

4.1 Coordinate freedom

The physically relevant properties of a spacetime do not depend on the choice of coordinates. Hence, some solutions will be redundant because they are related to another solution through a coordinate transformation.

The metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are physically identical if there is a diffeomorphism ϕ such that $\tilde{g}_{\mu\nu} = \phi^* g_{\mu\nu}$. Consider the one-parameter family of metrics $g_{\mu\nu}(\lambda) = \bar{g}_{\mu\nu} + \lambda\gamma_{\mu\nu}$, and consider an arbitrary family of diffeomorphisms ϕ_λ. Since we intend to describe $\phi^*_\lambda g_{\mu\nu}(\lambda)$ as a small perturbation on a fixed background we require the transformed perturbation to again be small. Hence, we require $\phi^*_0 g_{\mu\nu} = \bar{g}_{\mu\nu}$. Then $\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu}$ are related through:

$$\gamma_{\mu\nu} - \tilde{\gamma}_{\mu\nu} = \frac{d}{d\lambda}\left(g_{\mu\nu}(\lambda) - \phi^*_\lambda g_{\mu\nu}(\lambda)\right)\Big|_{\lambda=0} = \lim_{\lambda\to 0}\frac{g_{\mu\nu}(\lambda) - \bar{g}_{\mu\nu} - \phi^*_\lambda g_{\mu\nu}(\lambda) + \bar{g}_{\mu\nu}}{\lambda} = \lim_{\lambda\to 0}\frac{\phi^*_{-\lambda}g_{\mu\nu} - g_{\mu\nu}}{\lambda}, \tag{4.1}$$

where in the last equality we used that $\phi^*_\lambda \equiv -\phi^*_{-\lambda}$. Here we recognize the Lie derivative of $g_{\mu\nu}$ along a vector field generated by the flow ϕ_λ. We can thus rewrite Eq. (4.1) as:

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu} = \gamma_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)}. \tag{4.2}$$

Hence, if we can find a ξ that solves Eq. (4.2), then $\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu}$ are physically identical.

Suppose we are given a background metric $g_{\mu\nu}$ in Bondi-Sachs coordinates. We would now like to find a covector field ξ_μ such that the perturbed metric remains in Bondi-Sachs form. This coordinate condition is called the *Bondi-gauge*. In this gauge, $\gamma_{\tilde{r}\tilde{r}} = \gamma_{\tilde{r}A} = h^{AB}\gamma_{AB} = 0$. The final condition ensures that $\det(\tilde{r}^2 h_{AB} + \gamma_{AB}) = \tilde{r}^4 \det(q_{AB})(1+\tilde{r}^{-2}h^{AB}\gamma_{AB}) + \mathcal{O}(\gamma^2) = \tilde{r}^4\det(q_{AB})$. For computational convenience, we factor out the scale factor and set $\lambda = 1$ by rescaling $\gamma_{\mu\nu}$ such that the perturbed metric and its coordinate freedom are given by

$$ds^2 = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}} (g_{\mu\nu}+\gamma_{\mu\nu})dx^\mu dx^\nu \tag{4.3}$$

$$\text{and} \quad \tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \mathscr{L}_\xi g_{\mu\nu} - \frac{2s}{1-s}\frac{1}{\tilde{r}} g_{\mu\nu}\mathscr{L}_\xi \tilde{r}. \tag{4.4}$$

First, we set $\tilde{\gamma}_{\tilde{r}\tilde{r}} = 0$:

$$\begin{aligned} \gamma_{\tilde{r}\tilde{r}} &= 2\nabla_{\tilde{r}}\xi_{\tilde{r}} = 2(\partial_{\tilde{r}} - \Gamma^{\tilde{r}}{}_{\tilde{r}\tilde{r}})\xi_{\tilde{r}} \\ \implies \xi_{\tilde{r}} &= f_{\tilde{r}}(u,x^A)e^{\int \Gamma^{\tilde{r}}{}_{\tilde{r}\tilde{r}}d\tilde{r}} + e^{\int \Gamma^{\tilde{r}}{}_{\tilde{r}\tilde{r}}d\tilde{r}}\int e^{-\int \Gamma^{\tilde{r}}{}_{\tilde{r}\tilde{r}}d\tilde{r}}\frac{1}{2}\gamma_{\tilde{r}\tilde{r}}d\tilde{r}, \end{aligned} \tag{4.5}$$

where all other connection coefficients vanish since $\Gamma^\mu{}_{\tilde{r}\tilde{r}} = g^{\tilde{r}\mu}\partial_{\tilde{r}}g_{\mu\tilde{r}}$ is only non-zero if $\mu = \tilde{r}$. Next, we set $\tilde{\gamma}_{\tilde{r}A} = 0$:

$$\gamma_{\tilde{r}A} = 2\nabla_{(\tilde{r}}\xi_{A)} = \partial_{\tilde{r}}\xi_A - 2\Gamma^B{}_{\tilde{r}A}\xi_B + (\partial_A - 2\Gamma^{\tilde{r}}{}_{\tilde{r}A})\xi_{\tilde{r}}. \tag{4.6}$$

This system of equations has no closed form solution for a general $\Gamma^B{}_{\tilde{r}A}$. If we assume that h_{AB} is diagonal however, which is the case for FLRW, then $\Gamma^B{}_{\tilde{r}A}$ is diagonal and ξ_A has the solution

$$\xi_A = f_A(u,x^A)e^{2\int\Gamma^A{}_{\tilde{r}A}d\tilde{r}} + e^{2\int\Gamma^A{}_{\tilde{r}A}d\tilde{r}}\int e^{-2\int\Gamma^A{}_{\tilde{r}A}d\tilde{r}}\left(\gamma_{\tilde{r}A} - (\partial_A - 2\Gamma^{\tilde{r}}{}_{\tilde{r}A})\xi_{\tilde{r}}\right)d\tilde{r}. \tag{4.7}$$

Here, $\Gamma^A{}_{\tilde{r}A}$ refers to the component and is not being contracted. Lastly, we set $h^{AB}\tilde{\gamma}_{AB} = 0$ by choosing an appropriate ξ_u, which is our one remaining component of ξ_μ. Since $g^{AB}\Gamma^u{}_{AB} = -\frac{1}{2}g^{u\tilde{r}}g^{AB}\partial_{\tilde{r}}g_{AB} = -\frac{1}{2}g^{u\tilde{r}}\frac{\partial_{\tilde{r}}\det(g_{AB})}{\det(g_{AB})}$ and since per definition $0 = \partial_{\tilde{r}}\det(h_{AB}) = \frac{1}{\tilde{r}^4}\partial_{\tilde{r}}\det(g_{AB}) - \frac{4}{\tilde{r}^5}\det(g_{AB})$, we can conclude that $g^{AB}\Gamma^u{}_{AB} = -\frac{2g^{u\tilde{r}}}{\tilde{r}} \neq 0$. Hence, we can solve

$$\begin{aligned} h^{AB}\gamma_{AB} &= 2h^{AB}\nabla_A\xi_B + \frac{4s\tilde{r}}{1-s}\xi_{\tilde{r}} = 2h^{AB}\left(D_A\xi_B - \Gamma^u{}_{AB}\xi_u + \left(\frac{2s\tilde{r}}{1-s}h_{AB} - \Gamma^{\tilde{r}}{}_{AB}\right)\xi_{\tilde{r}}\right) \\ \implies \xi_u &= -h^{AB}\frac{D_A\xi_B + \left(\frac{2s\tilde{r}}{1-s}h_{AB} - \Gamma^{\tilde{r}}{}_{AB}\right)\xi_{\tilde{r}} - \frac{1}{2}\gamma_{AB}}{2\tilde{r}g^{u\tilde{r}}}. \end{aligned} \tag{4.8}$$

The leftover freedom is captured by the functions $f_{\tilde{r}}(u,x^A)$ and $f_A(u,x^A)$. Sumarizing, we find that for FLRW, the transformations that preserve the Bondi-gauge are generated by

$$\xi_u = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}}\left(\frac{1}{2}(1-s)rD^Af_A(u,x^A) + ((1-s)D^AD_A + 2 - 4s)f_{\tilde{r}}(u,x^A) + s(1-s)uD^Af_A(u,x^A) + \mathcal{O}\left(\frac{1}{\tilde{r}}\right)\right) \tag{4.9a}$$

$$\xi_{\tilde{r}} = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}} f_{\tilde{r}}(u,x^A)\left(2 + \frac{4su}{\tilde{r}} - \frac{3s(1-3s)u^2}{\tilde{r}^2} + \mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right) \tag{4.9b}$$

$$\xi_A = \left(\frac{\tilde{r}^s}{1-s}\right)^{\frac{2}{1-s}}\left(r^2 f_A(u,x^A) + f_{\tilde{r}}(u,x^A)\left(2 + \frac{4su}{\tilde{r}} - \frac{3s(1-3s)u^2}{\tilde{r}^2} + \mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)\right). \tag{4.9c}$$

4.2 Separation of variables

Next, we use separation of variables to reduce the linearized field equations to a form we may reasonably expect to be able to solve. Specifically, we choose a function basis for the angular part of the perturbations. We are justified in doing this because FLRW is spherically symmetric.

The set of scalar spherical harmonics Y^{lm} form an orthogonal function basis on the two-sphere. We can also use these to define an orthogonal vector field basis on the two-sphere, consisting of even-parity harmonics $Y_A^{lm} = D_A Y^{lm}$ and odd-parity harmonics $X_A^{lm} = -\epsilon^B{}_A D_B Y^{lm}$. Extending this one step further, we define an orthogonal traceless 2-tensor basis on the two-sphere, consisting out of even-parity harmonics $Y_{AB}^{lm} = (D_A D_B + \frac{1}{2}l(l+1)q_{AB})Y^{lm}$ and odd-parity harmonics $X_{AB}^{lm} = -\frac{1}{2}(\epsilon^C{}_A D_B + \epsilon^C{}_B D_A)D_C Y^{lm}$. We expand our perturbations into these harmonics:

$$\gamma_{ab} = \sum_{l,m} p_{ab}^{lm} Y^{lm} \tag{4.10a}$$

$$\gamma_{aA} = \sum_{l,m} j_a^{lm} Y_A^{lm} + p_a^{lm} X_A^{lm} \tag{4.10b}$$

$$\gamma_{AB} = \sum_{l,m} K^{lm} q_{AB} Y^{lm} + G^{lm} Y_{AB}^{lm} + p^{lm} X_{AB}^{lm}, \tag{4.10c}$$

where $p_{ab}^{lm}, j_a^{lm}, p_a^{lm}, K^{lm}, G^{lm}$and p^{lm} are functions of u and $\tilde{r}$ only. The lowercase latin indices index the u and $\tilde{r}$ components. In the Bondi gauge, $p_{\tilde{r}\tilde{r}}^{lm} = j_{\tilde{r}}^{lm} = p_{\tilde{r}}^{lm} = K^{lm} = 0$. The linearized part of the Einstein tensor, $\delta G_{\mu\nu}$, can be written as:

$$\delta G_{ab} = \sum_{l,m} Q_{ab}^{lm} Y^{lm} \tag{4.11a}$$

$$\delta G_{aA} = \sum_{l,m} Q_a^{lm} Y_A^{lm} + P_a^{lm} X_A^{lm} \tag{4.11b}$$

$$\delta G_{AB} = \sum_{l,m} Q_\flat^{lm} q_{AB} Y^{lm} + Q_\sharp^{lm} Y_{AB}^{lm} + P^{lm} X_{AB}^{lm}. \tag{4.11c}$$

Of course, since any scalar, vector, or tensor on the two-sphere can be decomposed into these harmonics it is not obvious that decomposing the Einstein tensor in this manner results in a simpler expression. It turns out, however, that the coefficients lm in the sums (4.11) are linear functions of $p_{ab}^{lm}, j_a^{lm}, p_a^{lm}, K^{lm}, G^{lm}$and p^{lm}. This is a consequence of the combined facts that 1) the objects (4.11) are scalars, vectors, and tensors on the two-sphere, 2) the linearized field equations are, as their name implies, linear in (4.10), and 3) the harmonics are eigenfunctions of the Laplacian Δ.

Finally, since these harmonics are orthogonal, we can solve these equations for each lm separately, and split them into an even-parity ($Y^{lm}, Y_A^{lm}, Y_{AB}^{lm}$) and an odd-parity ($X_A^{lm}, X_{AB}^{lm}$) sector.

4.3 Perturbations near infinity

We are left with one final roadblock before we can start solving Eq. (4.11). The linearized field equations involve the curvature tensors of the background metric. Computing these seems simple, albeit cumbersome, since we could either perform a coordinate transformation on the curvature tensors of the metric in the usual coordinates, or perform a calculation involving only differentiation, addition, and multiplication of the metric coefficients in Eq. (3.13). While possible, the results are not "nice" because certain terms, $(\partial_u A)^2$ for example, cannot be reduced to a single sum. Instead we will expand $\gamma_{\mu\nu}$ and the background curvature tensors in inverse powers of $\tilde{r}$, and solve Eq. (4.11) term by term recursively.

To subleading order, we then find the even-parity Einstein tensor

$$\begin{aligned}
Q_{uu} &= \left(\frac{1-2s}{(1-s)\tilde r}\partial_{\tilde r} + \frac{l(l+1)(1-s)^2+2+2s(2-5s)}{2(1-s)^2\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)p_{uu} + \left(-\frac{1}{(1-s)\tilde r} + \frac{2su}{(1-s)\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)\partial_u p_{uu} \\
&\quad + \left(-\frac{2(1-2s)^2}{(1-s)\tilde r}\partial_{\tilde r} - \frac{l(l+1)(1-s)^2(1-2s)+2(1-2s+3s^2-5s^3)}{(1-s)^2\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)p_{u\tilde r} \\
&\quad + \left(\frac{2(1-s-2s^2)}{(1-s)\tilde r} + \frac{4s^2(1+s)u}{(1-s)\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)\partial_u p_{u\tilde r} + \left(\frac{l(l+1)(1-2s)}{\tilde r^2}\partial_{\tilde r} + \frac{l(l+1)(1+s-4s^2)}{(1-s)\tilde r^3}\right)j_u \\
&\quad - \frac{l(l+1)}{\tilde r^2}\partial_u j_u + \left(-\frac{(l-1)l(l+1)(l+2)(1-2s)}{4\tilde r^4} - \frac{(l-1)l(l+1)(l+2)s(2-3s)u}{2\tilde r^5} + \mathcal{O}\left(\frac{1}{\tilde r^6}\right)\right)G,
\end{aligned} \tag{4.12a}$$

$$\begin{aligned}
Q_{u\tilde r} &= \left(\frac{1}{1-s}\frac{1}{\tilde r}\partial_{\tilde r} - \frac{2su}{\tilde r^2}\partial_{\tilde r} + \frac{1}{(1-s)^2}\frac{1}{\tilde r^2} + \frac{2s(1-2s)u}{(1-s)^2\tilde r^3} + \mathcal{O}\left(\frac{1}{\tilde r^4}\right)\right)p_{uu} \\
&\quad + \left(-\frac{2(1-2s)}{(1-s)\tilde r}\partial_{\tilde r} - \frac{4s^2u}{\tilde r^2}\partial_{\tilde r} - \frac{l(l+1)(1-s)^2+2(2-4s+s^2)}{2(1-s)^2\tilde r^2} - \frac{2s(2-6s+5s^2)u}{(1-s)^2\tilde r^3} + \mathcal{O}\left(\frac{1}{\tilde r^4}\right)\right)p_{u\tilde r} \\
&\quad + \left(\frac{l(l+1)}{2\tilde r^2}\partial_{\tilde r} + \frac{1+s}{1-s}\frac{l(l+1)}{\tilde r^3}\right)j_u + \left(-\frac{l(l+1)(l-1)(l+2)}{4\tilde r^4} - \frac{(l-1)l(l+1)(l+2)su}{2\tilde r^5} + \mathcal{O}\left(\frac{1}{\tilde r^6}\right)\right)G,
\end{aligned} \tag{4.12b}$$

$$Q_{\tilde r\tilde r} = \left(-\frac{2}{1-s}\frac{1}{\tilde r}\partial_{\tilde r} + \frac{4su}{1-s}\frac{1}{\tilde r^2}\partial_{\tilde r} - \frac{4su}{1-s}\frac{1}{\tilde r^3} + \mathcal{O}\left(\frac{1}{\tilde r^4}\right)\right)p_{u\tilde r}, \tag{4.12c}$$

$$\begin{aligned}
Q_u &= \left(-\frac{1}{2}\partial_{\tilde r} + \frac{su}{\tilde r}\partial_{\tilde r} - \frac{s}{1-s}\frac{1}{\tilde r} + \frac{2s^2u}{(1-s)\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\right)p_{uu} + \left(\frac{1}{2}(1-2s)\partial_{\tilde r} - \frac{s^2}{(1-s)\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\right)p_{u\tilde r} \\
&\quad + \left(\frac{1}{2} - \frac{su}{\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\right)\partial_u p_{u\tilde r} + \left(-\frac{1}{2}(1-2s)\partial_{\tilde r}\partial_{\tilde r} - \frac{s(2-3s)}{(1-s)\tilde r}\partial_{\tilde r} + \frac{s(4-9s+4s^2)}{(1-s)^2\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)j_u \\
&\quad + \left(\frac{1}{2}\partial_{\tilde r} - \frac{su}{\tilde r}\partial_{\tilde r} - \frac{1}{\tilde r} + \frac{2su}{\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r}\right)\right)\partial_u j_u - \frac{(l-1)(l+2)}{4\tilde r^2}\partial_u G,
\end{aligned} \tag{4.12d}$$

$$\begin{aligned}
Q_{\tilde r} &= \left(\frac{1}{2}\partial_{\tilde r} - \frac{su}{\tilde r}\partial_{\tilde r} - \frac{1}{(1-s)\tilde r} + \frac{s(3-s)u}{\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)p_{u\tilde r} \\
&\quad + \left(-\frac{1}{2}\partial_{\tilde r}\partial_{\tilde r} + \frac{su}{\tilde r}\partial_{\tilde r}\partial_{\tilde r} - \frac{s}{1-s}\frac{1}{\tilde r}\partial_{\tilde r} - \frac{s(1-3s)u}{(1-s)\tilde r^2}\partial_{\tilde r} + \frac{1+s}{(1-s)\tilde r^2} - \frac{4s^2u}{(1-s)\tilde r^3} + \mathcal{O}\left(\frac{1}{\tilde r^4}\right)\right)j_u \\
&\quad + \left(-\frac{(l-1)(l+2)}{4\tilde r^2}\partial_{\tilde r} + \frac{(l-1)(l+2)}{2\tilde r^3}\right)G,
\end{aligned} \tag{4.12e}$$

$$\begin{aligned}
Q_\flat &= \left(-\frac{1}{2}\tilde r^2\partial_{\tilde r}\partial_{\tilde r} + 2su\tilde r\partial_{\tilde r}\partial_{\tilde r} - \frac{1+s}{1-s}\tilde r\partial_{\tilde r} - \frac{s(3+5s)u}{1-s}\partial_{\tilde r} - \frac{s^2}{(1-s)^2} - \frac{2s(1-3s^2)u}{(1-s)^2\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\right)p_{uu} \\
&\quad + \left(\frac{2(1-3s)}{1-s}\tilde r\partial_{\tilde r} + l(l+1) + \frac{4s(2-3s)}{(1-s)^2} + \mathcal{O}\left(\frac{1}{\tilde r}\right)\right)p_{u\tilde r} \\
&\quad + \left(-\frac{1}{2}l(l+1)\partial_{\tilde r} + \frac{l(l+1)}{\tilde r}\partial_{\tilde r} - \frac{s}{1-s}\frac{l(l+1)}{\tilde r} + \frac{2s^2u}{1-s}\frac{l(l+1)}{\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)j_u,
\end{aligned} \tag{4.12f}$$

$$\begin{aligned}
Q_\sharp &= \left(-1 + \frac{2su}{\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\right)p_{u\tilde r} + \left(\partial_{\tilde r} - \frac{2su}{\tilde r}\partial_{\tilde r} + \frac{2s}{(1-s)\tilde r} - \frac{2s^2u}{(1-s)\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)j_u \\
&\quad + \left(-\frac{1}{2}(1-2s)\partial_{\tilde r}\partial_{\tilde r} - \frac{(1-2s)^2}{(1-s)\tilde r}\partial_{\tilde r} - \frac{1+8s-14s^2+6s^3}{(1-s)^2\tilde r^2}\right)G \\
&\quad + \left(\partial_{\tilde r} - \frac{2su}{\tilde r}\partial_{\tilde r} - \frac{1-2s}{(1-s)\tilde r} + \frac{2s(1-2s)u}{(1-s)\tilde r^2} + \mathcal{O}\left(\frac{1}{\tilde r^3}\right)\right)\partial_u G.
\end{aligned} \tag{4.12g}$$

We dropped the lm-superscripts to make the expressions more readable.

To subleading order, the odd-parity Einstein tensor has the following components:

$$P_u = \left(\frac{l(l+1)(1-s)^2+2s(4-s(9-4s)}{2(1-s)^2\tilde{r}^2}+\mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)p_u+\left(-\frac{1-2s}{2}\partial_{\tilde{r}}\partial_{\tilde{r}}+\frac{1}{2}\partial_{\tilde{r}}-\frac{su}{\tilde{r}}\partial_{\tilde{r}}-\frac{1}{\tilde{r}}+\frac{2su}{\tilde{r}^2}+\mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)\partial_u p_u$$
$$+\left(-\frac{(l-1)(l+2)}{4\tilde{r}^2}\right)\partial_u p, \tag{4.13a}$$

$$P_{\tilde{r}} = \left(-\frac{1}{2}\partial_{\tilde{r}}\partial_{\tilde{r}}+\frac{su}{\tilde{r}}\partial_{\tilde{r}}\partial_{\tilde{r}}-\frac{s}{(1-s)\tilde{r}}\partial_{\tilde{r}}-\frac{s(1-3s)u}{(1-s)\tilde{r}^2}\partial_{\tilde{r}}+\frac{1+s}{(1-s)\tilde{r}^2}-\frac{4s^2u}{(1-s)\tilde{r}^3}+\mathcal{O}\left(\frac{1}{\tilde{r}^4}\right)\right)p_u$$
$$+\left(-\frac{(l-1)(l+2)}{4\tilde{r}^2}\partial_{\tilde{r}}+\frac{(l-1)(l+2)}{2\tilde{r}^3}\right)p, \tag{4.13b}$$

$$P = \left(-\partial_{\tilde{r}}+\frac{2su}{\tilde{r}}\partial_{\tilde{r}}-\frac{2s}{1-s}\frac{1}{\tilde{r}}+\frac{4s^2u}{(1-s)\tilde{r}^2}+\mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)p_u+\left(-\frac{1-2s}{2}\partial_{\tilde{r}}\partial_{\tilde{r}}+\frac{(1-2s)^2}{(1-s)\tilde{r}}\partial_{\tilde{r}}-\frac{1-8s+14s^2-6s^3}{(1-s)^2\tilde{r}^2}+\mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)p$$
$$+\left(\partial_{\tilde{r}}-\frac{2su}{\tilde{r}}\partial_{\tilde{r}}-\frac{1-2s}{1-s}\frac{1}{\tilde{r}}+\frac{2s(1-2s)u}{(1-s)\tilde{r}^2}+\mathcal{O}\left(\frac{1}{\tilde{r}^3}\right)\right)\partial_u p. \tag{4.13c}$$

$\gamma_{u\tilde{r}} = C(u,x^A)$ and $\gamma_{uA} = \tilde{r}^2 C_A(u,x^A)$ are pure gauge and can both be eliminated with a transformation (4.9) generated by $f_{\tilde{r}} = \frac{1}{2}\int C(u,x^A) - \frac{1}{2}(1-s)\left(\int D^A C_A(u,x^A)du\right)du$ and $f_A = \int C(u,x^A)du$. Only in the asymptotically flat case, $s=0$, are the mass- and the angular momentum aspect — the $\frac{1}{\tilde{r}}$-coefficients of γ_{uu} and γ_{uA} — constants of integration.

The stress-energy perturbation, $\delta T_{\mu\nu}$, cannot be arbitrary, and must fall off sufficiently fast as $\tilde{r}\to\infty$ to ensure that the perturbed spacetime has a cosmological null asymptote. The leading order diverging terms fall off as

$$\delta T_{u\tilde{r}} = \mathcal{O}(\tilde{r}^{-1}), \qquad \delta T_{\tilde{r}\tilde{r}} = \mathcal{O}(\tilde{r}^{-3}). \tag{4.14}$$

Note that the stress-energy diverges at infinity despite the fact that all component functions have a well-defined limit, since the basis one-form $d\tilde{r}$ diverges at infinity as $\mathcal{O}(\tilde{r}^2)$. The remaining stress-energy components must be finite at infinity. Similar to how we treated the metric perturbations and the linearized field equations, we decompose the stress-energy perturbation $\delta T_{\mu\nu}$ into spherical harmonics:

$$\delta T_{ab} = \sum_{l,m} t^{lm}_{ab} Y^{lm}, \tag{4.15a}$$
$$\delta T_{aA} = \sum_{l,m} t^{lm}_{a} Y^{lm}_{A} + S^{lm}_{a} X^{lm}_{A}, \tag{4.15b}$$
$$\delta T_{AB} = \sum_{l,m} T^{lm} q_{AB} Y^{lm} + t^{lm} Y^{lm}_{AB} + S^{lm} X^{lm}_{AB}. \tag{4.15c}$$

Similar to Eq. (2.9d) we write $\gamma_{AB} = \tilde{r}(C^{even}Y_{AB}+C^{odd}X_{AB}) + d^{even}Y_{AB} + d^{odd}X_{AB} + \mathcal{O}(\frac{1}{\tilde{r}})$ and find p_{uu}, $p_{u\tilde{r}}$, j_u, p_u, and the evolution $\partial_u p_{uu}$, $\partial_u j_u$, and $\partial_u p_u$ in terms of the stress-energy terms and C^{even} and C^{odd}. The leading and subleading order odd-parity solutions are

$$p_u = 8\pi(1-s)S^{(1)}_{\tilde{r}}\tilde{r} + 8\pi s(1-s)uS^{(1)}_{\tilde{r}} - \frac{1}{4}(l+2)(l-1)\frac{1-s}{1+s}C^{odd} + 8\pi\frac{1-s}{1+s}S^{(2)}_{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.16a}$$
$$\partial_u p_u = -16\pi t^{(0)}_u\tilde{r} - \frac{1}{4}(l+2)(l-1)\partial_u C^{odd} - 8\pi S^{(1)}_u + 8\pi\frac{s(2-4s+s^2)}{1-s}S^{(1)}_{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.16b}$$
$$p = C^{odd}\tilde{r} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.16c}$$

and the leading and subleading order even-parity solutions are

$$p_{uu} = -\frac{(1-s)^2}{s^2}(8\pi t^{(0)} + 4\pi(1-s)\partial_u t^{(3)}_{\tilde{r}\tilde{r}} + 4\pi l(l+1)(1+s) t^{(1)}_{\tilde{r}}) + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.17a}$$

$$\begin{aligned}\partial_u p_{uu} = &-8\pi(1-s) t^{(1)}_{uu} - 8\pi l(l+1)(1-s)^2 \partial_u t^{(1)}_{\tilde{r}} + (-8\pi(1-s) t^{(2)}_{uu} + 2su\partial_u p^{(0)}_{uu} - l(l+1)(1-s)\partial_u j^{(0)}_u - \frac{1}{2}(1-s)\Delta p^{(0)}_{uu} \\ &+ (1-s)(1-2s) 8\pi\partial_u t^{(3)}_{\tilde{r}\tilde{r}} - l(l+1)(1-s)(s^2+s-1)16\pi t^{(1)}_{\tilde{r}} + \frac{1+2s-5s^2}{1-s} p^{(0)}_{uu})\frac{1}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right),\end{aligned} \tag{4.17b}$$

$$p_{u\tilde{r}} = 4\pi(1-s) t^{(3)}_{\tilde{r}\tilde{r}}\frac{1}{\tilde{r}} + (2\pi(1-s) t^{(4)}_{\tilde{r}\tilde{r}} + 8\pi s(1-s) u t^{(3)}_{\tilde{r}\tilde{r}})\frac{1}{\tilde{r}^2} + \mathcal{O}\left(\frac{1}{\tilde{r}^3}\right), \tag{4.17c}$$

$$j_u = 8\pi(1-s) t^{(1)}_{\tilde{r}}\tilde{r} + 8\pi s(1-s) u t^{(1)}_{\tilde{r}} + 2\pi\frac{3-s}{1+s}(1-s) t^{(3)}_{\tilde{r}\tilde{r}} - \frac{1}{4}(l+2)(l-1)\frac{1-s}{1+s}C^{even} + 8\pi\frac{1-s}{1+s} t^{(2)}_{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.17d}$$

$$\partial_u j_u = -16\pi t^{(0)}_u \tilde{r} - \frac{s}{1-s} p^{(0)}_{uu} - \frac{1}{4}(l+2)(l-1)\partial_u C^{even} + 2\pi(1-s)\partial_u t^{(3)}_{\tilde{r}\tilde{r}} - 8\pi t^{(1)}_u + 8\pi\frac{s(2-4s+s^2)}{1-s} t^{(1)}_{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{4.17e}$$

$$G = C^{even}\tilde{r} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right). \tag{4.17f}$$

It should be noted that it is not clear whether or not the series solution to the expanded linearized field equations is the same as the expanded solution to the (complete) linearized field equations. There are reasons to expect that it does, however. Most notably a direct computation shows that the order of expansion is irrelevant to linear first order differential equations, and the linearized field equations on a Schwarzschild background as a specific example.

Secondly, it should be remarked that simply solving Eq. (4.13) and Eq. (4.12) as we did here is suboptimal. It is possible to reduce the number of equations down to six using the Bianchi identities. Additionally, it would be easier to check if the linear solutions Eq. (4.16) and Eq. (4.17) are consistent with the full non-linear solutions of Einstein's equations for arbitrary spacetimes with a cosmological null asymptote if we solved both in the same manner. However, since one of the purposes of this section is to provide us with a way to check the non-linear solution it makes the comparison more convincing if different steps are involved to reach the same conclusion.

5 Einstein's equations and mass loss

It is well known that there is no electromagnetic monopole radiation as a consequence of charge conservation. Since Maxwell's equations are linear, this further implies the much stronger result that waves cannot have any spherically symmetric component at all due to the superposition principle. A similar result exists for gravitational waves. Birkhoff's theorem states that spherically symmetric (vacuum) solutions to Einstein's equations are static, and hence do not contain waves. However, since general relativity is a non-linear theory, the superposition principle does not apply, and the theorem therefore does not exclude the possibility of gravitational waves having a spherically symmetric component. Indeed, in §1 we saw the Bondi mass loss formula for asymptotically flat spacetimes (2.5) which shows that the gravitational monopole, the Bondi mass, is not conserved. With the advent of new methods which generalize the Bondi-Sachs formalism used to determine this famous formula, a natural question to ask is how this result generalizes to spacetimes with a cosmological null asymptote. The purpose of this section is therefore to solve Einstein's equations for the generalised Bondi-Sachs metric (2.1). In §4.1 we solve the main equations in order to determine the asymptotic solution of the metric coefficients and in §4.2 we evaluate the evolution equations and obtain generalized mass- and angular momentum loss formulas. Along the way we check for consistency with the solutions of the linearized field equations on an FLRW background found in the previous section.

5.1 The main equations

We can drastically simplify our calculation by computing the curvature tensors of the unphysical metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where $\Omega = (\frac{\tilde{r}}{1-s})^{-\frac{1}{1-s}}$. Unlike in the asymptotically flat case, Ω is not a good coordinate at infinity since it is not analytic there for a general s. We therefore define $\tilde{\Omega} = \frac{1}{\tilde{r}}$ and $n_\mu = \tilde{\nabla}_\mu \tilde{\Omega}$. Eq. (2.6) then becomes

$$8\pi T_{\mu\nu} - \tilde{G}_{\mu\nu} = \frac{2\tilde{\Omega}^{-1}}{1-s}(\tilde{\nabla}_\mu n_\nu - \tilde{g}_{\mu\nu}\tilde{\nabla}^\lambda n_\lambda) + \frac{\tilde{\Omega}^{-2}}{(1-s)^2}(2s n_\mu n_\nu + (3-2s)\tilde{g}_{\mu\nu} n^\lambda n_\lambda). \tag{5.1}$$

The $\tilde{\Omega}\tilde{\Omega}$ and $\tilde{\Omega}A$ components of the unphysical Ricci tensor are relatively simple expressions:

$$\tilde{R}_{\tilde{\Omega}\tilde{\Omega}} = -\frac{1}{4}\partial_{\tilde{\Omega}} h_{AB} \partial_{\tilde{\Omega}} h_{CD} h^{AC} h^{BD}, \tag{5.2}$$

$$\tilde{R}_{\tilde{\Omega}A} = -\frac{1}{2}\partial_{\tilde{\Omega}}(e^{-2\beta} h_{AB}\partial_{\tilde{\Omega}} U^B) - \partial_{\tilde{\Omega}}\eth_A\beta + \frac{1}{2}\eth^B(\partial_{\tilde{\Omega}} h_{AB}), \tag{5.3}$$

where $\eth_A$ is the covariant derivative compatible with h_{AB}. The corresponding components of Einstein's equations can then be cast into the following form

$$\partial_{\tilde{\Omega}}\beta = \frac{s}{1-s}\tilde{\Omega}^{-1} - (1-s)4\pi T_{\tilde{\Omega}\tilde{\Omega}} - \frac{1}{8}(1-s)\partial_{\tilde{\Omega}} h_{AB}\partial_{\tilde{\Omega}} h_{CD} h^{AC} h^{BD}, \tag{5.4}$$

$$\partial_{\tilde{\Omega}}(e^{-2\beta} h_{AB}\partial_{\tilde{\Omega}} U^B) - \frac{2}{1-s}\tilde{\Omega}^{-1} e^{-2\beta} h_{AB}\partial_{\tilde{\Omega}} U^B = -16\pi T_{\tilde{\Omega}A} - \frac{4}{1-s}\tilde{\Omega}^{-1}\eth_A\beta - 2\partial_{\tilde{\Omega}}\eth_A\beta + \eth^B(\partial_{\tilde{\Omega}} h_{AB}), \tag{5.5}$$

which can be integrated to obtain β and U^A. Contracting Eq. (2.7) with $\tilde{g}^{\mu\nu}$ and using that $\tilde{g}^{\mu\nu}T_{\mu\nu} = \mathscr{T} + \mathscr{O}(\tilde{\Omega})$ reveals that $n^\lambda \tau_\lambda = \mathscr{O}(\tilde{\Omega})$ which means that $\tau_u = 0$. We can therefore write $\tau_\mu = \tau n_\mu + \tau_A \nabla_\mu x^A$. For the remainder of this section angular indices are raised and lowered with q_{AB}. We then obtain the leading and subleading order solutions

$$\beta = -\frac{1}{2}s\tau\tilde{\Omega} + \left(-(1-s)\pi T^{(0)}_{\tilde{\Omega}\tilde{\Omega}} - (1-s)\frac{1}{32}C^{AB}C_{AB}\right)\tilde{\Omega}^2 + \mathscr{O}(\tilde{\Omega}^3), \tag{5.6}$$

$$U^A = s\tau^A\tilde{\Omega} + \left(-\frac{1}{2}s^2\tau\tau^A - \frac{1}{2}sC^{AB}\tau_B - \frac{1}{2}s\frac{3-s}{1+s}D^A\tau - \frac{1}{2}\frac{1-s}{1+s}D_B C^{AB} - \frac{1-s}{1+s}8\pi q^{AB} T^{(0)}_{\tilde{\Omega}B}\right)\tilde{\Omega}^2 + U^{(3)A}\tilde{\Omega}^3 + \mathscr{O}(\tilde{\Omega}^4). \tag{5.7}$$

The superscript in brackets $X^{(n)}$ denotes the $\tilde{\Omega}^n$ coefficient of the series expansion of X at infinity. We have eliminated the constants of integration $\beta^{(0)}$ and $U^{(0)A}$ by normalizing $n^\lambda\tilde{\nabla}_\lambda u = 1 + \mathscr{O}(\tilde{\Omega})$ and $n^\lambda\tilde{\nabla}_\lambda x^A = \mathscr{O}(\tilde{\Omega})$ at infinity. Notice that in the asymptotically flat case $s = 0$, the left hand side of Eq. (5.5) can be written as $\tilde{\Omega}^2\partial_{\tilde{\Omega}}(\tilde{\Omega}^{-2}e^{-2\beta}h_{AB}\partial_{\tilde{\Omega}}U^B)$ which implies that the angular momentum aspect — the third order solution of U^A, $U^{(3)A}$ — is not uniquely determined by Einstein's equations. Intuitively, this can be understood through the fact that in asymptotically flat spacetimes, the stress-energy vanishes at infinity and an observer located there could only see the rate at which angular momentum is being radiated, but never the actual amount. In cosmological spacetimes, the situation is quite different. There may be angular momentum in the form $\tilde{r}^2 T^u{}_A = \frac{1}{8\pi}\tilde{r}s\tau_A + \frac{1}{8\pi}s^2\tau\tau_A + T^{(2)}_{\tilde{r}A} + \mathscr{O}(\frac{1}{\tilde{r}})$ at infinity. For a general $0 \le s < 1$, the third order solution of Eq. (5.5) is

$$\begin{aligned} sU^{(3)A} = &-\frac{2}{3}sC^{AB}U^{(2)}_B - \frac{1}{3}s^2 d^{AB}\tau_B - \frac{2}{3}s^2\tau U^{(2)A} - \frac{1}{3}s^3\tau\tau_B C^{AB} + \frac{2}{3}s^2\beta^{(2)}\tau^A + \frac{1}{6}s^3\tau^2\tau^A \\ &+ \frac{1}{8}s(1-s)D^A(C^{BC}C_{BC}) + 8\pi(1-s)q^{AB}T^{(1)}_{\tilde{\Omega}B} - 2\pi(1-s)(2-s)D^A T^{(0)}_{\tilde{\Omega}\tilde{\Omega}}. \end{aligned} \tag{5.8}$$

In the asymptotically flat case $\Omega^{-2}T_{\mu\nu}$ has a smooth limit to infinity, and hence the right side of Eq. (5.8) also vanishes when $s = 0$. It is customary to integrate $G_{u\Omega} = 8\pi T_{u\Omega}$ to obtain V. The authors of [4] chose instead to use the trace of Einstein's equations, $8\pi T + R = 0$. Solving this equation is equivalent to solving $G_{u\Omega} = 8\pi T_{u\Omega}$.

Let us prove this by writing Einstein's equations as $E_{\mu\nu} := G_{\mu\nu} - 8\pi T_{\mu\nu}$. Then the $\tilde{r}$ component of the Bianchi identities is

$$\begin{aligned}
0 = \nabla_\lambda E^\lambda{}_{\tilde{r}} &= \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}E^\lambda{}_{\tilde{r}}) + \frac{1}{2}(\partial_{\tilde{r}} g^{\mu\nu})E_{\mu\nu} \\
&= \frac{1}{\sqrt{-g}}\partial_{\tilde{r}}(\sqrt{-g}g^{u\tilde{r}}E_{u\tilde{r}}) + (\partial_{\tilde{r}} g^{u\tilde{r}})E_{u\tilde{r}} + \frac{1}{2}(\partial_{\tilde{r}} g^{AB})E_{AB} \\
&= (\Omega^2\partial_{\tilde{r}}\Omega^{-2})g^{u\tilde{r}}E_{u\tilde{r}} + g^{u\tilde{r}}\partial_{\tilde{r}}E_{u\tilde{r}} + (\partial_{\tilde{r}} g^{u\tilde{r}})E_{u\tilde{r}} + \frac{1}{2}(\partial_{\tilde{r}}\Omega^2)h^{AB}E_{AB} \\
&= \partial_{\tilde{r}}(g^{u\tilde{r}}E_{u\tilde{r}}),
\end{aligned} \tag{5.9}$$

where in the second line we used that $E_{\tilde{r}\tilde{r}} = E_{\tilde{r}A} = 0$. In the third line, we used the determinant condition $\sqrt{-g} = g_{u\tilde{r}}\Omega^{-2}\sqrt{q}$ together with the fact that $g_{u\tilde{r}}g^{u\tilde{r}} = 1$. Finally, in the fourth line we used that $E = 2g^{u\tilde{r}}E_{u\tilde{r}} + g^{AB}E_{AB} = 0$. Hence, if $E = 0$, then Bianchi's identities guarantee that $E_{u\tilde{r}} = 0$ since $g^{u\tilde{r}}E_{u\tilde{r}} = \mathcal{O}(\Omega^2)$. In terms of the unphysical metric the trace of Einstein's equations becomes

$$8\pi T + \tilde{R} = -\frac{6\tilde{\Omega}^{-1}}{1-s}\tilde{\nabla}^\lambda n_\lambda + \frac{6(2-s)\tilde{\Omega}^{-2}}{(1-s)^2}n^\lambda n_\lambda. \tag{5.10}$$

The unphysical Ricci scalar is quite unwieldy, but to leading order in $\tilde{\Omega}$ it becomes a relatively simple expression. Since $\tilde{R}_{\mu\nu} = \mathcal{O}(1)$ and since the only inverse metric coefficients that are $\mathcal{O}(1)$ are $\tilde{g}^{u\tilde{\Omega}} = 1 + \mathcal{O}(\tilde{\Omega})$ and $\tilde{g}^{AB} = q^{AB} + \mathcal{O}(\tilde{\Omega})$, $\tilde{R} = 2\tilde{R}_{u\tilde{\Omega}} + q^{AB}\tilde{R}_{AB} + \mathcal{O}(\tilde{\Omega})$. A handful of connection coefficients are relevant. To lowest order they are:

$$\tilde{\Gamma}^u{}_{uA} = \frac{1}{2}s\tau_A + \mathcal{O}(\tilde{\Omega}) \qquad \tilde{\Gamma}^u{}_{AB} = -\frac{1}{2}C_{AB} + \mathcal{O}(\tilde{\Omega}) \qquad \tilde{\Gamma}^{\tilde{\Omega}}{}_{\tilde{\Omega}\tilde{\Omega}} = -s\tau + \mathcal{O}(\tilde{\Omega}) \tag{5.11}$$

$$\tilde{\Gamma}^{\tilde{\Omega}}{}_{\tilde{\Omega}A} = -\frac{1}{2}s\tau_A + \mathcal{O}(\tilde{\Omega}) \qquad \tilde{\Gamma}^A{}_{u\tilde{\Omega}} = -\frac{1}{2}s\tau^A + \mathcal{O}(\tilde{\Omega}) \qquad \tilde{\Gamma}^A{}_{\tilde{\Omega}B} = \frac{1}{2}C^A{}_B + \mathcal{O}(\tilde{\Omega}) \tag{5.12}$$

$$\tilde{\Gamma}^{\tilde{\Omega}}{}_{u\tilde{\Omega}} = -\tilde{V}^{(-1)}\tilde{\Omega} + \frac{1}{2}s^2\tau^A\tau_A\tilde{\Omega} + \mathcal{O}(\tilde{\Omega}^2) \qquad \tilde{\Gamma}^{\tilde{\Omega}}{}_{AB} = -\frac{1}{2}N_{AB}\tilde{\Omega} - sD_{(A}\tau_{B)}\tilde{\Omega} + \mathcal{O}(\tilde{\Omega}^2). \tag{5.13}$$

Additionally, $\tilde{\Gamma}^A{}_{BC}$ are to lowest order equal to the connection coefficients of q_{AB}. A complete list of the connection coefficients can be found in Appendix A. The relevant leading order Ricci tensor components are therefore:

$$\tilde{R}_{u\tilde{\Omega}} = -\tilde{V}^{(-1)} + \frac{1}{2}s^2\tau^A\tau_A - \frac{1}{2}sD_A\tau^A + s\partial_u\tau + \mathcal{O}(\tilde{\Omega}) \tag{5.14}$$

$$q^{AB}\tilde{R}_{AB} = 2 - sD_A\tau^A - \frac{1}{2}s^2\tau^A\tau_A + \mathcal{O}(\tilde{\Omega}). \tag{5.15}$$

To first order, Eq. (5.10) and its solution become

$$8\pi\mathcal{T} - 2\tilde{V}^{(-1)} + 2 + 2s\partial_u\tau - 2sD_A\tau^A + \frac{1}{2}s^2\tau^A\tau_A + \mathcal{O}(\tilde{\Omega}) = \frac{6s}{(1-s)^2}\tilde{V}^{(-1)} - \frac{s}{1-s}D_A\tau^A + \mathcal{O}(\tilde{\Omega}) \tag{5.16}$$

$$\implies \tilde{V}^{(-1)} = \frac{(1-s)^3}{2(1-s^3)}\left(8\pi\mathcal{T} + 2 + 2s\partial_u\tau + \frac{2s(2+s)}{1-s}D_A\tau^A + \frac{1}{2}s^2\tau^A\tau_A\right). \tag{5.17}$$

The mass aspect — which is related to the subleading order solution of $\tilde{V}$ as $M = -\frac{1}{2}\tilde{V}^{(0)}$ — like the angular momentum aspect, is not uniquely determined by Einstein's equations for a general $0 < s < 1$. This can be deduced directly from Eq. (5.10) by observing that the components of n_μ and $\tilde{g}_{\mu\nu}$ are independent of s, and that 1, $\frac{1}{1-s}$ and $\frac{2-s}{(1-s)^2}$ are linearly independent. Since the mass aspect coefficient in $[n^\lambda n_\lambda]^{(3)} = \tilde{V}^{(0)} + s\tau\tilde{V}^{(-1)}$ is non-vanishing, the mass aspect coefficient in Eq. (5.10) must be non-vanishing for a general s. Indeed, a direct computation shows that the subleading order of Eq. (5.10) is trivially satisfied if and only if $g_{\mu\nu}$ is asymptotically flat:

$$s^2\tilde V^{(0)} = \frac{4\pi}{3}(1-s)^2\mathscr{T}^{(1)} - s(1-s+s^2)\tau\tilde V^{(-1)} - \frac{4}{3}(1-s)^2\partial_u\beta^{(2)} - \frac{1}{3}s^2(1-s)(7-s)\tau\partial_u\tau - \frac{1}{3}s^2(1-s)(2+s)\tau D_A\tau^A$$
$$+\frac{1}{3}(1-s)(1+2s)D_AU^{(2)A} + \frac{5}{12}s^2(1-s)^2C_{AB}\tau^A\tau^B + \frac{11}{12}s(1-s)^2\tau_AU^{(2)A} + \frac{5}{24}s^3(1-s)^2\tau\tau^A\tau_A$$
$$-\frac{1}{2}s^2(1-s)\tau^AD_A\tau + \frac{1}{3}s(1-s)^2\Delta\tau + \frac{1}{6}s(1-s)^2C^{AB}D_A\tau_B - \frac{1}{12}(1-s)^2C_{AB}N^{AB} + \frac{1}{6}(1-s)^2D_AD_BC^{AB}. \tag{5.18}$$

β, U^A and $\tilde V$ reduce to the known solutions for asymptotically flat spacetimes (2.3) when $s=\tau=\tau_A=0$ and $\Omega^{-2}T_{\mu\nu}$ has a smooth limit to infinity. Their leading order coefficients are the same as those found in [4].

5.2 The evolution equations

We have eliminated four independent components of the metric using four components of Einstein's equations. The final two independent components of the metric are encoded in the two degrees of freedom of h_{AB} whose time evolution can be found by solving the AB components of Einstein's equations. Note that the u-component of Bianchi's identities ensure that $g^{AB}E_{AB}=0$ and hence the trace-free part of E_{AB}, $E_{AB}-\frac{1}{2}g_{AB}g^{CD}E_{CD}$, consists of two independent components, as required. Given that $E=E_{\tilde r A}=E_{\tilde r\tilde r}=E_{AB}-\frac{1}{2}g_{AB}g^{CD}E_{CD}=0$, the remaining components of Bianchi's identities, in terms of the physical metric, reduce to $\partial_{\tilde r}(\Omega^{-4}e^{2\beta}E^{\tilde r}{}_u)=\partial_{\tilde r}(\Omega^{-4}e^{2\beta}E^{\tilde r}{}_A)=0$. Hence, in the asymptotically flat case, $E^{\tilde r}{}_u=E^{\tilde r}{}_A=0$ reduce to $E^{\tilde r(4)}_u=E^{\tilde r(4)}_A=0$ (since $R^{\tilde r}{}_u$, $R^{\tilde r}{}_A$, and β are order $\mathscr{O}(\Omega^2)$) which contain the mass- and angular momentum evolution formulas. In the general case when $0<s<1$, we cannot pick out any non-trivial coefficients, however, and the full set of equations $E^{\tilde\Omega}{}_u=E^{\tilde\Omega}{}_A=0$ are trivially satisfied. The mass- and angular momentum evolution can instead be obtained by differentiating Eqs. (5.8) and (5.18) with respect to u. We will compute this (trivial) set of equations, to make it easier to compare the mass- and angular momentum evolution formulae to those of asymptotically flat spacetimes. The leading and subleading order solutions of $E^{\tilde\Omega}{}_u=0$ ($E^{\tilde\Omega(1)}_u=E^{\tilde\Omega(2)}_u=0$) are

$$\partial_u\tilde V^{(-1)} = \frac{1}{2}s^2(1-s)\tau^A\partial_u\tau_A + s(1-s)D_A\partial_u\tau^A + (1-s)8\pi T^{\tilde\Omega(0)}_u, \tag{5.19}$$

$$\partial_u\tilde V^{(0)} = -2s\partial_u(\tau\tilde V^{(-1)}) + \frac{1}{2}s^2(1+s)\tau^A\tau^BN_{AB} - \frac{1}{2}s(1-s)D_A(\tilde V^{(-1)}\tau^A) + \frac{1}{2}s^3(1-s)\partial_u(\tau\tau_A\tau^A) - \frac{1}{2}s^3(1-s)D_A(\tau^A\tau_B\tau^B)$$
$$+s^2(1-s)(\partial_u\tau)(D_A\tau^A) + (1-s)D_A\partial_uU^{(2)A} - \frac{1}{2}(1-s)\Delta\tilde V^{(-1)} + \frac{1}{2}s^2(1-s)\tau_A\Delta\tau^A + \frac{1}{4}(1-s)N_{AB}N^{AB}$$
$$+\frac{1}{2}s^2(1-s)(D_A\tau^B)(D_B\tau^A) - \frac{1}{2}s^2(1-s)(D_A\tau^B)(D^A\tau_B) + \frac{1}{2}s^2(1-s)\tau^AD_A\partial_u\tau + s^2(1-s)\tilde V^{(-1)}\tau_A\tau^A$$
$$+\frac{1}{2}s^2(1-s)C_{AB}\tau^A\partial_u\tau^B - \frac{1}{2}s(1-s)\tau^AD^BN_{AB} + 3s(1-s)U^{(2)A}\partial_u\tau_A + \frac{5}{2}s^2(1-s)\tau^A\partial_u\tau_A$$
$$+s^2(1-s)\tau D_A\partial_u\tau^A + 8\pi(1-s)T^{\tilde\Omega(1)}_u. \tag{5.20}$$

In the asymptotically flat case, $s=\tau=\tau_A=0$ and $\Omega^{-2}T_{\mu\nu}=\mathscr{O}(1)$. The leading order equation is then trivially satisfied, and the subleading order reduces to

$$\partial_uV^{(0)} = D_A\partial_uU^{(2)A} - \frac{1}{2}\Delta V^{(-1)} + \frac{1}{4}N_{AB}N^{AB}. \tag{5.21}$$

Since in the asymptotically flat case $V=\Omega^{-1}-2M+\mathscr{O}(\Omega)$ and $U^{(2)A}=-\frac{1}{2}D_BC^{AB}$, this is equal to

$$\partial_uM = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_AD_BN^{AB}, \tag{5.22}$$

which is simply Eq. (2.4a), as required. Using the three leading order terms of $E^{\tilde{\Omega}}_{A} = 0$ ($E^{\tilde{\Omega}(0)}_{A} = E^{\tilde{\Omega}(1)}_{A} = E^{\tilde{\Omega}(2)}_{A} = 0$) we find that $\partial_u U_A$ has the following asymptotic solution:

$$
\begin{aligned}
\partial_u U^{(1)}_A &= 16\pi T^{(0)}_{uA}, && (5.23)\\
\partial_u U^{(2)}_A &= -\frac{1}{2}D^B N_{AB} - \frac{1}{2}sD_A\partial_u\tau - \frac{s}{1-s}D_A\tilde{V}^{(-1)} - \frac{1}{2}s\tau_A\tilde{V}^{(-1)} + sD^B D_{[A}\tau_{B]} - s^2\tau\partial_u\tau_A - \frac{1}{2}s^2\tau\tau_A - \frac{1}{4}s\tau^B N_{AB}\\
&\quad - \frac{1}{2}s^3\tau_A\tau^B\tau_B + \frac{1}{4}s\tau_B N^{BC}C_{AC} - \frac{1}{2}s^2 C_{AB}\tau_C D^{[B}\tau^{C]} - \frac{1}{2}s\partial_u(C_{AB}\tau^B) + \frac{1}{2}s^2\tau^B D_{[A}\tau_{B]} + 8\pi T^{\tilde{\Omega}(1)}_{A}, && (5.24)\\
\partial_u U^{(3)}_A &= \frac{16\pi}{3}T^{(2)}_{uA} - \frac{1}{3}(U^{(2)}_A + sC_{AB}\tau^B)(2 - 2\tilde{V}^{(-1)} + 2s\partial_u\tau - 2sD_A\tau^A + \frac{1}{2}s^2\tau^A\tau_A) + \frac{8\pi}{3}\tau_A\mathscr{T}^{(1)} + \frac{2}{3}D_A\partial_u\beta^{(2)} + D_A V^{(0)}\\
&\quad - \frac{1}{3}s\tau_A\left(\frac{6(2s-1)}{(1-s)^2}\tilde{V}^{(-1)} - \frac{6}{1-s}D_B(U^{(2)B} + s^2\tau\tau^B) - \tau\tilde{V}^{(-1)}\right) + s\tau_A\tilde{V}^{(0)} + \frac{2}{3}\tau D_A\tilde{V}^{(-1)} + \frac{1}{6}C_{AB}D^B\tilde{V}^{(-1)}\\
&\quad - \frac{2}{3(1-s)}(D_A\tilde{V}^{(0)} - 5s\tau_A\tilde{V}^{(0)} - sD_A(\tau\tilde{V}^{(-1)}) + (3s^2\tau\tau_A - 2U^{(2)}_A - 2sC_{AB}\tau^B)\tilde{V}^{(-1)} + (s^2\tau\tau^B + U^{(2)B})(N_{AB} - 2sD_{[A}\tau_{B]})\\
&\quad + \frac{1}{2}s\tau_A C^{BC}N_{BC} - 2s\tau^B D_{[A}U^{(2)}_{B]} - 2s^2\tau^B\tau^C D_{[A}C_{B]C} - 2sD_B\tau^B(U^{(2)}_A - sC_{AB}\tau^B) - 2s\tau_A(D_B U^{(2)B} + s^2 D_B(\tau\tau^B)))\\
&\quad + \frac{2(3-2s)}{3(1-s)^2}(s\tau_A\tilde{V}^{(-1)} + (U^{(2)}_A + sC_{AB}\tau^B)\tilde{V}^{(-1)}) + \frac{1}{3}C^C{}_{AB}(N^B{}_C + 2s^2\tau^B\tau_C - sD_C\tau^B + sD^B\tau_C) + \frac{1}{3}sD_B(C^B{}_{AC}\tau^C)\\
&\quad + (sD_A\tau - \frac{2}{3}U^{(2)}_A - \frac{4}{3}sC_{AB}\tau^B - \frac{2}{3}s^2\tau\tau_A - \frac{1}{6}sN_{AB}\tau^B)\tilde{V}^{(-1)} + (\frac{1}{3}C_{AB} - s\tau q_{AB})\partial_u U^{(2)B} - (N_{AB} + \frac{2}{3}s\partial_u\tau q_{AB})U^{(2)B}\\
&\quad - s\partial_u((\frac{1}{6}s^2\tau^2 + \frac{2}{3}\beta^{(2)})\tau_A) + \frac{1}{3}s(2d_{AB} - s\tau C_{AB})\partial_u\tau^B - \frac{1}{2}sC^{BC}N_{BC}\tau_A - \frac{1}{2}s^2\tau\tau^B N_{AB} - \frac{1}{3}s^2\partial_u\tau\tau^B C_{AB}\\
&\quad + s(\frac{8}{3}U^{(2)} + s^2\tau\tau^B + \frac{1}{3}sD^B\tau + sC^{BC}\tau_C)D_{[A}\tau_{B]} + \frac{5}{3}s\tau^B D_{[A}U^{(2)}_{B]} + \frac{1}{3}s^2 D_A(\tau^B D_A\tau) - \frac{1}{3}s^2 D_B(\tau^B\tau_A)\\
&\quad + \frac{1}{3}D_B(C^{BC}N_{AC}) - \frac{1}{6}D_A(C^{BC}N_{BC}) + \frac{2}{3}D^B D_{[A}U^{(2)}_{B]} - \frac{2}{3}sD_B(C^{BC}D_{[A}\tau_{C]}) - \frac{2}{3}s\tau^B D^C D_{[A}C_{B]C} + 2s^2\tau^B\tau^C D_{[A}C_{B]C}\\
&\quad - s^2(\frac{2}{3}sD^B\tau + \frac{1}{6}s^2\tau\tau^B + \frac{7}{6}sC^{BC}\tau_C + \frac{5}{3}U^{(2)B})\tau_A\tau_B + s^2(sC_{AC}\tau^C - \frac{1}{2}s^2\tau\tau_A + \frac{2}{3}U^{(2)}_A)\tau^B\tau_B - \frac{1}{3}s^2 N_{AB}\tau_C D^{[B}\tau^{C]}\\
&\quad + \frac{1}{6}s^2 C_{AB}D^B(\tau^C\tau_C) + s(\frac{2}{3}U^{(2)B} + \frac{2}{3}sC^{BC}\tau_C - \frac{1}{3}s^2\tau\tau^B - \frac{1}{3}D^B\tau)D_{(A}\tau_{B)} - \frac{1}{3}s\tau^B D_{(A}U^{(2)}_{B)} - \frac{1}{3}s^2\tau^B\tau^C D_{(A}C_{B)C}\\
&\quad - \frac{1}{3}s^2 C_{AB}\tau_C D^{[B}\tau^{C]} - \frac{1}{3}sD^B(q_{AC}C^C{}_{BD}\tau^D) - \partial_u(C_{AB}U^{(2)B} + s\tau_A C^{BC}C_{BC}), && (5.25)
\end{aligned}
$$

where we defined $C^C{}_{AB} := \frac{1}{2}q^{CD}(D_A C_{DB} + D_B C_{CD} - D_D C_{AB})$ such that $\eth_A V^C = D_A V^C + C^C{}_{AB}V^B\tilde{\Omega} + \mathscr{O}(\tilde{\Omega}^2)$. In the asymptotically flat case, the first equation is trivially satisfied and the second equation reduces to

$$\partial_u U^{(2)}_A = -\frac{1}{2}D^B N_{AB}, \tag{5.26}$$

which is consistent with Eq. (2.3) and thus this equation does not provide additional constraints on the asymptotically flat metric coefficients.

In the asymptotically flat case, we find the much simpler evolution equation for the angular momentum aspect

$$
\begin{aligned}
\partial_u U^{(3)}_A = &-\frac{2}{3}D_A M + \frac{1}{8}D_A(C^{BC}N_{BC}) + \frac{1}{6}D^B D^C D_{[A}C_{B]C} - \frac{1}{6}D_B(N^{BC}C_{AC})\\
&+ \frac{1}{3}C_{AB}D_C N^{BC} - \frac{1}{3}C_{AB}D_C N^{BC} + \frac{1}{3}N_{AB}D_C C^{BC} - \frac{1}{3}N^{BC}D_B C_{AC}. \qquad (5.27)
\end{aligned}
$$

Unfortunately, this equation does not match any of the existing equations found in the literature [1, 7, 8, 9]. In fact, all four of the expressions obtained in the listed sources contradict each other! In appendix C we therefore include a brief derivation of the asymptotically flat angular momentum aspect evolution equation, and in appendix D we list and compare some of the different expressions found elsewhere.

To illustrate that, in general, none of these equations provide additional constraints let us explicitly compute Eq. (5.23) through the fact that $E^{\tilde{\Omega}}{}_A = 0$ follows from the A components of Bianchi's identities. The A component of the conservation equation is

$$\nabla_\lambda T^\lambda{}_A = \tilde{\nabla}_\lambda T^\lambda{}_A - \frac{2\tilde{\Omega}^{-1}}{1-s} n_\lambda T^\lambda{}_A + \frac{\tilde{\Omega}^{-1}}{2(1-s)} (\delta^\mu{}_\nu n_A + \delta^\mu{}_A n_\nu - \tilde{g}_{\nu A} n^\mu) T^\nu{}_\mu \tag{5.28}$$

$$= \frac{s\tilde{\Omega}^{-1}}{8\pi(1-s)} \partial_u \tau_A - \frac{2\tilde{\Omega}^{-1}}{1-s} T^{(0)}_{uA} + \mathcal{O}(1) = 0 \tag{5.29}$$

$$\implies s\partial_u \tau_A = 16\pi T^{(0)}_{uA}. \tag{5.30}$$

Recall that $U^{(1)}_A = s\tau_A$. Hence, we have rediscovered (5.23), as required.

5.3 The linearized solutions

If $g_{\mu\nu}$ is the perturbed FLRW spacetime of the previous section, we can find $\delta g_{\mu\nu}$ using $\tau = -2u + \frac{1-s}{s} 4\pi\delta T^{(3)}_{\tilde{r}\tilde{r}}$ and $\tau_A = \frac{1-s}{s} 8\pi\delta T^{(1)}_{\tilde{r}A}$ and only keeping terms of order $\mathcal{O}(\delta)$:

$$\delta g_{uu} = \frac{(1-s)^2}{1+s^2} \left(-4\pi q^{AB} \delta T^{(0)}_{AB} + 8\pi\delta T^{(2)}_{u\tilde{r}} + 4\pi(1-s)\partial_u \delta T^{(3)}_{\tilde{r}\tilde{r}} - 8\pi(2+s) D^A \delta T^{(1)}_{\tilde{r}A}\right) + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{5.31a}$$

$$\delta g_{u\tilde{r}} = 4\pi(1-s)\delta T^{(3)}_{\tilde{r}\tilde{r}} \frac{1}{\tilde{r}} + (2\pi(1-s)\delta T^{(4)}_{\tilde{r}\tilde{r}} + 8\pi s(1-s) u\delta T^{(3)}_{\tilde{r}\tilde{r}}) \frac{1}{\tilde{r}^2} + \mathcal{O}\left(\frac{1}{\tilde{r}^3}\right), \tag{5.31b}$$

$$\delta g_{uA} = 8\pi(1-s)\delta T^{(1)}_{\tilde{r}A} \tilde{r} + 8\pi s(1-s) u\delta T^{(1)}_{\tilde{r}A} + 2\pi \frac{3-s}{1+s}(1-s) D_A \delta T^{(3)}_{\tilde{r}\tilde{r}} + \frac{1}{2}\frac{1-s}{1+s} D^B C_{AB} + 8\pi \frac{1-s}{1+s} \delta T^{(2)}_{\tilde{r}A} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right), \tag{5.31c}$$

$$\delta g_{AB} = C_{AB} \tilde{r} + \mathcal{O}\left(\frac{1}{\tilde{r}}\right). \tag{5.31d}$$

Note that we switched coordinates from $\tilde{\Omega}$ to $\tilde{r}$, and thus transformed tensor components accordingly (for example: $T^{(2)}_{u\tilde{r}} \frac{1}{\tilde{r}^2} = (\partial_{\tilde{r}} \tilde{\Omega}) T^{(0)}_{u\tilde{\Omega}}$). In the first line we used that $\delta\tilde{V} = -\tilde{r}\delta g_{uu}$ and $\delta\mathcal{T} = -\frac{2s}{(1-s)^2} \delta g^{(0)}_{uu} + 32\pi s u\delta T^{(2)}_{u\tilde{r}} + 8\pi q^{AB} T^{(0)}_{AB}$. Clearly, $\delta g_{u\tilde{r}}$ and δg_{uA} are consistent with Eqs. (4.16a), (4.17c) and (4.17d). In order for δg_{uu} to be consistent with Eq. (4.17a), we require

$$8\pi\delta T^{(2)}_{u\tilde{r}} = -\frac{1}{(1-s)^2} \delta g_{uu} + 4\pi(3+s) D^A \delta T^{(1)}_{\tilde{r}A}. \tag{5.32}$$

Indeed, using Eq. (5.14) we find

$$8\pi T_{u\tilde{\Omega}} = -1 - \frac{1}{4} s^2 \frac{3+s}{1-s} \tau^A \tau_A - \frac{1}{2} s \frac{3+s}{1-s} D_A \tau^A + \frac{1}{(1-s)^2} \tilde{V}^{(-1)} + \mathcal{O}(\tilde{\Omega}). \tag{5.33}$$

Upon linearizing, this is equal to Eq. (5.32).

The leading and subleading order of the evolution equation $\partial_u V$ take the following form:

$$\partial_u \delta g^{(0)}_{uu} = 8\pi(1-s)^2 D^A \partial_u \delta T^{(1)}_{\tilde{r}A} - 8\pi(1-s)\delta T^{(0)}_{uu}, \tag{5.34}$$

$$\begin{aligned} \partial_u \delta g^{(1)}_{uu} = {} & 4su\partial_u \delta g^{(0)}_{uu} + \frac{4s-5s^2}{1-s} \delta g^{(0)}_{uu} + 8\pi(1-s)(1-2s)\partial_u \delta T^{(3)}_{\tilde{r}\tilde{r}} - 4\pi(1-s)^2(1+2s) D^A \delta T^{(1)}_{\tilde{r}A} \\ & + (1-s) D^A \partial_u \delta g^{(0)}_{uA} - \frac{1}{2}(1-s)\Delta\delta g^{(0)}_{uu} - 16\pi s(1-s)^2 u D^A \partial_u \delta T^{(1)}_{\tilde{r}A} - 8\pi(1-s)\delta T^{(2)}_{uu} \\ & + 16\pi s(1-s) u\partial_u \delta T^{(1)}_{uu} - 8\pi(1-s)(1-2s)\delta T^{(2)}_{u\tilde{r}}. \end{aligned} \tag{5.35}$$

The 'mass' perturbation (5.35) is equal to the solution (4.17b) we found earlier, which can be seen by rewriting $2su\partial_u \delta g^{(0)}_{uu}$ using the first equation, and using Eq. (5.32) to rewrite $\delta T_{u\tilde{r}}$.

5.4 The mass loss formula

Eq. (5.20) looks rather complicated, so let us examine the case in which the background stress-energy tensor is spherically symmetric. After we set $D_A T_{\mu\nu} = 0$ and $T_{\mu A} = 0$, we get

$$
\begin{aligned}
\partial_u \tilde{V}^{(0)} &= -2s\partial_u(\tau\tilde{V}^{(-1)}) + (1-s)D_A\partial_u U^{(2)A} + \frac{1}{4}(1-s)N_{AB}N^{AB} + 8\pi(1-s)T^{\tilde{\Omega}(1)}_{u} \\
&= \frac{1}{4}(1-s)N_{AB}N^{AB} - \frac{1}{2}\frac{(1-s)^2}{1+s}D_A D_B N^{BC} - 16\pi(1-s)\partial_u(\tilde{V}^{(-1)}T^{u(-1)}_{u}) + 8\pi(1-s)T^{\tilde{\Omega}(1)}_{u}.
\end{aligned} \tag{5.36}
$$

In the second line we reordered some terms and used that the only surviving term of $U^{(2)A}$ is $U^{(2)A} = -\frac{1}{2}\frac{1-s}{1+s}D_B C^{AB}$. We rewrote $s\tau = 8\pi(1-s)T^{(-1)}_{u\tilde{\Omega}} = 8\pi(1-s)T^{u(-1)}_{u}$ using Eq. (2.7). If we interpret $m = -\frac{1}{8\pi}\oint \tilde{V}^{(0)}d^2S$ as the mass, similar to the asymptotically flat case, we get the following mass loss formula:

$$
\partial_u m = -\frac{1-s}{32\pi}\oint N^{AB}N_{AB}d^2S + 8\pi(1-s)\partial_u(\tilde{V}^{(-1)}T^{u(-1)}_{u}) - 4\pi(1-s)T^{\tilde{\Omega}(1)}_{u}. \tag{5.37}
$$

We recognize the first term as a gravitational radiation term. To interpret the second term, we first observe that $\tilde{V}$ measures the length of the vectors n^λ. These are null at infinity in the unphysical spacetime, but in the physical spacetime they are spacelike and orthogonal to surfaces of constant $\tilde{r}$. If spacetime is asymptotically flat, $\tilde{V}^{(-1)} = 1$. In general, spacetime will not be empty near infinity, and $\tilde{V}^{(-1)}$ will measure the density of space at infinity due to the non-zero Ricci curvature. Hence, we can make sense of the second term, as being proportional to the rate at which the energy at infinity changes. The last term contains the subleading order contribution to the energy flux through null infinity. The leading order contribution $T^{\tilde{\Omega}(0)}_{u}$ also contributes to the mass evolution through the second term, since $\partial_u\tilde{V}^{(-1)} = 8\pi(1-s)T^{\tilde{\Omega}(0)}_{u}$.

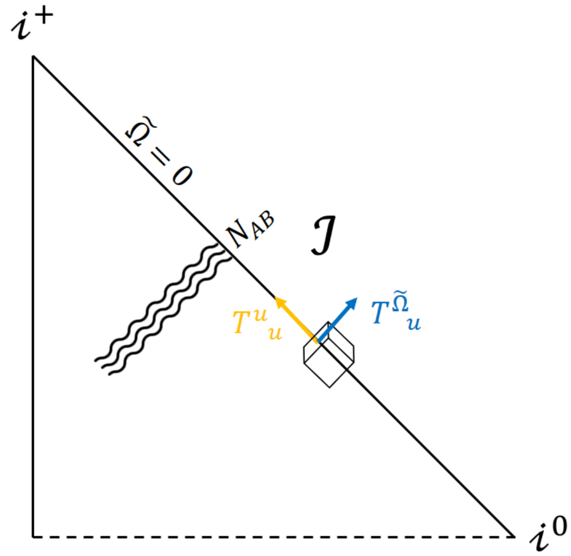

Figure 1: Penrose diagram of a spacetime with a cosmological null asymptote. An observer traveling along null infinity sees an energy density $T^u{}_u$, and an energy flux $T^{\tilde{\Omega}}{}_u$ across the boundary. The presence of Gravitational radiation at null infinity is seen as $N_{AB} \neq 0$.

At the present moment, it remains unclear how we should define the mass of cosmological spacetimes. We will discuss some of the difficulties in the next section.

6 Discussion

The full non-linear field equations were employed to find the asymptotic metric coefficients of spacetimes with a cosmological null asymptote. Along the way we demonstrated that, in the appropriate limit, these metric coefficients reduce to the known asymptotically flat solutions. A second, entirely different check was done to boost our confidence in these new solutions by computing the linear perturbations of decelerating FLRW spacetimes in Bondi-Sachs coordinates, using the techniques used to study linear perturbations of the Schwarzschild spacetime as described in a paper by Martel & Poisson [11].

The generalizations of the mass- and angular momentum evolution equations form the highlight of this thesis. We found that given the geometry and the news tensor at null infinity, the mass- and angular momentum aspects of spacetimes with a cosmological null asymptote are uniquely determined through Einstein's equations. This is different from the asymptotically flat case, where in order to determine the mass- and angular momentum aspects one additionally requires initial data in the form of $M(u_0, x^A)$ and $N_A(u_0, x^A)$ on some hypersurface $u = u_0$.

The identification of the Bondi mass with the mass of an *asymptotically flat spacetime* is supported by several results. Firstly, if a spacetime has a timelike Killing vector field, then this Killing vector field gives rise to a conserved charge, the Komar mass. It can be shown that the Komar mass coincides with the Bondi mass in asymptotically flat spacetimes (see chapter 11.2 of [10]). Notable examples of such spacetimes include the Schwarzschild and Kerr spacetimes. Secondly, the Bondi mass can be expressed as an integral over the 'Coulomb' part of the asymptotic Weyl tensor (see chapter 7 of [5]). Thirdly, the difference between the Bondi masses evaluated at two different cross sections of infinity can be expressed as an integral over null infinity between these cross sections. The integrand then contains precisely the energy flux across null infinity if we relax the condition that $\Omega^{-2} T_{\mu\nu} = \mathcal{O}(1)$ somewhat, alongside a geometric term which we recognize as gravitational radiation (again, see chapter 7 of [5]). Lastly, the Bondi mass is the gravitational Hamiltonian belonging to asymptotic u-translations (see for a pedagogical description chapter 4 of [12]). Unfortunately, none of these arguments generalize in a straightforward manner to spacetimes with a cosmological null asymptote. These spacetimes do not have timelike Killing vector fields, and we do not have exact solutions that, for example, describe a mass in an FLRW spacetime. Neither can we identify a 'Coulomb' part of the Weyl tensor since it does not have the peeling property (see remark 5.1 of [4]). As we saw in §5.5, there is no simple way to interpret the mass loss.

A Hamiltonian description of spacetimes with a cosmological null asymptote might be worth pursuing. A possible starting point for further investigations could be the Einstein-perfect fluid Lagrangian described in §2 of [13].

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A List of Connection Coefficients

The unphysical Bondi-Sachs metric is

$$ds^2 = -We^{2\beta}du^2 + 2e^{2\beta}dud\Omega + h_{AB}(dx^A - U^A du)(dx^B - U^B du), \tag{A.1}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu = 2e^{-2\beta}\partial_u\partial_\Omega + We^{-2\beta}\partial_\Omega^2 + 2e^{-2\beta}U^A\partial_\Omega\partial_A + h^{AB}\partial_A\partial_B. \tag{A.2}$$

We will partially compute its connection coefficients. To aid calculations, we will replace all partial derivatives ∂_A by covariant derivatives $\eth_A$ compatible with the metric h_{AB}, such that the resulting 'Christoffel symbols' are tensorial in A. With this definition of Γ, you can essentially replace all ∂_A's by $\eth_A$'s in all tensor computations, as long as you compensate by adding the appropriate curvature tensors belonging to h_{AB}: wherever you encounter a ∂_A in your computation you will necessarily have to find some combination of Christoffel symbols belonging to h_{AB} such that the resulting expression is tensorial in A, because the line element (A.1) is tensorial in A. After symplifying, we would find that all ∂_A's and Christoffel symbols could be recombined into covariant derivatives and curvature tensors belonging to h_{AB}. The rule can be summarized as follows: replace all ∂_A's by $\eth_A$'s, and replace all R_{ABCD}'s by $R_{ABCD} + \mathscr{R}_{ABCD}$, where R_{ABCD} is computed from the 'Christoffel symbols' listed below, and where $\mathscr{R}_{ABCD}$ is the Riemann tensor belonging to h_{AB}. For example, the simplified expression for R_{AB} becomes: $R_{AB} = \partial_u\Gamma^u{}_{AB} + \partial_\Omega\Gamma^\Omega{}_{AB} + \eth_C\Gamma^C{}_{AB} - \eth_A\Gamma^\lambda{}_{B\lambda} + \Gamma^\lambda{}_{\rho\lambda}\Gamma^\rho_{AB} - \Gamma^\lambda{}_{A\rho}\Gamma^\rho{}_{B\lambda} + \mathscr{R}_{AB}$. We obtain the following results:

$$\Gamma^u{}_{u\Omega} = \Gamma^u{}_{\Omega\Omega} = \Gamma^u{}_{\Omega A} = \Gamma^A{}_{\Omega\Omega} = 0 \tag{A.3a}$$

$$\Gamma^u{}_{uu} = 2\partial_u\beta + \frac{1}{2}e^{-2\beta}\partial_\Omega W - \frac{1}{2}e^{-2\beta}\partial_\Omega(U^A U_A) \tag{A.3b}$$

$$\Gamma^u{}_{uA} = \eth_A\beta + \frac{1}{2}e^{-2\beta}\partial_\Omega U_A \tag{A.3c}$$

$$\Gamma^u{}_{AB} = -\frac{1}{2}e^{-2\beta}\partial_\Omega h_{AB} \tag{A.3d}$$

$$\begin{aligned}\Gamma^\Omega{}_{uu} = &-\frac{1}{2}e^{-2\beta}\partial_u W - \frac{1}{2}e^{-2\beta}U^A U^B\partial_u h_{AB} + 2e^{-2\beta}W\partial_u\beta + \frac{1}{2}e^{-2\beta}W\partial_{\tilde{\Omega}}W - \frac{1}{2}e^{-2\beta}W\partial_{\tilde{\Omega}}(U^A U_A) \\ &+ \frac{1}{2}e^{-2\beta}U^A\eth_A W - \frac{1}{2}e^{-2\beta}U^A\eth_A(U^B U_B)\end{aligned} \tag{A.3e}$$

$$\Gamma^\Omega{}_{u\Omega} = -\frac{1}{2}e^{-2\beta}\partial_\Omega W + \frac{1}{2}e^{-2\beta}U^A\partial_\Omega U_A - U^A\eth_A\beta \tag{A.3f}$$

$$\Gamma^\Omega{}_{uA} = -\frac{1}{2}e^{-2\beta}\eth_A W + W\eth_A\beta + \frac{1}{2}e^{-2\beta}W\partial_\Omega U_A + \frac{1}{2}e^{-2\beta}U^B\partial_u h_{AB} - e^{-2\beta}U^B\eth_{[A}U_{B]} \tag{A.3g}$$

$$\Gamma^\Omega{}_{\Omega\Omega} = 2\partial_\Omega\beta \tag{A.3h}$$

$$\Gamma^\Omega{}_{\Omega A} = \eth_A\beta - \frac{1}{2}e^{-2\beta}h_{AB}\partial_\Omega U^B \tag{A.3i}$$

$$\Gamma^\Omega{}_{AB} = -e^{-2\beta}\eth_{(A}U_{B)} - \frac{1}{2}e^{-2\beta}W\partial_\Omega h_{AB} - \frac{1}{2}e^{-2\beta}\partial_u N_{AB} \tag{A.3j}$$

$$\Gamma^A{}_{uu} = -e^{2\beta}\partial_u(e^{-2\beta}U^A) + \frac{1}{2}e^{-2\beta}U^A\partial_{\tilde{\Omega}}W - \frac{1}{2}e^{-2\beta}U^A\partial_\Omega(U^B U_B) + \frac{1}{2}\eth^A W - \frac{1}{2}\eth^A(U^B U_B) \tag{A.3k}$$

$$\Gamma^A{}_{u\Omega} = -\frac{1}{2}h^{AB}\partial_\Omega U_B - \frac{1}{2}\eth^A e^{2\beta} \tag{A.3l}$$

$$\Gamma^A{}_{uB} = U^A\eth_B\beta + \frac{1}{2}e^{-2\beta}U^A\partial_\Omega U_B + \frac{1}{2}h^{AC}\partial_u h_{BC} - h^{AC}\eth_{[B}U_{C]} \tag{A.3m}$$

$$\Gamma^A{}_{\Omega B} = \frac{1}{2}h^{AC}\partial_\Omega h_{BC} \tag{A.3n}$$

$$\Gamma^A{}_{BC} = -\frac{1}{2}e^{-2\beta}U^A\partial_\Omega h_{BC} \tag{A.3o}$$

B Identities involving h_{AB}

Here, we collect some useful identities involving h_{AB} which are used throughout this thesis. We expand h_{AB} and its inverse as

$$h_{AB} = q_{AB} + C_{AB}\Omega + d_{AB}\Omega^2 + \mathcal{O}(\Omega^3), \tag{B.1}$$
$$h^{AB} = q^{AB} - C^{AB}\Omega - (d^{AB} - C^{AC}C^{B}{}_{C})\Omega^2 + \mathcal{O}(\Omega^3). \tag{B.2}$$

The determinant condition then yields:

$$\det h_{AB} = \det q_{AB}(1 + q^{AB}C_{AB}\Omega + (q^{AB}d_{AB} - \frac{1}{2}C^{AB}C_{AB})\Omega^2) + \mathcal{O}(\Omega^3) := \det q_{AB}, \tag{B.3}$$

i.e.

$$q^{AB}C_{AB} = 0. \tag{B.4}$$
$$q^{AB}d_{AB} = \frac{1}{2}C^{AB}C_{AB}. \tag{B.5}$$

In fact, Mädler & Winicour [8] show that the trace-free part of d_{AB} vanishes, which implies

$$d_{AB} = \frac{1}{4}q_{AB}C^{CD}C_{CD}. \tag{B.6}$$

Furthermore, we have the following two identities:

$$N^{BC}D_AC_{BC} = N^{BC}D_BC_{AC} + N_{AB}D_CC^{BC}, \tag{B.7a}$$
$$C^{BC}D_AN_{BC} = C^{BC}D_BN_{AC} + C_{AB}D_CN^{BC}. \tag{B.7b}$$

The covariant derivative and curvature tensors belonging to the metric h_{AB} are:

$$\eth_A V^B = D_AV^B + \frac{1}{2}V^C(D_AC^{B}{}_{C} + D_CC^{B}{}_{A} - D^BC_{AC})\Omega + \mathcal{O}(\Omega^2), \tag{B.8}$$
$$\mathscr{R}_{ABCD} = q_{AC}q_{BD} - q_{AB}q_{CD} + \mathcal{O}(\Omega), \tag{B.9}$$
$$\mathscr{R}_{AB} = q_{AB} + (D^CD_{(A}C_{B)C} - \frac{1}{2}D^CD_CC_{AB})\Omega + \mathcal{O}(\Omega^2), \tag{B.10}$$
$$\mathscr{R} = 2 + D_AD_BC^{AB}\Omega + \mathcal{O}(\Omega^2). \tag{B.11}$$

C Some curvature tensor components

We are after $R^{(2)}_{uA}$ which contains the angular momentum aspect evolution $\partial_u U^{(3)}_A$. Let us first check for consistency by computing two field equation components containing $R^{(2)}_{uA}$, $G^{(2)}_{uA} = G^{\Omega(2)}{}_{A} = 0$. Note that $G^{(2)}_{uA} = R^{(2)}_{uA}$ since $R = \mathcal{O}(\Omega)$ and $g_{uA} = -h_{AB}U^B = \mathcal{O}(\Omega^2)$. Einstein's equations in terms of the unphysical metric $g_{\mu\nu}$ in Eq. (2.6) are

$$\begin{aligned} R^{(2)}_{uA} &= [2\Omega^{-1}(g_{uA}\nabla^\lambda n_\lambda - \nabla_u n_A) - 3\Omega^{-2}\tilde{g}_{uA}n^\lambda n_\lambda]^{(2)} \\ &= 2D_AM - \frac{1}{2}D^BC_{AB} - \frac{1}{2}N_{AB}D_CC^{BC}, \end{aligned} \tag{C.1}$$

$$\text{and} \quad \begin{aligned} R^{(2)}_{uA} + R^{(0)}_{\Omega A} - \frac{1}{2}D_BC^{BC}R^{(0)}_{AC} &= -[2\Omega^{-1}\nabla_u n^\Omega]^{(2)} \\ &= 2D_AM, \end{aligned} \tag{C.2}$$

using $\nabla_u n_A = -\Gamma^{\Omega}{}_{uA} = -(D_A M - \frac{1}{2}D^B C_{AB} - \frac{1}{4}N_{AB}D_C C^{BC})\Omega^3 + \mathcal{O}(\Omega^4)$, $\nabla^\lambda n_\lambda = 2\Omega + \mathcal{O}(\Omega^2)$ and $n^\lambda n_\lambda = \Omega^2 + \mathcal{O}(\Omega^3)$ in the first equation. Note that

$$R^{(0)}_{AB} = -N_{AB} + q_{AB}, \tag{C.3}$$

$$\text{and recall that} \quad R_{\Omega A} = -\frac{1}{2}\partial_\Omega(e^{-2\beta}h_{AB}\partial_\Omega U^B) - \partial_\Omega \eth_A \beta + \frac{1}{2}\eth^B(\partial_\Omega h_{AB})$$
$$= D^B C_{AB} + \mathcal{O}(\Omega), \tag{C.4}$$

which means that (C.1) and (C.2) are consistent. Let us calculate $R^{(2)}_{uA}$ carefully in steps. The ingredients are:

$$R^{(2)}_{uA} = [\partial_\lambda \Gamma^\lambda{}_{uA}]^{(2)} - [\partial_u \Gamma^\lambda{}_{A\lambda}]^{(2)} + [\Gamma^\lambda{}_{\rho\lambda}\Gamma^\rho{}_{uA}]^{(2)} - [\Gamma^\lambda{}_{u\rho}\Gamma^\rho{}_{A\lambda}]^{(2)}, \tag{C.5a}$$

$$[\partial_\lambda \Gamma^\lambda{}_{uA}]^{(2)} = -\frac{1}{16}D_A(C^{BC}N_{BC}) + \frac{3}{2}\partial_u U^{(3)}_A - \frac{3}{4}C_{AB}D_C C^{BC} + 3D_A M - \frac{3}{2}D^B C_{AB} - \frac{3}{2}N_{AB}D_C C^{BC}$$
$$+ \frac{1}{2}D^B\partial_u d_{AB} - \frac{1}{2}D_B(C^{BC}N_{AC}) + \frac{1}{2}N^{BC}D_B C_{AC} - \frac{1}{4}N^{BC}D_A C_{BC} + \frac{1}{4}D^B D^C D_{[A}C_{B]C}, \tag{C.5b}$$

$$[\partial_u \Gamma^\lambda{}_{A\lambda}]^{(2)} = -\frac{1}{8}D_A(C^{BC}N_{BC}), \tag{C.5c}$$

$$[\Gamma^\lambda{}_{\rho\lambda}\Gamma^\rho{}_{uA}]^{(2)} = 0, \tag{C.5d}$$

$$[\Gamma^\lambda{}_{u\rho}\Gamma^\rho{}_{A\lambda}]^{(2)} = -D^B C_{AB} - \frac{1}{2}N_{AB}D_C C^{BC} - \frac{1}{4}C_{AB}D_C N^{BC}. \tag{C.5e}$$

It should be remarked that the fact that $\eth_A \neq D_A$ (see Eq. (B.8)) contributes a term. $\eth_B \Gamma^B{}_{uA}$ contains the term

$$\frac{1}{2}\eth_B(h^{BC}\partial_u h_{AC}) = \frac{1}{2}D^B N_{AB}\Omega + (\frac{1}{2}D^B\partial_u d_{AB} - \frac{1}{2}D_B(C^{BC}N_{AC}) + \frac{1}{2}N^{BC}D_B C_{AC} - \frac{1}{4}N^{BC}D_A C_{BC})\Omega^2 + \mathcal{O}(\Omega^3). \tag{C.6}$$

Secondly, note that

$$h_{AB}U^B = -\frac{1}{2}D^B C_{AB}\Omega^2 + (U^{(3)}_A - \frac{1}{2}C_{AB}D_C C^{BC})\Omega^3 + \mathcal{O}(\Omega^4) \neq -\frac{1}{2}D^B C_{AB}\Omega^2 + q_{AB}U^{(3)B}\Omega^3 + \mathcal{O}(\Omega^4). \tag{C.7}$$

Finally, after simplifying using a few identities from appendix B, we obtain

$$R^{(2)}_{uA} = \frac{1}{16}D_A(C^{BC}N_{BC}) + \frac{3}{2}\partial_u U^{(3)}_A + 3D_A M - \frac{1}{2}D^B C_{AB} - N_{AB}D_C C^{BC} - \frac{1}{2}C_{AB}D_C N^{BC} + \frac{1}{4}C^{BC}D_A N_{BC}$$
$$- \frac{1}{2}D_B(C^{BC}N_{AC}) + \frac{1}{2}N^{BC}D_B C_{AC} + \frac{1}{4}D^B D^C D_{[A}C_{B]C}. \tag{C.8}$$

The field equation (C.1) (or equivalently (C.2)) becomes:

$$\partial_u U^{(3)}_A = -\frac{2}{3}D_A M + \frac{1}{8}D_A(C^{BC}N_{BC}) + \frac{1}{6}D^B D^C D_{[A}C_{B]C} - \frac{1}{6}D_B(N^{BC}C_{AC}) + \frac{1}{3}N_{AB}D_C C^{BC} - \frac{1}{3}N^{BC}D_B C_{AC}. \tag{C.9}$$

D Comparison of angular momentum evolution equations

We remarked in §5.2 that our expression for the evolution of the angular momentum aspect (Eq. (5.27)), and several others found in the literature, all contradict each other. In this appendix, we will compare these different expressions.

Compère et al. [7] define

$$\bar{N}_A = -\frac{3}{2}U_A^{(3)} + \frac{3}{32}D_A(C_{BC}C^{BC}). \tag{D.1}$$

We then find its evolution:

$$\begin{aligned}\partial_u \bar{N}_A = D_A M - \frac{1}{4}D^B D^C D_{[A}C_{B]C} + \frac{1}{4}D_B(N^{BC}C_{AC}) - \frac{1}{2}C_{AB}D_C N^{BC} \\ + \left[\frac{1}{4}C_{AB}D_C N^{BC} - \frac{3}{4}N_{AB}D_C C^{BC} + \frac{1}{2}N^{BC}D_B C_{AC}\right].\end{aligned} \tag{D.2}$$

Their result is similar to ours, except they do not find the last three terms in the square brackets.

Mädler & Winicour [8] define

$$-3L_A = -\frac{3}{2}U_A^{(3)} + \frac{1}{2}C_{AB}D_C C^{BC}. \tag{D.3}$$

Their News tensor also differs from ours by a factor. They define $N_{AB} := \frac{1}{2}\partial_u C_{AB}$. The evolution equation is then (sticking to our own convention, $N_{AB} := \partial_u C_{AB}$)

$$\begin{aligned}-3\partial_u L_A = D_A M - \frac{1}{4}D^B D^C D_{[A}C_{B]C} + \frac{1}{16}D_A(C^{BC}N_{BC}) - \frac{1}{2}D_B(C^{BC}N_{AC}) + \frac{1}{4}C^{BC}D_A N_{BC} \\ + \left[\frac{1}{2}N^{BC}D_B C_{AC}\right].\end{aligned} \tag{D.4}$$

Their result is almost consistent with ours, except they do not find the last term in square brackets. Comparing the expressions in [7] and [8], we note that L_A is related to $\bar{N}_A$ through

$$-3L_A = \bar{N}_A - \frac{3}{32}D_A(C^{BC}C_{BC}) + \frac{1}{2}C_{AB}D_C C^{BC}. \tag{D.5}$$

Their evolution equations also contradict each other:

$$\begin{aligned}&-3\partial_u L_A - \partial_u \bar{N}_A + \frac{3}{16}D_A(C^{BC}N_{BC}) - \frac{1}{2}\partial_u(C_{AB}D_C C^{BC}) \\ &= \frac{1}{2}N^{BC}D_B C_{AC} - \frac{1}{2}C^{BC}D_B N_{AC} - \frac{1}{4}\partial_u(C_{AB}D_C C^{BC}) \neq 0.\end{aligned} \tag{D.6}$$

Finally, Bonga & Poisson [9] simply define

$$B^A = -U^{(3)A}. \tag{D.7}$$

We find its evolution:

$$\begin{aligned}\partial_u B^A = D_B\left(\left(\frac{2}{3}M - \frac{1}{16}\partial_u(C^{CD}C_{CD})\right)q^{AB} - \frac{1}{6}D_C D^{[A}C^{B]C}\right) - \frac{1}{2}N^A{}_B D_C C^{BC} + \frac{1}{6}N^B{}_C D_B C^{AC} \\ + \left[\frac{1}{6}\partial_u(C^A{}_B D_C C^{BC}) + \frac{1}{3}N^B{}_C D_B C^{AC}\right].\end{aligned} \tag{D.8}$$

They find this expression except for the terms in square brackets. Their expression also contradicts those of Compère et al. [7] and Mädler & Winicour [8], since

$$\begin{aligned}
&\partial_u B_A - \frac{2}{3}\partial_u \bar{N}_A - \frac{1}{16} D_A \partial_u (C^{BC} C_{BC}) \\
&= -\frac{1}{2} N_{BC} D_A C^{BC} - \frac{1}{2} D_B (C^{BC} N_{AC}) \neq 0, \qquad \text{(D.9)}
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad &\partial_u B_A + 2\partial_u L_A + \frac{1}{3}\partial_u (C_{AB} D_C C^{BC}) \\
&= \frac{1}{3} C_{AB} D_C N^{BC} \neq 0. \qquad \text{(D.10)}
\end{aligned}$$

Hence, our expression for the evolution of the angular momentum aspect in asymptotically flat spacetimes, and several other expressions found in the literature [7, 8, 9], all contradict each other.