RADBOUD UNIVERSITY NIJMEGEN

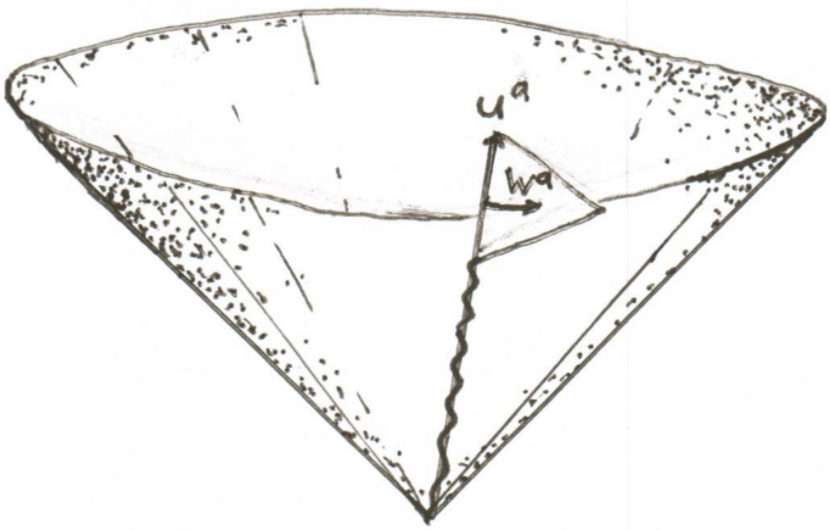
FACULTY OF SCIENCE

Reading the Cosmic Palm: Spinors, the Weyl Tensor Fingerprint, and Mass in General Relativity

THESIS MSC PARTICLE & ASTROPHYSICS

Author:
Berend SCHNEIDER

Supervisors:
dr. Béatrice BONGA
prof. dr. Klaas LANDSMAN



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Abstract

In this thesis, spinors are applied to a wide variety of problems in classical General Relativity. The three main results of this work are divided in three chapters. First, we introduce the two-spinor concept and use it to provide a novel visualisation of the distortion of the light-cone due to Weyl curvature. In the second part we apply spinors to study asymptotically flat space-times and discuss a recently proposed derivation of Einstein's equations using a symmetry argument [Freidel et al., 2021, Freidel & Pranzetti, 2022]. We show that this argument fails and explain in detail why. The final chapter discusses the geometric origin of energy in General Relativity, and we use this to provide a novel argument that the Bondi-mass meaningfully represents the total energy of the space-time at null infinity. Finally, we define Penrose's energy-momentum and angular momentum in detail and provide, to my knowledge, the first non-trivial example of Penrose's angular momentum by computing the mass and spin of the Kerr space-time at null infinity.

Contents

1 Preface

Spinors provide a remarkably efficient and beautiful description of space-time, which in many cases is more appropriate than the conventional tensor approach. Tensor calculus is, however, the formalism of choice for most physicists, in part due to some conceptual difficulties. For these reasons, the first chapter of this work is dedicated to buiding literal pictures of a spin-vector, the Maxwell spinor, and the Weyl spinor. Along the way all the necessary ideas are introduced. Our treatment is far from comprehensive, but no prior knowledge on spinors is required to be able to read this work. The first chapter leads up to a novel technique allowing us to visualise the distortion of the light-cone due to Weyl curvature, which lets us *see* the Weyl spinor in a literal sense.

The second chapter is dedicated to Penrose's conformal treatment of asymptotically flat space-times. These are space-times that represent isolated systems in General Relativity. We may add to these space-times a boundary, whose points represent 'points at infinity' in a rigorous way. Because the boundary is null, spinors will prove to be quite useful when describing the asymptotics and dynamics of fields at infinity. For a large part, this chapter is meant to cover the prerequisites for the final chapter. The final part of this chapter is a response to a recently proposed symmetry argument from which Einstein's equations could supposedly be derived [Freidel et al., 2021, Freidel & Pranzetti, 2022]. I demonstrate that this is not the case, and that part of my perceived confusion stems from the fact that the coordinate-based formalism that was chosen by the authors is ill-suited to the problem.

In the final chapter we discuss the geometric origin of mass in General Relativity. Providing a rigorous quasi-local definition of energy-momentum and angular momentum is an outstanding problem that has remained unsolved for over a hundred years. Here, we will discuss a proposed definition due to Penrose, which is defined in the context of Twistor theory. Twistors, as used in this work, should be seen as a tool used to solve problems in standard physics (much like spinors). We will not discuss some of Twistor theory's more sophisticated or speculative ideas. Penrose's definition, in cases where it is applicable, provides a remarkable notion of mass which is in line with our physical intuition in all space-times where it can be computed. Null infinity is one such special case. Here, Penrose's angular momentum manages to avoid some serious problems that plague more conventional BMS based definitions. This chapter will provide a basic introduction to Twistors, along with detailed explanation of Penrose's Twistorial definition of energy-momentum and angular momentum, with a special focus on null infinity. I have also striven to provide many clarifying remarks, figures, and examples. In particular, the end of this chapter will contain a computation of the mass and spin of the Kerr space-time at null infinity which, surprisingly, appears to be the first explicit example of Penrose's angular momentum that is non-trivial.

2 A short introduction to two-spinors

In modern times, tensors have found widespread application across all disciplines of theoretical physics. Because tensors are a direct representation of geometric quantities without reference to any particular coordinate system, tensors provide an elegant and effective description of the physics. In 3+1-dimensions there exists, by some miraculous coincidence, a more primitive description of geometry in terms of two-spinors. This description is remarkably efficient and is in many cases a considerable simplification over tensors. The two main reasons for this simplification are, in the first place, the fact that two-spinors only have two-components, which is half the amount of four-vectors. An important example of this is the Weyl curvature (which provides a complete description of vacuum curvature in general relativity) which is described by a 256-dimensional trace-free four index tensor C_{abcd} having complicated symmetries, $C_{abcd} = C_{[ab][cd]}$ and $C_{[abc]d} = 0$, reducing the number of independent real components down to ten. By contrast, the Weyl curvature can be represented by a 16-dimensional fully symmetric four index spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$, which has five independent complex components. In the second place, spinors utilize complex numbers which effectively halves the amount of real components one needs to keep track of. For example, the 40 real components of the Christoffel symbol $\Gamma^c{}_{ab}$ needed to describe the covariant derivative (of which only 24 are independent) get compacted down to just 12 complex spin coefficients.

It has been 110 years since Cartan introduced spinors, 95 years since Dirac used them to formulate his equation describing spin 1/2 particles, and 60 years since Penrose introduced spinors to General Relativity [Penrose, 1965]. Despite having been around for quite a long time[1] they are still relatively unknown, and rarely see use in general relativity. One of the reasons for this is that compared to vectors, the geometric interpretation of spinors is not quite as straightforward. At the same time, pure mathematicians, who are generally less discouraged by abstract concepts, prefer tensors in most cases since they work in any dimension.[2] For these reasons, the first chapter is dedicated to developing a literal picture of a spinor, along with all the tools necessary to understand the later chapters. I have tried to strike a balance between rigour and informal discussions, interrupting dry pieces of exposition with intuition, examples, or clarifying remarks.

2.1 The null flag

To start off this chapter, we will first develop a geometric picture of the simplest type of spinor, a *spin-vector*. Spinors are perfectly valid geometric objects in their own right, but average minds like my own are mostly equiped to think visually in terms of vectors and tensors. This is for good reasons, since virtually all objects one deals with in their everyday experience are tensorial in nature. Fortunately, spin-vectors may be understood almost entirely, up to a single sign ambiguity, by a flag.

Let us start by examing the following useful coincidence. Let $\vec{V} = (t, x, y, z)$ be a Minkowski vector. We may represent $\vec{V}$ as a matrix

$$\vec{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix}. \tag{2.1.1}$$

The Minkowski norm is conveniently given by the determinant of this matrix

$$|\vec{V}|^2 = 2 \det \vec{V} = t^2 - x^2 - y^2 - z^2. \tag{2.1.2}$$

Formally, we are using a basis of 2×2 hermitian matrices for Minkowski vectors to translate between the vector and matrix representation.

$$V^{AA'} = V^\mu \sigma_\mu{}^{AA'}, \tag{2.1.3}$$

[1] In fact, spinors pre-date GR by two years.

[2] The concept of a spinor may also be generalised to n dimensions, but the dimension of spin vectors increases exponentially in n, and therefore they are not nearly as useful in higher dimensions.

where, in this particular example, the $\mu = 0$ component of ${\sigma_\mu}^{AA'}$ is the identity matrix, and the $\mu = 1,2,3$ components are the Pauli-matrices.

We can write the Minkowski metric as a tensor[3], acting on the row and column vectors of the matrix $V^{AA'}$:

$$
\begin{aligned}
2\det V^{AA'} &= 2(V^{00'}V^{11'} - V^{01'}V^{10'}) = 2\epsilon_{AB}V^{A0'}V^{B1'} \\
&= \epsilon_{AB}\epsilon_{A'B'}V^{AA'}V^{BB'},
\end{aligned}
\tag{2.1.4}
$$

where $\epsilon_{01} = -\epsilon_{10} = 1$ and $\epsilon_{00} = \epsilon_{11} = 0$.

The row and column vectors of the matrix $V^{AA'}$ belong to a two-dimensional complex vector spaces S and $\bar{S}$, which are called ***spin spaces****. Elements of these vector space are called* ***spin vectors****.*

The reason we need two spin spaces, S and $\bar{S}$, is that in order to be able to speak of *real* vectors $V^{AA'} = \overline{V^{AA'}}$, we need a notion of complex conjugation on S. Suppose $\kappa^A \in S$, then if $\overline{\kappa^A}$ were also an element of S there would be a notion of a real spinor $\kappa^A = \overline{\kappa^A}$. However, the real and imaginary parts of the spinor *should* be on equal footing since a Lorentz transformation may mix the two. For example, a rotation about the z-axis will clearly mix the real and imaginary parts of $x + iy$. For this reason, $\overline{\kappa^A} = \bar{\kappa}^{A'}$ belongs to a *different* vector space, whose elements are indicated by a primed indices A'.

The tensor ϵ_{AB}, being anti-symmetric, does not define a metric on spin space. However, since

$$
V_{AA'}V^{AA'} = g_{ABA'B'}V^{AA'}V^{BB'} = \epsilon_{AB}\epsilon_{A'B'}V^{AA'}V^{BB'}, \tag{2.1.5}
$$

$$
\text{and similarly} \qquad V_{AA'}V^{AA'} = g^{ABA'B'}V_{AA'}V_{BB'} = \epsilon^{AB}\epsilon^{A'B'}V_{AA'}V_{BB'}, \tag{2.1.6}
$$

ϵ_{AB} raises and lowers spinor indices

$$
\epsilon_{AB}\kappa^A := \kappa_B \qquad \text{and} \qquad \epsilon^{AB}\kappa_B := \kappa^A. \tag{2.1.7}
$$

Remark. Of course, since $V_{AA'}V^{AA'} = V^{AA'}V_{AA'}$ we may have also chosen to raise or lower indices with $-\epsilon_{AB}$ or $-\epsilon^{AB}$. In spinor terms, this can be understood as a consequence of the fact that $g_{ABA'B'} = \epsilon_{AB}\epsilon_{A'B'} = (-\epsilon_{AB})(-\epsilon_{A'B'})$.

Remark. Note that, because ϵ_{AB} is anti-symmetric, $\kappa_A\lambda^A \neq \kappa^A\lambda_A$, but instead $\kappa_A\lambda^A = -\kappa^A\lambda_A$. When contracting spinor indices, one should proceed with caution to take into account possible factors of -1.

Let us finally build a geometric picture of a spinor. The simplest kind of vector one can build from a spin-vector is a null vector:

The spinor κ^A corresponds to a unique null vector $u^{AA'} = \kappa^A\overline{\kappa^A} = \kappa^A\bar{\kappa}^{A'}$.

$u^{AA'}$ is *real* since $\overline{u^{AA'}} = \overline{\kappa^A\bar{\kappa}^{A'}} = \bar{\kappa}^{A'}\kappa^A = \kappa^A\bar{\kappa}^{A'} = u^{AA'}$, and *null* since $u_{AA'}u^{AA'} = \kappa_A\bar{\kappa}_{A'}\kappa^A\bar{\kappa}^{A'} = |\kappa_A\kappa^A|^2 = 0$. Put differently, the determinant of a matrix $u^{AA'}$ which is the outer product of two spin-vectors $u^{AA'} = \kappa^A\bar{\kappa}^{A'}$ is always zero.

Conversely, a null vector $u^{AA'}$ can always be expressed as the product of two spinors $u^{AA'} = \kappa^A\lambda^{A'}$, but *not* always of a product of the form $u^{AA'} = \kappa^A\bar{\kappa}^{A'}$. To see this, consider two null vectors $u^{AA'} = \kappa^A\bar{\kappa}^{A'}$ and $v^{AA'} = \lambda^A\bar{\lambda}^{A'}$. Because $u_{AA'}v^{AA'} = \kappa_A\bar{\kappa}_{A'}\lambda^A\bar{\lambda}^{A'} = |\kappa_A\lambda^A|^2 > 0$, $u^{AA'}$ and $v^{AA'}$ are either both future-null or both past-null. Hence, *all* null vectors of the form $\kappa^A\bar{\kappa}^{A'}$ are either future-null *or* past-null. We can use this fact to define future- and past-causal vectors without reference to any particular time orientation vector field:

Definition 1. A vector $t^{AA'}$ is *future-causal* if and only if $t^{AA'}\kappa_A\bar{\kappa}_{A'} \geq 0$ for all κ^A. $t^{AA'}$ is *past-causal* if and only if $t^{AA'}\kappa_A\bar{\kappa}_{A'} \leq 0$ for all κ^A.

[3]Note that this is not a tensor on the space of Minkowski vectors. The determinant is a map that acts multilinearly on the rows and columns of the matrix.

All null vectors $u^{AA'}$ can be written as $u^{AA'} = \pm\kappa^A\bar{\kappa}^{A'}$ for some κ^A, where the sign is positive if $u^{AA'}$ is future-null and negative if $u^{AA'}$ is past-null. Any spinor that is a phase multiple of κ^A determines the same null vector, since $e^{i\varphi}\kappa^A\overline{e^{i\varphi}\kappa^A} = e^{i\varphi}e^{-i\varphi}\kappa^A\overline{\kappa^A} = \kappa^A\bar{\kappa}^{A'}$. Hence,

A null vector $u^{AA'}$ determines a spinor κ^A uniquely up to a phase: $u^{AA'} \leftrightarrow e^{i\varphi}\kappa^A$.

The phase can be given a geometric meaning. Let p_{ab} be a bivector constructed from κ^A as follows:

$$p_{ABA'B'} = \kappa_A\kappa_B\epsilon_{A'B'} + \bar{\kappa}_{A'}\bar{\kappa}_{B'}\epsilon_{AB}. \tag{2.1.8}$$

That the form p_{ab} is a bivector can be seen by writing $\epsilon_{AB} = \kappa_A\lambda_B - \lambda_A\kappa_B$ for some λ_A satisfying $\kappa_A\lambda^A = 1$. We can then write p_{ab} as

$$\begin{aligned} p_{ABA'B'} &= \kappa_A\kappa_B(\bar{\kappa}_{A'}\bar{\lambda}_{B'} - \bar{\lambda}_{A'}\bar{\kappa}_{B'}) + \bar{\kappa}_{A'}\bar{\kappa}_{B'}(\kappa_A\lambda_B - \lambda_A\kappa_B) && (2.1.9)\\ &= \kappa_A\bar{\kappa}_{A'}(\kappa_B\bar{\lambda}_{B'} + \lambda_B\bar{\kappa}_{B'}) - (\kappa_A\bar{\lambda}_{A'} + \lambda_A\bar{\kappa}_{A'})\kappa_B\bar{\kappa}_{B'} = u_{AA'}w_{BB'} - w_{AA'}u_{BB'}, && (2.1.10)\end{aligned}$$

where $w_{AA'} = \kappa_A\bar{\lambda}_{A'} + \lambda_A\bar{\kappa}_{A'}$ is a real (space-like[4]) vector. w_a is determined uniquely up to a multiple of u_a, since $p_{ab} = 2u_{[a}w_{b]} = 2u_{[a}(w_{b]} + ru_{b]})$ (or, put spinorially, since $\kappa_A(\lambda^A + r\kappa^A) = \kappa_A\lambda^A$). The bivector p_{ab} determines a unique null half plane spanned by $ru_a + sw_a$, where $r \in \mathbb{R}$ and $s \in \mathbb{R}_{>0}$. Conversely, the null half plane determines the spinor κ^A up to a sign, since

$$p_{ABA'B'} = \kappa_A\kappa_B\epsilon_{A'B'} + \bar{\kappa}_{A'}\bar{\kappa}_{B'}\epsilon_{AB} = (-\kappa_A)(-\kappa_B)\epsilon_{A'B'} + (-\bar{\kappa}_{A'})(-\bar{\kappa}_{B'})\epsilon_{AB}. \tag{2.1.11}$$

This concludes our geometric reconstruction of a spin-vector.

*A spinor up to a sign $\pm\kappa^A$ corresponds uniquely to a **null flag** $\{u^a, p_{ab}\}$. The **flag pole** of κ^A is given by the null vector $u^{AA'} = \kappa^A\bar{\kappa}^{A'}$, and the **flag** lies in the plane spanned by the vectors u^a and w^a determined by the bivector $p_{ab} = 2u_{[a}w_{b]}$. The flag points in the spatial direction w^a. See figure 1.*

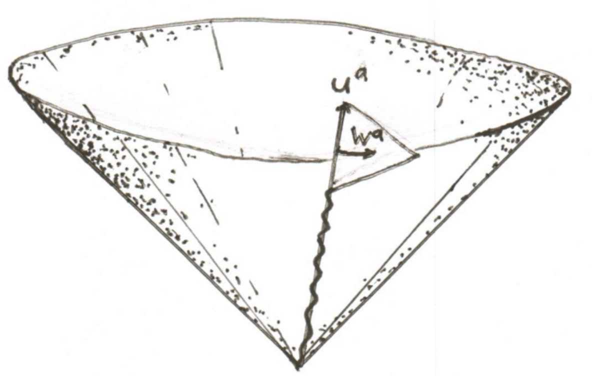


Figure 1: The null flag of a spinor κ^A with flag pole $\kappa^A\bar{\kappa}^{A'} = u^{AA'}$ and flag plane $\kappa_A\kappa_B\epsilon_{A'B'} + \bar{\kappa}_{A'}\bar{\kappa}_{B'}\epsilon_{AB} = 2u_{[a}w_{b]}$.

Remark. We may also asign a 'shrew sense' to the null flag by drawing the flag pole as a corkskrew rotating clockwise when the flag represents a spinor κ^A, and counter clockwise when the flag represents a complex conjugate spinor $\bar{\kappa}^{A'}$, since their flag planes rotate in the opposite direction under a phase change $\kappa^A \mapsto e^{i\varphi}\kappa^A$, $\bar{\kappa}^{A'} \mapsto e^{-i\varphi}\bar{\kappa}^{A'}$.

[4] $w_{AA'}w^{AA'} = (\kappa_A\bar{\lambda}_{A'} + \lambda_A\bar{\kappa}_{A'})(\kappa^A\bar{\lambda}^{A'} + \lambda^A\bar{\kappa}^{A'}) = \kappa_A\bar{\lambda}_{A'}\lambda^A\bar{\kappa}^{A'} + \lambda_A\bar{\kappa}_{A'}\kappa^A\bar{\lambda}^{A'} = -2.$

The famous fact that it takes two full rotations to return a spinor to its original state manifests itself as the fact that one full rotation of the spinor κ^A rotates the flag by 4π about the flagpole, because a rotation $\kappa^A \mapsto e^{i\varphi}\kappa^A$, rotates

$$p_{ABA'B'} \mapsto (e^{i\varphi}\kappa_A)(e^{i\varphi}\kappa_B)\epsilon_{A'B'} + (e^{-i\varphi}\bar{\kappa}_{A'})(e^{-i\varphi}\bar{\kappa}_{B'})\epsilon_{AB} = e^{2i\varphi}\kappa_A\kappa_B\epsilon_{A'B'} + e^{-2i\varphi}\bar{\kappa}_{A'}\bar{\kappa}_{B'}\epsilon_{AB}, \tag{2.1.12}$$

so that a full continuous rotation of the flag through an angle 2π about the flagpole changes the phase of the spinor κ^A by π.

2.2 Spinor algebra

Because a spinor index can only take on two values, any spinor that is simultaneously anti-symmetric in three or more indices must vanish; $\phi_{[ABC]D\ldots Z} = 0$ for any spinor $\phi_{A\ldots Z}$. As a special case of particular interest, consider

$$\epsilon_{[AB}\epsilon_{C]D} = 0, \tag{2.2.1}$$

$$\text{so that} \qquad \epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D = \epsilon_{AB}\epsilon^{CD}. \tag{2.2.2}$$

We can use this identity (2.2.2) to decompose any spinor into products of fully symmetric spinors and ϵ_{AB}. As a concrete example, consider an arbitrary two-index spinor ϕ_{AB}, which can be written as

$$\begin{aligned}\phi_{AB} &= \phi_{(AB)} + \phi_{[AB]} \\ &= \phi_{(AB)} + \tfrac{1}{2}(\epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D)\phi_{CD} = \phi_{(AB)} + \tfrac{1}{2}\epsilon_{AB}\epsilon^{CD}\phi_{CD} \\ &= \phi_{(AB)} + \tfrac{1}{2}\phi_C{}^C\epsilon_{AB}.\end{aligned} \tag{2.2.3}$$

Of course, this procedure also works with spinors with both unprimed and primed indices. Let us consider another important example by performing this decomposition on a skew tensor $F_{ab} = F_{[ab]}$:

$$\begin{aligned}F_{ABA'B'} &= F_{(AB)A'B'} + \tfrac{1}{2}F_C{}^C{}_{A'B'}\epsilon_{AB} \\ &= F_{(AB)(A'B')} + \tfrac{1}{2}F_{(AB)C'}{}^{C'}\epsilon_{A'B'} + \tfrac{1}{2}F_C{}^C{}_{(A'B')}\epsilon_{AB} + \tfrac{1}{2}F_C{}^C{}_{C'}{}^{C'}\epsilon_{AB}\epsilon_{A'B'}.\end{aligned} \tag{2.2.4}$$

Since $F_{ab} = F_{[ab]}$ is skew,

$$F_{(AB)A'B'} = -F_{(BA)B'A'} = -F_{(AB)B'A'} \qquad \text{so that} \qquad F_{(AB)A'B'} = F_{(AB)[A'B']}, \tag{2.2.5a}$$

$$\text{and similarly} \qquad F_{[AB]A'B'} = -F_{[BA]B'A'} = F_{[AB]B'A'} \qquad \text{so that} \qquad F_{[AB]A'B'} = F_{[AB](A'B')}. \tag{2.2.5b}$$

It follows that $F_{(AB)(A'B')} = 0$ and $F_C{}^C{}_{C'}{}^{C'} = F_{CDC'D'}\epsilon^{CD}\epsilon^{C'D'} = F_{[CD][C'D']}\epsilon^{CD}\epsilon^{C'D'} = 0$, so that

$$F_{ABA'B'} = \phi_{AB}\epsilon_{A'B'} + \psi_{A'B'}\epsilon_{AB}, \tag{2.2.6}$$

where $\phi_{AB} = \phi_{(AB)} = \frac{1}{2}F_{(AB)C'}{}^{C'}$ and $\psi_{A'B'} = \frac{1}{2}F_C{}^C{}_{(A'B')}$. If F_{ab} is a real tensor, $\psi_{A'B'} = \bar{\phi}_{A'B'}$, so that any real two-form is equivalent to a symmetric spinor $\phi_{AB} = \phi_{(AB)}$.

2.2.1 Decomposition of the Riemann tensor

As a final example of the decomposition procedure, let us apply what we have just learned to the Riemann tensor R_{abcd}. Using the anti-symmetry in the first two indices,

$$R_{ABCDA'B'C'D'} = \tfrac{1}{2}R_{(AB)CDE'}{}^{E'}{}_{C'D'}\epsilon_{A'B'} + \tfrac{1}{2}R_E{}^E{}_{CDA'B'C'D'}\epsilon_{AB}. \tag{2.2.7}$$

Using anti-symmetry in the last two indices,

$$R_{ABCDA'B'C'D'} = X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}, \tag{2.2.8}$$

where $X_{ABCD} = X_{(AB)(CD)} := \frac{1}{4}R_{ABCDA'}{}^{A'}{}_{B'}{}^{B'}$ and $\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} := \frac{1}{4}R_{ABC}{}^{C}{}_{C'}{}^{C'}{}_{A'B'}$. Given skew symmetries $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$, the algebraic Bianchi identity $R_{[abc]d} = 0$ is equivalent to the interchange symmetry $R_{abcd} = R_{cdab}$, which implies that $X_{ABCD} = X_{CDAB}$ and $\Phi_{ABA'B'} = \bar{\Phi}_{A'B'AB}$. Hence, Φ_{ab} corresponds to a *real* tensor, which is symmetric and trace-free. The symmetries of X_{ABCD} imply that $X_{CAB}{}^{C} = X_{C[AB]}{}^{C}$, so that $X_{CAB}{}^{C} = 3\Lambda\epsilon_{AB}$ where[5] $\Lambda := \frac{1}{6}X_{AB}{}^{AB}$ is real. Finally, we can isolate the fully symmetric part of X_{ABCD}:

$$\begin{aligned}
X_{ABCD} &= \tfrac{1}{3}(X_{ABCD} + X_{ACDB} + X_{ADBC}) + \tfrac{1}{3}(X_{ABCD} - X_{ACBD}) + \tfrac{1}{3}(X_{ABCD} - X_{ADCB}) \\
&= X_{(ABCD)} + \tfrac{1}{3}X_{AE}{}^{E}{}_{D}\epsilon_{BC} + \tfrac{1}{3}X_{AEC}{}^{E}\epsilon_{BD} \\
&= \Psi_{ABCD} + \Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}),
\end{aligned} \tag{2.2.9}$$

where we defined $\Psi_{ABCD} := X_{(ABCD)}$.

The Ricci tensor $R_{ab} := R_{acb}{}^{c}$ is given by

$$\begin{aligned}
R_{ABA'B'} &= X_{ACB}{}^{C}\epsilon_{A'B'} + 2\Phi_{ABA'A'} + \bar{X}_{A'C'B'}{}^{C'}\epsilon_{AB} \\
&= 6\Lambda\epsilon_{AB}\epsilon_{A'B'} + 2\Phi_{ABA'A'}.
\end{aligned} \tag{2.2.10}$$

It follows that $\Lambda\frac{1}{24}R$ where R is the Ricci scalar. Summarizing, we find that

$$\begin{aligned}
R_{ABCDA'B'C'D'} = {}& \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\
&+ 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}),
\end{aligned} \tag{2.2.11}$$

where we used the ϵ identity (2.2.2) to simplify the Λ part.

2.2.2 Canonical decomposition of fully symmetric spinors

There is a way to further decompose fully symmetric spinors into (one index) spinors, which is made possible by the fact that spinors are two-dimensional objects. This further simplification will later allow us to classify fully symmetric spinors (and, in fact, any spinor although the general scheme is much more complicated).

Theorem 1. *Let $\phi_{A...Z} = \phi_{(A...Z)}$ be a fully symmetric spinor. Then there exist a set $\alpha_A, \beta_B, ..., \lambda_Z$ which is unique up to proportionality, so that $\phi_{A...Z} = \alpha_{(A}\beta_B...\lambda_{Z)}$.*

Definition 2. The decomposition $\phi_{A...Z} = \alpha_{(A}\beta_B...\lambda_{Z)}$ of $\phi_{A...Z}$ is called its *canonical decomposition*. The spinors $\alpha_A, \beta_A, ..., \lambda_A$ are called *principle spinors* of $\phi_{A...Z}$. The flag pole directions of $\alpha_A, \beta_A, ..., \lambda_A$ are called the *principle null directions* of $\phi_{A...Z}$.

Before we can prove theorem 1, we will have to develop a bit of machinery, starting with the following useful lemma:

Lemma 1. *Let $T_{a...z} = T_{(a...z)}$ be a fully symmetric tensor. Then $T_{a...z}$ is determined by the function $T(X^a) = T_{a...z}X^a...X^z$.*

Especially in the final chapter, this lemma will prove to be quite useful since 'diagonal' terms $A_{ab}Z^aZ^b$ of a symmetric bilinear form $A_{ab} = A_{(ab)}$ may in some cases be a lot simpler than 'off-diagonal' terms $A_{ab}X^aY^b$.

Proof. We will first show that $T(X^a) = 0$ for all X^a if and only if $T_{a...z} = 0$. Write $X^a = Y^a + \lambda Z^a$. Then, if $T_{a...z}$ has n indices,

$$T(X^a) = T_{a...z}Y^a...Y^z + n\lambda T_{ab..z}Z^aY^b...Y^z + ... + \lambda^n T_{a...z}Z^a...Z^z. \tag{2.2.12}$$

$T(X^a)$ vanishes if and only if each of the coefficients $T_{a...z}Y^a...Y^z, T_{ab..z}Z^aY^b...Y^z, ..., T_{a...z}Z^a...Z^z$ vanish. We can repeat the same steps on $T_{ab..z}Z^aY^b...Y^z$, writing $Y^a = U^a + \mu V^a$ to find that $T_{abc...z}Z^aU^bV^c...V^z = 0$, etcetera. Hence, $T(X^a) = 0$ for all X^a if and only if $T_{a...z} = 0$. To complete the proof, notice that function $(T - T')(X^a) = 0$ if and only if $T_{a...z} = T'_{a...z}$, so that $T(X^a)$ determines $T_{a...z}$ uniquely. $\square$

[5]The numerical factor is chosen for later convenience.

In the special case of two indices, proposition 1 follows from the more well-known polarization identity, which expresses the value of $T_{ab}Y^aZ^b$ explicitly in terms of the function $T(X^a)$.

Proposition 1. *(The polarization identity.) Let $T_{ab} = T_{(ab)}$ be a symmetric tensor. Then*

$$\begin{aligned} T_{ab}X^aY^b &= \tfrac{1}{4}(T_{ab}(X^a+Y^a)(X^b+Y^b) - T_{ab}(X^a-Y^a)(X^b-Y^b)) \\ &= \tfrac{1}{4}(T(X^a+Y^a) - T(X^a-Y^a)). \end{aligned} \tag{2.2.13}$$

In fact, a generalised polarization identity may be derived, expressing $T_{a...z}Y^a...Z^a$ in terms of the function $T(X^a)$. We will be implicitly using lemma 1 throughout this work. For our immediate purposes, we may use it to finally prove theorem 1:

Proof of theorem 1. Let $\xi^A = \omega^A + z\pi^A$ be an arbitrary spinor, where $\omega_A\pi^A \neq 0$. Then $\phi_{A...Z}\xi^A...\xi^Z$ is a polynomial in z which, by the fundamental theorem of algebra, may be refactored by its roots:

$$\begin{aligned} \phi_{A...Z}\xi^A...\xi^Z &= \phi_{A...Z}\omega^A...\omega^Z + nz\phi_{AB..Z}\pi^A\omega^B...\omega^B + ... + z^n\phi_{A...Z} \\ &= (\alpha_0+\alpha_1 z)(\beta_0+\beta_1 z)...(\lambda_0+\lambda_1 z). \end{aligned} \tag{2.2.14}$$

The factors are unique up to proportionality and re-ordering. Define $\alpha_A = \alpha_0(\pi_A\omega^A)^{-1}\pi_A + \alpha_1(\omega_A\pi^A)^{-1}\omega_A$, so that

$$\alpha_A\xi^A = (\alpha_0(\pi_A\omega^A)^{-1}\pi_A + \alpha_1(\omega_A\pi^A)^{-1}\omega_A)\xi^A = \alpha_0 + \alpha_1 z. \tag{2.2.15}$$

Define $\beta_A, ..., \lambda_A$ in a similar manner. Then $\phi_{A...Z}\xi^A...\xi^Z = \alpha_A\beta_B...\lambda_Z\xi^A\xi^B...\xi^Z$. By proposition 1,

$$\phi_{A...Z} = \alpha_{(A}\beta_B...\lambda_{Z)}. \tag{2.2.16}$$

□

2.2.3 Classification of fully symmetric spinors

Theorem 1 provides a wonderfully simple classification scheme for fully symmetric spinors (and tensors equivalent to fully symmetric spinors). These spinors are defined, up to scale, by their principle null directions (PNDs). The classes in the classification scheme correspond to the pattern of coincidences of these PNDs. At the end of this chapter we will explore this classification scheme applied to the Weyl curvature, which is of particular interest to us, because it will turn out to have a rather satisfying geometric interpretation.

Definition 3. A fully symmetric spinor $\phi_{A...Z}$ is called *algebraically special* if two or more of its principle null directions coincide. A principle null direction ξ^A is called an n-fold principle null direction if $\phi_{A_1...A_nB...Z} = \xi_{(A_1}...\xi_{A_n}\beta_B...\lambda_{Z)}$. $\phi_{A...Z}$ is called *null* if all of its principle null directions coincide.

Notice that ξ^A is an n-fold principle null direction of $\phi_{A...Z}$ iff $\xi^{A_n}\xi^B...\xi^Z\phi_{A_1...A_{n-1}A_nB...Z} = 0$. In particlar, $\phi_{A...Z}$ is null iff $\phi_{A...Z}\xi^Z = 0$.

Example 1. (Classification of the electromagnetic field strength.) Because the Maxwell spinor ϕ_{AB} only has two indices, it is algebraically special iff it is null. We can identify three distinct classes:

Type I $\phi_{AB} = \alpha_{(A}\beta_{B)}$. This is the generic case. ϕ_{AB} is not algebraically special.

Type N $\phi_{AB} = \xi_A\xi_B$. In this case, the field strength tensor is *simple*: $F_{ab} = v_{[a}w_{b]}$ for some null vector v^a and space-like vector w^a, as can be seen from equation (2.1.8) and the discussion thereafter. v^a is tangent to the (unique) principle null direction iff $v^bF_{ab} = 0$.

Type O $\phi_{AB} = 0$.

The naming convention for the classes is as follows: a vanishing spinor $\phi_{A...Z} = 0$ is **type O**. A null spinor $\phi_{A...Z} = \xi_{(A}...\xi_{Z)}$ is **type N** (the **N** stands for **N**ull). If the spinor has a single repeating principle null direction with multiplicity n, its class will be denoted by the roman numeral n, for example $\phi_{A...Z} = \xi_{(A}\xi_B\xi_C\beta_D...\lambda_{Z)}$ where $\beta_A, ..., \lambda_A$ are all distinct null directions is **type III**. Spinors of valence four or more may have multiple distinct repeated principle null directions, for example $\psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$ is **type D** (the **D** stands for **D**ouble).

2.3 What is a spinor, physically?

In the previous section, we have seen that spinors (and tensors) can usefully be decomposed into fully symmetric spinors. If these spinors have an even amount of (combined unprimed and primed) indices, these corresponds to tensors. For example, from a fully symmetric even-indexed spinor having only one type of index (unprimed *or* primed), we can construct the tensor $P_{ab...yz} = \phi_{AB...YZ}\epsilon_{A'B'}...\epsilon_{Y'Z'}$ which is trace-less, and has algebraic 'Bianchi' symmetries: anti-symmetrization over any three indices yields zero; $P_{a...[ijk]...z} = 0$.

I would, at this point, like to stress that spinors are not merely a useful tool only to be used to simplify algebraic manipulations of tensor expressions. Spinors may also be used *instead* of tensors, and interpreted directly without relying on their correspondence to tensors. In some situations, a purely spinorial approach is more appropriate. The purpose of this section is to introduce this more direct geometric interpretation of spinors, as applied to two-index spinors (which includes the electromagnetic spinor, which is an instructive example). In a way, this section is a 'warm-up', so that at the end of this chapter we may apply many of the ideas put forth here to the Weyl spinor Ψ_{ABCD}.

To start with, recall that symmetric spinors $\phi_{AB} = \phi_{(AB)}$ are determined by their fully symmetric contractions $\phi_{AB}\kappa^A\kappa^B$. The spinor κ^A corresponds to a null flag, and conversely a null flag corresponds to two spinors, $\pm\kappa^A$. Notice that $\phi_{AB}\kappa^A\kappa^B = \phi_{AB}(-\kappa^A)(-\kappa^B)$, so that

The null flag of κ^A is sufficient to determine $\phi_{AB}\kappa^A\kappa^B$.

This should not come as a surprise, since the spinor ϕ_{AB} is equivalent to an anti-symmetric tensor $F_{ab} = F_{[ab]}$, which does not need spinorial elements to exist. We may therefore characterise two-index symmetric spinors in the following way:

ϕ_{AB} maps null flags to complex numbers,

$$\phi_{AB} : \{u^a, p_{ab}\} \mapsto \mathbb{C}. \tag{2.3.1}$$

In the special case that $\phi_{AB} = \xi_A\xi_B$ is null, this mapping is linear and ϕ_{AB} may actually be represented by a null flag.

Let us now consider the physical significance of the Maxwell spinor ϕ_{AB}. Consider $\phi_{AB}\kappa^A\kappa^B$, and choose $\kappa^A = o^A$ to be one of our basis spinors. The flag plane of o^A is given by $p_{ab} = l_a(m_b + \bar{m}_b) - (m_a + \bar{m}_a)l_b$. In a Cartesian basis, $l^a = \frac{1}{\sqrt{2}}(t^a - z^a)$, and $p_{ab} = 2\sqrt{2}l_{[a}x_{b]}$ so that the flag pole of o^A points in the z-direction and the flag plane points in the x-direction. From the definition of $\phi_{AB} = \frac{1}{2}F_{ABA'B'}\epsilon^{A'B'}$ we find that $\phi_0 = \phi_{AB}o^Ao^B = \frac{1}{2}F_{ab}l^am^b = \frac{1}{2}(E_x - iE_y - B_y - iB_x)$. Define the electromagnetic field vector as the complex three-vector $\mathbf{E} - i\mathbf{B}$. Then, $|\phi_{AB}o^Ao^B|$ tells us what the electromagnetic field strength is in the spatial direction orthogonal to the flag pole direction l^a. $-\arg\phi_{AB}o^Ao^B$ tells us the angle the electromagnetic field makes with the flag plane p_{ab} in the space-like two-plane perpendicular to the flag pole direction. The basis we chose was arbitrary so we can summarise the important points with the following proposition:

Proposition 2. *Let ϕ_{AB} be the Maxwell spinor, and let κ^A be arbitrary. Let Π be the space-like two-plane perpendicular to the flag pole $\kappa^A\bar{\kappa}^{A'}$. Let $\boldsymbol{C} := \boldsymbol{E} - i\boldsymbol{B}$ be the electromagnetic three-vector tangent to Π, and let $\boldsymbol{x}$ be the flag plane direction tangent to Π. Then $|\phi_{AB}\kappa^A\kappa^B| = \frac{1}{2}|\boldsymbol{C}|$ and $-\arg\phi_{AB}o^Ao^B = \arccos\frac{\boldsymbol{x}\cdot\boldsymbol{E}}{|\boldsymbol{x}||\boldsymbol{E}|}$. See figure 2.*

Proof. To prove the last statement, first perform a boost such that $\mathbf{B} = 0$. Then, $-\arg\phi_{AB}\kappa^A\kappa^B$ gives the angle between the electric vector $\mathbf{E}$ and the flag plane direction $\mathbf{x}$ of κ^A. Lorentz transformations are *conformal* on the celestial sphere (see definition 4), hence this angle is invariant. □

Definition 4. Consider a fully symmetric spinor ϕ_{AB} at some point p. Consider the *celestial sphere* S^+, the space of all future-null directions, at p. Given any null direction κ^A, $\phi_{AB}\kappa^A\kappa^B$ defines a vector[6] tangent to S^+.

[6] The components of this vector are given by the real and imaginary parts of $\phi_{AB}\kappa^A\kappa^B \in \mathbb{C}$.

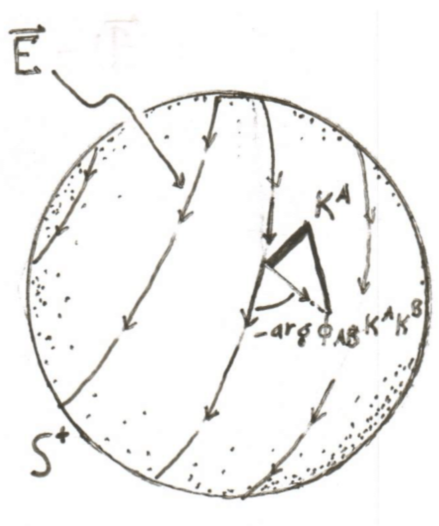


Figure 2: The electromagnetic spinor $\phi_{AB}\kappa^A\kappa^B$ along a null direction κ^A yields the direction of the electric field tangent to the celestial sphere S^+.

The vector field on S^+ that arises as one varies κ^A over the whole of S^+ is called the *fingerprint of* ϕ_{AB}. The fingerprint of the Maxwell spinor yields, at every point, the direction of the electromagnetic field tangent to S^+.

We may similarly describe higher valence fully symmetric spinors using a fingerprint. For example, a four-index spinor $\phi_{ABCD} = \phi_{(ABCD)}$ defines a line field on S^+, since $\phi_{ABCD}\kappa^A\kappa^B\kappa^C\kappa^D = \phi_{ABCD}(-\kappa^A)(-\kappa^B)(-\kappa^C)(-\kappa^D) = \phi_{ABCD}(\pm i\kappa^A)(\pm i\kappa^B)(\pm i\kappa^C)(\pm i\kappa^D)$. The spinors $\pm\kappa^A$ and $\pm i\kappa^A$ have opposite flag planes.

2.4 Covariant differentiation of spinors

The final ingredient we need in order to study geometry using spinors is a spinor covariant derivative. Fortunately, the Levi-Civita connection ∇_a can canonically extended to spinor fields by demanding that ∇_a is compatible with ϵ_{AB}, so that $\nabla_a\epsilon_{BC} = 0$. Intuitively, the fact that ∇_a may easily be extended can be understood from the fact that spinors are, up to a sign, equivalent to ordinary tensorial objects. ∇_a contains information about differential properties of fields, and the sign ambiguity does not cause any trouble.

A covariant, purely spinorial treatment of differentiation and curvature would take us too far afield and is not relevant for our immediate purposes. Instead, we will derive most of the relevant expressions from the corresponding tensorial expressions.

2.4.1 Spinor curvature

The trick to finding the commutator $\Delta_{ab} := \nabla_a\nabla_b - \nabla_b\nabla_a$ as applied to a spinor κ^A, is to consider the bivector $k^{ab} = \kappa^A\kappa^B\epsilon^{A'B'}$. From the ordinary Riemannian definition of the curvature tensor R_{abcd} we find that

$$\Delta_{ab}k^{cd} = R_{abe}{}^{c}k^{ed} + R_{abe}{}^{d}k^{ce}. \tag{2.4.1}$$

Using the Leibniz rule on the left-hand side, and using the curvature spinors (2.2.11) on the right-hand side we find that

$$\begin{aligned}\kappa^D\epsilon^{C'D'}\Delta_{ab}\kappa^C + \kappa^C\epsilon^{C'D'}\Delta_{ab}\kappa^D = {} & X_{ABE}{}^{C}\kappa^D\kappa^E\epsilon_{A'B'}\epsilon^{C'D'} + \Phi_{E}{}^{C}{}_{A'B'}\kappa^D\kappa^E\epsilon_{AB}\epsilon^{C'D'} \\ & + X_{ABE}{}^{D}\kappa^C\kappa^E\epsilon_{A'B'}\epsilon^{C'D'} + \Phi_{E}{}^{D}{}_{A'B'}\kappa^C\kappa^E\epsilon_{AB}\epsilon^{C'D'},\end{aligned} \tag{2.4.2}$$

so that

$$\kappa^{(C}\Delta_{ABA'B'}\kappa^{D)} = (X_{ABE}{}^{(C}\epsilon_{A'B'} + \Phi_E{}^{(C}{}_{A'B'}\epsilon_{AB})\kappa^{D)}\kappa^E. \tag{2.4.3}$$

Note that $\phi^{(A}{}_{C\ldots M}\psi^{B)}{}_{N\ldots Z} = 0$ iff $\phi^A{}_{C\ldots M} = 0$ or $\psi^B{}_{N\ldots Z} = 0$, hence, since κ^A is arbitrary,

$$\Delta_{ABA'B'}\kappa^C = (X_{ABE}{}^C\epsilon_{A'B'} + \Phi_E{}^C{}_{A'B'}\epsilon_{AB})\kappa^E. \tag{2.4.4}$$

Finally, let us decompose Δ_{ab} into

$$\begin{aligned}\Delta_{ABA'B'} &= \epsilon_{A'B'}\nabla_{C'(A}\nabla_{B)}{}^{C'} + \epsilon_{AB}\nabla_{C(A'}\nabla_{B')}{}^{C'}\\ &:= \epsilon_{A'B'}\Box_{AB} + \epsilon_{AB}\Box_{A'B'},\end{aligned} \tag{2.4.5}$$

which combined with our expression for $\Delta_{ab}\kappa^C$ yields

$$\Box_{AB}\kappa^C = X_{ABE}{}^C\kappa^E = \Psi_{ABE}{}^C\kappa^E + 2\Lambda\kappa_{(A}\epsilon_{B)}{}^C, \tag{2.4.6a}$$

$$\Box_{A'B'}\kappa^C = \Phi_E{}^C{}_{A'B'}\kappa^E. \tag{2.4.6b}$$

2.4.2 The spinor form of the Bianchi identities and Einstein's equations

The spinor form of the Bianchi identities may be found in a straightforward manner by substituting (2.2.11) into the tensorial expression $\nabla_{[a}R_{bc]de} = 0$. It is, perhaps, slightly more instructive to re-derive it using the commutators (2.4.6) we just found. Starting from the ϵ identity (2.2.2) we find

$$\begin{aligned}0 &= \epsilon_{AB}\epsilon_C{}^D - \epsilon_{AC}\epsilon_B{}^D + \epsilon_{BC}\epsilon_A{}^D\\ &= 2\epsilon_{A(B}\epsilon_{C)}{}^D - 2\epsilon_{AC}\epsilon_B{}^D + \epsilon_{BC}\epsilon_A{}^D,\end{aligned} \tag{2.4.7}$$

so that

$$\epsilon_{A(B}\epsilon_{C)}{}^D = \epsilon_{AC}\epsilon_B{}^D - \tfrac{1}{2}\epsilon_{BC}\epsilon_A{}^D, \tag{2.4.8}$$

from which it follows (by expanding all the terms using the identity (2.4.8) we just derived) that

$$\epsilon^{A(B}\epsilon_D{}^{C)}\epsilon_{D'}{}^{A'}\epsilon^{B'C'} - \epsilon_D{}^A\epsilon^{BC}\epsilon^{A'(B'}\epsilon_{D'}{}^{C')} - \epsilon^{AB}\epsilon_D{}^C\epsilon_{D'}{}^{(A'}\epsilon^{B')C'} + \epsilon_D{}^{(A}\epsilon^{B)C}\epsilon^{A'B'}\epsilon_{D'}{}^{C'} = 0. \tag{2.4.9}$$

Applying this to $\nabla_{AA'}\nabla_{BB'}\nabla_{CC'}\kappa^E$, we find, after a lengthy but straightforward computation, that

$$(\nabla^B_{D'}X_{BDC}{}^E - \nabla^{B'}_D\Phi_C{}^E{}_{B'D'})\kappa^C = 0, \tag{2.4.10}$$

but since κ^A is arbitrary,

$$\nabla^D_{A'}X_{ABCD} = \nabla^{B'}_A\Phi_{BCA'B'}. \tag{2.4.11}$$

The Bianchi identities in spinor form split naturally into two parts, a part symmetric in BC and skew in BC:

$$\nabla^D_{A'}\Psi_{ABCD} = \nabla^{B'}_{(A}\Phi_{BC)A'B'}, \tag{2.4.12a}$$

$$\nabla^{BB'}\Phi_{ABA'B'} + 3\nabla_{AA'}\Lambda = 0. \tag{2.4.12b}$$

The second of these (2.4.12b) is the spinor form of the contracted Bianchi identities $\nabla_b G^{ab} = 0$ where $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 2\Phi_{ab} - 6\Lambda g_{ab}$ is the Einstein tensor (i.e. the trace-reversed Ricci tensor). Similarly, Einstein's equations split into a trace part and a trace-free part:

$$\Phi_{ab} = 4\pi G(T_{ab} - \tfrac{1}{4}Tg_{ab}), \tag{2.4.13a}$$

$$\Lambda = \tfrac{1}{3}\pi GT + \tfrac{1}{6}\lambda, \tag{2.4.13b}$$

where λ is the cosmological constant. The contracted Bianchi identities imply local conservation of energy $\nabla_b T^{ab} = 0$, while the remaining part (2.4.12a) implies

$$\nabla^D_{A'} \Psi_{ABCD} = 4\pi G \nabla^{B'}_{(A} T_{BC)A'B'}. \tag{2.4.14}$$

In vacuum, the curvature is encoded entirely in a single four-index fully symmetric spinor Ψ_{ABCD}. Comparing the Bianchi identities with Maxwell's equations (in Gaussian units)

$$\nabla^B_{A'} \phi_{AB} = 2\pi j_{AA'}, \tag{2.4.15}$$

we see that Ψ_{ABCD} satisfies a wave equation similar to the electromagnetic field ϕ_{AB}, so that in Einstein's theory of General Relativity, Ψ_{ABCD} plays the role of the *gravitational field*.

2.5 The GHP formalism

Up until this point, we have almost entirely avoided explicit coordinate descriptions. There are distinct advantages and disadvantages to using covariant expressions and coordinates. Unless there exist four natural coordinates, there will be a considerable amount of freedom in choosing coordinates. A covariant approach avoids this problem altogether, and carries immediate geometric significance as is. On the other hand, scalar quantities are much easier to work with. As an added bonus, spinors employ complex numbers, and so a coordinate or frame approach effectively cuts the number of scalar quantities in half.

In this section we will develop the spin-coefficient formalism. This approach assumes a spin frame $\{o^A, \iota^A\}$ spanning spin space in every point. The formalism is most advantageous when a space-time singles out two null directions. In that case, only two gauge degrees of freedom are left, corresponding to the Lorentz transformations that leave these two null directions invariant[7]. A second important use-case for spin coefficients is provided when there may be only one null direction singled out, when these null directions are geodesic. In this case, some of the spin coefficients may be given a precise geometric meaning.

Let us start by introducing the spin frame $\{o^A, \iota^A\}$, satisfying $o_A \iota^A = 1$ so that $\epsilon_{AB} = o_A \iota_B - \iota_A o_B$. The metric can we written in terms of our spin frame as

$$\begin{aligned} g_{ABA'B'} &= \epsilon_{AB}\epsilon_{A'B'} \\ &= (o_A \iota_B - \iota_A o_B)(o_{A'} \iota_{B'} - \iota_{A'} o_{B'}) \\ &= o_A o_{A'} \iota_B \iota_{B'} - o_A \iota_{A'} \iota_B o_{B'} - \iota_A o_{A'} o_B \iota_{B'} + \iota_A \iota_{A'} o_B o_{B'}. \end{aligned} \tag{2.5.1}$$

There is a canonical null-tetrad associated with the spin frame, whose members are given by

$$l^{AA'} = o^A o^{A'}, \tag{2.5.2a}$$

$$m^{AA'} = o^A \iota^{A'}, \tag{2.5.2b}$$

$$\bar{m}^{AA'} = \iota^A o^{A'}, \tag{2.5.2c}$$

$$n^{AA'} = \iota^A \iota^{A'}, \tag{2.5.2d}$$

satisfying $l_a l^a = m_a m^a = \bar{m}_a \bar{m}^a = n_a n^a = 0$ and $l_a n^a = -m_a \bar{m}^a = 1$, and which span the tangent space at any point; $g_{ab} = 2 l_{[a} n_{b]} - 2 m_{[a} \bar{m}_{b]}$. The spin connection is given by sixteen complex numbers

$$\boxed{\begin{array}{cccc} \kappa & \epsilon & \gamma' & \tau' \\ \rho & \alpha & \beta' & \sigma' \\ \sigma & \beta & \alpha' & \rho' \\ \tau & \gamma & \epsilon' & \kappa' \end{array}} = \boxed{\begin{array}{cccc} o^A D o_A & \iota^A D o_A & -o^A D \iota_A & -\iota^A D \iota_A \\ o^A \delta o_A & \iota^A \delta o_A & -o^A \delta \iota_A & -\iota^A \delta \iota_A \\ o^A \delta' o_A & \iota^A \delta' o_A & -o^A \delta' \iota_A & -\iota^A \delta' \iota_A \\ o^A D' o_A & \iota^A D' o_A & -o^A D' \iota_A & -\iota^A D' \iota_A \end{array}} \tag{2.5.3}$$

[7] That there are precisely two can be seen as follows: perform, first, a boost such that the two null directions lie in opposite directions on the celectial sphere (say, the $\pm z$-directions). Then, the Lorentz transformations preserving these null directions are boosts along the null directions (i.e. in the z-direction) and rotations about the null directions (i.e. about the z-axis).

where

$$D = \nabla_{00'} = o^A o^{A'} \nabla_{AA'} = l^a \nabla_a \tag{2.5.4a}$$

$$\delta = \nabla_{01'} = o^A \iota^{A'} \nabla_{AA'} = m^a \nabla_a \tag{2.5.4b}$$

$$\delta' = \nabla_{10'} = \iota^A o^{A'} \nabla_{AA'} = \bar{m}^a \nabla_a \tag{2.5.4c}$$

$$D' = \nabla_{11'} = \iota^A \iota^{A'} \nabla_{AA'} = n^a \nabla_a. \tag{2.5.4d}$$

Note that $\epsilon = \iota^A D o_A = D(\iota^A o_A) - o^A D\iota_A = D(1) + \gamma' = \gamma'$. Similarly, $\alpha = \beta'$, $\beta = \alpha'$, and $\gamma = \epsilon'$ so that only twelve of the spin coefficients are independent. The prime ’ operation changes $o^A \mapsto i\iota^A$ and $\iota^A \mapsto i o^A$. This preserves $o_A \iota^A$. Priming achieves incredible notational economy, since it, again, effectively almost halves the amount of equations to be considered. The other half being obtained by priming the first half.

Remark. All of the spin coefficients can also be expressed in terms of the null tetrad $\{l^a, m^a, \bar{m}^a, n^a\}$. For example, $m^a D l_a = o^A \iota^{A'} D o_A o_{A'} = \cancel{o^A o_A} \iota^{A'} D o_{A'} + \iota^{A'} o_{A'} o^A D o_A = \kappa$. This also constitutes a proof of the assertion we made earlier, that the spin connection may be constructed canonically from the Levi-Civita connection.

2.5.1 The compacted spin coefficient formalism

As mentioned before, the spin coefficient formalism is best used when the space-time has two preferred null directions. The gauge transformations that leave the flag pole directions of o^A and ι^A invariant are rather simple, being given by $o^A \mapsto \lambda o^A$ and $\iota^A \mapsto \lambda^{-1}\iota^A$. We will assign a weight $\{p, q\}$ to any tensor η defined relative to the tetrad, if they transform according to $\eta \mapsto \lambda^p \bar{\lambda}^q \eta$. For example, the Riemann tensor R_{abcd} has weight $\{p, q\} = \{0, 0\}$, since it is independent of our spin frame. The scalar $\Psi_0 := \Psi_{ABCD} o^A o^B o^C o^D$ has weight $\{p, q\} = \{4, 0\}$. The spin coefficients are not tensors, and can therefore not be expected to have a well defined weight. Nevertheless, some of them do. For example,

$$\kappa = o^A D o_A \mapsto \lambda^2 \bar{\lambda} o^A D(\lambda o_A) = \lambda^3 \bar{\lambda} o^A D o_A + \lambda^2 \bar{\lambda} \cancel{o^A o_A} D\lambda \tag{2.5.5}$$

$$= \lambda^3 \bar{\lambda} \kappa, \tag{2.5.6}$$

so that κ has weight $\{p, q\} = \{3, 1\}$. The full list of spin coefficients having a well-defined weight are

$$\kappa \quad \text{with weight} \quad \{3, 1\}, \tag{2.5.7a}$$

$$\sigma \quad \text{with weight} \quad \{3, -1\}, \tag{2.5.7b}$$

$$\rho \quad \text{with weight} \quad \{1, 1\}, \tag{2.5.7c}$$

$$\tau \quad \text{with weight} \quad \{1, -1\}, \tag{2.5.7d}$$

and their primed variants. Priming changes the weights according to $\{p, q\}' = \{-p, -q\}$ and complex conjugating interchanges $\overline{\{p, q\}} = \{q, p\}$. For example, κ' has weight $\{-3, -1\}$ and $\bar{\kappa}$ has weight $\{1, 3\}$.

The remaining spin coefficients do not have a well defined weight. For example,

$$\begin{aligned} \epsilon = \iota^A D o_A \mapsto \bar{\lambda} \iota^A D(\lambda o_A) &= \lambda \bar{\lambda} \iota^A D o_A + \bar{\lambda} \iota^A o_A D\lambda \\ &= \lambda \bar{\lambda} \epsilon + \bar{\lambda} D\lambda. \end{aligned} \tag{2.5.8}$$

Another problem is that ∇_a acting on a weighted quantity does, in general, not produce another weighted quantity. There a nice workaround to both of these problems, however. Notice that the combination $D - p\epsilon - q\bar{\epsilon}$ acting on a weighted quantity always produces another weighted quantity, since, if η has weight $\{p, q\}$

$$\begin{aligned} (D - p\epsilon - q\bar{\epsilon})\eta \mapsto (\lambda\bar{\lambda} D - p\lambda\bar{\lambda}\epsilon - p\bar{\lambda} D\lambda - q\lambda\bar{\lambda}\bar{\epsilon} - q\lambda D\bar{\lambda})\lambda^p \bar{\lambda}^q \eta \\ = \lambda^{p+1}\lambda^{q+1}(D - p\epsilon - q\bar{\epsilon})\eta + (\lambda\bar{\lambda} D(\lambda^p \bar{\lambda}^q) - p\bar{\lambda} D\lambda - q\lambda D\bar{\lambda})\eta \\ = \lambda^{p+1}\lambda^{q+1}(D - p\epsilon - q\bar{\epsilon})\eta. \end{aligned} \tag{2.5.9}$$

We define the weighted operators

$$\text{þ} = D - p\epsilon - q\bar{\epsilon} \qquad \text{with weight} \qquad \{1,1\} \tag{2.5.10a}$$
$$\text{ð} = \delta - p\beta + q\bar{\beta}' \qquad \text{with weight} \qquad \{1,-1\} \tag{2.5.10b}$$
$$\text{ð}' = \delta + p\beta' - q\bar{\beta} \qquad \text{with weight} \qquad \{-1,1\} \tag{2.5.10c}$$
$$\text{þ}' = D' + p\epsilon + q\bar{\epsilon} \qquad \text{with weight} \qquad \{-1,-1\}. \tag{2.5.10d}$$

Using these weighted operators we further cut down the number of spin coefficients to keep track off by only considering the eight weighted spin coefficients! The resulting formalism, first developed by Geroch, Held, and Penrose, is called the *compacted spin coefficient formalism*, or *GHP formalism*.

Finally, let us consider the curvature spinors. We define the curvature scalars

$$\begin{array}{ccc}
\Phi_{00} := \Phi_{ABA'B'}o^A o^B o^{A'} o^{B'} & \Phi_{01} := \Phi_{ABA'B'}o^A o^B o^{A'} \iota^{B'} & \Phi_{02} := \Phi_{ABA'B'}o^A o^B \iota^{A'} \iota^{B'} \\
\Phi_{10} := \Phi_{ABA'B'}o^A \iota^B o^{A'} o^{B'} & \Phi_{11} := \Phi_{ABA'B'}o^A \iota^B o^{A'} \iota^{B'} & \Phi_{12} := \Phi_{ABA'B'}o^A \iota^B \iota^{A'} \iota^{B'} \\
\Phi_{20} := \Phi_{ABA'B'}\iota^A \iota^B o^{A'} o^{B'} & \Phi_{21} := \Phi_{ABA'B'}\iota^A \iota^B o^{A'} \iota^{B'} & \Phi_{22} := \Phi_{ABA'B'}\iota^A \iota^B \iota^{A'} \iota^{B'} \\
\Psi_0 := \Psi_{ABCD}o^A o^B o^C o^D & \Psi_1 := \Psi_{ABCD}o^A o^B o^C \iota^D & \Psi_2 := \Psi_{ABCD}o^A o^B \iota^C \iota^D \\
\Psi_3 := \Psi_{ABCD}o^A \iota^B \iota^C \iota^D & \Psi_4 := \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D &
\end{array} \tag{2.5.11}$$

The curvature scalars have the following weights

$$\Lambda \qquad \text{has weight} \qquad \{0,0\}, \tag{2.5.12a}$$
$$\Phi_{rs} \qquad \text{has weight} \qquad \{2-2r, 2-2s\}, \tag{2.5.12b}$$
$$\Psi_r \qquad \text{has weight} \qquad \{4-2r, 0\}. \tag{2.5.12c}$$

Expressions for the curvature spinors in terms of the spin coefficients can be found from the commutators (2.4.6). For example,

$$\Psi_{ABCD} = \epsilon_{D\mathbf{E}}\Box_{(AB}\epsilon_{C)}{}^{\mathbf{E}}, \tag{2.5.13}$$

where $\epsilon_A{}^{\mathbf{B}} = \{o_A, \iota_A\}$ is the spin basis. Naturally, they may also be found from the tensorial expression

$$2W^a X^b Y^c \nabla_{[a}\nabla_{b]} Z_c = R_{abcd} W^a X^b Y^c Z^d, \tag{2.5.14}$$

by substituting our tetrad vectors for W^a, X^a, Y^a, Z^a. The weighted components of (2.5.14) are

$$\text{þ}\rho - \text{ð}'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa + \Phi_{00} \tag{2.5.15a}$$
$$\text{þ}\sigma - \text{ð}\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0 \tag{2.5.15b}$$
$$\text{þ}\tau - \text{þ}'\kappa = (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Phi_1 + \Phi_{01} \tag{2.5.15c}$$
$$\text{ð}\rho - \text{ð}'\sigma = (\rho - \bar{\rho})\tau + (\rho' - \bar{\rho}')\kappa - \Psi_1 + \Phi_{01} \tag{2.5.15d}$$
$$\text{ð}\tau - \text{þ}'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \Phi_{02} \tag{2.5.15e}$$
$$\text{þ}'\rho - \text{ð}'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 - 2\Lambda. \tag{2.5.15f}$$

Notice, again, the incredible economy of the compacted spin coefficient formalism: only six equations are left from the initial 256 components!

Not all curvature scalars can be computed this way. The remaining scalars show up in the weighted commutators

$$\text{þþ}' - \text{þ}'\text{þ} = (\bar{\tau} - \tau')\text{ð} + (\tau - \bar{\tau}')\text{ð}' - p(\kappa\kappa' - \tau\tau' + \Psi_2 + \Phi_{11} - \Lambda) - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2 + \Phi_{11} - \Lambda) \tag{2.5.16a}$$
$$\text{þð} - \text{ðþ} = \bar{\rho}\text{ð} + \sigma\text{ð}' - \bar{\tau}'\text{þ} - \kappa\text{þ}' + p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar{\sigma}\bar{\kappa} - \bar{\rho}\bar{\tau}' + \Phi_{01}) \tag{2.5.16b}$$
$$\text{ðð}' - \text{ð}'\text{ð} = (\bar{\rho}' - \rho')\text{þ} + (\rho - \bar{\rho})\text{þ}' + p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda) - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2 - \Phi_{11} - \Lambda). \tag{2.5.16c}$$

2.5.2 Space-like two-surfaces

One important situation where two null directions are singled out is on two-dimensional submanifolds. A space-like two surface singles out the null directions orthogonal to it, i.e. the outgoing rays perpendicular to the front and back of the surface at every point. A time-like two-surface singles out the null directions tangent to the surface.

We will concern ourselves here with space-like surfaces $\mathscr{S}$, since they will be heavily involved in the later chapters. Having chosen our spin frame $\{o^A, \iota^A\}$ orthogonal to $\mathscr{S}$, the vectors m^a and $\bar{m}^a$ are tangent to $\mathscr{S}$. Let D_a be the covariant derivative intrinsic to $\mathscr{S}$, and let V^a be a $\{p,q\} = \{0,0\}$ vector field in $\mathscr{S}$. Then since $ðm^a = -\sigma n^a - \bar{\sigma}' l^a$ and $ð\bar{m}^a = -\bar{\rho} n^a - \rho' l^a$, the components of $D_a V^b$ are given by

$$m_a ð V^a, \qquad \bar{m}_a ð V^a, \qquad m_a ð' V^a, \qquad \bar{m}_a ð' V^a. \tag{2.5.17}$$

More generally, the components of $D_a T^{b...z}$ for any non-weighted tensor field intrinsic to $\mathscr{S}$ are given by

$$m_a ... m_z ð T^{a...z}, \qquad m_a ... \bar{m}_z ð T^{a...z}, \qquad ..., \qquad \bar{m}_a ... \bar{m}_z ð' T^{a...z}. \tag{2.5.18}$$

We may commute ð and ð′ with m^a and $\bar{m}^a$ (since $ðm^a$ and $ð\bar{m}^a$ have no components intrinsic to $\mathscr{S}$), and since m^a and $\bar{m}^a$ have weight $p = -q = 1$ we see that

The operator ð acting on weighted quantities with weights $p = -q$ is intrinsic to $\mathscr{S}$.

The reason for this is that $p = -q$ weighted quantities arise as contractions $T_{a...ij...z} m^a ... m^i \bar{m}^j ... \bar{m}^z$. Which are components of $T_{a...z}$ projected onto $\mathscr{S}$.

As a simple consequence, consider the commutator

$$ðð' - ð'ð = (\bar{\rho}' - \rho')þ + (\rho - \bar{\rho})þ' + p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda) - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2 - \Phi_{11} - \Lambda). \tag{2.5.19}$$

The operators þ and þ′ involve derivatives away from $\mathscr{S}$, and so we find that

Proposition 3. *Suppose the spin frame $\{o^A, \iota^A\}$ is orthogonal to a space-like two-surface. Then $\rho = \bar{\rho}$ and $\rho' = \bar{\rho}'$.*

Furthermore,

Proposition 4. *$K + \bar{K}$, where*

$$K := \rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda, \tag{2.5.20}$$

is the Gaussian curvature of $\mathscr{S}$.

This can be proven by writing $2D_{[a}D_{b]}V^c = k(h_{ad}h_b{}^c - h_{bd}S_a{}^c)V^d$ where k is the Gaussian curvature and $h_{ab} = 2m_{[a}m_{b]}$ is the metric instrinsic to $\mathscr{S}$, and using the commutator (2.5.19).

2.5.3 Integration on space-like two-surfaces

Here, I will briefly describe how to integrate on space-like two-surfaces and null hypersurfaces in the spin coefficient formalism. Let us start with space-like two-surfaces. The area form $\mathscr{S}$ is given by

$$\begin{aligned} \mathscr{S} &= X_a dx^a \wedge Y_a dx^a \\ &= \tfrac{1}{\sqrt{2}}(m_a + \bar{m}_a)dx^a \wedge \tfrac{i}{\sqrt{2}}(\bar{m}_a - m_a)dx^a \\ &= i\bar{m}_a m_b dx^a \wedge dx^b. \end{aligned} \tag{2.5.21}$$

Note that we are using the same symbol to denote both the surface $\mathscr{S}$ and its area element. Integrating a form F_{ab} over $\mathscr{S}$ yields

$$\int_{\mathscr{S}} F_{ab} dx^a \wedge dx^b = \int F\mathscr{S}, \tag{2.5.22}$$

where $F := -2iF_{ab}m^a\bar{m}^b$. Here, $\mathscr{S}$ serves a dual purpose: it denotes both the area element and the domain of integration. If F_{ab} is exact, i.e. if it can be written as $F_{ab} = \nabla_{[a}A_{b]}$, we can write F in GHP form as

$$iF = \eth(\bar{m}^a A_a) - \eth'(m^a A_a). \tag{2.5.23}$$

Set $A_a = -A_b\bar{m}^b m_a - A_b m^b \bar{m}_a := Am_a + \tilde{A}\bar{m}_a$ (note that, if A_a is real, $\tilde{A} = \bar{A}$) so that

$$-iF = \eth A - \eth'\tilde{A}. \tag{2.5.24}$$

By Stoke's theorem,

$$\begin{aligned}\int F\mathscr{S} &= i\int(\eth A - \eth'\tilde{A})\mathscr{S} \\ &= \oint A_a\end{aligned} \tag{2.5.25}$$

In particular, if $\mathscr{S}$ is closed,

$$\oint \eth A\mathscr{S} = 0 = \oint \eth'\tilde{A}\mathscr{S}, \tag{2.5.26}$$

from which we can derive the following useful integration by parts formulae:

$$\oint f\eth' g\mathscr{S} = -\oint g\eth' f\mathscr{S} \qquad \oint \bar{f}\eth\bar{g}\mathscr{S} = -\oint \bar{g}\eth\bar{f}\mathscr{S}, \tag{2.5.27}$$

where fg is a $\{p,q\} = \{1,-1\}$ weighted quantity.

On null hypersurfaces $\mathscr{N}$ we again have an obvious null direction, namely the direction n^a generating $\mathscr{N}$. Stoke's theorem on null hypersurfaces in GHP form may be derived in a manner similar to our derivation on two-surface. It is given by

$$\int((þ' - 2\rho')\mu - (\eth - \tau)\nu)\mathscr{N} = \oint \mu\mathscr{S}, \tag{2.5.28}$$

where μ is a $\{0,0\}$ scalar and ν is a $\{-2,0\}$ weighted scalar. $\mathscr{S} = \partial\mathscr{N}$ is assumed to be *space-like* and orthogonal to the spin frame. If $\partial\mathscr{N}$ contains a non-space-like part, (2.5.28) will also contain a term involving ν on the right-hand side.

2.5.4 Spin weighted spherical harmonics

To end this section I will give a very brief summary of some key results involving spin-weighted spherical harmonics. The theory of spin-weighted spherical harmonics can be derived from straightforward spinorial arguments, but a complete discussion is somewhat lengthy.

Theorem 2. *There exists a complete set of* $\{p,q\} = \{s,-s\}$ *weighted functions* ${}_sY_{j,m}$ *on the unit sphere, called spin-weighted spherical harmonics. The label* j *takes on the values* $|s|+n$*, where* $n \in \mathbb{N}_{\geq 0}$*, and* $m \in \{-j,-j+1,\ldots,j\}$*. When* $s=0$*, there harmonics reproduce the ordinary spherical harmonics. The harmonics are orthogonal:*

$$\oint {}_sY_{j,m}\,\overline{{}_sY_{j',m'}}\,\mathscr{S} = \oint (-1)^{m'}\,{}_sY_{j,m}\,{}_{-s}Y_{j',-m'}\,\mathscr{S} := \delta_{j,j'}\delta_{m,m'}. \tag{2.5.29}$$

The operators ð *and* ð′ *map spin* s *harmonics to spin* $s+1$ *and* $s-1$ *harmonics, respectively:*

$$ð\,{}_sY_{j,m} = -\sqrt{\tfrac{1}{2}(j+s+1)(j-s)}\,{}_{s+1}Y_{j,m}, \tag{2.5.30a}$$

$$ð'\,{}_sY_{j,m} = \sqrt{\tfrac{1}{2}(j-s+1)(j+s)}\,{}_{s-1}Y_{j,m}. \tag{2.5.30b}$$

Remark. We will sometimes omit the m label on spin-weighted spherical harmonics. For example, the phrase

"The function f consists of ${}_{-1}Y_1$ harmonics."

Means, more precisely,

"The function f is a linear combination of ${}_{-1}Y_{1,-1}$, ${}_{-1}Y_{1,0}$ and ${}_{-1}Y_{1,1}$ harmonics."

The main application of spherical harmonics is to provide solutions to differential equations on the two-sphere. The theorem listed above, together with the number of spherical harmonics given the values of s and j, can be used derive all important results. Virtually all their properties can be read from table 1:

$j=0$							1						
$\frac{1}{2}$						2		2					
1			$\overleftarrow{ð'}$		3		3		3		$\overrightarrow{ð}$		
$\frac{3}{2}$				4		4		4		4			
2			5		5		5		5		5		
$\frac{5}{2}$		6		6		6		6		6		6	
$\vdots$	$\iddots$												$\ddots$
	$s=\cdots$	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\cdots$

Table 1: Table of the number of spin-weighted spherical harmonics for each value of s *and* j*. If an entry is empty there are no harmonics of this type.*

First, we see that all non-zero spin-weighted spherical harmonics have $|s| \leq j$. Secondly, notice that $ðf = 0$ where f is a spin s weighted function if and only if f consists of $j = s$ harmonics. We may slightly generalise this, to obtain:

Proposition 5. *Let* f *be a spin* s *weighted function satisfying* $ð^n f = 0$*. Then* f *must consist of* $j \leq s+n-1$ *harmonics. In particular, if* f *has negative spin weight, then* $ðf = 0$ *if and only if* $f = 0$*.*

Thirdly, we can use the fact that ð is injective everywhere except when $s = j$, and surjective everywhere except when $-1-s = j$ to derive the following:

Proposition 6. *Let* f *be a spin* s *weighted function satisfying* $ð^n f = g$*. Then* g *satisfies* $ð^r g = 0$ *where* $r \geq n+2s$*.*

From this we can see at a glance how many solutions f equations of the type $ð^n f = g$ have. Let us go over a few examples:

Example 2. In the following, let ${}_s f$ denote a spin s weighted function.

- $ð\, {}_0 f = 0$ if and only if f is proportional to ${}_0Y_0$, which is constant. $ð^2\, {}_0 f = 0$ if and only if f consists of ${}_0Y_0$ and ${}_0Y_1$ harmonics.
- The real part of the operator $ðð'$ working on functions with no spin weight is the ordinary two-sphere Laplacian Δ. A classical result is that a spin 0 function ${}_0 f$ is constant on the sphere if and only if $\Delta\, {}_0 f = 0$. Similarly, ${}_0 f$ is constant if and only if $ð\, {}_0 f = 0 = ð'\, {}_0 f$. In some situations it is easier to compute $ðð'\, {}_0 f$, however. Let us re-derive the classical result using proposition 5. Suppose $ðð'\, {}_0 f = 0$, then, because $ð'\, {}_0 f$ has spin weight -1, $ð'\, {}_0 f = 0$ by proposition 5. Applying proposition 5 a second time, we find that ${}_0 f$ is proportional to ${}_0Y_0$. Hence, $ðð'\, {}_0 f = 0$ if and only if ${}_0 f$ is constant.
- $ð\, {}_1 f = {}_2 g$ has a three dimensional solution. Let ${}_1\tilde{f}$ be a solution, then

  $$f = {}_1\tilde{f} + A\, {}_1Y_{1,-1} + B\, {}_1Y_{1,0} + C\, {}_1Y_{1,1}, \tag{2.5.31}$$

  spans the solution space.
- Given any function ${}_s f$ where s is an integer, there exists a potential ${}_0A$ such that either ${}_s f = ð^s\, {}_0A$ if $s \geq 0$ or ${}_s f = ð'^{-s}\, {}_0A$ if $s \leq 0$.

The following theorem states that $ð$ and $ð'$ may sometimes preserve conformal invariance.

Theorem 3. *Let f be a spin weight s function. Suppose that under a conformal transformation $\mathscr{S} \mapsto \Theta^2\mathscr{S}$, f transforms as $f \mapsto \Theta^w f$. Then the functions $ð^{w-s+1} f$ and $ð'^{w+s+1} f$ are conformal.*

As an example, suppose W is a spin 0, conformal weight $w = -1$ function satisfying $ð^2 W$. Then W consists of ${}_0Y_0$ and ${}_0Y_1$ harmonics. A conformal transformation will mix these harmonics together, but the resulting function will again consist of ${}_0Y_0$ and ${}_0Y_1$ harmonics.

2.6 Null congruences

A *congruence* is a family of curves such that there is exactly one curve that passes through every point in space-time. Here, we will consider *null congruences*, and in particular *geodetic* null congruences, which are congruences whose curves are geodesics. In this section we will see that these congruences are efficiently described by the spin coefficient formalism. When a basis spinor o^A is chosen to be tangent to null geodesics (which we will refer to as *rays*), the spin coefficients σ, ρ and the vanishing of ϵ will attain a geometric meaning.

The vector l^a is *geodetic* (i.e. tangent to a geodesic) if and only if

$$l^a \nabla_a l^b = Dl^b \propto l^b. \tag{2.6.1}$$

The curves are *affinely parametrized* if and only if

$$Dl^a = 0. \tag{2.6.2}$$

We will refer to a spinor as being geodetic if its flag pole is geodetic. Let us see what this means in terms of spin coefficients. From their definition,

$$Do^A = \epsilon o^A - \kappa \iota^A, \tag{2.6.3}$$

so that

$$\begin{aligned} Dl^{AA'} &= (\epsilon o^A - \kappa \iota^A) o^{A'} + o^A(\bar{\epsilon} o^{A'} - \bar{\kappa} \iota^{A'}) \\ &= (\epsilon + \bar{\epsilon}) l^{AA'} - \kappa \iota^A o^{A'} - \bar{\kappa} o^A \iota^{A'}. \end{aligned} \tag{2.6.4}$$

Hence,

$$o^A \textit{ is geodetic} \iff \kappa = 0.$$
$$o^A \textit{ is affinely parametrized} \iff \kappa = 0 = \epsilon + \bar{\epsilon}.$$

Note that ϵ can always locally be set to zero by reparametrizing $o^A \mapsto \lambda o^A$ where $D\lambda = \epsilon\lambda$. $\epsilon = 0$ if the null flag o^A is parallelly propagated. If $\epsilon - \bar{\epsilon} = 0$, ϵ can be made to vanish by re-scaling $o^A \mapsto \lambda o^A$ with a *real* λ, which only affects the flag pole. We can therefore interpret $\epsilon - \bar{\epsilon} = 0$ as the condition for parallelly propagating flag planes.

From this point onward, let us assume o^A is geodetic. Spinorially, we can write this condition as

$$o^A o^B o^{A'} \nabla_{AA'} o_B = 0, \tag{2.6.5}$$

so that

$$o^A o^{A'} \nabla_{AA'} o_B = \epsilon o_B, \tag{2.6.6a}$$
$$o^B o^{A'} \nabla_{AA'} o_B = \rho o_A, \tag{2.6.6b}$$
$$o^A o^B \nabla_{AA'} o_B = \sigma o_{A'}. \tag{2.6.6c}$$

The three complex numbers ϵ, ρ, and σ are fully defined in terms of the congruence o^A. We have already seen that ϵ can be made to vanish by re-scaling λo^A, but since ρ and σ are weighted quantities they cannot.

Remark. If o^A were not geodetic, it *would* be possible to make ρ or σ vanish. In this case, $o^A o^B o^{A'} \nabla_{AA'} o_B \neq 0$ so that we can choose

$$\iota_A \propto o^B o^{A'} \nabla_{AA'} o_B, \tag{2.6.7a}$$
$$\text{or} \qquad \iota_{A'} \propto o^A o^B \nabla_{AA'} o_B, \tag{2.6.7b}$$

to set $\rho = 0$ or $\sigma = 0$.

To interpret ρ and σ geometrically, let the spin frame be parallelly propagated by o^A, so that $Do^A = D\iota^A = 0$, and consider two nearby points separated by a small vector

$$q^a = g l^a + \bar{\zeta} m^a + \zeta \bar{m}^a + h n^a. \tag{2.6.8}$$

Let us drag this vector along the congruence:

$$\mathscr{L}_l q^a = 0 \tag{2.6.9a}$$
$$\text{so that} \qquad D\zeta = -\rho\zeta - \sigma\bar{\zeta} - \tau h, \tag{2.6.9b}$$
$$Dh = 0, \tag{2.6.9c}$$
$$Dg = (\bar{\beta} - \beta')\zeta + (\beta - \bar{\beta}')\bar{\zeta} - (\epsilon' + \bar{\epsilon}')h. \tag{2.6.9d}$$

We will ignore g, since it turns out to not be physically meaningful. We may consider neighbouring rays with $h = 0$, since this condition is preserved along the rays. Such rays are called *abreast*.

To summarise, our setup is as follows: we are considering a congruence of geodetic, affine, abreast rays. We saw in the previous section that if $\rho - \bar{\rho} = 0$, m^a and $\bar{m}^a$ are tangent to a space-like two-surface and the congruence is hypersurface forming. In this case, we may interpret ζ as a coordinate on a small complex plane as it is dragged along l^a. Each value of ζ corresponds to a ray, so that (2.6.9b) tells us how this plane is distorted as it moves along its generators. More generally, if $\rho - \bar{\rho} \neq 0$, ζ represents a vector connecting neighbouring points on the congruence seperated by an infinitesimal distance $|\zeta|$.

Write $\rho = c + it$ and $\sigma = se^{2i\vartheta}$ where $s \geq 0$, then

$$\text{if } \sigma = 0, \qquad D\zeta = -(c + it)\zeta, \tag{2.6.10a}$$
$$\text{if } \rho = 0, \qquad D\zeta = -se^{2i\vartheta}\bar{\zeta}. \tag{2.6.10b}$$

From the first equation we can see that c, representing the real part of ρ measures the convergence of the congruence. t leaves the distance $|\zeta|$ between neighbouring rays invariant but changes the phase of ζ, so that the imaginary part of ρ measures the twist of the congruence. When $\rho = 0$, $se^{2i\vartheta}\bar{\zeta}$ is a positive real multiple of ζ when $\arg\zeta = \vartheta$ or $\arg\zeta = \vartheta + \pi$ and a negative real multiple of ζ when $\arg\zeta = \vartheta \pm \frac{1}{2}\pi$. Hence σ measures the shear of the congruence, where $\frac{1}{2}\arg\sigma$ and $\frac{1}{2}\arg\sigma + \pi$ measures the directions of maximal focusing.

2.7 The fingerprint of the Weyl tensor

To wrap up this chapter, we will apply some of the results and techniques we just found to the Weyl spinor Ψ_{ABCD}. We will find that it is possible to construct a picture of Ψ_{ABCD} on the celestial sphere. This picture will represent, in a rather satisfying way, how an observer *sees* the distortion of his light cone in a small neighbourhood. We saw in §2.3 that the Maxwell spinor ϕ_{AB} may be represented as a vector field on S^+. To construct a similar representation of Ψ_{ABCD} on S^+, we proceed from a slightly different starting point.

Consider a rotation-free geodetic congruence ξ^A, and let ρ and σ be the convergence and shear of ξ^A. Then, by (2.5.15b),

$$D\sigma = 2\rho\sigma + \Psi, \tag{2.7.1}$$

where $\Psi := \Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D$. The effect of Ψ is to induce shear in the congruence. The shear will, in turn, distort a circular arrangement of nearby abreast rays into an ellipse. Represent the axis tangent to S^+ along the direction of maximum focusing by a line segment on S^+. Doing this for every point on S^+, we can define a *line field* on S^+ representing Ψ.

Proposition 7. *Along every null direction ξ^A, $\Psi := \Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D$ is defined by a line segment with magnitude $|\Psi|$ which makes an angle of $\frac{1}{2}\arg\Psi$ and $\frac{1}{2}\arg\Psi + \pi$ with the flag plane of ξ^A.*

Definition 5. The line field on S^+ arising from Ψ_{ABCD} is called *the fingerprint of the Weyl tensor.*

Using what we have learned in §2.6, we can construct a visual representation of the distortion due to the Weyl curvature. Let o^A represent a direction on S^+. To see how the curvature along o^A affects nearby abreast rays ξ^A we differentiate (2.6.9b) again to obtain

$$D^2\zeta = -\Phi\zeta - \Psi\bar{\zeta}, \tag{2.7.2}$$

where $\Phi := \Phi_{ABA'B'}\xi^A\xi^B\xi^{A'}\xi^{B'}$. We will only consider vacuum curvature, so that $\Phi = 0$. To first order in ζ, $\Psi = \Psi_0$ is constant, and $\zeta(u) \approx -\frac{1}{2}u^2\Psi\bar{\zeta}(0) + \mathcal{O}(u^3)$ where u is a parameter along o^A. See figure 3:

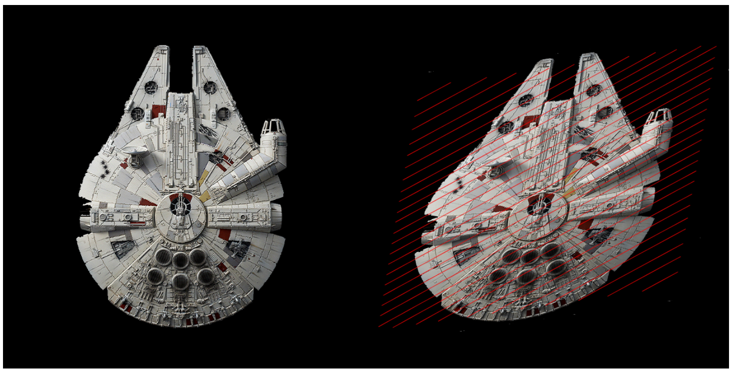


Figure 3: The visible distortion due to the presence of Weyl curvature at leading order in ζ and affine distance along the rays u. The right picture is undistorted. The red lines in the left picture indicate the direction of the fingerprint.

If o^A is a principle null direction of Ψ_{ABCD}, $\Psi_0 = 0$, so we shall need a better approximation. To higher orders in ζ, we need to take into account that $\xi^A = o^A + \mathcal{O}(\zeta)$, so that $\Psi = \Psi_0 + \mathcal{O}(\zeta)$. Notice that we have

$$\begin{aligned} \xi^A \xi^{A'} - o^A o^{A'} &\approx q^{AA'} \\ &= \bar{\zeta} o^A \iota^{A'} + \zeta \iota^A o^{A'}, \end{aligned} \tag{2.7.3}$$

so that $\xi^A = o^A + \zeta\iota^A + \mathcal{O}(\zeta^2)$. We find that if o^A is an n-fold principle null direction,

$$\Psi(\zeta) = \binom{4}{n} \Psi_n \zeta^n + \mathcal{O}(\zeta^{n+1}). \tag{2.7.4}$$

2.7.1 The Petrov-Pirani-Penrose classification

In chapter §2.2.3 we saw that there is a neat way to classify fully symmetric spinors by the multiplicities of their principle null directions. Historically, the first use of part of this classification scheme was due to Petrov and later Pirani, who originally distinguished only three cases[8]. The full classification with spinorial arguments was found later by Penrose. The different classes are

Type I The generic case; no repeated principle null directions.

Type II One two-fold principle null direction $\Psi_{ABCD} = \xi_{(A}\xi_B\alpha_C\beta_{D)}$.

Type D Two repeated principle null directions $\Psi_{ABCD} = \xi_{(A}\xi_B\eta_C\eta_{D)}$.

Type III A triple repeated principle null direction $\Psi_{ABCD} = \xi_{(A}\xi_B\xi_C\eta_{D)}$.

Type N A quadruple repeated principle null direction $\Psi_{ABCD} = \xi_A\xi_B\xi_C\xi_D$.

Type O $\Psi_{ABCD} = 0$.

In figure 3 we saw the generic picture one would see when Weyl curvature is present. In most other direction, we would see an almost identical image, with the only differences being the degree of distortion (determined by $|\Psi|$) and the axis along which the ellipsoidal distortion takes place (determined by $\arg\Psi$). If an observer is looking in a principle null direction, we get a more exciting image, as shown in figure 4 below:

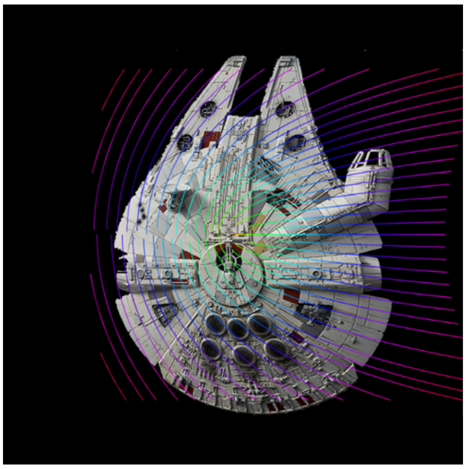


Figure 4: The visible distortion due to the presence of Weyl curvature along a principle null direction. The line field is the fingerprint, and the color represents the magnitude of Ψ, where purple represents smaller $|\Psi|$ than blue.

[8] Petrov's type 1 corresponds to type O, I and D, type 2 corresponds to type II and N, and type 3 corresponds to type III.

The fingerprint of an algebraically special Weyl spinor will have one or more of these simple type I singularities merged. For example, the fingerprint around a two-fold repeated principle null direction will either look like concentric circles, or as a 'sink' type pattern where all surrounding lines terminate at the principle null direction. This type of principle null direction is present in type II or D Weyl spinors. See figure 5 below:

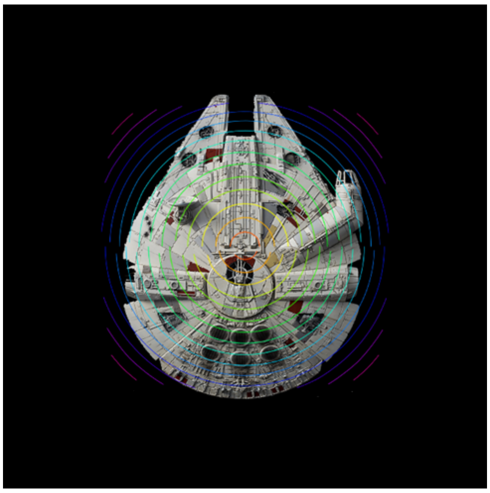


Figure 5: The visible distortion due to the presence of type II or D Weyl curvature along a repeated principle null direction. The line field is the fingerprint.

Type D curvature is associated with point masses. The Kerr-Newman space-time, which is a family of stationary black hole space-times, is type D everywhere. Stationary isolated systems (which we will define in the next chapter) are approximately type D at large distances. The two remaining types of principle null directions are type III and type N, shown in figure 6:

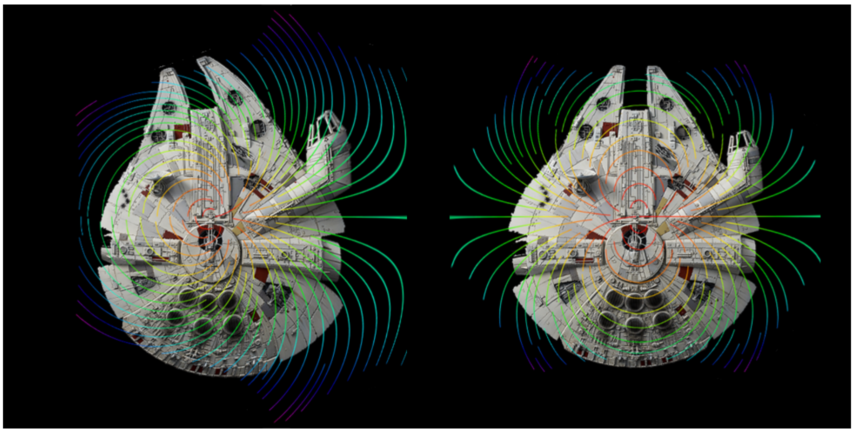


Figure 6: The visible distortion due to the presence of Weyl curvature along a principle null direction. The Left picture shows a neighbourhood of a triple repeated principle null direction. The right picture shows a neighbourhood of a quadruple repeated principle null direction. The line field is the fingerprint.

At large distances in non-stationary regions of isolated systems, the curvature is approximately type N. It represents gravitational radiation.

3 Asymptotically flat space-times

In physics, one is often interested in the asymptotic behaviour of fields. The asymptotics of General Relativity are particularly interesting since dynamical systems can have a wildly complex geometry. For example, no analytic solution to Einstein's equations representing a dynamic binary black hole system has been found, even though the corresponding Newtonian two-body problem is fairly simple. As one moves far away from an isolated system, however, we might expect these space-times to approach a simpler space-time. In this chapter, we will define and explore a large class of space-times describing isolated systems in General Relativity, called *asymptotically flat space-times.* These space-times allow for additional 'boundary points at infinity', denoted by $\mathscr{I}$, to be added to the manifold $\mathscr{M}$, similar to how one may add $\{z^{-1}=0\}$ to the complex plane to obtain the Riemann sphere. The asymptotic behaviour of fields can be studied at — and in a neighbourhood of — infinity in a coordinate-free way, without having to resort to using inelegant limiting arguments involving coordinate representations of field components. At infinity, the curvature is zero[9], making rigorous the intuitive idea that the metric should approach the Minkowski metric at large distances.

In order for the points at $\mathscr{I}$ to meaningfully represent infinity, we require the metric $\hat{g}_{ab}$ of $\mathscr{M}$ to diverge at $\mathscr{I}$, since we would like points in $\mathscr{M}$ lying in an infinitesimal neighbourhood of $\mathscr{I}$ to be separated by an infinite distance to points in $\mathscr{I}$. *The conformal geometry*, however, is only concerned with ratios of distances between neighbouring points, and so we should be able to define a conformal metric $g_{ab}=\Omega^2\hat{g}_{ab}$ which is well defined at $\mathscr{I}$. Infinity, here, must be located at $\Omega=0$ because if Ω^{-2} were smooth at $\mathscr{I}$, $\hat{g}_{ab}:=\Omega^{-2}g_{ab}$ could be smoothly extended to $\mathscr{I}$.

The efficiency and elegance of Penrose's conformal description of asymptotic flatness is quite remarkable, especially when combined with his spinorial treatment of General Relativity. Indeed, a large number of concepts are conformal. Some important examples include causality, null geodesics, Weyl curvature, the vacuum Bianchi identities $\nabla^{AA'}(\Omega^{-1}\Psi_{ABCD})=0$ and the source-free Maxwell's equations $\nabla^{AA'}(\Omega^{-1}\phi_{AB})=0$. (The last two of these are examples of *mass-less field equations* with arbitrary spin $\nabla^{AA'}(\Omega^{-1}\phi_{A\dots Z})=0$.)

Concretely, asymptotic flatness is typically defined as follows [Penrose, 1965, Penrose & Rindler, 1986]:

Definition 6 (Asymptotic flatness)**.** A space-time $\mathscr{M}$ with metric $\hat{g}_{ab}$ is *asymptotically flat* at future [past] null infinity if there exists a space-time $\bar{\mathscr{M}}:=\mathscr{M}\cup\mathscr{I}^+$ [$\bar{\mathscr{M}}:=\mathscr{M}\cup\mathscr{I}^-$] with boundary $\partial\bar{\mathscr{M}}:=\mathscr{I}^+$ [$\partial\bar{\mathscr{M}}:=\mathscr{I}^-$] and metric g_{ab}, and a smooth *conformal factor* Ω such that:

1. $g_{ab}=\Omega^2\hat{g}_{ab}$ in $\mathscr{M}$.
2. $\Omega\approx 0$, $N_a:=-\nabla_a\Omega\not\approx 0$ is future-null [past-null], where we define "$\approx$" to mean "equal when evaluated on $\mathscr{I}^\pm$".
3. $\mathscr{I}^\pm\cong\mathbb{S}^2\times\mathbb{R}$ is complete.
4. Einstein's equations hold in $\mathscr{M}$, $\hat{G}_{ab}=-8\pi\hat{T}_{ab}$, and $\Omega^{-2}\hat{T}_{ab}$ can be smoothly extended to $\mathscr{I}^\pm$.

Remark. Modern definitions of asymptotic flatness do not require $\mathscr{I}^\pm$ to be complete. A space-time satisfying conditions 1-4 is called *asymptotically Minkowskian.* Here, we will adopt Penrose's definition of asymptotic flatness, which is equivalent to asymptotically Minkowskian.

An unphysical space-time $(\bar{\mathscr{M}},g_{ab})$ satisfying condition 1 of definition 6 is called a *conformal completion* of the physical space-time $(\mathscr{M},\hat{g}_{ab})$. We will use the shorthand notation "$\mathscr{I}$" to refer to either $\mathscr{I}^+$ or $\mathscr{I}^-$.

We have introduced four more conditions. Firstly, by requiring $d\Omega\not\approx 0$ we ensure that $\mathscr{I}$ is a non-singular

[9] Presently, it is not clear that the curvature will be well-defined at infinity since, as we will see shortly, the metric will diverge here. We will later show, however, that the curvature *can* be uniquely extended to infinity, where it is seen to vanish. This justifies the term 'asymptotically flat'.

hypersurface, and that the conformal geometry of $\bar{\mathscr{M}}$ is *uniquely determined* by the geometry of the physical space-time $(\mathscr{M}, \hat{g}_{ab})$:

Proof. Let $(\bar{\mathscr{M}}, g_{ab})$ and $(\bar{\mathscr{M}}', g'_{ab})$ be two conformal completions of $(\mathscr{M}, \hat{g}_{ab})$ with conformal factors Ω, Ω'. In $\mathscr{M}$, $g'_{ab} = \Omega'^2\Omega^{-2}g_{ab}$. In $\bar{M}$, applying $\nabla_c\nabla_d$ to $\Omega^2 g'_{ab} = \Omega'^2 g_{ab}$ we find

$$2(\nabla_c\Omega\nabla_d\Omega + \Omega\nabla_c\nabla_d\Omega + \Omega\nabla_{(c}\Omega\nabla_{d)} + \tfrac{1}{2}\Omega^2\nabla_c\nabla_d)g'_{ab} = 2(\nabla_c\Omega'\nabla_d\Omega' + \Omega'\nabla_c\nabla_d\Omega')g_{ab}, \tag{3.0.1}$$

$$\text{whence} \qquad \nabla_c\Omega\nabla_d\Omega g'_{ab} \approx \nabla_c\Omega'\nabla_d\Omega' g_{ab}, \tag{3.0.2}$$

so that on $\partial\mathscr{M}$, $g'_{ab} = (\frac{d\Omega'}{d\Omega})^2 g_{ab}$ where $\frac{d\Omega'}{d\Omega}$ is well-defined since $d\Omega$ and $d\Omega'$ are both orthogonal to $\partial\mathscr{M}$ (and non-vanishing), and hence co-linear. □

Secondly, the topological condition $\mathscr{I} \cong \mathbb{S}^2 \times \mathbb{R}$ ensures that $\mathscr{I}$ is 'as big' as the future or past conformal boundary of Minkowski space. Originally, Penrose defined a smaller class of space-times, called *asymptotically simple* space-times, where the topological condition 3 is replaced by the condition that all null geodesics start and end at $\mathscr{I}$. If $\mathscr{I}$ is space-like or null, $\mathscr{I}$ splits into two pieces, $\mathscr{I}^+$ and $\mathscr{I}^-$ consisting of the future and past end points of null geodesics respectively. Penrose then proved that for asymptotically simple space-times with null $\mathscr{I}$, $\mathscr{I}^\pm \cong \mathbb{S}^2 \times \mathbb{R}$:

Proof sketch. Let p be a point in $\mathscr{M}$, and let $\mathscr{N}^+$ be its future light-cone. The rays generating $\mathscr{N}^+$ intersect each generator of $\mathscr{I}^+$ exactly once in some cut $\mathscr{C}^+$. Let λ be an affine parameter on $\mathscr{N}^+$ starting at p, which is scaled such that $\mathscr{C}_1 = \mathscr{C}^+$ where $\mathscr{C}_\lambda$ are the cuts of $\mathscr{N}^+$ of constant λ. $\mathscr{C}_\lambda$ may be singular at at most countably infinite λ, So that the function $\oint_{\mathscr{C}_\lambda} K(\lambda)d\mathscr{C}_\lambda = 4\pi\chi(\mathscr{C}_\lambda)$, where $K(\lambda)$ is the Gaussian curvature of $\mathscr{C}_\lambda$, can be extended to a continuous function. Hence $\chi(\mathscr{C}_\lambda)$ is constant, so that the non-singular $\mathscr{C}_\lambda$'s are topological spheres. $\mathscr{I}^+ \cong \mathscr{C}^+ \times \mathbb{R}$, and since $d\Omega \not\approx 0$, $\mathscr{C}^+$ is non-singular and thus $\mathscr{C}^+ \cong \mathbb{S}^2$. □

Hence, the class of asymptotically simple space-times is a subset of asymptotically flat space-times, but they are not equivalent: some very relevant examples of isolated systems, such as the Schwarzschild space-time, are asymptotically flat but not asymptotically simple, due to the existence of photon orbits around the black hole. One could try to broaden the definition by merely requiring a neighbourhood of $\mathscr{I}$ to be isometric to an asymptotically simple space-time, but even this requirement might be too restrictive. In any case, the topology of $\mathscr{I}$ will be a crucial ingredient in the proofs of several important (physically motivated) theorems, so any condition replacing 3 should have the current condition as a consequence. There does not seem to be any advantage to further restrict the definition. Completeness of $\mathscr{I}$ allows for the existence of a large asymptotic symmetry group, which we will explore shortly.

Finally, Einstein's equations are assumed to hold, and the fall-off condition on the stress-energy is weak enough to allow for radiation, but strong enough so that the resulting geometry of $\mathscr{I}$ is equivalent to the geometry we would have found if we had instead required the stress energy to have compact support. We can therefore intuitively understand asymptotically flat space-times as describing isolated matter sources which may emit radiation.

Evoking Einstein's equations makes sense from a physicist's perspective, but if we are only interested in the geometry of $\mathscr{I}$ we can simplify condition 4 in several ways, the most obvious of which is replace it with a fall-off condition for the physical Ricci curvature. We will be revisiting this problem at the end of this chapter.

3.1 The geometry of $\mathscr{I}$

Having set the stage, let us study the geometry of $\mathscr{I}$. Under a conformal transformation, the scalar curvature transforms as

$$\begin{aligned}\Omega^{-2}\hat{\Lambda} &= \Lambda + \tfrac{1}{4}\Omega\Box\Omega^{-1} \\ &= \Lambda + \tfrac{1}{4}\Omega^{-1}\nabla_a N^a + \tfrac{1}{2}\Omega^{-2}N_a N^a.\end{aligned} \tag{3.1.1}$$

We can use this to determine the character of $\mathscr{I}$:

Proposition 8. *Let* $(\mathscr{M}, \hat{g}_{ab})$ *be a space-time satisfying conditions 1-3 of definition 6. Then* $\mathscr{I}$ *is space-like, time-like, or null depending on if* $\lambda + 2\pi\hat{T}$ *is positive, negative, or zero at* $\mathscr{I}$. *In particular, if* $\mathscr{M}$ *is asymptotically flat,* $\mathscr{I}$ *is null.*

Proof. Einstein's equations yield

$$\tfrac{1}{3}\pi\hat{T} + \tfrac{1}{6}\lambda = \Omega^2\Lambda + \tfrac{1}{4}\Omega\nabla_a N^a + \tfrac{1}{2}N_a N^a. \tag{3.1.2}$$

Both sides are smooth at $\mathscr{I}$, so that this expression extends smoothly to $\mathscr{I}$. Hence, $\frac{1}{3}\pi\hat{T} + \frac{1}{6}\lambda \approx \frac{1}{2}N_a N^a$. In particular, in asymptotically flat space-times $\frac{1}{3}\pi\hat{T} + \frac{1}{6}\lambda \approx 0$ so that $\mathscr{I}$ is null. □

Note that if $\mathscr{I}$ is non-null, $N_a \not\approx 0$ so that the condition $d\Omega \not\approx 0$ is automatically satisfied in this case.

A further important property of $\mathscr{I}$ is the following:

Theorem 4 (Penrose, 1965)**.** *Let* $(\mathscr{M}, \hat{g}_{ab})$ *be an asymptotically flat space-time. Then* $C_{abcd} \approx 0$.

Proof. The physical Bianchi identity is

$$\hat{\nabla}^A_{A'}\hat{\Psi}_{ABCD} = 8\pi\hat{\nabla}^{B'}_B\hat{T}_{CDA'B'}. \tag{3.1.3}$$

In terms of unphysical quantities, this becomes

$$\Omega\nabla^A_{A'}\Psi_{ABCD} - N^A_{A'}\Psi_{ABCD} = 4\pi\Omega^2\nabla^{B'}_B T_{CDA'B'} - 12\pi\Omega N^{B'}_B T_{CDA'B'}. \tag{3.1.4}$$

Because g_{ab} (and therefore also its curvature) and T_{ab} are smooth at $\mathscr{I}$, we find that $N^A_{A'}\Psi_{ABCD} \approx 0$. If the matrix formed by the components of $N^A_{A'}$ is invertible, we immediately find that $\Psi_{ABCD} \approx 0$. This will be the case when $\det N^A_{A'} \neq 0$, i.e. if and only if $N^A_{A'} \neq \kappa^A\lambda_{A'}$, i.e. if and only if N^a is non-null. Since $\mathscr{M}$ is asymptotically flat, however, $\mathscr{I}$ is null, so that $N^A_{A'}\Psi_{ABCD} \approx \xi^A\bar{\xi}_{A'}\Psi_{ABCD} \approx 0$ implies that $\Psi_{ABCD} \approx \Psi\xi_A\xi_B\xi_C\xi_D$ for some function Ψ. Hence, Ψ_{ABCD} is type N at $\mathscr{I}$. Choose a spin frame with ι^A orthogonal to the principle null direction. Then $\Psi_1 \approx \Psi_2 \approx \Psi_3 \approx \Psi_4 \approx 0$. The $0'000$ component of the physical Bianchi identity then becomes $\hat{\eth}'\hat{\Psi}_0 \approx 0$. Ψ_0, having positive spin weight, must therefore also vanish. □

The proof crucially relied on the spherical topology of the cross-sections of $\mathscr{I}$ because in the final step we used that there are no positive spin-weight solutions Ψ to $\eth'\Psi = 0$. Intuitively, we can understand this part of the proof as a consequence of the non-existence of purely spherically symmetric gravitational waves: Consider a field Ψ orthogonal to a spherical surface S, and assume Ψ is type N in a neighbourhood of S. Ψ needs to be constant on S since otherwise it would induce a gradient in the other components through the Bianchi identities. Because there are no purely spherically symmetric gravitational waves (for the same reason there are no spherically symmetric electromagnetic waves), Ψ must vanish.

As a corollary to the vanishing of the conformal curvature, the physical curvature is seen to vanish on $\mathscr{I}$. This justifies the term 'asymptotic flatness':

Corollary 1. *Let* $(\mathscr{M}, \hat{g}_{ab})$ *be an asymptotically flat space-time. Then* $\hat{R}^a{}_{bcd}$ *can be smoothly extended to* $\mathscr{I}$, *where it vanishes.*

Proof. Under a conformal rescaling $g_{ab} = \Omega^2\hat{g}_{ab}$, $\epsilon_{AB} = \Omega\hat{\epsilon}_{AB}$ and $\Psi_{ABCD} = \hat{\Psi}_{ABCD}$. By condition 4 the right-hand side of Einstein's equations has a smooth limit to $\mathscr{I}$, so that $\hat{G}_{ab}$ can be smoothly extended to $\mathscr{I}$. Furthermore, $\hat{R}_{ab} \approx 0$ and $\Omega^{-2}\hat{R} = g^{ab}\hat{R}_{ab} \approx 0$ so that

$$\begin{aligned}\hat{R}^{AA'}{}_{BCDB'C'D'} &= \hat{\Psi}_{BCDE}\hat{\epsilon}^{AE}\hat{\epsilon}^{A'}{}_{B'}\hat{\epsilon}_{C'D'} + \hat{\bar{\Psi}}_{B'C'D'E'}\hat{\epsilon}^{A'E'}\hat{\epsilon}^{A}{}_{B}\hat{\epsilon}_{CD} + \hat{\Phi}_{B'E'CD}\hat{\epsilon}^{A'E'}\hat{\epsilon}^{A}{}_{B}\hat{\epsilon}_{C'D'} + \hat{\bar{\Phi}}_{BEC'D'}\hat{\epsilon}^{AE}\hat{\epsilon}^{A'}{}_{B'}\hat{\epsilon}_{CD} \\ &\quad + 2\hat{\Lambda}(\hat{\epsilon}^{A}{}_{C}\hat{\epsilon}_{BD}\hat{\epsilon}^{A'}{}_{C'}\hat{\epsilon}_{B'D'} - \hat{\epsilon}^{A}{}_{D}\hat{\epsilon}_{BC}\hat{\epsilon}^{A'}{}_{D'}\hat{\epsilon}_{B'C'}) \\ &= \Psi_{BCDE}\epsilon^{AE}\epsilon^{A'}{}_{B'}\epsilon_{C'D'} + \bar{\Psi}_{B'C'D'E'}\epsilon^{A'E'}\epsilon^{A}{}_{B}\epsilon_{CD} + \hat{\Phi}_{B'E'CD}\epsilon^{A'E'}\epsilon^{A}{}_{B}\epsilon_{C'D'} + \hat{\bar{\Phi}}_{BEC'D'}\epsilon^{AE}\epsilon^{A'}{}_{B'}\epsilon_{CD} \\ &\quad + 2\Omega^{-2}\hat{\Lambda}(\epsilon^{A}{}_{C}\epsilon_{BD}\epsilon^{A'}{}_{C'}\epsilon_{B'D'} - \epsilon^{A}{}_{D}\epsilon_{BC}\epsilon^{A'}{}_{D'}\epsilon_{B'C'}) \\ &\approx 0. \end{aligned} \tag{3.1.5}$$

□

Finally, let us consider the trace-free Ricci curvature. Under a conformal rescaling Φ_{ab} transforms as:

$$\hat{\Phi}_{ABA'B'} = \Phi_{ABA'B'} + \Omega^{-1}\nabla_{A'(A}N_{B)B'}. \tag{3.1.6}$$

By condition 4, $\nabla_{A'(A}N_{B)B'} \approx 0$. Φ_{ab} is highly dependent on the particular choice of Ω, but, as we will see, only two components will have physical significance. In order to simplify the analysis, we will assume $\mathscr{I} = \mathscr{I}^+$ is future null which will allow us to efficiently employ the GHP formalism.

Because $\mathscr{I}^+$ is null, we can choose one of our tetrad vectors to be proportional to N^a on $\mathscr{I}$: $N^a \approx An^a$ for some $\{1,1\}$ scalar A, which is positive since $\mathscr{I}^+$ is future null. $\iota^{A'}\nabla_{A'(A}N_{B)B'} \approx 0$ then yields

$$\sigma' \approx 0, \qquad \rho' \approx \bar{\rho}', \qquad \kappa' \approx 0, \qquad \eth A \approx 0, \quad \text{and} \quad (\text{þ}' + \rho')A \approx 0. \tag{3.1.7}$$

Note that $\rho' \approx \bar{\rho}'$ and $\kappa' \approx 0$ also follows from the fact that $\mathscr{I}^+$ is a null hypersurface, and $\sigma' \approx 0$ also follows from the vanishing of Ψ_0 and Ψ_1 and (a slightly modified version of) the Goldberg Sachs theorem. As a consequence, cross sections of $\mathscr{I}^+$ are mapped *conformally* along its generators: choose a scaling for n^a so that the generators of $\mathscr{I}^+$ are affine, and complete the tetrad with m^a tangent to its cross-sections. Then:

$$D'(m_{(a}\bar{m}_{b)}) \approx \bar{\sigma}' m_a m_b + (\rho' + \bar{\rho}')m_{(a}\bar{m}_{b)} + \sigma'\bar{m}_a\bar{m}_b \approx (\rho' + \bar{\rho}')m_{(a}\bar{m}_{b)}. \tag{3.1.8}$$

By the uniformization theorem, each cross section is conformal to the unit sphere, so we may choose Ω so that $K \approx \frac{1}{2}$, where K is half the Gaussian curvature of the cross-sections. We will not make this specialisation just yet, but instead we will simply set $\rho' + \bar{\rho}' \approx 0$ so that cross sections of $\mathscr{I}^+$ are mapped *isometrically* along its generators. This can be achieved simply by choosing Ω such that $d\Omega$ is null in a small neighbourhood of $\mathscr{I}^+$. We can then set $N^a := A\iota^A\iota^{A'}$. The components of $o^{A'}\nabla_{A'(A}N_{B)B'} \approx 0$ then read:

$$\tau' \approx 0, \qquad \rho' \approx 0, \qquad \text{þ}A \approx 0. \tag{3.1.9}$$

Finally, we can use the remaining freedom in Ω, and an appropriate choice of o^A, to set all but one of the remaining spin coefficients to zero in $\mathscr{I}^+$: let u be a future increasing parameter along the generators of $\mathscr{I}^+$ satisfying $\text{þ}'u \approx A^{-1}$, and choose o^A to be orthogonal to cross-sections of constant u so that $\eth u \approx 0$. We find that $0 \approx (\text{þ}'\eth - \eth\text{þ}')u \approx \tau\text{þ}'u$ so that $\tau \approx 0$. At $\mathscr{I}^+$, o^A is hypersurface orthogonal so that $\rho \approx \bar{\rho}$ and $\kappa \approx 0$. Finally, under a conformal transformation $\iota^A \mapsto \iota^A$, $o^A \mapsto \Theta o^A$, where $\eth\Theta \approx 0 \approx \text{þ}'\Theta$ all spin coefficients transform conformally, except for $\rho \mapsto \Theta^2\rho + \Theta\text{þ}\Theta$. Choose Θ such that $\rho \approx 0$ and scale the spin frame such that $\text{þ}\rho \approx 0$.

Essentially, the entire geometry at $\mathscr{I}^+$ is encoded in a single complex function σ, which we will soon learn is related to the outgoing flux of gravitational radiation at $\mathscr{I}^+$. The Ricci identities yield:

$$\Phi_{00} \approx -\sigma\bar{\sigma}, \qquad \Phi_{01} \approx -\eth'\sigma, \qquad \Phi_{02} \approx -\text{þ}'\sigma, \qquad \Phi_{12} \approx 0 \approx \Phi_{22}. \tag{3.1.10}$$

The remaining component of the Ricci curvature is given by the Gaussian curvature of the cross-sections of $\mathscr{I}^+$ through $K \approx \Phi_{11} + \Lambda$.

3.2 The BMS group

The fact that space-time has no isometries in general is an important feature of General Relativity. In flat space-time, isometries are an essential ingredient for the definition of conserved energy-momentum and angular momentum. Indeed, the ten Killing fields k^a of Minkowski space together with local conservation of energy yield ten globally conserved charges which arise from the conserved currents $J_a = T_{ab}k^b$. The stress energy contains information about the local matter content, and because gravity interacts with matter and exchanges energy with it, we cannot expect the stress energy to give rise to a globally conserved energy. In the weak field limit, the local conservation law $\nabla_a T^{ab} = 0$ will express the non-conservation of matter energy with respect to the flat background, and to remedy this a gravitational stress energy may be introduced. Gravity in General Relativity is non-local, however, as is expressed by the equivalence principle. It is therefore *impossible* to find a gravitational energy density. A full discussion of gravitational energy will have to wait until the next chapter. Presently, we will be concerned with the problem of finding a Poincaré group which is a suitable generalisation of isometries in Minkowski space.

What we have just seen in the previous section, is that $\mathscr{I}^+$ has a delightfully simple structure compared to the geometry of the bulk space-time. In summary, the intrinsic conformal geometry of $\mathscr{I}^+$ is given by the conformal metric

$$dl^2 = 0 \cdot du^2 - \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}, \tag{3.2.1}$$

while the extrinsic geometry of $\mathscr{I}^+$ is given by a single complex scalar field σ. The simple structure, which we will soon define and call the *strong conformal geometry*, allows us to cut down the diffeomorphism group to give rise to a large symmetry group of diffeomorphisms that leave this structure invariant. This group contains a unique four parameter subgroup of translations, which can be used to define the total energy-momentum of the space-time.

Let us start in Minkowski space $\mathbb{M}$ and its compactification $\bar{\mathbb{M}}$ with metrics

$$d\tilde{s}^2 = du^2 - 2du dr - \frac{4r^2 d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} \qquad \text{and} \qquad ds^2 = \Omega^2 du^2 + 2du d\Omega - \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}, \tag{3.2.2}$$

The surfaces of constant u are future light cones in $\mathbb{M}$, so that the cuts of constant u on $\mathscr{I}^+$ are celestial spheres of the points at $r = 0$. Poincaré transformations of $\mathbb{M}$ are seen on $\mathscr{I}^+$ as transformations mapping a set *good cuts* of $\mathscr{I}^+$ to each other. In $\bar{\mathbb{M}}$, a cut is a good cut if it corresponds to a light cone in $\mathbb{M}$, which can be expressed in terms of quantities defined on $\mathscr{I}^+$ by the condition $\sigma \approx 0$ where l^a is chosen tangent to the surfaces of constant u. Specifically, the transformations $\zeta \mapsto \frac{a\zeta+b}{c\zeta+d}$ are Lorentz transformations, and the transformations $u \mapsto u + W$, where W is composed of $l = 0$ and $l = 1$ spherical harmonics are translations. To see this, you could use coordinates, but it is instructive to use tetrads: choose $l_a = A\nabla_a u$ and $n_a \approx A^{-1}\nabla_a\Omega$ for some constant $\{-1,-1\}$ scalar A. Translating $u \mapsto u + W(\zeta,\bar{\zeta})$ transforms

$$l^a \mapsto l^a - m^a ð'W - \bar{m}^a ðW - n^a ðW ð'W, \qquad \text{and} \qquad m^a \mapsto m^a - n^a ðW, \qquad \text{so that} \tag{3.2.3}$$

$$\sigma = m_a \delta l^a \mapsto (m_a - n_a ðW)(\delta - ðWD')(l^a - m^a ð'W - \bar{m}^a ðW - n^a ðWð'W) \tag{3.2.4}$$

$$\begin{aligned} &= \sigma + ð^2 W - \tau ðW - \rho'(ðW)^2 - 2\bar{\sigma}' ðW ð'W + \kappa'(ðW)^3 + 2\bar{\kappa}'(ðW)^2 ð'W \\ &\approx \sigma + ð^2 W - \tau ðW. \end{aligned} \tag{3.2.5}$$

In this tetrad, $\sigma = 0 \approx \tau$. Hence, this transformation maps good cuts to good cuts iff $ð^2 W = 0$, i.e. iff W consists of $l = 0$ and $l = 1$ spherical harmonics.

Next, let us return to a general asymptotically flat space-time. First, we would like to find a function u on the generators of $\mathscr{I}^+$ that generalises the retarded time coordinate u in $\mathbb{M}$.

Definition 7. A future-increasing real parameter u on $\mathscr{I}^+$ attaining the full range $(-\infty,\infty)$ on each generator is a *Bondi parameter* if $(\text{þ}'-2\rho')\text{þ}'u \approx 0$.

The condition $(\text{þ}'-2\rho')\text{þ}'u \approx 0$ is conformal, in other words u being a Bondi parameter is independent of the choice of conformal factor Ω. In particular, if Ω is chosen such that $\rho' \approx 0$, and if we scale our tetrad such that n^a is affine, then u is a Bondi parameter iff $D'^2 u \approx 0$. Hence, if u is a Bondi parameter all other possible Bondi parameters are of the form $Gu+H$, where $G>0$ and H are arbitrary functions on the cross sections of constant u.

Definition 8. If Ω is chosen such that the cross-sections of $\mathscr{I}^+$ are unit spheres, then a Bondi parameter u is called a *Bondi retarded time coordinate*.

Given N^a we can fix a definite scaling for u on each generator by choosing a constant ν and setting $N^a\nabla_a u = \nu$. Note that $\nu > 0$ on $\mathscr{I}^+$ since N^a is future null and du is future causal (and not proportional to N_a) because u is future-increasing. Consequently, applying $(\text{þ}'+\rho')$ to $A\text{þ}'u - \nu \approx 0$ and using that $(\text{þ}'+\rho')A \approx 0$ we find that $0 \approx A(\text{þ}'+\rho')\text{þ}'u - \nu\rho' \approx A(\text{þ}'-2\rho')\text{þ}'u + 2\rho' \approx 2\rho'$ if u is a Bondi parameter. Hence, $\rho' \approx 0$ and the cross-sections of $\mathscr{I}^+$ are mapped isometrically along its generators. Note that choosing $N^a\nabla_a u \approx \nu$ is therefore equivalent to choosing a divergence free frame $\nabla_a N^a \approx 0$ (and hence also $\nabla_a N_b \approx 0$).

Definition 9. A constant $N^a\nabla_a u = \nu$ is called a *null angle*. The conformal geometry of an asymptotically flat space-time $\mathscr{M}$, together with a choice of null angle defines the *strong conformal geometry* of $\mathscr{M}$.

Henceforth we will choose $\nu = 1$, and Ω such that the cross-sections of $\mathscr{I}^+$ are unit spheres. The metric of $\mathscr{I}^+$ is now given by (3.2.1).

Definition 10. The transformations $\zeta \mapsto \frac{a\zeta+b}{c\zeta+d}$, $u \mapsto \frac{1+\zeta\bar{\zeta}}{|a\zeta+b|^2+|c\zeta+d|^2}u + H(\zeta,\bar{\zeta})$ that preserve the Bondi retarded time coordinate and strong conformal geometry form a group $\mathscr{B}$, called the *BMS group*. The BMS group is the semi-direct product $\mathscr{B} = \mathscr{R} \rtimes \mathscr{U}$ of Lorentz rotations $\mathscr{R}$, given by $\zeta \mapsto \frac{a\zeta+b}{c\zeta+d}$, $u \mapsto \frac{1+\zeta\bar{\zeta}}{|a\zeta+b|^2+|c\zeta+d|^2}u$ and *supertranslations* $\mathscr{U}$ given by $u \mapsto u + H(\zeta,\bar{\zeta})$.

As we have seen from the Minkowski example, the BMS group contains the Poincaré group $\mathscr{P}$ as a subgroup. $\mathscr{B}$ is much bigger than $\mathscr{P}$, being infinite dimensional due to the fact that there are infinitely many smooth functions H on the sphere. Fortunately, similar to how the translation subgroup is the unique four-parameter normal subgroup of $\mathscr{P}$, we have

Theorem 5 ([Sachs, 1962b])**.** *The translation subgroup is the unique four-parameter normal subgroup of $\mathscr{B}$.*

I will not give the full proof here, but to show that translations are a normal subgroup of $\mathscr{B}$ uses the fact that under a conformal transformation, $l=0$ and $l=1$ spherical harmonics transform among each other. Concretely, if f is conformal with spin weight s and boost- and conformal weight w, then the equations

$$\eth^{w-s+1} f = g \qquad \text{and} \qquad \eth'^{w+s+1} f = h \tag{3.2.6}$$

are Lorentz invariant. A supertranslation $u \mapsto u+H$ is a translation if $\eth^2 H = 0$. Since H has no spin weight and a boost- and conformal weight $w=1$, this condition is Lorentz invariant.

3.3 The gravitational field at $\mathscr{I}^+$

Recall that the mass-less field equations $\hat{\nabla}^{AA'}\hat{\phi}_{A\ldots Z} = 0$ are conformal if $\phi_{A\ldots Z} = \Omega^{-1}\hat{\phi}_{A\ldots Z}$, and that the vacuum Bianchi identities are given by $\hat{\nabla}^A_{A'}\hat{\Psi}_{ABCD} = 0$. Furthermore, because Ψ_{ABCD} is conformal and vanishing on $\mathscr{I}$, we are motivated to define the following field on $\bar{\mathscr{M}}$:

Definition 11. Let $(\mathscr{M}, \hat{g}_{ab})$ be asymptotically flat. Define the *gravitational field* ψ_{ABCD} as

$$\psi_{ABCD} = \Omega^{-1}\Psi_{ABCD}. \tag{3.3.1}$$

ψ_{ABCD} is smooth at $\mathscr{I}$.

At $\mathscr{I}$, half the components of the gravitational field are given by the Bianchi identities:

Proposition 9. *On $\mathscr{I}$, the Bianchi identities are given by*

$$A\psi_{1ABC}\epsilon_{A'B'}\iota_{C'} + A\epsilon_{AB}\iota_C\bar{\psi}_{1'A'B'C'} \approx 2\nabla_{[a}P_{b]c}, \tag{3.3.2}$$

where $P_{ab} := \frac{1}{12}Rg_{ab} - \frac{1}{2}R_{ab}$ is the Schouten tensor. In GHP form its components are

$$\psi_4 \approx þ'N, \tag{3.3.3}$$

$$\psi_3 \approx ðN - ð'K, \tag{3.3.4}$$

$$\psi_2 - \bar{\psi}_2 \approx ð'^2\sigma - ð^2\bar{\sigma} + \sigma N - \bar{\sigma}\bar{N}. \tag{3.3.5}$$

Proof. Starting from the Bianchi identities[10] $\nabla^d R_{abcd}$ and the decomposition $R_{ab}{}^{cd} = C_{ab}{}^{cd} + 4P_{[a}{}^{[c}g_{b]}{}^{d]}$, we find that the Bianchi identities are equivalent to

$$\nabla^d C_{abcd} = 2\nabla_{[a}P_{b]c}. \tag{3.3.6}$$

It follows directly from the definition that $\nabla_{AA'}\Psi_{BCDE} \approx -N_{AA'}\psi_{BCDE}$ so that
$\nabla^{AA'}(\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}) \approx A\iota^A\psi_{ABCD}\iota_{B'}\epsilon_{C'D'}$. Hence,

$$A\psi_{1ABC}\epsilon_{A'B'}\iota_{C'} + A\epsilon_{AB}\iota_C\bar{\psi}_{1'A'B'C'} \approx 2\nabla_{[a}P_{b]c} \tag{3.3.7}$$

□

3.4 Einstein's field equations at $\mathscr{I}^+$

In our derivation of the asymptotic geometry of space-time, we made use of Einstein's equations, but nowhere did we *need* them. By construction, the Einstein tensor $\hat{G}_{ab}$ has precisely the same symmetry and divergence-free properties as the stress-energy $\hat{T}_{ab}$. Without further restrictions on the stress-energy, any Lorentzian manifold is a solution to Einstein's equations. As far as the geometry is concerned, the only constraint assumption 4 of definition 6 imposes on the space-time geometry is the fall-off $\hat{R}_{ab} = \mathscr{O}(\Omega^2)$ of the Ricci tensor, which is a consequence of Einstein's equations with the fall-off $\hat{T}_{ab} = \mathscr{O}(\Omega^2)$ of the stress-energy.

In this section, I will first dispense with condition 4 of definition 6 and replace it by a more direct, local condition on the geometry of $\mathscr{I}$. This new condition implies a weaker fall-off rate of the physical Ricci curvature. It is tempting to try to constrain the possible asymptotic field equations of gravity by demanding that they are invariant under BMS and conformal transformations. This was the premise of two recent papers [Freidel et al., 2021, Freidel & Pranzetti, 2022]. Indeed, it is easy to show, (and somewhat trivial, when using the right tools) that Einstein's equations at $\mathscr{I}$ enjoy BMS and conformal invariance. I will prove, however, that this invariance is not only generic but *weaker* than diffeomorphism invariance. In fact, I will provide two examples of field equations that are, in addition to being BMS and conformally invariant, invariant at $\mathscr{I}$ under conformal transformations of the *physical* metric.

The single most important property of $\mathscr{I}$ in asymptotically flat space-times, which is a consequence of condition 4, is the fact that $\mathscr{I}$ is a shear-free null hypersurface. It is this fact that not only gives rise to the simple intrinsic conformal metric of $\mathscr{I}$ (3.2.1), and therefore to the existence of BMS symmetries, but it is also a crucial ingredient in the proof of the peeling theorem. We will call the condition that $\mathscr{I}$ is a shear-free null hypersurface, written spinorially as $\nabla_{A'(A}N_{B)B'} \approx 0$, the *asymptotic Einstein condition.* A second major consequence of condition 4 is the vanishing of the Weyl curvature at $\mathscr{I}$. This allowed us to define the gravitational field ψ_{ABCD} on $\bar{\mathscr{M}}$, which in vacuum satisfies the conformal mass-less spin two field equations. The condition $\Psi_{ABCD} \approx 0$ in

[10] These obtained from the standard form of the Bianchi identities, $\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$ by contracting the index a with e.

addition to the asymptotic Einstein condition is called the *strong asymptotic Einstein condition*. This condition is a good replacement for condition 4 of definition 6, and had we chosen to define asymptotic flatness with the strong asymptotic Einstein condition the preceding sections of this chapter would have been nearly identical. Although this condition is weaker, it does similarly imply some fall-off rate on the physical Ricci tensor, as the following theorem proves:

Theorem 6. *Let $(\mathscr{M}, \hat{g}_{ab})$ be a space-time satisfying conditions 1-3 of definition 6, and the strong asymptotic Einstein condition. Then $\hat{\Phi}_{ab}$ can be smoothly extended to $\mathscr{I}$, and has the following peeling-like property:*

$$\hat{\Phi}_{ab} = \mathscr{O}(1), \tag{3.4.1a}$$

$$\hat{\Phi}_{ABA'B'}\iota^{B'} = \mathscr{O}(\Omega), \tag{3.4.1b}$$

$$\hat{\Phi}_{ABA'B'}\iota^{A'}\iota^{B'} = \mathscr{O}(\Omega^2), \tag{3.4.1c}$$

which can be compactly written in GHP form as $\hat{\Phi}_{rs} = \mathscr{O}(\Omega^s)$.

Proof. On $\mathscr{M}$,

$$\hat{\Phi}_{ABA'B'} = \Phi_{ABA'B'} + \Omega^{-1}\nabla_{A'(A}N_{B)B'}. \tag{3.4.2}$$

By the strong asymptotic Einstein condition, $\nabla_{A'(A}N_{B)B'} \approx 0$ so that $\Omega^{-1}\nabla_{A'(A}N_{B)B'}$ is smooth on $\bar{\mathscr{M}}$. Therefore, the right hand side of (3.4.2) is smooth on $\bar{\mathscr{M}}$ and we can smoothly extend $\hat{\Phi}_{ABA'B'}$ to $\bar{\mathscr{M}}$.

Starting from the physical Bianchi identities,

$$\hat{\nabla}^{A}_{A'}\hat{\Psi}_{ABCD} = \hat{\nabla}^{B'}_{(B}\hat{\Phi}_{CD)A'B'}, \tag{3.4.3}$$

we find upon conformally rescaling:

$$\Omega\nabla^{A}_{A'}\Psi_{ABCD} - N^{A}_{A'}\Psi_{ABCD} = \Omega\nabla^{B'}_{(B}\hat{\Phi}_{CD)A'B'} - N^{B'}_{(B}\hat{\Phi}_{CD)A'B'}, \tag{3.4.4}$$

$$\text{so that} \qquad \hat{\Phi}_{ABA'B'}\iota^{B'} \approx 0. \tag{3.4.5}$$

Multiplying (3.4.4) by $\Omega^{-1}N^{A'}_{E}$, we find

$$N^{A'}_{E}\nabla^{A}_{A'}\Psi_{ABCD} - N^{A'}_{E}N^{A}_{A'}\psi_{ABCD} = N^{A'}_{E}\nabla^{B'}_{(B}\hat{\Phi}_{CD)A'B'} - \Omega^{-1}N^{A'}_{E}N^{B'}_{(B}\hat{\Phi}_{CD)A'B'}. \tag{3.4.6}$$

The left-hand-side vanishes on $\mathscr{I}$ since $N^{A'}_{E}\nabla^{A}_{A'}$ is intrinsic to $\mathscr{I}$ at $\mathscr{I}$. Furthermore,

$$\begin{aligned} N^{EA'}\nabla^{B'}_{(B}\hat{\Phi}_{CD)A'B'} &\approx \nabla^{B'}_{(B}(\Omega\Omega^{-1}N^{EA'}\hat{\Phi}_{CD)A'B'}) \\ &= \Omega\nabla^{B'}_{(B}(\Omega^{-1}N^{EA'}\hat{\Phi}_{CD)A'B'}) - \Omega^{-1}N^{EA'}N^{B'}_{(B}\hat{\Phi}_{CD)A'B'} \\ &\approx -\Omega^{-1}N^{EA'}N^{B'}_{(B}\hat{\Phi}_{CD)A'B'}, \end{aligned} \tag{3.4.7}$$

by the strong asymptotic Einstein condition, so that

$$\Omega^{-1}\hat{\Phi}_{ABA'B'}\iota^{A'}\iota^{B'} \approx 0. \tag{3.4.8}$$

□

Finally, let turn to issue of deriving Einstein's equations. The gravitational field equations, whichever one we choose, should be invariant under asymptotic symmetries at $\mathscr{I}$ and independent of the conformal completion. At first glance, Einstein's equations seem to be involved in a wonderful coincidence. Consider, once more, the physical Bianchi identities (3.4.4), written in terms of the unphysical curvature:

$$\begin{aligned} N^{A'}_{A}\nabla^{E}_{A'}\psi_{BCDE} &\approx \Omega^{-2}N^{A'}_{A}N^{B'}_{(B}\Phi_{CD)A'B'} + \Omega^{-3}N_{AA'}N_{BB'}\nabla^{A'}_{C}N^{B'}_{D} \\ &\approx \Omega^{-2}N^{A'}_{A}N^{B'}_{(B}\hat{\Phi}_{CD)A'B'}. \end{aligned} \tag{3.4.9}$$

The left-hand-side is *conformal*, because the gravitational field ψ_{ABCD} has conformal weight -1. The right-hand-side (3.4.9) is not conformal, however. Miraculously, when using Einstein's equations, we find

$$N_A^{A'}\nabla_{A'}^{E}\psi_{BCDE} \approx 4\pi G N_A^{A'} N_{(B}^{B'}\hat{T}_{CD)A'B'}, \tag{3.4.10}$$

which *is* conformal. Additionally, (3.4.10) is BMS invariant. Indeed, we have seen before that a supertranslation $u \mapsto u+H$ transforms a tetrad with $n^a := A^{-1}N^a$ fixed as follows:

$$l^a \mapsto l^a - m^a \eth' H - \bar{m}^a \eth H - n^a \eth H \eth' H, \qquad \text{and} \qquad m^a \mapsto m^a - n^a \eth H. \tag{3.4.11}$$

In an appropriately chosen tetrad[11], $\sigma \mapsto \sigma + \eth^2 H$, and it is easy to show that $\psi_r \mapsto \sum_{s=r}^4 \binom{4-r}{s-r}\psi_s(-\eth H)^{s-r}$. A somewhat lengthy combinatorical computation then shows

$$\begin{aligned}
\eth\psi_{r+1} \mapsto & \sum_{s=r+1}^{4}\binom{3-r}{s-r-1}(\eth - \eth H \text{þ}')\psi_s(-\eth H)^{s-r-1} \\
=& \sum_{s=r+1}^{4}\binom{3-r}{s-r-1}[(-\eth H)^{s-r}\text{þ}'\psi_s + (-\eth H)^{s-r-1}\eth\psi_s - (s-r-1)(-\eth H)^{s-r-2}\psi_s \eth^2 H] \\
=& \sum_{s=r}^{4}\left[\binom{4-r}{s-r}-\binom{3-r}{s-r}\right](-\eth H)^{s-r}\text{þ}'\psi_s + \sum_{s=r}^{3}\binom{3-r}{s-r}(-\eth H)^{s-r}\eth\psi_{s+1} - \sum_{s=r}^{2}\binom{2-r}{s-r}(3-r)(-\eth H)^{s-r}\psi_{s+2}\eth^2 H \\
=& \text{þ}'\sum_{s=r}^{4}\binom{4-r}{s-r}(-\eth H)^{s-r}\psi_s - \sum_{s=r}^{3}\binom{3-r}{s-r}(-\eth H)^{s-r}[\text{þ}'\psi_s - \eth\psi_{s+1}] - \sum_{s=r}^{2}(-\eth H)^{s-r}\binom{2-r}{s-r}(3-r)\eth^2 H\psi_{s+2} \\
=& \text{þ}'\sum_{s=r}^{4}\binom{4-r}{s-r}(-\eth H)^{s-r}\psi_s - (3-r)\eth^2 H\left(\sum_{s=r}^{2}\binom{2-r}{s-r}(-\eth H)^{s-r}\psi_{s+2}\right) - \sum_{s=r}^{3}\binom{3-r}{s-r}(-\eth H)^{s-r}[\text{þ}'\psi_s - \eth\psi_{s+1}] \\
=& -\text{þ}'\psi_r + \eth\psi_{r+1} + (3-r)\sigma\psi_{r+2} + \text{þ}'\sum_{s=r}^{4}\binom{4-r}{s-r}(-\eth H)^{s-r}\psi_s - (3-r)(\sigma+\eth^2 H)\left(\sum_{s=r}^{2}\binom{2-r}{s-r}(-\eth H)^{s-r}\psi_{s+2}\right) \\
& - \sum_{s=r+1}^{3}\binom{3-r}{s-r}(-\eth H)^{s-r}[\text{þ}'\psi_s - \eth\psi_{s+1} - (3-s)\sigma\psi_{s+2}],
\end{aligned} \tag{3.4.12}$$

so that

$$\text{þ}'\psi_s - \eth\psi_{s+1} - (3-s)\sigma\psi_{s+2} \mapsto \text{þ}'\psi_s - \eth\psi_{s+1} - (3-s)\sigma\psi_{s+2} + \sum_{s=r+1}^{3}\binom{3-r}{s-r}(-\eth H)^{s-r}[\text{þ}'\psi_s - \eth\psi_{s+1} - (3-s)\sigma\psi_{s+2}]. \tag{3.4.13}$$

Hence, the vacuum equations $\text{þ}'\psi_s - \eth\psi_{s+1} - (3-s)\sigma\psi_{s+2} \approx 0$ are conformal and BMS invariant.

Perhaps this should not have been surprising. Given that the components of *any* spinor $\phi_{A\ldots HK'\ldots Q'}$ transform among each other linearly under a BMS transformation. Hence the equation $\phi_{A\ldots HK'\ldots Q'} \approx 0$ must also hold in the BMS transformed frame. Another way to make the same point is as follows: BMS transformations $u \mapsto Gu + H$, $\zeta \mapsto \frac{a\zeta+b}{c\zeta+d}$ are a subset of diffeomorphisms of $\bar{\mathscr{M}}$ as seen on $\mathscr{I}$. Therefore, clearly:

Remark. Any diffeomorphism invariant field in $\bar{\mathscr{M}}$ is BMS invariant.

We can thus conclude that BMS invariance is a *red herring*, since we already require all physical laws to be covariant.

The conformal invariance of the field equation for ψ_{ABCD} (3.4.10) is similarly unremarkable, although the problem is more subtle. The trace-free Einstein equations are, in terms of the unphysical and physical curvatures,

[11] For example, a tetrad in which $\eth u \approx 0$

$$\Phi_{ABA'B'} + \Omega^{-1}\nabla_{A'(A}N_{B)B'} = 4\pi G\hat{T}_{ABA'B'}, \tag{3.4.14a}$$
$$\hat{\Phi}_{ab} = 4\pi G\hat{T}_{ab}. \tag{3.4.14b}$$

The first of these (3.4.14a) is not conformal. The second (3.4.14b) is similarly non-conformal under conformal transformations of the physical metric, $\hat{g}_{ab} \mapsto \Theta^2\hat{g}_{ab}$, but it *is* conformal under transformations of the unphysical metric $\Omega \mapsto \Theta\Omega$. By the same logic, we may make the rather obvious remark:

Remark. Any function of the physical metric is independent of the conformal completion.

Even though the unphysical Ricci tensor (3.4.14a) is not conformal, the conformal gravitational field equation (3.4.10) is conformal under conformal transformations of *both* the physical *and* unphysical metrics. This fact, though remarkable, is far from unique to Einstein's equations.

Proposition 10. *Let $\mathscr{M}$ be asymptotically flat, or satisfy conditions 1-3 of definition 6 and the strong asymptotic Einstein condition. Then all but two of the leading order physical curvature tensor components on $\mathscr{I}$ can be expressed in terms of the unphysical curvature in a conformal manner,*

$$\Omega^{-1}\hat{\Psi}_{ABCD} = \Omega^{-1}\Psi_{ABCD} := \psi_{ABCD}, \tag{3.4.15a}$$
$$\Omega^{-2}N^{B'}_{(A}\hat{\Phi}_{BC)A'B'} \approx \nabla^{D}_{A'}\psi_{ABCD}. \tag{3.4.15b}$$

Note that, as an important example, the trace of Einstein's equations, $\hat{\Lambda} = \frac{1}{3}\pi\hat{T} + \frac{1}{6}\lambda$ cannot be be expressed in terms of the unphysical curvature in a conformal manner.

From proposition 10 it is clear that the majority of the components of most quantities constructed from the physical curvature tensor can be made manifestly conformal. As a non-trivial example, consider the following field equations:

$$\hat{B}_{ab} + 6\hat{\Lambda}\hat{g}_{ab} - \lambda\hat{g}_{ab} = -2\pi G\hat{T}_{ab}, \tag{3.4.16}$$

where $B_{ab} := (\nabla^c\nabla^d - \frac{1}{2}R^{cd})C_{acbd}$ is the symmetric, trace-free, divergence-free, and, most importantly, conformal *Bach tensor*. The trace (3.4.16) is equivalent to the trace of Einstein's equations. Coincidently, the De Sitter-Schwarzschild space-time is a vacuum solution to these equations (3.4.16). In vacuum, its trace-free components are given in terms of the curvature spinors as

$$B_{ABA'B'} = (\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD} + (\nabla^{C'}_{A}\nabla^{D'}_{B} + \Phi^{C'D'}_{AB})\bar{\Psi}_{A'B'C'D'} = 0, \tag{3.4.17}$$

from which it is easy to derive its asymptotics:

$$\begin{aligned}(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD} &\approx -\nabla^{C}_{A'}(N^{D}_{B'}\psi_{ABCD})\\ &\approx -N^{C}_{A'}\nabla^{D}_{B'}\psi_{ABCD} \approx -\Omega^{-2}N^{C}_{A'}N^{C'}_{(A}\hat{\Phi}_{BC)B'C'},\end{aligned} \tag{3.4.18}$$

by proposition 10. Hence $\Omega^{-2}\hat{\Phi}_{ABA'B'}\iota^{A'}\iota^{B'} \approx 0$, so that the vacuum Einstein equations tangential to $\mathscr{I}$ hold. The components of the Bach tensor tangential to $\mathscr{I}$ vanish, since $\Psi_{ABCD} \approx 0$ and $\iota^{A'}\nabla_{AA'}$ is intrinsic to $\mathscr{I}$: $\iota^{A'}\iota^{B'}(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD} \approx 0$. Dividing by Ω we find

$$\begin{aligned}\Omega^{-1}\iota^{A'}\iota^{B'}(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\Psi_{ABCD} &\approx \iota^{A'}\iota^{B'}(\nabla^{C}_{A'}\nabla^{D}_{B'} + \Phi^{CD}_{A'B'})\psi_{ABCD}\\ &\approx (\iota^{A'}\iota^{B'}\nabla^{C}_{A'}\nabla^{D}_{B'} + \bar{N}\iota^{C}\iota^{D})\psi_{ABCD}\\ &\approx \bar{N}\iota^{C}\iota^{D}\psi_{ABCD}.\end{aligned} \tag{3.4.19}$$

Therefore, either $N \approx 0$ or $\psi_2 \approx \psi_3 \approx \psi_4 \approx 0$. Recall from proposition 9 that $\psi_3 \approx \eth N$ with an appropriate Ω, which would then also imply that $N \approx 0$ since ψ_3 has negative spin weight. Hence, $N \approx 0 \approx \psi_3 \approx \psi_4$. The remaining field equations that are tangential to $\mathscr{I}$ simplify to

$$þ'\psi_2 \approx 0 \tag{3.4.20a}$$

$$þ'\psi_1 - \eth\psi_2 \approx 0 \tag{3.4.20b}$$

$$þ'^2\psi_0 - \eth^2\psi_2 \approx 0. \tag{3.4.20c}$$

4 Charges in General Relativity

Formulating a satisfactory precise definition of energy in General Relativity has proven to be extremely difficult, as the problem has remained unsolved for over a hundred years since GR's inception. Instead of entering directly into the stress-energy T_{ab}, gravitational energy manifests as an obstruction to integrating the local conservation laws $\nabla_b T^{ab} = 0$ to yield globally conserved charges. The problem is that energy is part of the energy-momentum four-vector $p^a = T^a{}_b t^b$ as seen locally by an observer with four-velocity t^a. There is, in general, no way to 'sum' all of these local four-vectors together because they are elements of different tangent spaces. In Minkowski space, parallel transport provides a unique way to identify vectors from different tangent spaces since without any curvature, parallel transport is path-independent. Another way to put this is that there exist *constant* vector fields k^a, which are translation Killing fields[12] whose components are constant in standard Cartesian coordinates. With these translation vector fields, we can construct the conserved currents $J^a = T^a{}_b k^b$, which may be integrated to yield four conserved charges.

It might seem like a senseless exercise to insist on defining an energy concept in curved space-time. After all, conservation of energy, which is energy's defining property in flat space-time, is also the property that makes energy physically interesting. However, there are several examples of space-times and special cases where a 'mass' parameter exists that plays a similar role to mass in Newtonian gravity. For example, it seems reasonable to say that in the Schwarzschild space-time with mass parameter m the regions of space containing the 'hole' have energy m, while all other regions have no energy. More notably, there exist similar 'mass' parameters in asymptotically flat space-times defined as integrals at spatial and null infinity, that quantify, in some well-defined sense how much the sources of these space-times gravitate. We would therefore like to redefine the mass or energy of a region of space-time as, roughly speaking, the degree to which it gravitates.[13]

Remarkably — even without a precise definition — gravitational energy can be understood in a wonderfully intuitive manner [Penrose, 1983]. In the next section, I will first elucidate the geometric origin of energy in General Relativity. Then, I will discuss a quasi-local definition of energy-momentum and angular momentum due to Penrose [Penrose, 1982], where spinors are once again proven to be invaluable. Along the way, I will use the geometric intuition we have built to derive the Bondi mass at null infinity. Unfortunately, Penrose's definition is not yet complete, but in the special cases where it does apply, its results are physically satisfying [Tod, 1983, Tod, 1986]. In particular, we will show that Penrose's definition of angular momentum at null infinity does not suffer from some of the problems of earlier attempts[14] at such a definition.

4.1 The geometric origin of gravitational energy

Consider a congruence l^a of hypersurface orthogonal affinely parameterised null geodesics (so that $þ = D$ and $\rho - \bar{\rho} = 0 = \kappa$). The dynamics of the congruence are described by the Raychaudhuri equations:

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}, \tag{4.1.1a}$$

$$D\sigma = 2\rho\sigma + \Psi_0. \tag{4.1.1b}$$

The effect of a localised source of Ricci curvature Φ_{00} along one of the generators γ, which may, for example, arise from a point mass intersecting γ, is to cause a jump in the convergence ρ. In other words, a localised energy source along a bundle of light rays has the effect of a *lens*.

Suppose there are two point masses m_1 and m_2 along γ separated by an affine distance $d < 1/m_1$ (see figure 7). Let r be an affine parameter along γ, and suppose m_1 is at $r = 0$, so that along γ, $\Phi_{00}(r) = m_1\delta(r) + m_2\delta(r-d)$.

[12] A vector field k^a is a *translation* Killing field if $\nabla_a k_b = 0$.

[13] If the distance between nearby parallel geodesics passing through some region gets smaller, this region has positive energy, even in the absence of matter energy.

[14] See, for example, [Bramson, 1975].

Assume, for simplicity, that $\Psi_0 = 0$ and the congruence consists of parallel rays before $r = 0$ (so that $\rho(r < 0) = 0 = \sigma$). In vacuum, (4.1.1a) has the shear-free solution[15]

$$\rho = \frac{\rho_0}{1 - \rho_0(r - r_0)}. \tag{4.1.2}$$

On the interval $0 < r < d$, $\rho = m_1(1 - m_1 r)^{-1}$, where we used the initial condition $\rho_0 = m_1$ at $r_0 = 0$ due to the fact that Φ_{00} causes a jump m_1 of the convergence at $r = 0$. At $r = d + \delta r$, we find from the Raychaudhuri equation (4.1.1a) that

$$\begin{aligned} \rho(d + \delta r) &= \rho(d - \delta r) + (\rho(d))^2 \delta r + m_2 \\ &= \frac{m_1}{1 - m_1 d} + m_2 + \mathcal{O}(\delta r) = \frac{m_1 + m_2 - m_1 m_2 d}{1 - m_1 d} + \mathcal{O}(\delta r), \end{aligned} \tag{4.1.3}$$

hence, using (4.1.2) with initial conditions (4.1.3) we find

$$\rho(r > d) = \frac{m_1 + m_2 - m_1 m_2 d}{1 - m_1 d - (m_1 + m_2 - m_1 m_2 d) r}. \tag{4.1.4}$$

The net focusing effect of the two masses, therefore, is equivalent to the focusing effect due to a single source[16] with mass $m_1 + m_2 - m_1 m_2 d$.

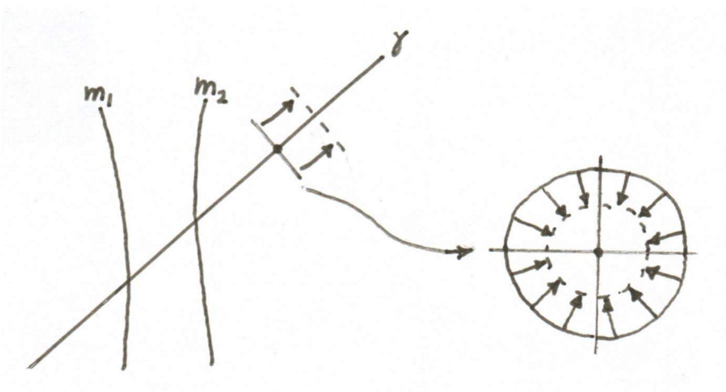


Figure 7: A light ray γ passes through two masses m_1 and m_2 (as seen on the left), causing nearby rays to get focused (as seen on the right).

Remark.

1. The way in which the two masses add up to effectively create a single mass is precisely the thin lens addition law

$$P_{tot} = P_1 + P_2 - P_1 P_2 d, \tag{4.1.5}$$

where $P = 1/f$ is the lens's power, which is the reciprocal of its focal length f.

2. The term $-m_1 m_2 d$ has the same form as the one-dimensional Newtonian gravitational potential energy of two masses m_1 and m_2 separated by a distance d.

3. The net focusing effect is a *non-local* phenomenon: the non-linear term $-m_1 m_2 d$ can be found by considering the combined two mass system, but cannot be found by examining the two masses separately.

[15] Which can be found, for example, through separation of variables: $\frac{d\rho}{\rho^2} = dr$, and then integrating. Note that $D = \frac{d}{dr}$.

[16] (4.1.4) has the form (4.1.2) with $\rho_0 = m_1 + m_2 - m_1 m_2 d$ and $r_0 = m_1 d(m_1 + m_2 - m_1 m_2 d)^{-1}$.

We can similarly consider the effect of a localised source of Weyl curvature Ψ_0 along γ, for example due to a burst of gravitational radiation intersecting γ. This burst will cause a jump in the shear σ, which will cause a circular bundle of rays around γ to be distorted into an ellipse. Along the major axis, the burst has the effect of a negatively focusing lens, while the rays along the minor axis are focused. For simplicity, let us examine the net focusing effect due to two such bursts separated by a distance d (as depicted in figure 8), the second of whose major axis is rotated by $\frac{\pi}{2}$ with respect to the first burst's major axis. This will ensure that a shear-free beam after passing through both bursts will have approximately zero shear. Suppose, once again, that $\rho(r<0)=0=\sigma(r<0)$. Let the first burst occur at $r=0$ and the second at $r=d$, so that Ψ_0 is given by $\Psi_0=\Psi\delta(r)-\Psi\delta(r-d)$ for some (complex) constant Ψ. Then for small distances,

$$D\sigma \approx \Psi\delta(r)-\Psi\delta(r-d), \tag{4.1.6}$$

$$\text{and} \qquad D^2\rho \approx \sigma(\bar{\Psi}\delta(r)-\bar{\Psi}\delta(r-d))+\bar{\sigma}(\Psi\delta(r)-\Psi\delta(r-d)). \tag{4.1.7}$$

Upon integrating, we find that

$$\sigma(0<r<d)\approx\Psi, \tag{4.1.8}$$

$$\text{so that} \qquad (D\rho)(0<r<d)\approx\Psi\bar{\Psi}. \tag{4.1.9}$$

Finally, integrating (4.1.9) yields

$$\rho(d)\approx\Psi\bar{\Psi}d. \tag{4.1.10}$$

After passing through both bursts, $\sigma(r>d)\approx 0$ so that

$$\rho(r>d)\approx\frac{\Psi\bar{\Psi}}{1-\Psi\bar{\Psi}d(r-d)}. \tag{4.1.11}$$

We therefore find that the net focusing effect due to the two bursts of gravitational radiation is equivalent to a single matter source with mass $\Psi\bar{\Psi}d$.

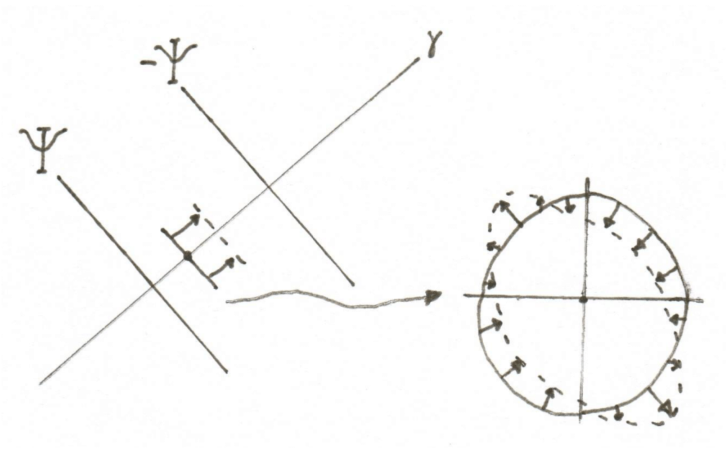


Figure 8: A light ray γ passes through a burst of gravitational radiation Ψ (as seen on the left), causing a circular arrangement of nearby rays to get distorted into an ellipse (as seen in the spatial picture on the right). After passing through a second burst $-\Psi$, these nearby rays will get distorted back into a spherical shape. There is a residual focusing effect.

Remark. The net focusing effect due to the Weyl curvature is

1. *entirely non-local*: The effect of local energy is to cause a jump in convergence in a bundle of rays, while leaving the shear unaffected. While traveling through the bursts of gravitational radiation, it is only after traveling through *both* bursts that the net effect is to cause a jump in convergence in a bundle of rays, while leaving the shear unaffected.

2. *positive.* This means that gravitiational radiation has positive energy. Conguences tangent to light cones deviate from flat-space light cones through the term $\sigma\bar{\sigma}+\Phi_{00}$ in (4.1.1a). Assuming the null energy condition holds, $\Phi_{00}\geq 0$, so that $D\rho \geq \rho^2+\sigma\bar{\sigma}$. Hence, the effect of matter is to increase the convergence. The shear term $\sigma\bar{\sigma}$ is also positive, $\sigma\bar{\sigma}\geq 0$, so that shear has the same effect as a positive energy density.

The focusing argument laid down here assumes a situation not far from Minkowski. In general, the shear does not enter (4.1.1a) as directly as Ricci curvature does, since the convergence also plays a role in the evolution of the shear. It is therefore not clear 'how much' of the focussing is due to the Weyl curvature, and how much is due to the shear. There is, however, an important situation where we can make this distinction, and where the small distance approximation (which is effectively equivalent to a small convergence and shear approximation) becomes exact. Let $\mathscr{C}_r$ be a family of geodesic null congruences n^a of rays running approximately parallel to null infinity in a small neighbourhood of $\mathscr{I}^+$, so that $\mathscr{C}_\infty := \mathscr{C}\subset\mathscr{I}^+$ (see figure 9). Consider the difference in total convergence between two cuts $\mathscr{S}'_r$ and $\mathscr{S}_r$ of $\mathscr{C}_r$, $\oint\rho'\mathscr{S}'_r - \oint\rho'\mathscr{S}_r$. Even though each integral diverges as $r\to\infty$, this difference is finite, and given by

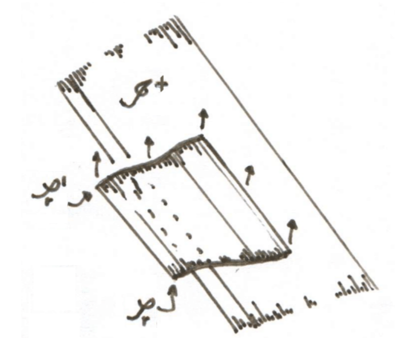


Figure 9: A small piece of $\mathscr{I}^+$. As $r\to\infty$, the null hypersurface bounded by $\mathscr{S}$ and $\mathscr{S}'$ approaches a piece of $\mathscr{I}^+$.

$$\oint\rho'\mathscr{S}'_r - \oint\rho'\mathscr{S}_r \approx \int(D'-2\rho')\rho'\mathscr{N} \approx \int(\sigma'\bar{\sigma}'+\Phi_{22}-\rho'^2)\mathscr{N}, \tag{4.1.12}$$

where $\mathscr{N}$ is the piece of $\mathscr{I}^+$ bounded by $\mathscr{S}'$ and $\mathscr{S}$. To leading ($\mathscr{O}(r^{-1})$) order, ρ' is the same for all asymptotically flat space-times. This is because at infinity, the congruence n^a generates $\mathscr{I}^+$, which is a conformally flat light-cone for all asymtotically flat space-times. First order deviations from flat space are given by $\int\sigma'\bar{\sigma}'+4\pi G T_{ab}n^a n^b\mathscr{N}$. The integrand is conformal, so that in the conformally re-scaled space-time of the previous chapter,

$$\int(\hat{\sigma}'\hat{\bar{\sigma}}'+4\pi G\hat{T}_{ab}n^a n^b)\hat{\mathscr{N}} = \int(\Omega^{-2}\sigma'\bar{\sigma}'+4\pi G T_{ab}n^a n^b)\mathscr{N} \approx \int(\text{þ}\sigma'\text{þ}\bar{\sigma}'+4\pi G T_{ab}n^a n^b)\mathscr{N}. \tag{4.1.13}$$

(Hatted quantities refer, here, to the physical space-time.) The Ricci identities furthermore give $\text{þ}\sigma'\approx -\Phi_{20} :\approx -N$, so that

Proposition 11. *The energy $m(\mathscr{N})$ of a region $\mathscr{N}\subset\mathscr{I}^+$ bounded by two cuts of $\mathscr{I}^+$ is*

$$m(\mathscr{N}) = \int((4\pi G)^{-1}N\bar{N}+T_{ab}n^a n^b)\mathscr{N}. \tag{4.1.14}$$

Perhaps it would have been more appropriate to label proposition 11 as a definition rather than a proposition, since we have not clearly defined energy. What makes this a good definition of energy on $\mathscr{I}$ is that in the limit, the dynamics of σ' are independent of ρ',[17] so that $\sigma'\bar{\sigma}'$ has precisely the same effect on the total convergence of $\mathscr{I}$ as Φ_{22}.

4.1.1 The non-locality of gravitational radiation

It is tempting to call $(4\pi G)^{-1}N\bar{N}$ 'the energy flux of gravitational radiation' in an attempt to generalise $T_{ab}n^a n^b$, which is the energy flux of non-gravitational radiation. There is a tantalising similarity with the electromagnetic energy flux $(2\pi)^{-1}\phi_2\bar{\phi}_2 = T^{EM}_{ab}n^a n^b$. The News N, however, is a non-local quantity. This means that N cannot be computed at any one point of $\mathscr{I}^+$. To see this, note that N is related to the Weyl curvature by $\text{þ}'N\approx\psi_4$, $\eth N\approx\psi_3$ so that N can be computed as $N\approx\eth^\dagger\psi_3$ or $N\approx\int_{-\infty}^{u}\psi_4\,du$ which are integrals over an entire cut or generator of $\mathscr{I}^+$. The News is also given as the component $N\approx\Phi_{20}$ of the unphysical Ricci curvature. However, the

[17] Indeed, $D'\sigma' = \Psi_4+\mathscr{O}(r^{-2})$ since both σ' *and* ρ' vanish on $\mathscr{I}$.

unphysical Ricci curvature R_{ab} is a function of the conformal factor Ω. Explicitly, Φ_{20} is given by $\Phi_{20} \approx \Omega^{-1}\eth'^2\Omega$ where Ω is defined by the *global* condition that the re-scaled metric on each cut of $\mathscr{I}$ is a unit sphere.

It is even possible to arrange for N to be non-zero in flat space, in some cases. Indeed, let U be a flat small neighbourhood of some point $p \in \mathscr{I}^+$. Then if N is constant on $U \cup \mathscr{I}^+$, $\psi_4 \approx 0 \approx \psi_3$ on U. It is for this reason that in proposition 11 we required Σ to be bounded by two cuts of $\mathscr{I}^+$: if the News N is non-zero somewhere on a cut $\mathscr{S}$, ψ_3 must necessarily also be non-zero somewhere on $\mathscr{S}$,[18] so that $\mathscr{S}$ cannot be a cut of flat space. This ensures that $m(\mathscr{N})$ is necessarily zero in flat space.

4.1.2 Mass at $\mathscr{I}^+$

From proposition 11 we can deduce a definition of mass on each cut of $\mathscr{I}^+$. Suppose $\mathscr{N}$ is bounded by two cuts $\mathscr{S}'$ and $\mathscr{S}$ of $\mathscr{I}^+$, where $\mathscr{S}'$ lies in the future of $\mathscr{S}$. Let $m(\mathscr{S})$ be the total energy of a non-time-like hypersurface intersecting $\mathscr{I}^+$ in $\mathscr{S}$. Suppose $m(\mathscr{S})$ is conserved in the sense that the energy does not depend on the particular choice of hypersurface, then $m(\mathscr{S}) = m(\mathscr{S}') + m(\mathscr{N})$. With these requirements, we can uniquely define an energy function intrinsic to $\mathscr{I}^+$ with these properties:

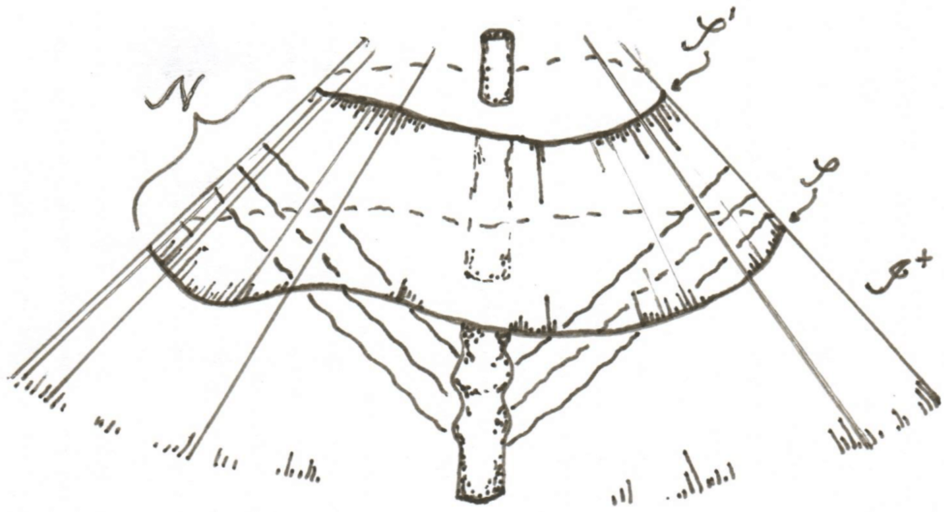


Figure 10: The mass difference $m(\mathscr{S}) - m(\mathscr{S}')$ between two cuts $\mathscr{S}$ and $\mathscr{S}'$ is given by the total amount of radiation $m(\mathscr{N})$ escaping through $\mathscr{N}$. Adapted from [Penrose & Rindler, 1986].

Theorem 7. *Let $m(\mathscr{S})$, the total energy of $\mathscr{S}$, be a family of real valued functions on cuts of $\mathscr{I}^+$ that vanish in flat space and are conserved in the sense that for all cuts $\mathscr{S}'$ and $\mathscr{S}$ bounding $\mathscr{N} \subset \mathscr{I}^+$, where $\mathscr{S}'$ lies in the future of $\mathscr{S}$, $m(\mathscr{S}') - m(\mathscr{S}) = -m(\mathscr{N})$, i.e.*

$$m(\mathscr{S}') - m(\mathscr{S}) = -\int ((4\pi G)^{-1} N\bar{N} + T_{ab}n^a n^b)\mathscr{N}. \tag{4.1.15}$$

Then

$$m(\mathscr{S}) = -\frac{1}{4\pi G}\oint (\psi_2 - \sigma N)\mathscr{S}. \tag{4.1.16}$$

[18] N has spin weight $s = -2$ and can therefore not be a non-zero constant on a sphere.

Proof. Let $\mathscr{S}$ be a cut with constant Bondi parameter u. The mass flux law (4.1.15) yields the following differential equation for $m(u)$:

$$\dot{m}(u) = -\oint ((4\pi G)^{-1} N\bar{N} + T_{ab} n^a n^b)\mathscr{S} \tag{4.1.17}$$

$$\text{so that} \quad \frac{d}{du}\left(m - \oint (4\pi G)^{-1} \sigma N \mathscr{S}\right) = -\oint ((4\pi G)^{-1} \sigma \psi_4 + T_{ab} n^a n^b)\mathscr{S}, \tag{4.1.18}$$

where we used integration by parts[19]

$$\int N\bar{N}\mathscr{N} = \int \sigma\psi_4 \mathscr{N} - \oint \sigma N \mathscr{S}' + \oint \sigma N \mathscr{S}. \tag{4.1.19}$$

The 0011 component of the asymptotic Einstein equations (3.4.10) is $\dot{\psi}_2 - \eth\psi_3 - \sigma\psi_4 \approx 4\pi G T_{ab} n^a n^b$ so that $m = -(4\pi G)^{-1} \oint \psi_2 - \sigma N \mathscr{S} + c$, where $c \in \mathbb{C}$ is an arbitrary constant, solves (4.1.18).[20] Since in Minkowski space $\psi_2 \approx 0 \approx N$, the constant of integration c must be zero in order to make the mass vanish in flat space. Hence

$$m(\mathscr{S}) = -\frac{1}{4\pi G} \oint (\psi_2 - \sigma N)\mathscr{S}. \tag{4.1.20}$$

Finally, to show that (4.1.20) is real, we use the asymptotic Bianchi identities (proposition 9), $\psi_2 - \bar{\psi}_2 - \sigma N + \bar{\sigma}\bar{N} \approx \eth'^2\sigma - \eth^2\bar{\sigma}$. Since the right-hand-side is a total divergence, its integral over a closed surface vanishes, so that

$$m(\mathscr{S}) - \bar{m}(\mathscr{S}) = -\frac{1}{4\pi G} \oint (\eth'^2\sigma - \eth^2\bar{\sigma})\mathscr{S} = 0. \tag{4.1.21}$$

□

This mass (4.1.20) is called the *Bondi mass*, and proposition 11 (or rather, (4.1.17)) is called *Bondi's mass loss theorem*. These were first found using very different arguments![21]

4.2 Penrose's quasi-local mass

As we mentioned at the start of this section, there is, in general, no way to 'integrate' the local energy-momentum density $p^a = T^a{}_b t^b$ as seen by a congruence of observers with four-velocity t^a. In Minkowski space, there does exist an absolute parallelism given by the translation Killing vector fields. We can obtain the energy-momentum and angular momentum as conserved charges associated with the ten conserved currents $J^a = T^a{}_b \xi^b$ where ξ^a is a Killing vector. The total energy-momentum vector $p_{\mathbf{a}}(\Sigma)$ and total angular momentum tensor $M_{\mathbf{ab}}(\Sigma, O)$ of a space-like hypersurface Σ with respect to some origin O is given by

$$k^{\mathbf{a}} p_{\mathbf{a}}(\Sigma) + \underset{1}{k}{}^{\mathbf{a}} \underset{2}{k}{}^{\mathbf{b}} M_{\mathbf{ab}}(\Sigma, O) := \int_\Sigma (k^a + \underset{1}{k}{}^{[a} \underset{2}{k}{}^{b]} x_b) T_a{}^c \epsilon_{cdef}, \tag{4.2.1}$$

where $k^{\mathbf{a}}$, $\underset{1}{k}{}^{\mathbf{a}}$ and $\underset{2}{k}{}^{\mathbf{a}}$ are translation Killing vector fields whose values at each point are given by k^a, $\underset{1}{k}{}^a$ and $\underset{2}{k}{}^a$. x^a is a position vector field, i.e. a solution of $\nabla_a x^b = g_a{}^b$ that vanishes at O. Note that k^a and $\underset{1}{k}{}^{[a} \underset{2}{k}{}^{b]} x_b$ span the full ten-dimensional space of Killing vector fields. (4.2.1) defines a linear map and a skew bilinear map on Σ relative to some origin O from translation Killing vector fields to real numbers. Hence, it defines a dual translation vector field $p_{\mathbf{a}}(\Sigma)$ and a skew translation tensor field $M_{\mathbf{ab}}(\Sigma, O)$.

[19] Recall that $\psi_4 \approx \dot{N}$ and $N = -\dot{\bar{\sigma}}$.
[20] Note that $\oint \eth\psi_3 \mathscr{S} = 0$.
[21] Compare with the original work by to Bondi and collaborators [Bondi et al., 1962, Sachs, 1962a].

Unlike the gravitational source T_{ab}, the conserved electromagnetic charge current j^a may be integrated over a three-volume to yield a conserved charge, and therefore does not require any Killing fields:[22]

$$q := \int_\Sigma \star \mathbf{j}. \tag{4.2.2}$$

We may express the charge as a surface integral over the field strength using Maxwell's equations. In terms of differential forms, they are given by $d \star \mathbf{F} = 4\pi \star \mathbf{j}$, so that

$$q := \int_\Sigma \star \mathbf{j} = \frac{1}{4\pi} \oint_{\partial\Sigma} \star \mathbf{F}. \tag{4.2.3}$$

The field strength $\mathbf{F}$ may be written as a symmetric two-spinor

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \phi_{A'B'}\epsilon_{AB}, \tag{4.2.4}$$

$$\text{and} \qquad (\star\mathbf{F})_{ab} = i\phi_{AB}\epsilon_{A'B'} - i\phi_{A'B'}\epsilon_{AB}, \tag{4.2.5}$$

$$\text{so that} \qquad F_{ab} - i(\star\mathbf{F})_{ab} = 2\phi_{AB}\epsilon_{A'B'} \tag{4.2.6}$$

Since $d\mathbf{F} = 0$ by the homogeneous Maxwell equations, $\oint \mathbf{F} = 0$ so that (4.2.3) can be rewritten as

$$q = \frac{i}{2\pi} \oint_{\partial\Sigma} \phi_{AB}\epsilon_{A'B'}. \tag{4.2.7}$$

We can use this alternative description of electric charge as a two-surface integral of the field to generate gravitational charges. The Weyl spinor Ψ_{ABCD} in vacuum satisfies a higher spin version of Maxwell's vacuum equations. The Bianchi identities are $\nabla^A_{A'}\Psi_{ABCD} = 0$ (compare Maxwell's vacuum equations $\nabla^A_{A'}\phi_{AB} = 0$). We can lower the spin of the Weyl spinor by contracting it with some spinor ω^A satisfying $\nabla^{(A}_{A'}\omega^{B)} = 0$, so that $\Psi_{ABCD}\omega^C\omega^D$ satisfies Maxwell's vacuum equations. Integrating these forms over a closed two-surface will then yield gravitational conserved charges:

$$q_{\mathbf{AB}}\omega^{\mathbf{A}}\omega^{\mathbf{A}} := \frac{i}{4\pi G} \oint_{\partial\Sigma} \Psi_{ABCD}\omega^C\omega^D\epsilon_{A'B'}. \tag{4.2.8}$$

(4.2.8), together with the equation $\nabla^{(A}_{A'}\omega^{B)} = 0$ defining ω^A, is the starting point of Penrose's definition of energy momentum and angular momentum in general relativity [Penrose, 1982]. Presently, $q_{\mathbf{AB}} \equiv 0$ since, by the commutator equations,

$$\nabla^{A'(A}\nabla^B_{A'}\omega^{C)} = \Psi^{ABCD}\omega_D, \tag{4.2.9}$$

so that $\nabla^{(A}_{A'}\omega^{B)} = 0$ has non-zero solution if and only if Ψ_{ABCD} is type N, in which case the integrand of (4.2.8) vanishes.

Before we try to generalise (4.2.8), let us first explore its connection to (4.2.1).

4.2.1 Twistors in $\mathbb{M}$

A *Twistor* is a solution ω^A to the *Twistor equation*

$$\nabla^{(A}_{A'}\omega^{B)} = 0. \tag{4.2.10}$$

[22] Recall that in order to integrate the gravitational source T_{ab} to yield globally conserved charges, we needed to construct conserved currents $J^a = T^a{}_b\xi^b$ where ξ is a Killing vector.

In Minkowski space $\mathbb{M}$, its solutions form a four complex dimensional vector space $\mathbb{T}^\alpha$. It is easy to see[23] that the solutions to (4.2.10) are given by

$$\omega^A = \mathring{\omega}^A - i x^{AA'} \pi_{A'}, \tag{4.2.11}$$

where we have introduced a circle above a spinor, for example $\mathring{\xi}^A$ to mean 'ξ^A evaluated at O'. $\mathring{\omega}^A$ and $\pi_{A'}$ are constant spinor fields (the factor $-i$ will be convenient to us later), and x^a is a position vector in $\mathbb{M}$ relative to some origin O. A Twistor $\mathsf{Z}^\alpha \in \mathbb{T}^\alpha$ is therefore given, at any point, by two spinor fields $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$, where $\pi_{A'} = \frac{1}{2} i \nabla_{AA'} \omega^A$.

The dual Twistor space $\mathbb{T}_\alpha$ consists of pairs $\mathsf{W}_\alpha = (\lambda_A, \mu^{A'})$ at O, so that the scalar product is

$$\mathsf{Z}^\alpha \mathsf{W}_\alpha = \omega^A \lambda_A + \pi_{A'} \mu^{A'}. \tag{4.2.12}$$

We similarly want to describe $\mathbb{T}_\alpha$ as spinor *fields* on $\mathbb{M}$. To achieve this, we demand that (4.2.12) holds not just at O, but at every point. We then find that

$$\begin{aligned} \mathsf{Z}^\alpha \mathsf{W}_\alpha &= \mathring{\omega}^A \mathring{\lambda}_A + \mathring{\pi}_{A'} \mathring{\mu}^{A'} \\ &= \omega^A \lambda_A + \pi_{A'} \mu^{A'} = (\mathring{\omega}^A - i x^{AA'} \mathring{\pi}_{A'}) \lambda_A + \mathring{\pi}_{A'} \mu^{A'}. \end{aligned} \tag{4.2.13}$$

It is easy to see that, in order to make (4.2.12) constant, we require $\mu^{A'} = \mathring{\mu}^{A'} + i x^{AA'} \lambda_A$ and $\lambda_A = \mathring{\lambda}_A$, which are the solutions to $\nabla_A^{A'} \mu^{B'} = i \epsilon^{A'B'} \lambda_A$, and in particular the complex conjugate of the Twistor equation, $\nabla_A^{(A'} \mu^{B')} = 0$. *Complex conjugation* therefore provides a map between $\mathbb{T}^\alpha$ and $\mathbb{T}_\alpha$.

4.2.2 The Kinematic Twistor

Notice that (4.2.8) maps two Twistors to a complex number bilinearly, and therefore defines a (symmetric) Twistor $\mathsf{Q}_{\alpha\beta} \in \mathbb{T}_{(\alpha\beta)}$. Of course, in Minkowski space $\mathsf{Q}_{\alpha\beta} = 0$, and in curved space-times $\mathbb{T}^\alpha$ is generally not well-defined. Consider therefore, instead, the linearized Einstein equations on $\mathbb{M}$. Let K_{abcd} be the linearized Riemann tensor. K_{abcd} satisfies the differential Bianchi identity $\nabla_{[a} K_{bc]de} = 0$ and the linearized Einstein equations $K_{acb}{}^c - \frac{1}{2} \eta_{ab} K_{cd}{}^{cd} = 8\pi G T_{ab}$. Let the totally symmetric spinor field ϕ_{ABCD} be the Linearized Weyl spinor. By the differential Bianchi identity, this spinor satisfies the mass-less field equations, so that $\phi_{ABCD} \omega^C \omega^D$ satisfies Maxwell's equations, which yield conserved gravitational charges. The following proposition relates these charges back to energy-momentum and angular momentum:

Proposition 12. *The form* $(\Xi)_{abc} = -\xi^e T_e{}^d \epsilon_{abcd}$ *is exact, i.e.* $\Xi = d\boldsymbol{\Theta}$*, where* $\boldsymbol{\Theta}$ *is given by*

$$(\boldsymbol{\Theta})_{ab} = K_{abcd} \epsilon^{cdef} Q_{ef}, \tag{4.2.14}$$

where $Q^{ab} = i\sigma^{AB} \epsilon^{A'B'} - i\bar{\sigma}^{A'B'} \epsilon^{AB}$ *is an anti-symmetric tensor, satisfying* $\nabla^{(a} Q^{b)c} - \nabla^{(a} Q^{c)b} + g^{a[b} \nabla_d Q^{c]d} = 0$. *It follows that* $\xi^a = \frac{1}{3} \nabla_b Q^{ab}$ *is a Killing vector field, and that* σ^{AB} *satisfies the valence two Twistor equation* $\nabla_{A'}^{(A} \sigma^{BC)} = 0$.

Proof. We will treat the first two and last two indices of K_{abcd} as differential form indices. The Hodge star operation on the first and last pair will be denoted by $\star\mathbf{K}$ and $\mathbf{K}\star$, respectively. By the differential Bianchi identity,

$$(d\boldsymbol{\Theta})_{abc} = \nabla_{[a} Q^{de} (\mathbf{K}\star)_{bc]de} \qquad \text{so that} \qquad (\star d\boldsymbol{\Theta})_a = \tfrac{1}{3} \nabla^b Q^{cd} H_{abcd}, \tag{4.2.15}$$

[23] $\nabla_{A'}^A \nabla_{B'}^B \omega^C$ is skew in BC and, since $\mathbb{M}$ is flat, also skew in AC because we may commute the covariant derivatives. Hence, being skew in three indices, $\nabla_{A'}^A \nabla_{B'}^B \omega^C = \nabla_{A'}^{[A} \nabla_{B'}^B \omega^{C]} = 0$, so that $\nabla_{AA'} \omega^B = -i\epsilon_A{}^B \pi_{A'}$, where $\pi_{A'}$ is an arbitrary contant spinor. This equation may then straightforwardly be integrated.

where we defined $\mathbf{H} = \star\mathbf{K}\star$. By the first Bianchi identity,

$$\nabla^b Q^{cd} H_{abcd} = 2\nabla^c Q^{bd} H_{abcd} \qquad \text{so that} \qquad \nabla^b Q^{cd} H_{abcd} = \tfrac{4}{3}\nabla^{(b} Q^{c)d} H_{abcd}. \tag{4.2.16}$$

Finally, using $\nabla^{(a}Q^{b)c} - \nabla^{(a}Q^{c)b} + g^{a[b}\nabla_d Q^{c]d} = 0$, we find

$$\begin{aligned}(\star d\boldsymbol{\Theta})_a = \tfrac{4}{9}\nabla^{(b}Q^{c)d}H_{abcd} &= -\tfrac{2}{9}g^{bc}\nabla_e Q^{de}H_{abcd}\\ &= -\tfrac{2}{3}g^{bc}\xi^d H_{abcd}\\ &= \tfrac{2}{3}\xi^d(K_{abd}{}^b - \tfrac{1}{2}\eta_{ad}K_{bc}{}^{bc}) = \tfrac{16\pi G}{3}T_{ab}\xi^b.\end{aligned} \tag{4.2.17}$$

In going to the last line, we used that H_{abcd} is related to K_{abcd} by multiplying the Weyl part by -1, and reversing the trace of the Ricci part. This means that $H_{abc}{}^b = K_{abc}{}^b - \frac{1}{2}\eta_{ac}K_{bd}{}^{bd}$. □

In vacuum, we therefore find that

$$\frac{1}{8\pi G}\oint \phi_{ABCD}\sigma^{CD}\epsilon_{A'B'} + \bar{\phi}_{A'B'C'D'}\bar{\sigma}^{C'D'}\epsilon_{AB} = \int \xi^a T_a{}^b \epsilon_{bcde}. \tag{4.2.18}$$

We can write σ^{AB} as the symmetric product of two Twistors, $\sigma^{AB} = \sum_{i,j} \underset{i}{\omega}^{(A}\underset{j}{\omega}^{B)}$, so that we may define a symmetric Twistor $\mathsf{A}_{\alpha\beta} \in \mathbb{T}_{(\alpha\beta)}$, which is given by

$$\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta := \frac{1}{8\pi G}\oint \phi_{ABCD}\omega^C\omega^D\epsilon_{A'B'}. \tag{4.2.19}$$

With (4.2.18), we can rewrite this as

$$\begin{aligned}\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta &= \int -\tfrac{2}{3}i(\nabla_{AA'}\omega^A\omega^B)T_B{}^{A'c}\epsilon_{cdef} = \int -2\omega^B\pi_{B'}T_B{}^{B'c}\epsilon_{cdef}\\ &= \int (\mathring{\omega}^B\pi^{B'} + i\epsilon^{AB}\pi^{A'}\pi^{B'}x_{AA'})T_{BB'}{}^c\epsilon_{cdef} = 2p_{\mathbf{A}}{}^{\mathbf{A'}}\omega^{\mathbf{A}}\pi_{\mathbf{A'}} + 2i\mu^{\mathbf{A'B'}}\pi_{\mathbf{A'}}\pi_{\mathbf{B'}},\end{aligned} \tag{4.2.20}$$

where in the last line we used (4.2.1). M^{ab} is given by $M^{ab} = \mu^{A'B'}\epsilon^{AB} + \bar{\mu}^{AB}\epsilon^{A'B'}$. (4.2.20) tells us what the different components of $\mathsf{A}_{\alpha\beta}$ are in terms of the standard basis $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$.

4.2.3 Two-surface Twistors

As we remarked earlier, the Twistor equation generally does not have non-zero solutions. We may regard this as a *feature* of General Relativity, since we would otherwise have a definition of energy in which vacuum regions don't contribute to the total energy, which is contrary to our physical intuition. We therefore should not expect an expression like $q_{\mathbf{AB}}$ (4.2.8) to yield conserved charges like it does in flat space. In order for $q_{\mathbf{AB}}$ (4.2.8) to provide us with a definition of energy-momentum and angular momentum at some two-surface $\mathscr{S}$, we merely need some four-dimensional space of spinor fields ω^A at $\mathscr{S}$, which may be identified with Twistors in the case that $\Psi_{ABCD} = 0$.

Penrose's suggestion [Penrose, 1982] is to consider only the components of the Twistor equation (4.2.10) involving derivatives tangential to $\mathscr{S}$. In GHP form, with l^a and n^a orthogonal to $\mathscr{S}$, these are

$$ð\omega^1 = \sigma\omega^0 \qquad \text{and} \qquad ð'\omega^0 = \sigma'\omega^1. \tag{4.2.21}$$

Its solution space can be shown to be at least four (complex) dimensional when $\mathscr{S}$ is a topological sphere, and may only have more than four (complex) dimensions in exceptional cases. When (4.2.21) has exactly four independent solutions, it defines a *two-surface Twistor space* which we will denote by $\mathbb{T}^\alpha(\mathscr{S})$.

Having found a Twistor concept that is usable in curved space-time, we may now simply adopt an expression of the form (4.2.20) as a definition of energy-momentum and angular momentum in General Relativity.

Definition 12. The *Kinematic Twistor* is a symmetric Twistor $\mathsf{A}_{\alpha\beta} \in \mathbb{T}_{(\alpha\beta)}$ given in GHP form by

$$\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta := -\frac{i}{4\pi G}\oint(\Psi_1-\Phi_{01})(\omega^0)^2+2(\Psi_2-\Phi_{11}-\Lambda)\omega^0\omega^1+(\Psi_3-\Phi_{21})(\omega^1)^2\mathscr{S}. \tag{4.2.22}$$

By construction, the Kinematic Twistor has the correct weak field limit.

Example 3. (Spherically symmetric space-times [Huggett & Tod, 1994])
Let $\mathscr{S}$ be a sphere of spherical symmetry, and let l^a and n^a be orthogonal to $\mathscr{S}$. Then the only non-zero spin coefficients are those with no spin weight, since there are no constant harmonics with non-zero spin weight. The two-surface Twistor equations become

$$ð\omega^1 = 0 \qquad \text{and} \qquad ð'\omega^0 = 0, \tag{4.2.23}$$

the solutions of which are multiples of ${}_{\frac{1}{2}}Y_{\frac{1}{2}}$ and ${}_{-\frac{1}{2}}Y_{\frac{1}{2}}$ respectively. These form a basis of Twistor space:

$$\omega^A = \mathsf{Z}^\alpha e^A_\alpha = \mathsf{Z}^0\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}}o^A+\mathsf{Z}^1\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}}o^A+\mathsf{Z}^2\,{}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}}\iota^A+\mathsf{Z}^3\,{}_{\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}}\iota^A. \tag{4.2.24}$$

The norm is given by

$$\begin{aligned}\{\mathsf{Z}^\alpha\bar{\mathsf{Z}}_\alpha\} &= i(\bar\omega^{1'}ð\omega^0-\omega^0ð\bar\omega^{1'}+\bar\omega^{0'}ð'\omega^1-\omega^1ð'\bar\omega^{0'})\\ &= -\frac{i}{2\sqrt{2}\pi r}(\mathsf{Z}^0\bar{\mathsf{Z}}_3-\mathsf{Z}^1\bar{\mathsf{Z}}_2+\mathsf{Z}^2\bar{\mathsf{Z}}_1-\mathsf{Z}^3\bar{\mathsf{Z}}_0),\end{aligned} \tag{4.2.25}$$

which is constant. r, here, is a standard radial coordinate such that ð is simply r^{-1} times the unit sphere ð. We may transform to a more convenient basis by setting $\mathsf{Z}^0=\Omega^0$, $\mathsf{Z}^1=\Omega^1$, $\mathsf{Z}^2=-2\sqrt{2}\pi r iP_{1'}$, and $\mathsf{Z}^3=-2\sqrt{2}\pi r iP_{0'}$, so that

$$\{\mathsf{Z}^\alpha\bar{\mathsf{Z}}_\alpha\} = \Omega^0\bar P_0+\Omega^1\bar P_1+\bar\Omega^{0'}P_{0'}+\bar\Omega^{1'}P_{1'} = \Omega^{\mathbf{A}}\bar P_{\mathbf{A}}+\bar\Omega^{\mathbf{A}'}P_{\mathbf{A}'}. \tag{4.2.26}$$

The only curvature components with no spin weight are Φ_{11}, Λ and Ψ_2. The Kinematic Twistor may then easily be computed as

$$\begin{aligned}\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta &= \frac{i}{2\pi G}\oint(\Phi_{11}+\Lambda-\Psi_2)\omega^0\omega^1\mathscr{S}\\ &= \frac{ir^2}{2\pi G}(\Phi_{11}+\Lambda-\Psi_2)\oint(\mathsf{Z}^0\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}}+\mathsf{Z}^1\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})(\mathsf{Z}^2\,{}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}}+\mathsf{Z}^3\,{}_{\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})dS\\ &= \frac{ir^2}{2\pi G}(\Phi_{11}+\Lambda-\Psi_2)(\mathsf{Z}^1\mathsf{Z}^2-\mathsf{Z}^0\mathsf{Z}^3)\\ &= \frac{\sqrt{2}r^3}{G}(\Phi_{11}+\Lambda-\Psi_2)(\Omega^1P_{0'}-\Omega^0P_{1'}) = 2p_{\mathbf{A}}{}^{\mathbf{A}'}\Omega^{\mathbf{A}}P_{\mathbf{A}'}+2i\mu^{\mathbf{A}'\mathbf{B}'}P_{\mathbf{A}'}P_{\mathbf{B}'}.\end{aligned} \tag{4.2.27}$$

Hence,

$$\mu^{\mathbf{A}'\mathbf{B}'} = 0 = p_0{}^{0'} = p_1{}^{1'} \qquad \text{and} \qquad p_1{}^{0'} = \frac{r^3}{\sqrt{2}G}(\Phi_{11}+\Lambda-\Psi_2) = -p_0{}^{1'}, \tag{4.2.28}$$

$$\text{so that the Penrose mass is given by} \qquad m_P = r^3G^{-1}(\Phi_{11}+\Lambda-\Psi_2). \tag{4.2.29}$$

As a specific example, consider the Reissner-Nordström black hole, for which, in Gaussian units, $\Phi_{11}=-\frac{1}{2}GQ^2r^{-4}$, $\Lambda=0$ and $\Psi_2=-Gmr^{-3}$, so that

$$m_P = m-\frac{Q^2}{2r}. \tag{4.2.30}$$

In flat space-time, the energy density of the electric field due to a stationary point charge Q is $\frac{Q^2}{8\pi r^4}$, so that the Penrose mass has the correct Newtonian limit. The Penrose mass, in this case, is different from the Komar mass, which is $m_K = m - \frac{Q^2}{r}$. Finally, it may be remarked that in this special case, the Penrose mass is equivalent to the Hawking mass, which is $m_H = \left(\frac{A}{16\pi G^2}\right)^{1/2}(1+\frac{A}{2\pi}\rho\rho')$. To see this, we first note that $\Phi_{11}+\Lambda-\Psi_2-\rho\rho' = K$, where K is half the Gaussian curvature, which by the Gauss-Bonnet theorem is simply $\frac{1}{2}r^{-2}$ so that

$$m_P = \left(\frac{A}{16\pi G^2}\right)^{1/2}(1+\frac{A}{2\pi}\rho\rho') = m_H. \tag{4.2.31}$$

Not every two-surface has a well-defined Penrose energy-momentum and angular momentum. Notice that $\mathsf{A}_{\alpha\beta}$ has, in general, ten *complex* (i.e. 20 real) components. In flat space-time, only 10 (4 energy-momentum and 6 angular momentum) components are non-zero. Twistorially, the 10 constraints are $\mathsf{A}_{\mathbf{AB}} = 0$ and $\mathsf{A}_{\mathbf{A}}{}^{\mathbf{A'}} = p_{\mathbf{A}}{}^{\mathbf{A'}}$ is *real*. We can rewrite these using the *infinity Twistor* $\mathsf{I}_{\alpha\beta}$, which is defined in flat space as

$$\mathsf{I}_{\alpha\beta}\underset{1}{\mathsf{Z}}^{\alpha}\underset{2}{\mathsf{Z}}^{\beta} := \underset{1}{\pi}_{0'}\underset{2}{\pi}_{1'} - \underset{1}{\pi}_{1'}\underset{2}{\pi}_{0'}, \tag{4.2.32}$$

so that the only non-zero components of $\mathsf{I}_{\alpha\beta}$ are $\mathsf{I}^{\mathbf{A'B'}} = \epsilon^{\mathbf{A'B'}}$. The non-zero components of $\mathsf{A}_{\alpha\gamma}\mathsf{I}^{\gamma\beta}$ are therefore $\mathsf{A}_{\mathbf{AC}}\epsilon^{\mathbf{CB}}$ and $\mathsf{A}^{\mathbf{A'}}{}_{\mathbf{C}}\epsilon^{\mathbf{CB}}$, and the non-zero components of $\mathsf{I}_{\gamma\alpha}\bar{\mathsf{A}}^{\beta\gamma}$ are $\epsilon_{\mathbf{C'A'}}\overline{\mathsf{A}_{\mathbf{BC}}}$ and $\epsilon_{\mathbf{C'A'}}\overline{\mathsf{A}^{\mathbf{C'}}{}_{\mathbf{B}}}$. Hence, the constraints can be written as $\mathsf{A}_{\alpha\gamma}\mathsf{I}^{\gamma\beta} = \mathsf{I}_{\gamma\alpha}\bar{\mathsf{A}}^{\beta\gamma}$, or alternatively, $\mathsf{A}_{\alpha\gamma}\mathsf{I}^{\gamma\beta} = \overline{\mathsf{A}_{\beta\gamma}\mathsf{I}^{\gamma\alpha}}$ so that

$$\mathsf{Z}^{\alpha}\mathsf{A}_{\alpha\gamma}\mathsf{I}^{\gamma\beta}\bar{\mathsf{Z}}_{\beta} \in \mathbb{R}. \tag{4.2.33}$$

In order to state (4.2.33), we need some kind of Twistor $\mathsf{I}^{\alpha\beta} \in \mathbb{T}^{[\alpha\beta]}(\mathscr{S})$ generalising the infinity Twistor in flat space, and a Twistor norm $\mathsf{Z}^{\alpha}\bar{\mathsf{Z}}_{\alpha}$ (or, if we are only after the mass, only the Twistor norm is sufficient to demand $m_P^2 = -\frac{1}{4}\bar{\mathsf{A}}^{\alpha\beta}\mathsf{A}_{\alpha\beta} \in \mathbb{R}$). Unfortunately, neither (4.2.32) nor (4.2.12) are constant on a general two-surface.

Remark. The norm (4.2.12) can be shown to be constant if and only if $\mathscr{S}$ can be embedded isometrically in a conformally flat space. The reason for this is that the Twistor equation (4.2.10) is conformal. Let $\hat{\epsilon}_{AB} = \Omega\epsilon_{AB}$, then

$$\hat{\nabla}_{AA'}\hat{\omega}^B = \nabla_{AA'}\hat{\omega}^B + \epsilon_A{}^B\Upsilon_{CA'}\hat{\omega}^C \qquad \text{so that} \qquad \hat{\nabla}_{A'}^{(A}\hat{\omega}^{B)} = \Omega^{-1}\nabla_{A'}^{(A}\hat{\omega}^{B)}. \tag{4.2.34}$$

Two-surfaces on which the norm (4.2.12) is constant are called *non-contorted*. In order for a surface to be non-contorted, the curvature will have to satisfy several constraints at $\mathscr{S}$, one of which is $\Psi_2 - \bar{\Psi}_2 = 0$.

Unfortunately, no solution has been found that is applicable to contorted two-surfaces, so that the Penrose mass is generally only defined on non-contorted two-surfaces. On the other hand, there exist a wide variety of exact solutions to Einstein's equations containing non-contorted two-surfaces, and in all cases the Penrose mass provides an appropriate energy concept. To name a few, the Penrose mass provides a notion of

1. gravitational potential energy [Tod, 1983]: any two-surface on time-symmetric vacuum initial data sets is non-contorted. In particular, data representing a set of point masses m_i yields

$$m_P = \sum_i m_i - \sum_{i\neq j}\frac{m_i m_j}{d_{ij}} + \mathscr{O}(d_{ij}^{-2}), \tag{4.2.35}$$

   where the sums range over all masses enclosed by the chosen two-surface, and d_{ij} is the distance between masses m_i and m_j.

2. total energy [Penrose, 1982], including (positive) gravitational wave energy: at spatial- and null infinity, m_P reduces to the ADM- and Bondi mass, respectively.

3. rest-mass energy [Tod, 1983]: in FLRW space-times, all two-surfaces are non-contorted and yield a mass of ρV, where $\rho = T_{ab}t^a t^b$ is the energy density of the fluid at rest.

4. electrostatic energy: in the Reissner-Nordström space-time, as shown in example 3 (4.2.30)
5. the irreducible mass of a black hole, in the case of a spherically symmetric marginally outer trapped surface, which can be seen from example 3 by noting that $\rho = 0$ on a marginally outer trapped surface, so that the mass (4.2.31) becomes

$$m_P = \left(\frac{A}{16\pi G^2}\right)^{1/2}. \tag{4.2.36}$$

4.3 Mass and Angular momentum at $\mathscr{I}$

Null infinity is a rather interesting special case that warrants closer inspection. By construction, the Kinematic Twistor reduces to the standard flat space description of energy-momentum and angular momentum in the weak field limit. In some sense, all fields at $\mathscr{I}$ are 'weak' and one might therefore expect $\mathscr{I}$ to be non-contorted. However, perhaps surprisingly, even though the Weyl curvature vanishes on $\mathscr{I}$, $\mathscr{I}$ is generally contorted. Indeed, in order for $\underset{0}{\mathrm{Z}}^\alpha \bar{\mathrm{Z}}_\alpha$ to be constant, $\eth\eth'\{\underset{0}{\mathrm{Z}}^\alpha \bar{\mathrm{Z}}_\alpha\}$ has to vanish[24]. After a short computation, we find

$$\begin{aligned}
\eth\eth'\{\underset{0}{\mathrm{Z}}^\alpha \bar{\mathrm{Z}}_\alpha\} &= i\eth\eth'(\bar{\omega}^{1'}\eth\underset{0}{\omega}^0 - \underset{0}{\omega}^0\eth\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth'\underset{0}{\omega}^1 - \underset{0}{\omega}^1\eth'\bar{\omega}^{0'}) \\
&= i\eth(\bar{\sigma}\bar{\omega}^{0'}\eth\underset{0}{\omega}^0 + \bar{\omega}^{1'}\eth'\eth\underset{0}{\omega}^0 - \underset{0}{\omega}^0\eth'\eth\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth'^2\underset{0}{\omega}^1) \\
&= i(\eth\bar{\sigma}\bar{\omega}^{0'}\eth\underset{0}{\omega}^0 + \eth\bar{\omega}^{1'}\eth'\eth\underset{0}{\omega}^0 + \bar{\omega}^{1'}\eth\eth'\eth\underset{0}{\omega}^0 - \eth\underset{0}{\omega}^0\eth'\eth\bar{\omega}^{1'} - \underset{0}{\omega}^0\eth\eth'\eth\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth\eth'^2\underset{0}{\omega}^1) \\
&\approx i(\eth\bar{\sigma}\bar{\omega}^{0'}\eth\underset{0}{\omega}^0 + \eth\bar{\omega}^{1'}(\eth\eth' + \tfrac{1}{2})\underset{0}{\omega}^0 + \bar{\omega}^{1'}\eth(\eth\eth' + \tfrac{1}{2})\underset{0}{\omega}^0 - \eth\underset{0}{\omega}^0(\eth\eth' + \tfrac{1}{2})\bar{\omega}^{1'} - \underset{0}{\omega}^0\eth(\eth\eth' + \tfrac{1}{2})\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth'^2\eth\underset{0}{\omega}^1) \\
&\approx i(\eth\bar{\sigma}\bar{\omega}^{0'}\eth\underset{0}{\omega}^0 + \tfrac{1}{2}\eth\bar{\omega}^{1'}\underset{0}{\omega}^0 + \tfrac{1}{2}\bar{\omega}^{1'}\eth\underset{0}{\omega}^0 - \eth\underset{0}{\omega}^0\eth(\bar{\sigma}\bar{\omega}^{0'}) - \tfrac{1}{2}\eth\underset{0}{\omega}^0\bar{\omega}^{1'} - \underset{0}{\omega}^0\eth^2(\bar{\sigma}\bar{\omega}^{0'}) - \tfrac{1}{2}\underset{0}{\omega}^0\eth\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth'^2(\sigma\underset{0}{\omega}^0)) \\
&\approx i\underset{0}{\omega}^0\bar{\omega}^0(\eth'^2\sigma - \eth^2\bar{\sigma}),
\end{aligned} \tag{4.3.1}$$

where we used the two-surface Twistor equations (4.2.21), which also imply that $\eth^2\omega^0 \approx 0$ since $\eth'\omega^0 \approx 0$. We also assume we are using a Bondi frame, in which the Gaussian curvature $K \approx \frac{1}{2}$, so that $(\eth\eth' - \eth'\eth)f \approx -sf$ for spin s weighted functions f. Hence the Twistor norm is constant if and only if the symptotic shear σ is *purely electric*, $\eth'^2\sigma \approx \eth^2\bar{\sigma}$.

Remark. The condition that σ is purely electric also guarantees the existence of a four parameter family of good cuts, related to each other by translations. Recall that under a supertranslation $u \mapsto u + H$,

$$\sigma(u) = \sigma(u + H) + \eth^2 H. \tag{4.3.2}$$

Recall, also, that $\eth \mapsto \eth - \eth H \text{þ}'$ so that

$$\begin{aligned}
\eth'\sigma \mapsto (\eth' - \eth' H\text{þ}')(\sigma(u+H) + \eth'^2\eth^2 H) &= (\eth'\sigma)(u+H) + (\text{þ}'\sigma)(u+H)\eth H - \eth H(\text{þ}'\sigma)(u+H) + \eth'\eth^2 H \\
&= (\eth'\sigma)(u+H) + \eth'\eth^2 H,
\end{aligned} \tag{4.3.3}$$

where we have used the chain rule. Similarly,

$$\eth'^2\sigma(u) \mapsto (\eth'^2\sigma)(u+H) + \eth'^2\eth^2 H. \tag{4.3.4}$$

The reason it is, in general, not possible to find a supertranslation so that $\sigma \mapsto 0$, is that H is real, while σ is complex and thus has two real degrees of freedom. As it turns out, the magnetic part of σ is invariant under supertranslations:

$$\eth'^2\sigma(u) - \eth^2\bar{\sigma}(u) \mapsto (\eth'^2\sigma)(u+H) + \eth'^2\eth^2 H - (\eth^2\bar{\sigma})(u+H) - \eth^2\eth^2 H = \eth'^2\sigma(u+H) - \eth^2\bar{\sigma}(u+H), \tag{4.3.5}$$

so that if $\eth'^2\sigma - \eth^2\bar{\sigma} \approx 0$, there always exists a supertranslation mapping a bad cut onto a good cut. The vanishing of the magnetic part of σ, therefore, singles out a unique Poincaré subgroup of the BMS group as the largest subgroup that maps good cuts to good cuts.

[24] $\eth\eth'$ is a kind of 'complex Laplacian'. Its real part $\eth\eth' + \eth'\eth$ acting on quantities with no spin weight is precisely the ordinary Laplacian.

Even though the Twistor norm is not well-defined at $\mathscr{I}^+$, it is possible to state (4.2.33) since it turns out that there exists a well-defined map $\mathsf{Z}^\alpha \mapsto \mathsf{I}^{\alpha\beta}\bar{\mathsf{Z}}_\beta$. In order to construct this map, let us first examine the infinity Twistor $\mathsf{I}_{\alpha\beta}$ at $\mathscr{I}^+$. In terms of spinor fields on the conformally completed space-time $(\bar{\mathscr{M}}, \epsilon_{AB} = \Omega\hat{\epsilon}_{AB})$, it is given by

$$\begin{aligned}
\mathsf{I}_{\alpha\beta}\underset{1}{\mathsf{Z}}^\alpha\underset{2}{\mathsf{Z}}^\beta &= \hat{\epsilon}^{A'B'}\underset{1}{\hat{\pi}}_{A'}\underset{2}{\hat{\pi}}_{B'} = -\tfrac{1}{4}\hat{\epsilon}^{A'B'}(\hat{\nabla}_{AA'}\underset{1}{\omega}^A)(\hat{\nabla}_{BB'}\underset{2}{\omega}^B) \\
&= -\Omega\epsilon^{A'B'}(\tfrac{1}{2}\nabla_{AA'}\underset{1}{\omega}^A + \Upsilon_{AA'}\underset{1}{\omega}^A)(\tfrac{1}{2}\nabla_{BB'}\underset{2}{\omega}^B + \Upsilon_{BB'}\underset{2}{\omega}^B) \\
&= -\Omega\epsilon^{A'B'}(\tfrac{1}{2}\nabla_{AA'}\underset{1}{\omega}^A - \Omega^{-1}N_{AA'}\underset{1}{\omega}^A)(\tfrac{1}{2}\nabla_{BB'}\underset{2}{\omega}^B + \Omega^{-1}N_{BB'}\underset{2}{\omega}^B) \\
&\approx iN_A^{A'}(\underset{1}{\pi}_{A'}\underset{2}{\omega}^A - \underset{2}{\pi}_{A'}\underset{1}{\omega}^A) \\
&\approx iA(\underset{1}{\pi}_{1'}\underset{2}{\omega}^0 - \underset{2}{\pi}_{1'}\underset{1}{\omega}^0) \approx A(\underset{1}{\omega}^0 \eth \underset{2}{\omega}^0 - \underset{2}{\omega}^0 \eth \underset{1}{\omega}^0).
\end{aligned} \tag{4.3.6}$$

Let us summarise some important properties in a proposition:

Lemma 2. *The infinity Twistor $I_{\alpha\beta} \in \mathbb{T}_{[\alpha\beta]}(\mathscr{S})$ is well defined on cuts of $\mathscr{I}^+$, being given by the constant expression*

$$I_{\alpha\beta}\underset{1}{Z}^\alpha\underset{2}{Z}^\beta \approx A(\underset{2}{\omega}^0 \eth \underset{1}{\omega}^0 - \underset{1}{\omega}^0 \eth \underset{2}{\omega}^0). \tag{4.3.7}$$

There exists a two-(complex)-dimensional linear subspace $I^{\alpha\beta}\bar{\mathbb{T}}_\beta \subset \mathbb{T}^\alpha$ that is annihilated by $I_{\alpha\beta}$, which has the property that the scalar product $W^\alpha \bar{Z}_\alpha$ is well defined for all $W^\alpha \in I^{\alpha\beta}\bar{\mathbb{T}}_\beta$ and $\bar{Z}_\alpha \in \mathbb{T}_\alpha$. Using this fact, we can construct a unique map $Z^\alpha \mapsto I^{\alpha\beta}\bar{Z}_\beta$ characterised by

$$\underset{0}{Z}^\alpha \overline{I^{\alpha\beta}\bar{Z}_\beta} = I_{\alpha\beta}\underset{0}{Z}^\alpha Z^\beta. \tag{4.3.8}$$

In terms of spinor fields, this map is given by

$$\{\omega^0, \omega^1\} \mapsto \{0, -iA\bar{\omega}^{0'}\}. \tag{4.3.9}$$

Proof. It is easy to see that applying ð to (4.3.7) annihilates it, so that (4.3.7) is constant, so that the $\mathsf{I}_{\alpha\beta}$ is well-defined at $\mathscr{I}^+$.

If either $\underset{1}{\omega}^0$ or $\underset{2}{\omega}^0$ vanishes, $\mathsf{I}_{\alpha\beta}\underset{1}{\mathsf{Z}}^\alpha\underset{2}{\mathsf{Z}}^\beta = 0$. If $\omega^0 = 0$, one of the components of the two-surface Twistor equations (4.2.21) at $\mathscr{I}^+$ is automatically satisfied, while the other is $\eth\omega^1 \approx 0$. This equation has a two-(complex)-dimensional solution space spanned by ${}_{\frac{1}{2}}Y_{\frac{1}{2}}$. Notice that, by (4.3.1), the Twistor scalar product with any other Twistor is constant.

Finally, we compute the map $\mathsf{Z}^\alpha \mapsto \mathsf{I}^{\alpha\beta}\bar{\mathsf{Z}}_\beta$, which maps $\{\omega^0, \omega^1\} \mapsto \{\eta^0, \eta^1\}$ by setting

$$i(\bar{\eta}^{1'} \eth \underset{0}{\omega}^0 - \underset{0}{\omega}^0 \eth \bar{\eta}^{1'} + \bar{\eta}^{0'} \eth' \underset{0}{\omega}^1 - \underset{0}{\omega}^1 \eth' \bar{\eta}^{0'}) :\approx A(\omega^0 \eth \underset{0}{\omega}^0 - \underset{0}{\omega}^0 \eth \omega^0), \tag{4.3.10}$$

from which is is easy to see that $\eta^0 = 0$ and $\eta^1 = -iA\bar{\omega}^{0'}$. □

Having found $\mathsf{I}^{\alpha\beta}\bar{\mathsf{Z}}_\beta$, we can now finally extract the energy-momentum from the Kinematic Twistor (4.2.20), and show that it is real:

Theorem 8. *At $\mathscr{I}^+$, the Kinematic Twistor satisfies*

$$Z^\alpha A_{\alpha\beta} I^{\beta\gamma}\bar{Z}_\gamma = -\frac{1}{4\pi G}\oint (\psi_2 - \sigma N A^{-1}) A\omega^0\bar{\omega}^{0'}\mathscr{S} \in \mathbb{R}, \tag{4.3.11}$$

so that only ten real components are non-zero. The term $A\omega^0\bar{\omega}^{0'}$ satisfies $\eth^2(A\omega^0\bar{\omega}^{0'}) = 0$, so that its span W is a linear combination of Y_0 and Y_1 spherical harmonics. In spherical coordinates, $W = 1$ and $W = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ yield the energy and momentum components, respectively, in an orthonormal basis.

Proof. Let us, for simplicity, consider $\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta$.[25] We can transform the $(\omega^1)^2$ term into a $\omega^0\omega^1$ term by using $\psi_3 \approx A^{-1}\eth N$ and integrating by parts:

$$\begin{aligned}\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta &= -\frac{i}{4\pi G}\oint(\psi_1(\omega^0)^2 + 2\psi_2\omega^0\omega^1 + \psi_3(\omega^1)^2)\mathscr{S}\\ &= -\frac{i}{4\pi G}\oint(\psi_1(\omega^0)^2 + 2\psi_2\omega^0\omega^1 - 2A^{-1}N\omega^1\eth\omega^1)\mathscr{S}\\ &= -\frac{i}{4\pi G}\oint(\psi_1(\omega^0)^2 + 2(\psi_2 - \sigma N A^{-1})\omega^0\omega^1)\mathscr{S}.\end{aligned} \tag{4.3.12}$$

Using (4.3.9) we see that the energy-momentum $\mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma$ is given by

$$\mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma = -\frac{1}{4\pi G}\oint(\psi_2 - \sigma N A^{-1})A\omega^0\bar{\omega}^{0'}\mathscr{S}. \tag{4.3.13}$$

Using the asymptotic Bianchi identities (proposition 9), $A\psi_2 - A\bar{\psi}_2 - \sigma N + \bar{\sigma}\bar{N} \approx \eth'^2\sigma - \eth^2\bar{\sigma}$. We can therefore write the imaginary part of the integrand on the right-hand-side, using integration by parts twice, as

$$\begin{aligned}\mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma - \overline{\mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma} &= -\frac{1}{4\pi G}\oint(\eth'^2\sigma - \eth^2\bar{\sigma})\omega^0\bar{\omega}^{0'}\mathscr{S}\\ &= -\frac{1}{4\pi G}\oint(\sigma\eth'^2(\omega^0\bar{\omega}^{0'}) - \bar{\sigma}\eth^2(\omega^0\bar{\omega}^{0'}))\mathscr{S} = 0.\end{aligned} \tag{4.3.14}$$

Hence,

$$\mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma \in \mathbb{R}. \tag{4.3.15}$$

It may be verified that

$$\begin{aligned}m^2 = \frac{1}{4\pi G^2}&\oint(\psi_2 - \sigma N A^{-1})(Y_{0,0} - \tfrac{1}{\sqrt{3}}Y_{1,0})\mathscr{S}\oint(\psi_2 - \sigma N A^{-1})(Y_{0,0} + \tfrac{1}{\sqrt{3}}Y_{1,0})\mathscr{S}\\ &- \frac{1}{6\pi G^2}\oint(\psi_2 - \sigma N A^{-1})Y_{1,-1}\mathscr{S}\oint(\psi_2 - \sigma N A^{-1})Y_{1,1}\mathscr{S}\end{aligned} \tag{4.3.16}$$

Is Lorentz invariant, in the sense that m^2 (4.3.16) does not depend on the two-sphere metric on $\mathscr{S}$. Hence

$$W^{\mathbf{a}}p_{\mathbf{a}}(\mathscr{S}) := -\frac{1}{4\pi G}\oint(\psi_2 - \sigma N A^{-1})W\mathscr{S}, \tag{4.3.17}$$

defines the total energy-momentum vector $p_{\mathbf{a}}$, where $W^{\mathbf{a}} = (W^0, W^1, W^2, W^3)$ and

$$W = \sqrt{4\pi}W^0 Y_{0,0} + \sqrt{\tfrac{2\pi}{3}}W^1(Y_{1,-1} + Y_{1,1}) + i\sqrt{\tfrac{2\pi}{3}}W^2(Y_{1,-1} - Y_{1,1}) + \sqrt{\tfrac{4\pi}{3}}W^3 Y_{1,0}. \tag{4.3.18}$$

In spherical coordinates, $W = W^0 + W^1\cos\vartheta\sin\varphi + W^2\sin\vartheta\sin\varphi + W^3\cos\varphi$. The Minkowski norm of $p_{\mathbf{a}}$ is given by $m^2 = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$. □

Presently, we cannot evaluate the energy momentum flux '$p_{\mathbf{a}}(\mathscr{S}') - p_{\mathbf{a}}(\mathscr{S})$' between two different cuts $\mathscr{S}'$ and $\mathscr{S}$ because the vectors $p_{\mathbf{a}}(\mathscr{S}')$ and $p_{\mathbf{a}}(\mathscr{S})$ belong to different vector spaces. To remedy this, we simply extend the two-surface Twistor $\mathsf{Z}^\alpha = \{\omega^0, \omega^1\} \in \mathbb{T}^\alpha(\mathscr{S})$ to the whole of $\mathscr{I}^+$ by setting $\text{þ}'\omega^0 \approx 0$. The resulting space is called *asymptotic spin space* $S^{\mathbf{A}}$. We can now interpret $p_{\mathbf{a}}(\mathscr{S}) \in S_{\mathbf{A}} \otimes \bar{S}_{\mathbf{A}'}$ as a vector field on the whole of $\mathscr{I}^+$.

[25] Since $\mathsf{A}_{\alpha\beta}$ is symmetric, we can construct any component $\mathsf{A}_{\alpha\beta}\underset{1}{\mathsf{Z}}^\alpha\underset{2}{\mathsf{Z}}^\beta$ from $\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta$ using the polarization identity.

Theorem 9 (Bondi's mass loss theorem)**.** *The energy momentum flux $p_{\boldsymbol{a}}(\mathscr{S}') - p_{\boldsymbol{a}}(\mathscr{S})$ through a region $\mathscr{N} \subset \mathscr{I}^+$ bounded by two cuts $\mathscr{S}'$ and $\mathscr{S}$, where $\mathscr{S}'$ lies in the future of $\mathscr{S}$, is given by*

$$W^{\boldsymbol{a}}(p_{\boldsymbol{a}}(\mathscr{S}') - p_{\boldsymbol{a}}(\mathscr{S})) = -\int ((4\pi G)^{-1} N\bar{N} + T_{ab}n^a n^b) W \mathscr{N}. \tag{4.3.19}$$

If $T_{ab}n^a n^b \geq 0$ on $\mathscr{I}^+$, the flux is past causal.

Proof. The energy-momentum flux law (4.3.19) follows immediately from proposition 11 (4.1.14).

W corresponds to a future null direction if $W = \omega\bar{\omega}$ where ω consists of ${}_{\frac{1}{2}}Y_{\frac{1}{2}}$. Hence, if W is future null, $W \geq 0$, and if $T_{ab}n^a n^b \geq 0$ on $\mathscr{I}^+$ then $((4\pi G)^{-1} N\bar{N} + T_{ab}n^a n^b)W \geq 0$. Hence, $p_{\mathbf{a}}(\mathscr{S}') - p_{\mathbf{a}}(\mathscr{S})$ is past causal. □

4.4 Angular momentum

Obtaining a satisfactory definition of angular momentum is a much more challenging problem than defining energy-momentum, since in order to define the latter we merely needed to associate a vector to some cut $\mathscr{S} \subset \mathscr{I}^+$. In order to define angular momentum we also need to provide a concept of 'origin' to $\mathscr{S}$. The earliest attempts at such a definition regard the cut itself to provide an origin. Indeed, in Minkowski space $\mathbb{M}$, good cuts[26] correspond to light cones. These light cones correspond to a unique point in $\mathbb{M}$ via their vertex.

In the dynamical regions of $\mathscr{I}^+$ — and these are generically all of $\mathscr{I}^+$ — there are generally only bad cuts. Even if a system starts out being stationary, and returns to being stationary after emitting gravitational radiation, a good cut will generally be translated into a bad cut by virtue of the fact that

$$\sigma(u) \mapsto \sigma(u+H) - \eth^2 H = \sigma(u) - \int_u^{u+H} \bar{N} \mathrm{d}u - \eth^2 H. \tag{4.4.1}$$

In order for a good cut to translate into another good cut, we need $\int_u^{u+T} N \mathrm{d}u$ to vanish, which is generally not the case.

In stationary space-times we may, nevertheless, use the existence of a family of good cuts whose members are related to each other through translations. The Bondi-Sachs definition of angular momentum on good cuts is

$$J_B := -\frac{1}{8\pi G} \oint Y \psi_1 \mathscr{S}, \tag{4.4.2}$$

where Y consists of ${}_{-1}Y_1$ spherical harmonics. When performing a supertranslation $u \mapsto u + H$, ψ_1 transforms as

$$\psi_1(u) \mapsto \psi_1(u+H) - 3\psi_2(u+H)\eth H. \tag{4.4.3}$$

In stationary space-times, by Einstein's equations,

$$\dot{\psi}_1 \approx \eth\psi_2 \qquad \text{and} \qquad \dot{\psi}_2 \approx 0, \tag{4.4.4}$$

$$\text{so that} \qquad \psi_2(u) \approx \psi_2(0) \equiv \psi_2 \qquad \text{and} \qquad \psi_1(u) \approx \psi_1(0) + u\dot{\psi}_1(0) \equiv \psi_1(0) + u\eth\psi_2. \tag{4.4.5}$$

We therefore find that under a supertranslation $u \mapsto u + H$ which maps $\mathscr{S} \mapsto \mathscr{S}'$,

$$J_B \mapsto -\frac{1}{8\pi G} \oint Y(\psi_1 + H\eth\psi_2 - 3\psi_2 \eth H)\mathscr{S}'. \tag{4.4.6}$$

[26] recall that *good cuts* are cuts of $\mathscr{I}^+$ on which the shear σ vanishes.

We can simplify the terms involving H through integration by parts:
$\oint Y(H\eth\psi_2 - 3\psi_2\eth H)\mathscr{S} = -\oint Y(H\eth\psi_2 - 3\psi_2\eth H)\mathscr{S} + 2\oint H\psi_2\eth Y\mathscr{S}$ so that

$$J_B \mapsto J_B + \frac{1}{4\pi G}\oint (2Y\eth H - H\eth Y)\psi_2\mathscr{S}'. \tag{4.4.7}$$

Notice that if $\eth^2 H = 0$ which is the case if H is a *translation*, $\eth^2(2Y\eth H - H\eth Y) = 0$, so that the effect of a translation is precisely to add multiples of the Bondi momentum to J_B.

Example 4. Bondi angular momentum of the Kerr space-time.
In a Bondi tetrad, the gravitational field ψ_{ABCD} of the Kerr space-time at $\mathscr{I}^+$ is given by [Bai et al., 2007]

$$\psi_0 \approx \sqrt{\frac{96\pi}{5}}\frac{GJ^2}{m}\,{}_2Y_2 \tag{4.4.8a}$$

$$\psi_1 \approx 2\sqrt{3\pi}iGJ\,{}_1Y_1 = \frac{3iGJ\sin\theta}{\sqrt{2}} \tag{4.4.8b}$$

$$\psi_2 \approx -Gm \tag{4.4.8c}$$

$$\psi_3 \approx 0 \approx \psi_4, \tag{4.4.8d}$$

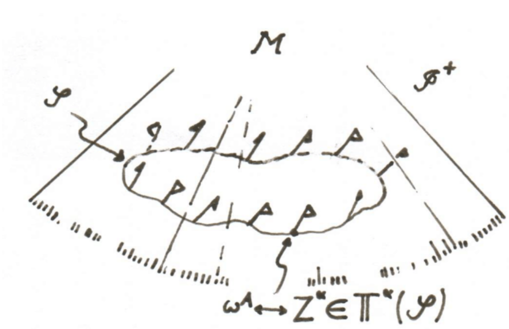
So that $J_{B,Kerr} = -\frac{1}{8\pi G}\oint \sin\theta\psi_1\mathscr{S} = -\frac{i}{\sqrt{2}}J$. Note that a translation adds a *real* multiple of the Bondi momentum to J_B, so that Kerr's Bondi angular momentum is purely spin.

4.4.1 Twistor geometry, and Penrose's angular momentum

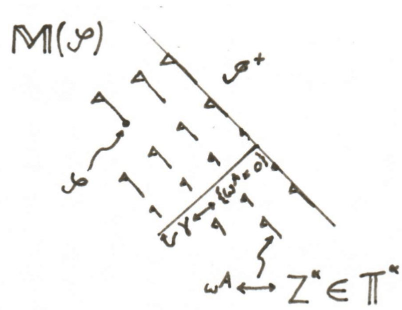
The Twistorial definition of angular momentum operates very differently. Rather than the cut itself providing an origin, the two-surface Twistor space $\mathbb{T}^\alpha(\mathscr{S})$ provides a *Minkowski space of origins* $\mathbb{M}(\mathscr{S})$. To construct this space, first assume $\mathscr{S}$ is non-contorted. This means that $\mathscr{S}$ can be embedded in Minkowski space $\mathbb{M}(\mathscr{S})$. (If $\mathscr{S}$ is contorted, we may instead embed $\mathscr{S}$ in *complexified Minkowski space* $\mathbb{CM}(\mathscr{S})$.) Two-surface Twistors in $\mathbb{T}^\alpha(\mathscr{S})$ correspond to spinor fields on $\mathscr{S} \subset \mathbb{M}(\mathscr{S})$, which may be extended to the whole Minkowski space $\mathbb{M}(\mathscr{S})$, so that two-surface Twistors in $\mathbb{T}^\alpha(\mathscr{S})$ correspond to Twistors in $\mathbb{M}(\mathscr{S})$. The points at which the spinor fields ω^A corresponding to null Twistors Z^α (Twistors with vanishing norm; $\mathsf{Z}^\alpha\bar{\mathsf{Z}}_\alpha = 0$) vanish, are null geodesics γ in $\mathbb{M}$:

$$\gamma := \{x^{AA'} \in \mathbb{M} \mid \mathring{\omega}^A - ix^{AA'}\pi_{A'} = 0\} = \{(i\bar{\mathring{\omega}}^A\bar{\pi}_{A'})^{-1}\mathring{\omega}^A\bar{\mathring{\omega}}^{A'} + \lambda\bar{\pi}^A\pi^{A'} \mid \lambda \in \mathbb{R}\}. \tag{4.4.9}$$

(If Z^α is non-null, we may instead identify Z^α with a null geodesic in complexified Minkowski space $\mathbb{CM}$.)



(a) The physical space-time M containing a non-contorted two-surface $\mathscr{S}$. $\mathscr{S}$ can be embedded into Minkowski space: see the figure on the right.



(b) A two-dimensional slice of the Minkowski space of origins $\mathbb{M}(\mathscr{S})$, into which $\mathscr{S}$ is embedded. Two-surface Twistors correspond to Twistors in $\mathbb{M}(\mathscr{S})$. A Null Twistor determines a null geodesic γ in $\mathbb{M}(\mathscr{S})$, defined as the points at which ω^A vanishes.

In summary, null two-surface Twistors $\mathsf{Z}^\alpha \in \mathbb{T}(\mathscr{S})$, where $\mathscr{S}$ is non-contorted, correspond to null geodesics $\gamma \subset \mathbb{M}(\mathscr{S})$, where $\mathbb{M}(\mathscr{S})$ is an (abstract) Minkowski space whose points are unrelated to points in the physical space-time M, except in the special case that $M = \mathbb{M}$ (or M is conformally flat.)

In the Minkowski space definition of angular momentum (4.2.1), the Killing vectors $\xi^a = \underset{1}{k}{}^{[a}\underset{2}{k}{}^{b]}x_b$ generating the components of the angular momentum also determine an origin $O = \{x^a = 0\}$ about which the angular momentum is defined: this origin corresponds to the point where ξ^a vanishes. Similarly, from the flat space definition of the Kinematic Twistor (4.2.20) we see that the origin is given by a point at which ω^A vanishes. This makes sense, since solutions ω^A to the Twistor equation determine solutions ξ^a to Killings equations. The points at which $\omega^A = 0$ are the points at which $\xi^a = 0$. This follows from proposition 12.

Hence *both* the component *and* the origin of the angular momentum in the expression $\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta$ are provided by the Twistor Z^α. What we have just shown is that Z^α singles out the null geodesic on which the origin lies, but not any one point. This may seem troublesome, but it is actually a nice feature of the Twistorial definition, as the following proposition demonstrates:

Proposition 13. *Let γ be a null geodesic in $\mathbb{M}$ with tangent vector $\bar{\pi}^A\pi^{A'}$. Then $\mu^{A'B'}\pi_{A'}\pi_{B'}$ is constant along γ.*

Proof. Under a change of origin $x^a \mapsto x^a + y^a$, the angular momentum tensor M^{ab} changes according to $M^{ab}(x^c) \mapsto M^{ab}(x^c + y^c) = M^{ab}(x^c) - 2y^{[a}p^{b]}$. In terms of the spinor $\mu^{A'B'} = \frac{1}{2}M^{ABA'B'}\epsilon_{AB}$, this change is given by $\mu^{A'B'}(x^c) \mapsto \mu^{A'B'}(x^c + y^c) = \mu^{A'B'}(x^c) - y^{A(A'}p_A^{B')}$. Let $y^{AA'} = \lambda\bar{\pi}^A\pi^{A'}$, then $\mu^{A'B'}\pi_{A'}\pi_{B'}$ transforms as

$$\mu^{A'B'}(x^{CC'} + \lambda\bar{\pi}^C\pi^{C'})\pi_{A'}\pi_{B'} = \mu^{A'B'}(x^{CC'})\pi_{A'}\pi_{B'} - \lambda\bar{\pi}^C\pi^{(A'}p_C^{B')}\pi_{A'}\pi_{B'} = \mu^{A'B'}(x^c)\pi_{A'}\pi_{B'}. \tag{4.4.10}$$

□

This proposition tells us that, while the angular momentum is defined with respect to some point, the component $\mu^{A'B'}\pi_{A'}\pi_{B'}$ is defined with respect to a null geodesic with tangent vector $\bar{\pi}^A\pi^{A'}$. To find the full angular momentum with respect to some (real) origin O, one computes $\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta$ for at least three linearly independent null Twistors Z^α passing through O. Let $\underset{1}{\gamma}$ and $\underset{2}{\gamma}$ be two null geodesics in $\mathbb{M}(\mathscr{S})$ represented by null Twistors $\underset{1}{\mathsf{Z}}{}^\alpha$ and $\underset{2}{\mathsf{Z}}{}^\alpha$. Suppose we have chosen our coordinate origin such that $\underset{2}{\mathsf{Z}}{}^\alpha$ passes through it. Then $\underset{2}{\mathsf{Z}}{}^\alpha = \{0, \underset{2}{\pi}{}_{A'}\}$, and in order for $\underset{2}{\mathsf{Z}}{}^\alpha$ to intersect $\underset{1}{\mathsf{Z}}{}^\alpha$, $\underset{2}{\pi}{}_{A'}$ has to be proportional to $\underset{1}{\bar{\omega}}{}_{A'}$. This then implies that $\{\underset{1}{\mathsf{Z}}{}^\alpha\underset{2}{\bar{\mathsf{Z}}}{}_\alpha\} = 0$. Since the coordinate origin is arbitrary, we conclude that $\underset{1}{\gamma}$ and $\underset{2}{\gamma}$ intersect if and only if $\{\underset{1}{\mathsf{Z}}{}^\alpha\underset{2}{\bar{\mathsf{Z}}}{}_\alpha\} = 0$. See figure 12.

Finally, let us examine the angular momentum part of the Kinematic Twistor at $\mathscr{I}^+$:

$$\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta = -\frac{i}{4\pi G}\oint \psi_1(\omega^0)^2 + 2(\psi_2 - \sigma N A^{-1})\omega^0\omega^1\mathscr{S}. \tag{4.4.11}$$

The Bondi-Sachs angular momentum has been a widely used measure of total angular momentum. It produces a sound definition in a few cases, and so it should not come as a surprise that it shares a few similarities with Penrose's definition.

Remark. (similarities with the Bondi-Sachs angular momentum)

1. $(\omega^0)^2$ satisfies $\eth'(\omega^0)^2 = 0$ and therefore consists of ${}_{-1}Y_1$ harmonics.
2. since ω^1 satisfies the Twistor equation $\eth\omega^1 = \sigma\omega^0$, so does $\omega^1 + \xi$ where $\eth\xi = 0$. The term $\omega^0\xi$ satisfies $\eth^2(\omega^0\xi) = 0$, and therefore consists of Y_0 and Y_1 harmonics. Hence, adding ξ to ω^1 results in multiples of the Bondi energy-momentum being added to (4.4.11).
3. On a good cut, $\omega^0\omega^1$ consists of Y_0 and Y_1 harmonics, so that Penrose's angular momentum has the same form as the Bondi angular momentum.

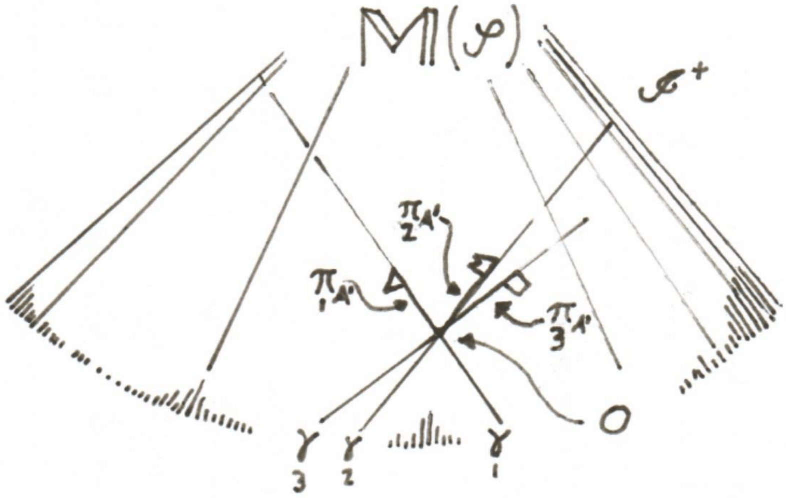


Figure 12: A set of orthogonal null Twistors $\underset{i}{Z}^\alpha = (\underset{i}{\omega}^A, \underset{i}{\pi}_{A'})$ correspond to null geodesics $\underset{i}{\gamma}$ intersecting at O. The angular momentum at O is given by $2i\mu^{A'B'}(O)\underset{i}{\pi}_{A'}\underset{i}{\pi}_{B'} = A_{\alpha\beta}\underset{i}{Z}^\alpha\underset{i}{Z}^\beta$.

On a bad cut, the $\omega^0\omega^1$ term separates Penrose's angular momentum from all prior definitions. Its structure is somewhat complicated, depending on integrals of σ over $\mathscr{S}$. It is interesting to gain some more insight into this term. We may solve the two-surface Twistor equations $\eth'\omega^0 = 0$ and $\eth\omega^1 = \sigma\omega^0$ by introducing the potential λ for σ, satisfying[27] $\eth^2\lambda = \sigma$. Given ω^0, ω^1 is

$$\omega^1 = \omega^0\eth\lambda - \lambda\eth\omega^0 + \xi \qquad \text{where} \qquad \eth\xi = 0. \tag{4.4.12}$$

We may substitute this into the Kinematic Twistor (4.4.11). Using integration by parts, we find the following expression for $\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta$:

$$\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta = -\frac{i}{4\pi G}\oint((\psi_1 + 3(\psi_2 - \sigma N A^{-1})\eth\lambda + \lambda\eth(\psi_2 - \sigma N A^{-1}))(\omega^0)^2 + 2(\psi_2 - \sigma N A^{-1})\omega^0\xi)\mathscr{S}. \tag{4.4.13}$$

It would be premature to call the $(\omega^0)^2$ part the Penrose angular momentum, even though this term seems similar to the Bondi definition. On a good cut, we *can* make this identification because $\{\omega^0, 0\}$ is a null Twistor *and* the Twistors $\underset{1}{\mathsf{Z}}^\alpha = \{\underset{1}{\omega}^0, 0\}$ and $\underset{2}{\mathsf{Z}}^\alpha = \{\underset{2}{\omega}^0, 0\}$ are orthogonal, $\underset{1}{\mathsf{Z}}^\alpha\underset{2}{\bar{\mathsf{Z}}}_\alpha = 0$, so that $\underset{1}{\mathsf{Z}}^\alpha$ and $\underset{2}{\mathsf{Z}}^\alpha$ intersect. Their point of intersection determines a *real* origin. The expression we are left with is identical to the Bondi definition. On bad cuts, the $(\omega^0)^2$ part of the Kinematic Twistor (4.4.13) yields the Penrose angular momentum with respect to a real point *only if* the cut is purely electric.[28] In particular, this is the case for stationary cuts.

Proposition 14. *Let $\mathscr{S}$ be a purely electric cut of $\mathscr{I}^+$. Then Twistors of the form $\{\omega^0, \omega^0\eth\lambda - \lambda\eth\omega^0\}$ are null and mutually orthogonal, so that they determine a real point in $\mathbb{M}(\mathscr{S})$. With respect to this point, the angular momentum is given by*

$$\mu^{A'B'}\pi_{A'}\pi_{B'} = -\frac{1}{8\pi G}\oint(\psi_1 + 3(\psi_2 - \sigma N A^{-1})\eth\lambda + \lambda\eth(\psi_2 - \sigma N A^{-1}))(\omega^0)^2\mathscr{S}. \tag{4.4.14}$$

Proof. The condition $\eth'^2\sigma = \eth^2\bar{\sigma}$ for a cut to be purely electric implies that λ is real. A lengthy computation then reveals that the norm (4.2.12) simplifies to

$$\{\underset{0}{\mathsf{Z}}^\alpha\bar{\mathsf{Z}}_\alpha\} = i(\bar{\xi}\eth\underset{0}{\omega}^0 - \underset{0}{\omega}^0\eth\bar{\xi} + \bar{\omega}^{0'}\eth'\underset{0}{\xi} - \underset{0}{\xi}\eth'\bar{\omega}^{0'}), \tag{4.4.15}$$

[27] That such a potential exists, follows from the fact that σ has spin weight $s = 2$.

[28] Recall that a cut is purely electric if $\eth'^2\sigma = \eth^2\bar{\sigma}$.

from which it is easy to see that Twistors of the form $\{\omega^0, \omega^0 ð\lambda - \lambda ð\omega^0\}$ are null and mutually orthogonal. □

As an interesting special case, consider a good cut $\mathscr{S}$. On this cut there exists a point where the Twistor $\mathsf{Z}^\alpha = \{\omega^0, 0\}$ vanish. Hence, the null geodesic in $\mathbb{M}(\mathscr{S})$ determined by Z^α intersects $\mathscr{I}^+$, and the real origin of proposition 14 is therefore given by the vertex of the light cone in $\mathbb{M}(\mathscr{S})$ intersecting $\mathscr{I}^+$ in $\mathscr{S}$.

Penrose's Twistorial definition of angular momentum is not as straightforward as some other more commonly used definitions, such as the Bondi-Sachs or Komar angular momentum. There are quite a few conceptual hoops that one needs to jump through, in particular regarding the Minkowski space of origins. It is then perhaps not too surprising that for practical purposes, the Bondi-Sachs and Komar angular momentum (when applicable) have always been favored over Penrose's angular momentum. This is unfortunate, because both the Bondi-Sachs definition and the Komar definition suffer from serious problems; neither have the correct weak field limit. The former yields unphysical results when computed on bad cuts, while the latter yields unphysical results in the presence of matter.

In order to demonstrate the results of this section, let us explicitly compute the Kinematic Twistor on an arbitrary cut of $\mathscr{I}^+$ in the Kerr space-time. Surprisingly, as far as I am aware, this is the first and only explicit (non-vanishing) example of Penrose's angular momentum.

Theorem 10 (Total energy-momentum and angular momentum of the Kerr space-time)**.** *Let $\mathscr{S}$ be an arbitrary cut of $\mathscr{I}^+$ in the Kerr space-time with mass- and angular momentum parameters m and $J = ma$. At $\mathscr{S}$, the Penrose mass and spin, given by the norm of the Pauli-Lubanski spin vector are*

$$m_P = m \qquad \textit{and} \qquad S^{\boldsymbol{a}} S_{\boldsymbol{a}} = -m^2 J^2, \qquad \textit{where} \qquad S^{\boldsymbol{a}} := i p^{\boldsymbol{A'}}_{\boldsymbol{B}} \bar{\mu}^{\boldsymbol{AB}} - i p^{\boldsymbol{A}}_{\boldsymbol{B'}} \mu^{\boldsymbol{A'B'}}. \tag{4.4.16}$$

Hence, as expected, the total mass and spin of the Kerr space-time are m and J.

Proof. A basis for $\mathbb{T}^\alpha(\mathscr{S})$ is given by

$$\mathsf{Z}^\alpha = \{\omega^0 = \mathsf{Z}^0 \, {}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} + \mathsf{Z}^1 \, {}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}}, \omega^1 = \omega^0 ð\lambda - \lambda ð\omega^0 + \mathsf{Z}^3 \, {}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} + \mathsf{Z}^3 \, {}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}}\}. \tag{4.4.17}$$

The norm is given by (4.2.25,4.4.15)

$$\{\mathsf{Z}^\alpha \bar{\mathsf{Z}}_\alpha\} = -\frac{i}{2\sqrt{2}\pi}(\mathsf{Z}^0\bar{\mathsf{Z}}_3 - \mathsf{Z}^1\bar{\mathsf{Z}}_2 + \mathsf{Z}^2\bar{\mathsf{Z}}_1 - \mathsf{Z}^3\bar{\mathsf{Z}}_0). \tag{4.4.18}$$

We can find an $\epsilon_{\mathbf{AB}}$ spinor by using that, in flat space, $\epsilon_{AB}\underset{1}{\omega}^A\underset{2}{\omega}^B = \overline{\mathsf{I}_{\alpha\beta}\underset{1}{\mathsf{X}}^\alpha\underset{2}{\mathsf{X}}^\beta}$. Where $\omega^A = \mathsf{Z}^\alpha \in \mathsf{I}^{\alpha\beta}\bar{\mathbb{T}}_\beta$ and X^α is defined by $\mathsf{Z}^\alpha = \mathsf{I}^{\alpha\beta}\bar{X}_\beta$. Using that map provided by lemma 2 (4.3.9), we find that

$$\epsilon_{\mathbf{AB}}\underset{1}{\omega}^{\mathbf{A}}\underset{2}{\omega}^{\mathbf{B}} = A(\underset{1}{\omega}^1 ð'\underset{2}{\omega}^1 - \underset{2}{\omega}^1 ð'\underset{1}{\omega}^1), \tag{4.4.19}$$

where $\underset{1}{\mathsf{Z}}^\alpha = \{0, \underset{1}{\omega}^1\}$ and $\underset{2}{\mathsf{Z}}^\alpha = \{0, \underset{2}{\omega}^1\}$. For our basis (4.4.18), we find

$$\epsilon_{\mathbf{AB}}\underset{1}{\omega}^{\mathbf{A}}\underset{2}{\omega}^{\mathbf{B}} = \frac{A}{2\sqrt{2}\pi}(\underset{1}{\mathsf{Z}}^2\underset{2}{\mathsf{Z}}^3 - \underset{2}{\mathsf{Z}}^2\underset{1}{\mathsf{Z}}^3). \tag{4.4.20}$$

Transforming to a more convenient basis $\mathsf{Z}^0 \mapsto i\sqrt{2\sqrt{2}\pi A}P_{1'}$, $\mathsf{Z}^1 \mapsto -i\sqrt{2\sqrt{2}\pi A}P_{0'}$, $\mathsf{Z}^2 \mapsto \sqrt{\frac{2\sqrt{2}\pi}{A}}\Omega^0$, $\mathsf{Z}^3 \mapsto \sqrt{\frac{2\sqrt{2}\pi}{A}}\Omega^1$, the Twistor norm and $\epsilon_{\mathbf{AB}}$ simplify to

$$\{\mathsf{Z}^\alpha\bar{\mathsf{Z}}_\alpha\} = \Omega^0\bar{P}_0 + \Omega^1\bar{P}_1 + \bar{\Omega}^{0'}P_{0'} + \bar{\Omega}^{1'}P_{1'} \qquad \text{and} \qquad \epsilon_{\mathbf{AB}}\underset{1}{\omega}^{\mathbf{A}}\underset{2}{\omega}^{\mathbf{B}} = \underset{1}{\Omega}^0\underset{2}{\Omega}^1 - \underset{2}{\Omega}^0\underset{1}{\Omega}^1. \tag{4.4.21}$$

Finally, let us consider the Kinematic Twistor. Because all cuts are purely electric, any cut $\mathscr{S}$ can be transformed into a good cut with supertranslations $u \mapsto u - \lambda + T$ where T is an arbitrary translation, since, if $u \mapsto u + H$,

$$\eth^2 \lambda = \sigma \mapsto \sigma + \eth^2 H = \eth^2(\lambda + H). \tag{4.4.22}$$

Note that σ (and hence also λ) are independent of u since $N = 0$. Using the expressions for ψ_{ABCD} in example 4, we find that on a bad cut

$$\psi_1 \approx 2\sqrt{3\pi} i G J A^{-1}\,{}_1Y_1 - 3GmA^{-1}\eth(\lambda - T), \tag{4.4.23a}$$
$$\psi_2 \approx -GmA^{-1}, \tag{4.4.23b}$$
$$\psi_3 \approx 0. \tag{4.4.23c}$$

The Kinematic Twistor becomes

$$\mathsf{A}_{\alpha\beta}\mathsf{Z}^\alpha\mathsf{Z}^\beta = -\frac{i}{4\pi}\oint((2\sqrt{3\pi} i G J_1 Y_1 - 3Gm\eth(\lambda - T))(\omega^0)^2 - 2m\omega^0\omega^1)A^{-1}\mathscr{S}. \tag{4.4.24}$$

The energy-momentum is given by

$$\begin{aligned} p_{\mathbf{A}}^{\mathbf{A'}}\bar{\pi}^{\mathbf{A}}\pi_{\mathbf{A'}} &= \mathsf{Z}^\alpha\mathsf{A}_{\alpha\beta}\mathsf{I}^{\beta\gamma}\bar{\mathsf{Z}}_\gamma = \frac{1}{4\pi}\oint m\omega^0\bar{\omega}^{0'}\mathscr{S} \\ &= \frac{1}{\sqrt{2}}\oint m(P_{1'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} - P_{0'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})(\bar{P}_1\,{}_{\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}} + \bar{P}_0\,{}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}})\mathscr{S} \\ &= -\frac{m}{\sqrt{2}}(\bar{P}_0P_{0'} + \bar{P}_1P_{1'}) \end{aligned} \tag{4.4.25}$$

Hence,[29]

$$p^{00'} = \frac{m}{\sqrt{2}} = p^{11'}, \qquad p^{01'} = 0 = p^{10'} \qquad \text{so that} \qquad m_P^2 = p^{\mathbf{a}}p_{\mathbf{a}} = m^2. \tag{4.4.26}$$

Notice that $p^{\mathbf{a}}$ is future causal if $m > 0$ and past causal if $m < 0$, hence $m_P = m$.

Next, let us consider the angular momentum. The spin is origin independent, so we are free to fix a particular (real) origin. The simplest choice is provided by proposition 14. Starting from (4.4.14) and using (4.4.23a) we find that the angular momentum is independent of the shear:

$$\begin{aligned} \mu^{\mathbf{A'B'}}\pi_{\mathbf{A'}}\pi_{\mathbf{B'}} &= -\frac{1}{8\pi}\oint(2\sqrt{3\pi} i J A^{-1}\,{}_1Y_1 + 3mA^{-1}\eth T)(\omega^0)^2\mathscr{S} \\ &= \frac{1}{2\sqrt{2}}\oint(2\sqrt{3\pi} i J\,{}_1Y_1 + 3m\eth T)(P_{1'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} - P_{0'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})^2\mathscr{S} \\ &= -iJP_{0'}P_{1'} + \frac{3m}{2}\oint T(P_{1'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} - P_{0'}\,{}_{-\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})(P_{1'}\,{}_{\frac{1}{2}}Y_{\frac{1}{2},-\frac{1}{2}} - P_{0'}\,{}_{\frac{1}{2}}Y_{\frac{1}{2},\frac{1}{2}})\mathscr{S} \\ &:= -iJP_{0'}P_{1'} + \mathscr{T}^{A'B'}P_{A'}P_{B'}, \end{aligned} \tag{4.4.27}$$

where $\mathscr{T}^{A'B'}$ is *real* since T is real. Hence,

$$\mu^{0'1'} = -\tfrac{1}{2}iJ + \mathscr{T}^{0'1'} = \mu^{1'0'}, \qquad \text{and} \qquad \mu^{0'0'}, \mu^{1'1'} \qquad \text{are } \textit{real}, \tag{4.4.28}$$

$$\text{from which we find} \qquad S^{00'} = ip_{\mathbf{A}}^{0'}\bar{\mu}^{0\mathbf{A}} - ip_{\mathbf{A'}}^{0}\mu^{0'\mathbf{A'}} = -\frac{mJ}{\sqrt{2}} = -S^{11'}, \qquad S^{01'} = 0 = S^{10'}, \tag{4.4.29}$$

$$\text{so that} \qquad S^{\mathbf{a}}S_{\mathbf{a}} = -m^2J^2. \tag{4.4.30}$$

□

[29] Recall that $p_{\mathbf{A}}^{\mathbf{A'}}\bar{\pi}^{\mathbf{A}}\pi_{\mathbf{A'}} = -p^{\mathbf{AA'}}\bar{\pi}_{\mathbf{A}}\pi_{\mathbf{A'}}$.

One may take the analysis of Kerr's angular momentum structure further by decomposing T into spherical harmonics in order to compute $\mathscr{T}^{A'B'}$, which yields $M^{\mathbf{ab}} = \mu^{\mathbf{A'B'}}\epsilon^{\mathbf{AB}} + \bar{\mu}^{\mathbf{AB}}\epsilon^{\mathbf{A'B'}}$. A different origin may be chosen by choosing a two (complex) parameter family of Twistors with $\xi \neq 0$ that are mutually orthogonal. This may be a helpful exercise to familiarise oneself with Twistors, but from a physical point of view the full angular momentum is of little interest. The reason for this is that $M^{\mathbf{ab}}$ depends on an origin, which in curved space-times is not a point in the physical space-time itself. In order for such an identification to exist, the space-time manifold M would need to have a vector space structure that can be identified with $\mathbb{M}$, since a change of origin is represented by a vector in the Minkowski space of origins $\mathbb{M}(\mathscr{S})$.

On a contorted surface $\mathscr{S}$, there still exists a *complex* Minkowski space of origins $\mathbb{CM}(\mathscr{S})$, but without a Twistor norm we cannot identify the subspace $\mathbb{M}(\mathscr{S}) \subset \mathbb{CM}(\mathscr{S})$ of real origins. At $\mathscr{I}^+$, there is a way (albeit a somewhat crude way) to overcome this problem. If $p^{\mathbf{a}}$ is causal, (so that the mass m_P is positive) then there exists a unique unit two-sphere metric on $\mathscr{S}$ such that $p^{\mathbf{a}} = (m_P, 0, 0, 0)$ (recall that the conformal transformations preserving the metric of $\mathscr{S}$ are Lorentz transformations on $S_{\mathbf{a}}$.) Having fixed the metric on $\mathscr{S}$ to be this unique metric, define the Twistor norm as the average

$$\{\underset{0}{\mathsf{Z}}^{\alpha}\bar{\mathsf{Z}}_{\alpha}\} = \frac{i}{4\pi}\oint (\bar{\omega}^{1'}\eth\underset{0}{\omega}^{0} - \underset{0}{\omega}^{0}\eth\bar{\omega}^{1'} + \bar{\omega}^{0'}\eth'\underset{0}{\omega}^{1} - \underset{0}{\omega}^{1}\eth'\bar{\omega}^{0'})\mathscr{S}. \tag{4.4.31}$$

5 Concluding remarks

Throughout this work I have attempted to demonstrate the remarkable utility and elegance of spinors, when applied to problems in classical General Relativity. The analysis carried out in the first two chapters could, in principle, also have been translated into a tensorial language, but in some cases (such as the fingerprint, Petrov classification and field equations at $\mathscr{I}$) such a translation would have lead to wild complications.

In the final chapter, spinors seem to have played an essential role. In principle, it might be possible to express the kinematic Twistor in a tensorial way, since energy-momentum and angular momentum are tensors, although it is not at all clear what a non-Twistorial Penrose angular momentum would look like. In any case, such an investigation does not seem productive.

Finally, I would like to speculate about future investigations into energy-momentum and angular momentum. Like Ψ_{ABCD}, the quantities $\nabla^{A}_{A'} \ldots \nabla^{V}_{V'}\phi_{WXYZ}$ all satisfy the mass-less field equations in vacuum, where ϕ_{ABCD} is the weak-field Weyl spinor. Hence, it should be possible to define corresponding *multipole Twistors* $\mathsf{Q}_{\alpha_1 \ldots \alpha_{2n}}$ as integrals over these spinors [Curtis, 1978]. In general, we should not expect these to be well behaved, since $\mathsf{A}_{\alpha\beta}$ is not well-defined everywhere either, but perhaps when these quantities are evaluated at null infinity they can be used to define the total multipole moments. Finding a satisfactory definition of multipole moments at null infinity is still an open problem. Another problem is that of finding a suitable definition of kinematic Twistor on contorted two-surfaces. Unfortunately, two-surfaces are generically contorted, even in 'nice' space-times like Kerr. Curiously, a spinorial definition closely resembling Penrose's kinematic Twistor has been found in the Kerr space-time, which uses solutions to the Twistor equation with a modified connection [Bergqvist & Ludvigsen, 1989, Bergqvist, 1991]. This connection has vanishing curvature and non-vanishing torsion, so that the resulting Twistor equations can be integrated globally (refer back to equation (4.2.9)). Perhaps, instead of finding a replacement for the kinematic Twistor and Twistor norm, we should look for a modified 'contorted two-surface Twistor space'. For now these problems remain unresolved.

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