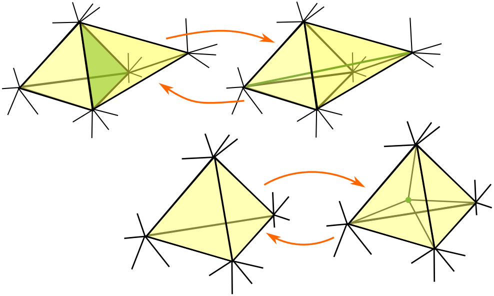
Making Quantum Gravity Computable, 22-06-2017

Monte Carlo methods in Dynamical Triangulations

Part II: Higher dimensions

Timothy Budd



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Outline

- Day 1: 2D random geometry
  - Combinatorial representation
  - Markov Chain Monte Carlo (MCMC) methods
  - Matter coupling
  - Observables
- Day 2: Dynamical Triangulations in higher dimensions
  - Quantum gravity
  - Combinatorial representation
  - MCMC methods
  - Phase diagram
  - Causal Dynamical Triangulations
- Tutorials: numerical analysis of various 2D random geometries
  - Measure observables for random geometries (produced by black box)
  - Extract critical exponents.
  - Experiment with (new?) observables.
  - Conclusions will be collected at the end and be discussed.

A space-time path integral?

$$\int_{\substack{\text{Lorentzian}\\ \text{metrics}}} \frac{\mathrm{d}\mu(g_{\alpha\beta})}{\mathrm{Diff}} e^{iS[g_{\alpha\beta}]}$$

Difficulties:

- QFT in perturbative regime: non-renormalizable
- Infinite-dimensional integral
- What is a good diffeo-invariant measure?
- Destructive interference is delicate
- How to interpret integrand?
- Numerical evaluation is hard

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Quantum → Statistical
Lorentzian → Euclidean

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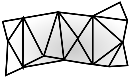
- Does the integral converge?
- Does it possess a continuum limit?

Cut-off



$$\sum_{\{\text{PL geometries } g_{ab}\}} \iint \frac{\mathrm{d}\mu(\ell_i)}{\text{Diff}} e^{-S[g_{ab}]}$$

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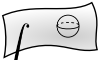

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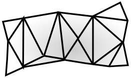

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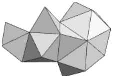
$\{$PL geometries $g_{ab}\}$

Discretize

$$\sum_{\{\text{Combinatorial geometries}\}} e^{-S[g_{ab}]}$$



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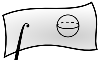

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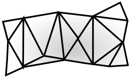

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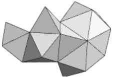

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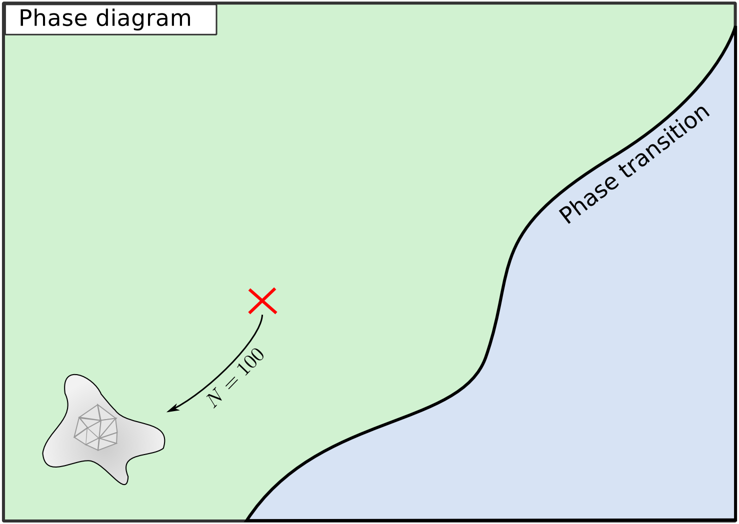

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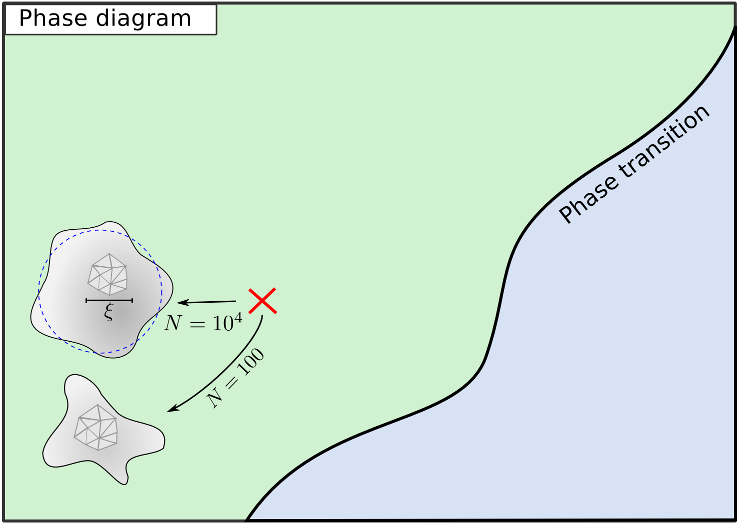

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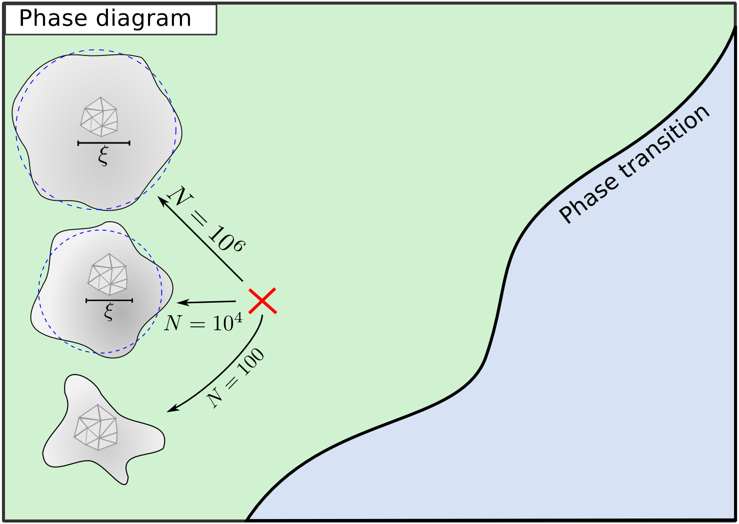

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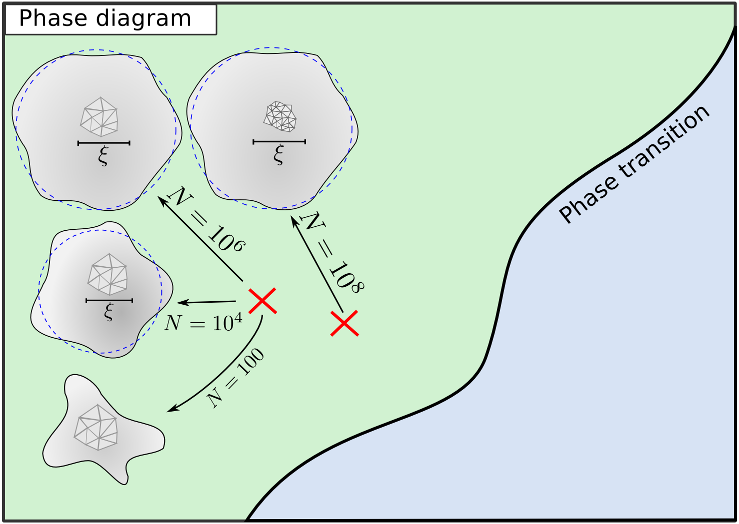

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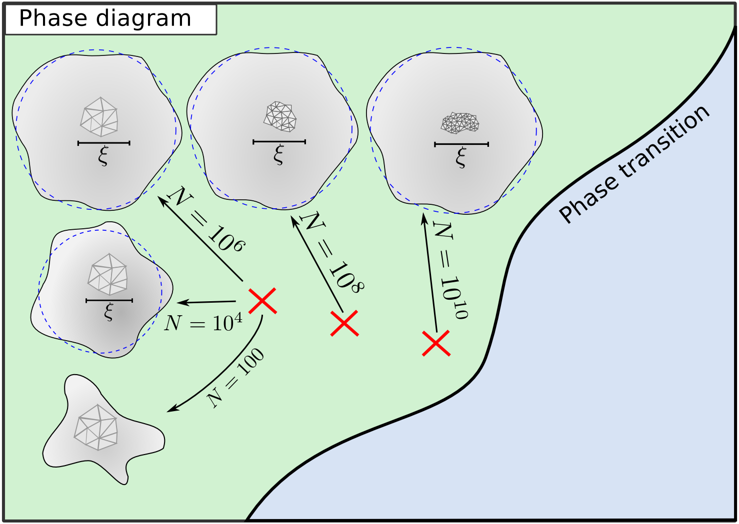

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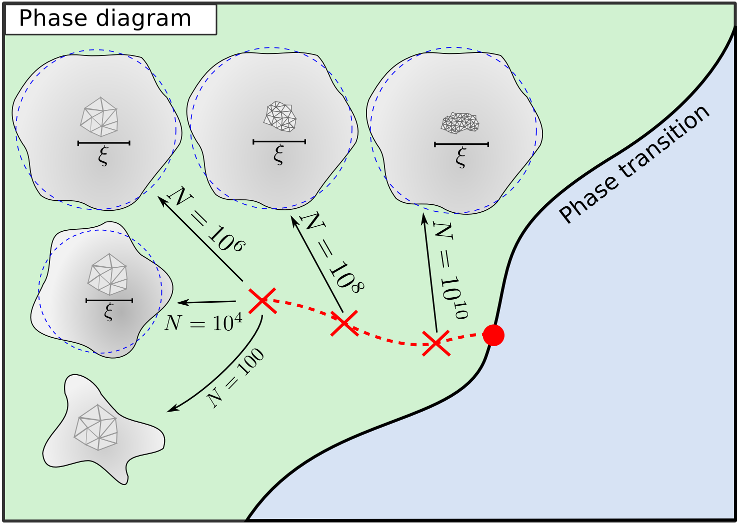

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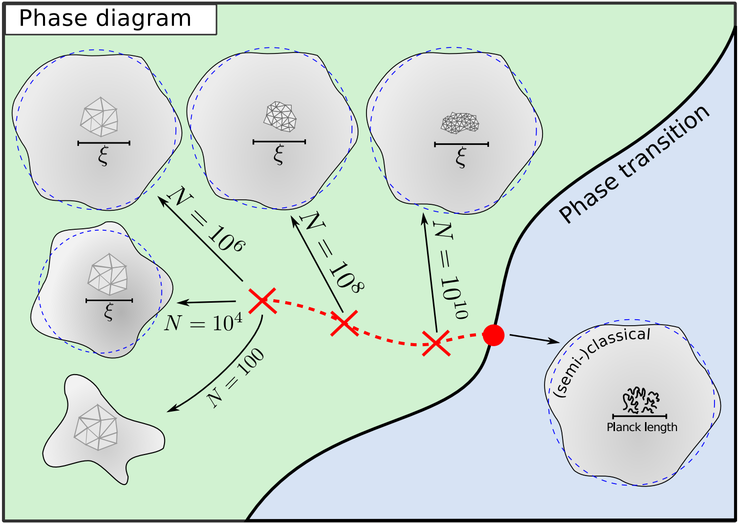
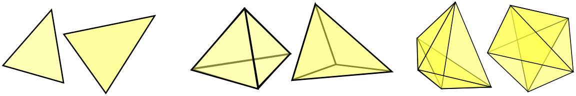


Piecewise linear geometry

- D-simplex: $\{\sum_{i=0}^{D} \lambda_i \mathbf{x}_i : \lambda_i \in [0,1], \sum \lambda_i = 1\} \subset \mathbb{R}^D$ with Euclidean geometry.



$D = 2$ $\quad$ $D = 3$ $\quad$ $D = 4$

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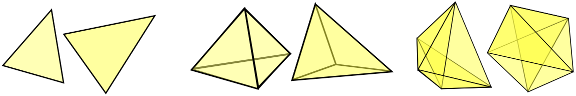
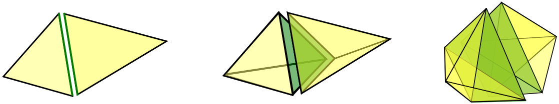


$D = 2$ $D = 3$ $D = 4$

- D-simplices can be glued into larger metric spaces along matching $(D-1)$-simplices.



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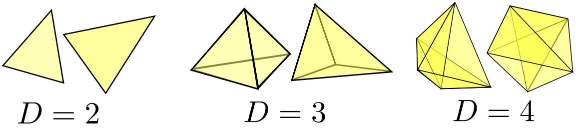

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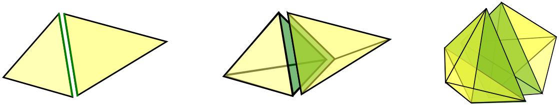
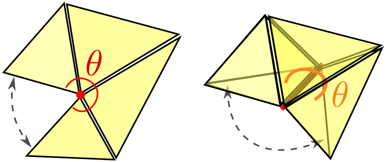


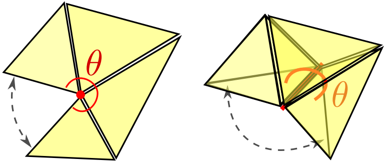
- Resulting geometry has curvature supported on $(D-2)$-simplices.



Einstein-Hilbert action

- Integrated curvature is naturally expressed in terms of deficit angles [Regge, '61]

$$\int \mathrm{d}^D x \sqrt{g} R \longrightarrow \sum_{(D-2)-\text{simplices } \sigma} |\sigma|(2\pi - \theta_\sigma)$$



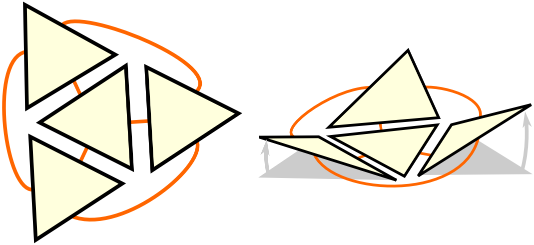
- If all simplices are taken of equal shape (say, equilateral) then linearity of Regge action implies that EH ($+\int \mathrm{d}^D x \sqrt{g} \Lambda$) is a simple linear combination

$$\kappa_D N_D - \kappa_{D-2} N_{D-2}.$$

- Makes sense to include in MCMC at least such two terms in Boltzmann weight.

Combinatorial representation

- Recall: need next and adjacent to navigate a map, or a polyhedron.



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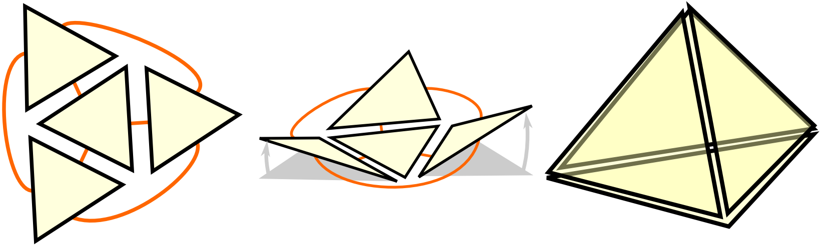

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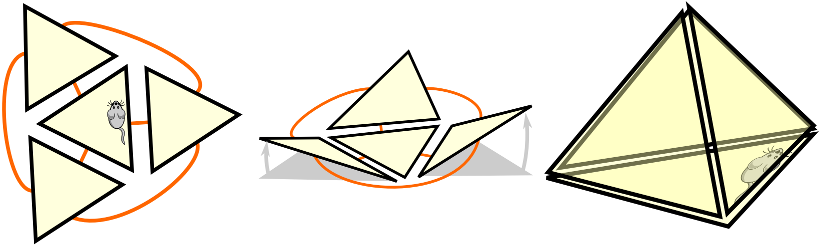

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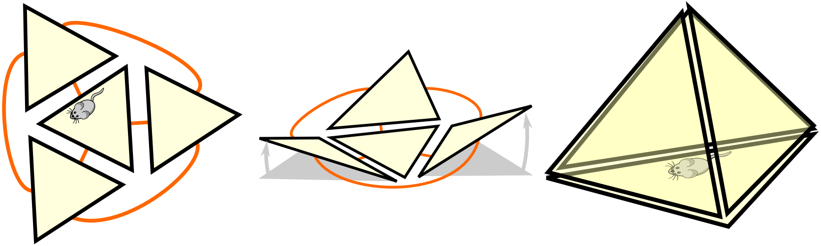

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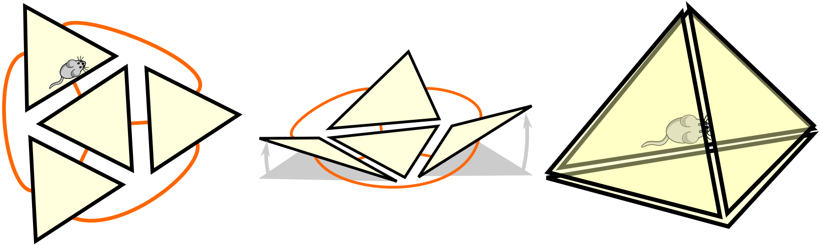

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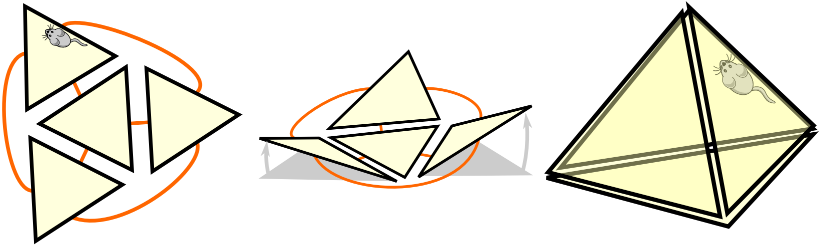

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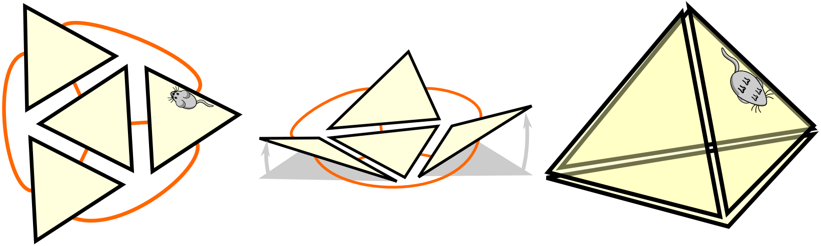

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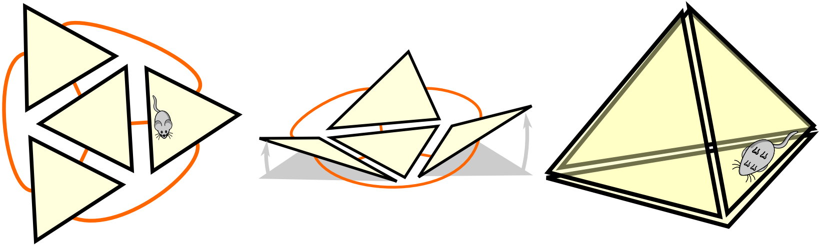

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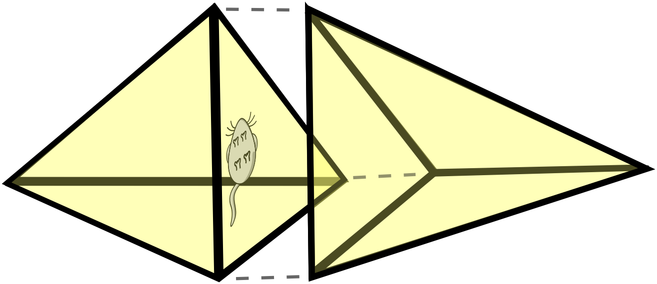

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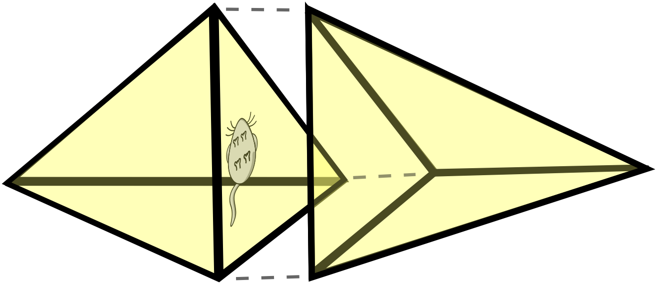
- Recall: need next and adjacent to navigate a map, or a polyhedron.
- Generalize: For D-dimensional geometry, $(n, a) \to (n, a_2, \cdots, a_D)$.
  a_d maps half-edge i to its d-dimensional neighbor $a_d(i)$.



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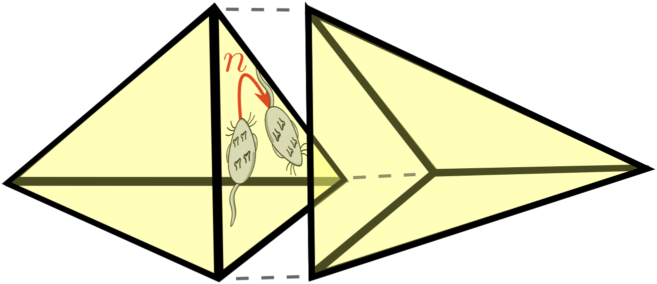

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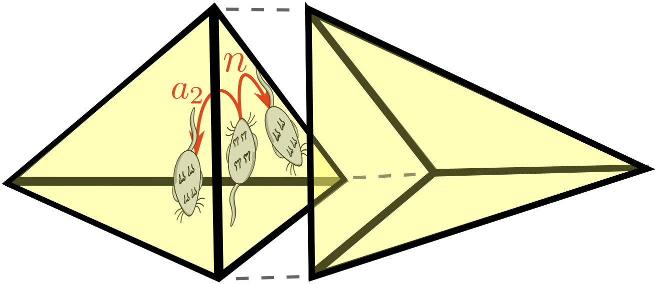

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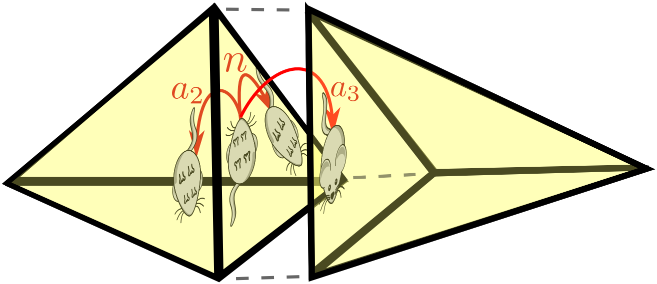

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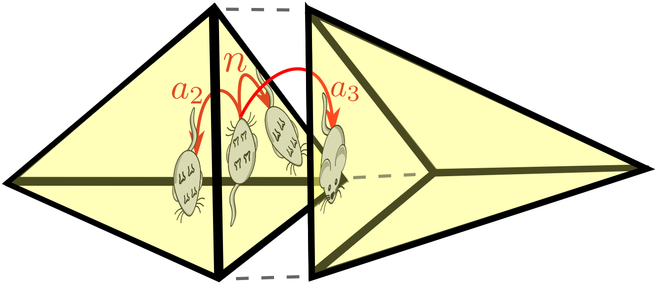
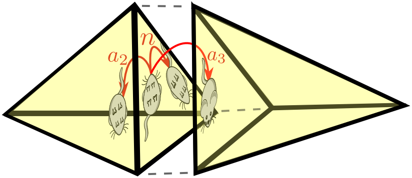
- Cells of various dimensions are identified as orbits. In 3D:
  $(n, a_2) \rightarrow$ polyhedra, $(n, a_3) \rightarrow$ faces, $(a_2, a_3) \rightarrow$ edges,
  $(n \circ a_2, n \circ a_3) \rightarrow$ vertices.



Remarks on combinatorics in 3D

- When does a triple of permutations (n, a_2, a_3) determine a topological 3-manifold?



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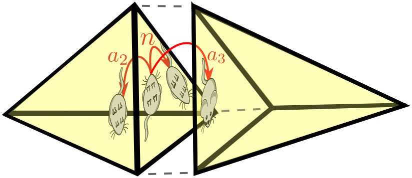
  - Proper gluing: $n \circ a_3 \circ n = a_3$.
  - Polyhedra (orbits under n, a_2) should have 3-ball topology (i.e. boundary S^2): Euler formula!
  - Neighbourhood of vertices (orbits under $n \circ a_2, n \circ a_3$) should have 3-ball topology: Euler formula!



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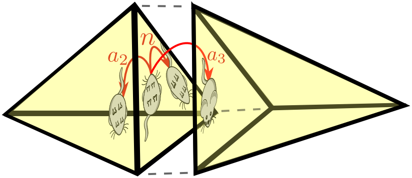

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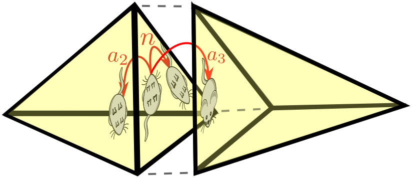
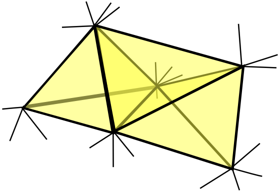
- What is the topology of the resulting 3-manifold?
  - Unfortunately, no simple combinatorial/algorithmic way to decide!
  - Luckily, any two geometries with equal topology are connected by a finite sequence of local moves!
- Situation very similar in 4D (and higher).



Simplicial manifolds

- *D-triangulation*: all D-cells are taken to be D-simplices (i.e. triangles in $2D$, tetrahedra in $3D$, 4-simplices in $4D$).



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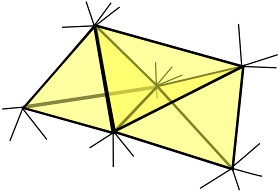
- Amounts to $(D+1)!/2$ numbers to store/update per D-simplex.



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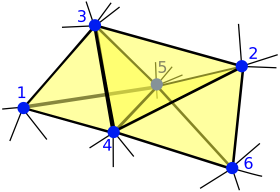
- In D-triangulations all dynamical information is stored in a_D.
- Amounts to $(D+1)!/2$ numbers to store/update per D-simplex.
- Label the vertices of a D-triangulation.
- *Simplicial D-triangulation*: each edge, face, ..., D-simplex must be uniquely characterized by its set of incident vertices.



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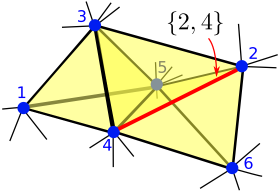

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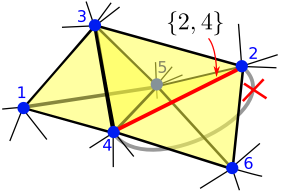

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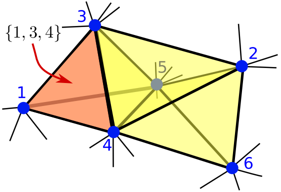

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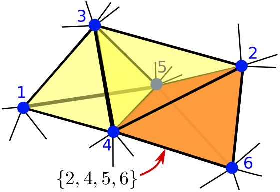

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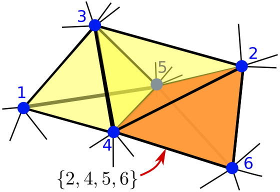
- Knowing the set $\{\{1,3,4,5\},\{2,3,4,5\},\{2,4,5,6\},\ldots\}$ of D-simplices, can reproduce the triple (n, a_2, a_3) up to relabeling (and orientation).



Triangulation size?

- What is the "size" of a D-triangulation?

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N &= \#\text{ of half-edges (size of } {\color{red}n}, {\color{red}a_d}) \\
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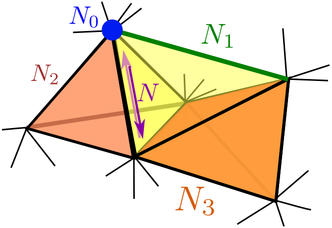

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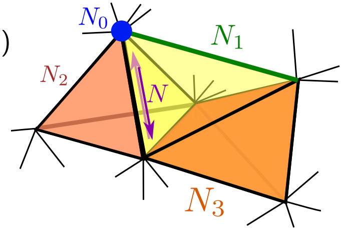

- Relations: $N = N_D(D+1)!/2$, $2N_{D-1} = N_D(D+1)$, $\sum_{k=0}^{d}(-1)^k N_k = \chi$ (Euler characteristic). In $D \geq 4$ more linear (Dehn-Sommerfield) relations.

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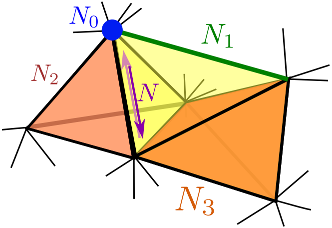

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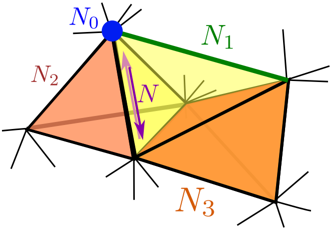

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- Only $\lfloor \frac{D+1}{2} \rfloor$ independent numbers. In 3D and 4D these are usually taken to be N_D and N_{D-2}, or N_D and N_0.
- Recall the EH action $S[N_D, N_{D-2}] = \kappa_D N_D - \kappa_{D-2} N_{D-2}$ is exactly a linear combination of these.
- As we will see: for fixed N_D, varying the ratio N_{D-2}/N_D has a large effect on the random geometries!

Labeling & symmetry

- Recall from yesterday: in 2D for fixed N_2 a uniform labeled triangulation $\mathfrak{t}$ with N_2 triangles is equivalent to an unlabeled triangulation $\tilde{\mathfrak{t}}$ with probability proportional to $1/|\mathrm{Aut}(\tilde{\mathfrak{t}})|$:

$$Z_{N_2} = \sum_{\substack{\text{labeled} \\ \text{triangulations } \mathfrak{t}}} 1 = (3N_2)! \sum_{\substack{\text{unlabeled} \\ \text{triangulations } \tilde{\mathfrak{t}}}} \frac{1}{|\mathrm{Aut}(\tilde{\mathfrak{t}})|}$$

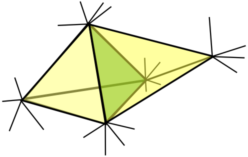
- No longer equivalent if N_2 (or N_D in dimension D) is allowed to vary.
- Settle upon convention that $S[N_D, N_0]$ is action for unlabeled triangulations:

$$Z = \sum_{\substack{\text{labeled} \\ \text{triangulations } \mathfrak{t}}} \frac{e^{-S[N_D,N_0]}}{(\#\text{labels})!} = \sum_{\substack{\text{unlabeled} \\ \text{triangulations } \tilde{\mathfrak{t}}}} \frac{e^{-S[N_D,N_0]}}{|\mathrm{Aut}(\tilde{\mathfrak{t}})|}$$

($\#\text{labels} = N_D(D+1)!/2$ for general and N_0 for simplicial triangulations)

Moves in 3D

- *23-move*: select a uniform random triangle, merge incident tetrahedra, split into 3 tetrahedra.



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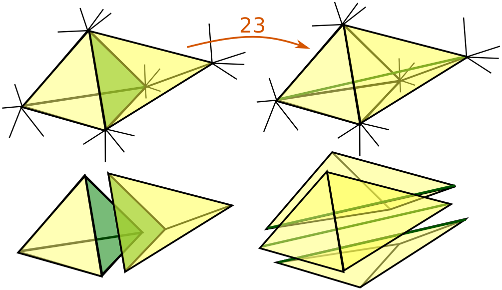

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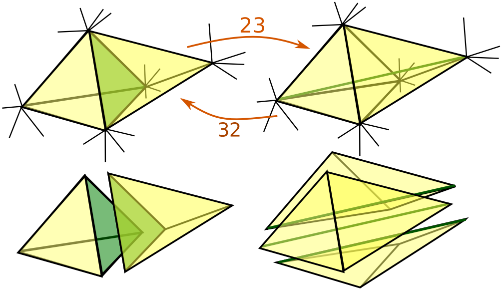

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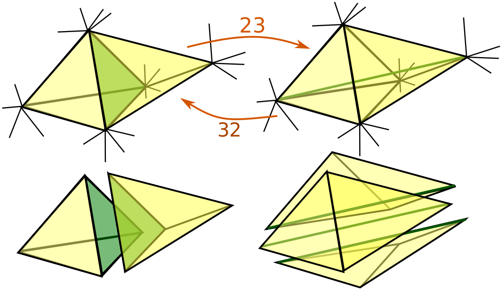

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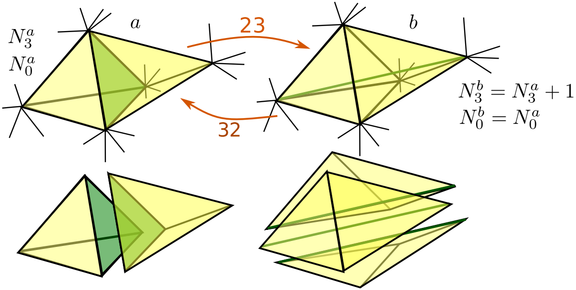

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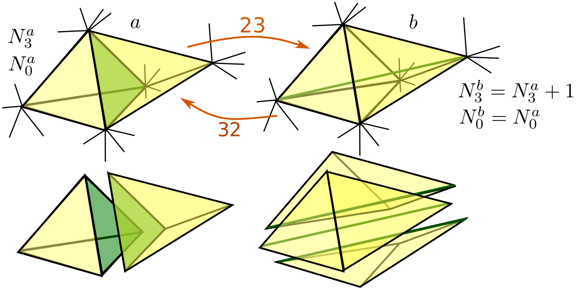

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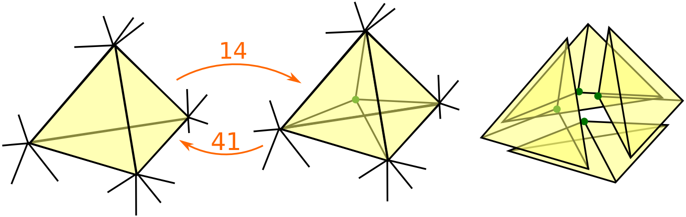

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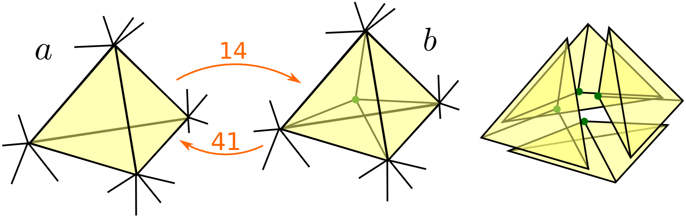

$N_3^b = N_3^a + 3$
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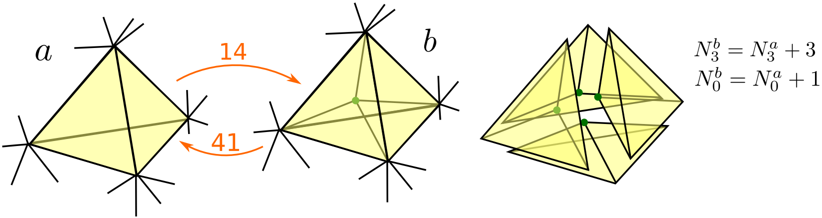

Grand canonical?

- The Markov step that attempts 23-, 32-, 14-, 41-move with probabilities $\frac{p}{2}, \frac{p}{2}, \frac{1-p}{2}, \frac{1-p}{2}$ $(0 < p < 1)$ satisfies detailed balance (w.r.t. Boltzmann weight $e^{-S[N_3,N_0]}$).

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  - Theoretically: $f(n) < e^{cn^2}$ [Mijatović,'03]
  - In practice: $f(n) \leq n + 2$ for all $n \leq 9$ (10^8 triangulations) [Burton,'11]

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- Need to use a grand-canonical ensemble in 3D/4D (contrary to 2D)!

- Why not just

$$Z = \sum_{\text{triang. } \mathfrak{t}} \frac{1}{|\mathrm{Aut}(\mathfrak{t})|} e^{-S[N_3, N_0]} \qquad , \quad S[N_3, N_0] = \kappa_3 N_3 - \kappa_0 N_0?$$

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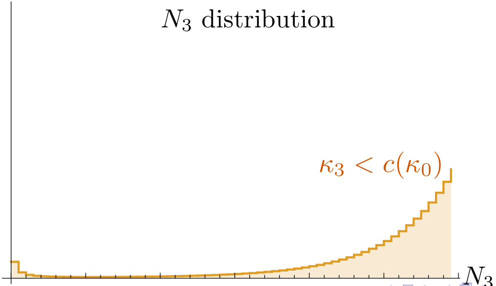

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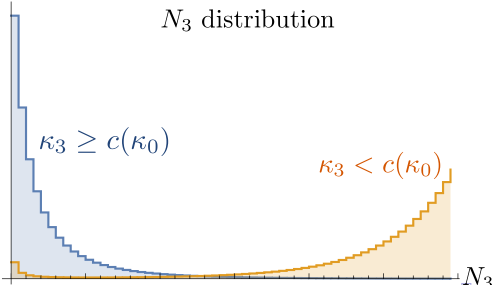

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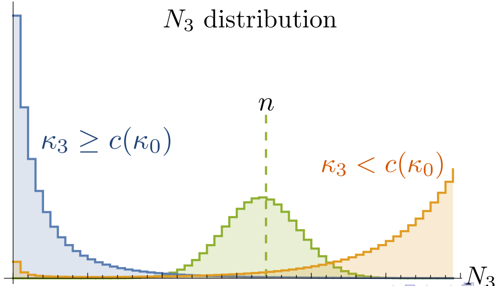

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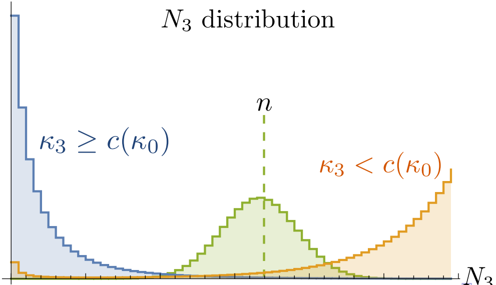
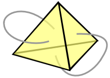
  - Rejection sampling of MCMC: effectively simulate $Z_{N_3=n}[\kappa_0] = \sum e^{\kappa_0 N_0}$. Need ϵ not too small.
  - Need ϵ not too large for ergodicity.



MCMC overview

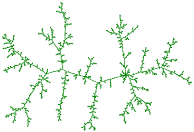
- Read parameters: desired size n, coupling κ_0.
- Initialize configuration: correct topology is sufficient.

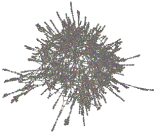


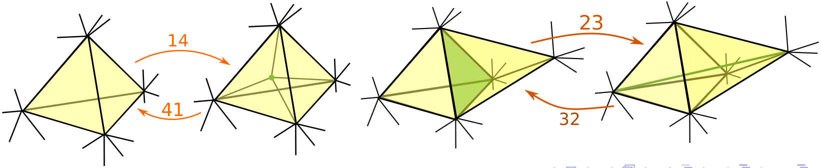
- Start performing Monte Carlo moves indefinitely
  - Thermalization phase
    - Parameter tuning (ϵ, κ_D, relative move frequency p)
    - Monitor thermalization with suitable observables.
  - Measurement phase
    - With predetermined frequency attempt measurement.
    - If desired, reject configuration if size outside window around n.
    - Add measurement data to list or histogram.

Phases

- By examining the moves we can already get an idea what the geometries will look like for κ_0 very small/large.
  - κ_0 large, maximize N_0 for fixed N_3: many 14-moves $\rightarrow$ tree-like structure.
  - κ_0 small, minimize N_0 for fixed N_3: many 23-moves $\rightarrow$ highly connected

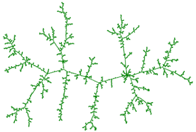




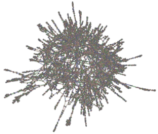


Phases

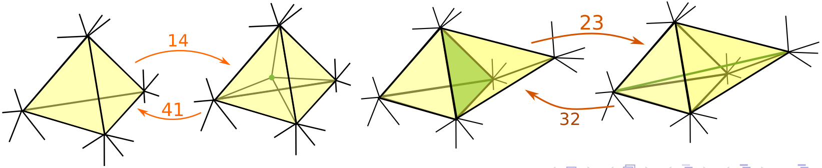
- By examining the moves we can already get an idea what the geometries will look like for κ_0 very small/large.
  - κ_0 large, maximize N_0 for fixed N_3:
    many 14-moves $\to$ tree-like structure.
    "Branched polymer phase"
    $d_{\mathrm{H}} = 2,\ d_s = 4/3$



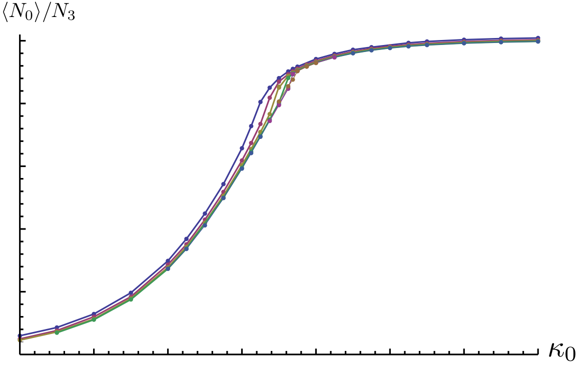
  - κ_0 small, minimize N_0 for fixed N_3:
    many 23-moves $\to$ highly connected
    "Crumpled phase"
    no conclusive scaling ($d_{\mathrm{H}} = d_s = \infty$?)



- Indeed these structures are characteristic for the two phases of DT in 3D and 4D. [Boulatov, Krzywicki, Ambjørn, Varsted, Agishtein, Migdal, Jurkiewicz, Renken, Catterall, Kogut, Thorleifsson, Bialas, Burda, Bilke, Thorleifsson, Petersson,. . . ,'90s]



Phase transition



- All is not lost: perhaps enhanced scaling at the phase transition?

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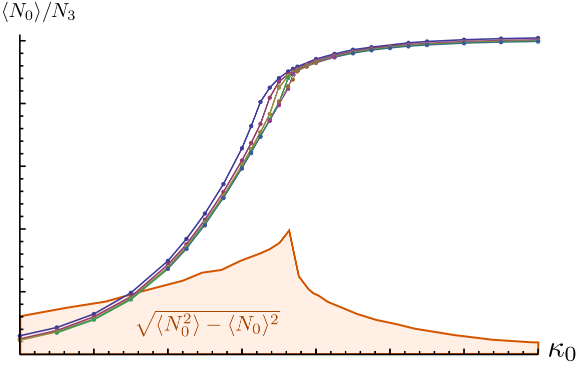

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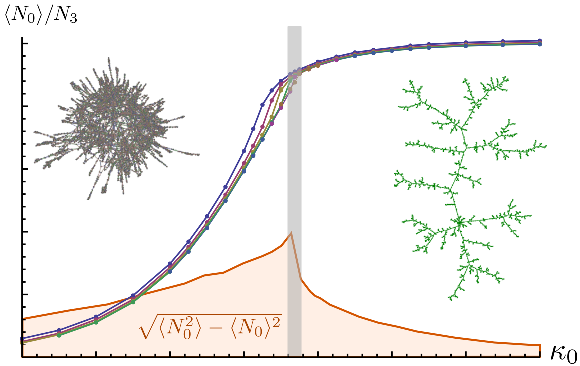
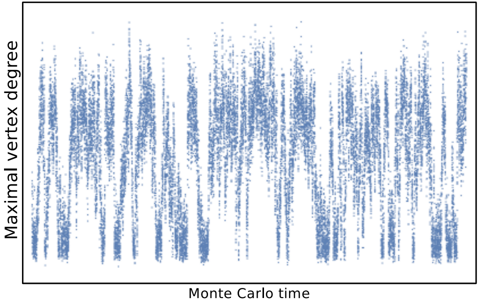


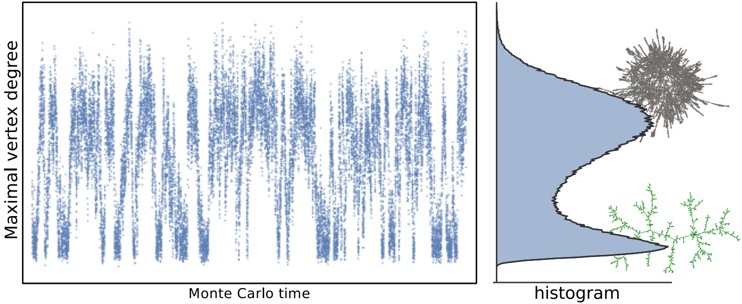
- All is not lost: perhaps enhanced scaling at the phase transition?
- Not clear from this plot whether transitions is discontinuous (1st order) or continuous (higher order).

Double peak structure



- When κ_0 is tuned to critical value: MCMC jumps between two meta-stable states.

Double peak structure



- When κ_0 is tuned to critical value: MCMC jumps between two meta-stable states.
- If double peak in histogram becomes more pronounced as $N_4 \to \infty$ then transition is discontinuous.

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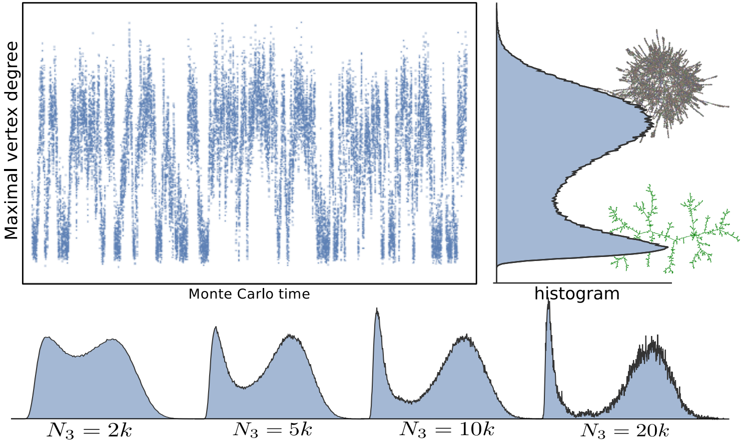

- When κ_0 is tuned to critical value: MCMC jumps between two meta-stable states.
- If double peak in histogram becomes more pronounced as $N_4 \to \infty$ then transition is discontinuous.
- It does. No hope of new scaling at transition.

How to proceed?

- 3D$\to$4D: Situation is similar, though discontinuity less pronounced.

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- Enlarge phase diagram with extra couplings or matter fields.
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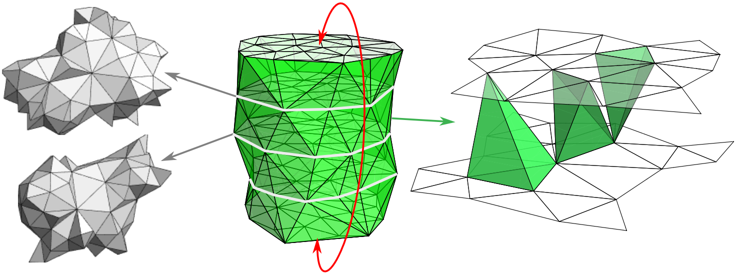
  - Gauge fields, Gaussian fields, Ising models.
- Change the ensemble of geometries.
  - Change topology.
  - Different polyhedra as building blocks.
  - Introduce foliation: Causal Dynamical Triangulations (CDT).

Causal Dynamical Triangulations in 3D

- Consider a (general or simplicial) 3-Triangulation of topology $S^1 \times S^2$.
- It is *causal* if it is "foliated" by triangulations of S^2 and all tetrahedra of two types (31-, 22-simplex).



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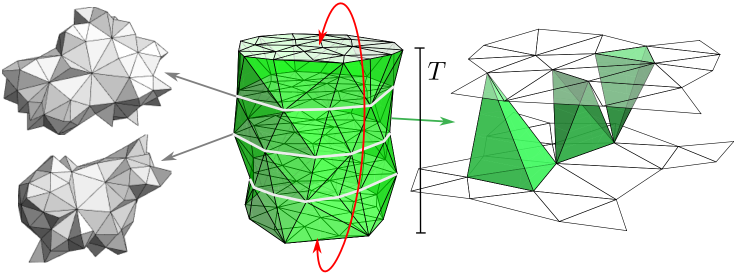
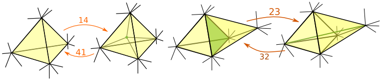
- Let's adapt our MCMC methods to sample such triangulations with

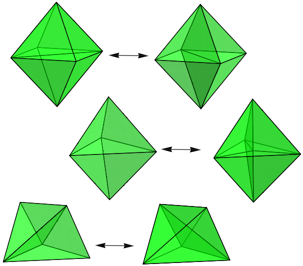
$$Z[N_3, N_0, T] := \sum_{\substack{\text{causal triangulations } \mathfrak{t} \\ \text{with } T \text{ layers}}} \frac{1}{|\mathrm{Aut}(\mathfrak{t})|} e^{-S[N_3,N_0]}.$$



Adaption to Causal triangulations

- Replace moves 

with a set that preserves the foliation and is ergodic in causal triangulations (with fixed T).



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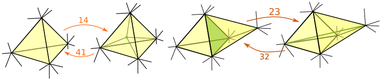
- Replace moves 

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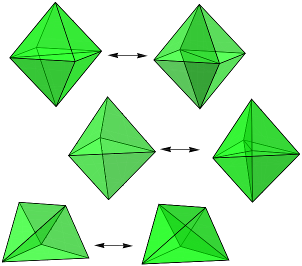
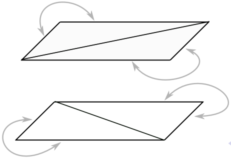


- Update detailed balance conditions.
- Construct by hand an initial configuration with correct topology.



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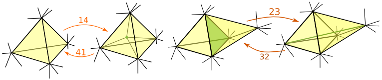
- Replace moves 

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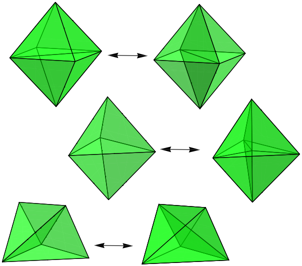

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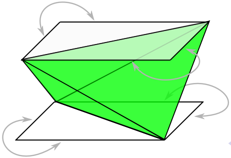

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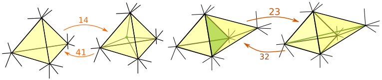
- Replace moves 

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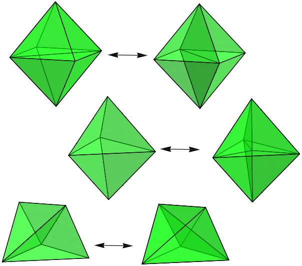

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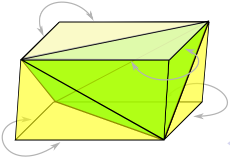
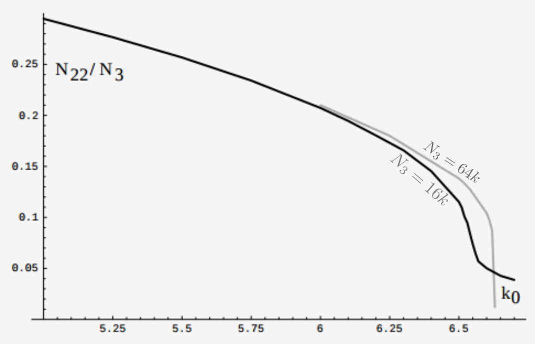


Phase diagram of CDT in 3D

- For fixed N_3
  - κ_0 large, maximize N_0, few 22-simplices
  - κ_0 small, minimize N_0, many 22-simplices

[Ambjorn, Jurkiewicz, Loll, hep-th/0011276]



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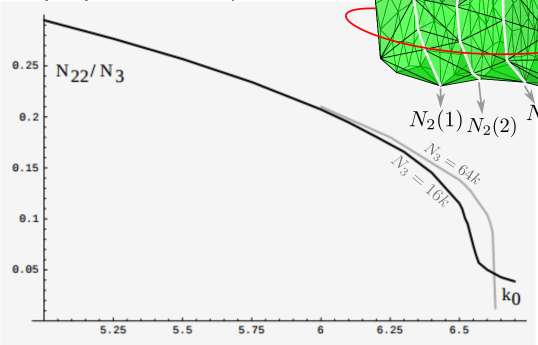

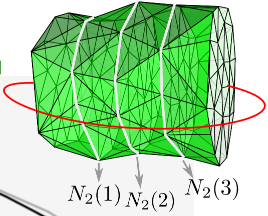

Phase diagram of CDT in 3D

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    Weak correlation between slices; collection of 2d random geometries
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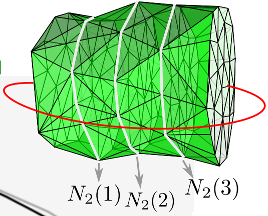

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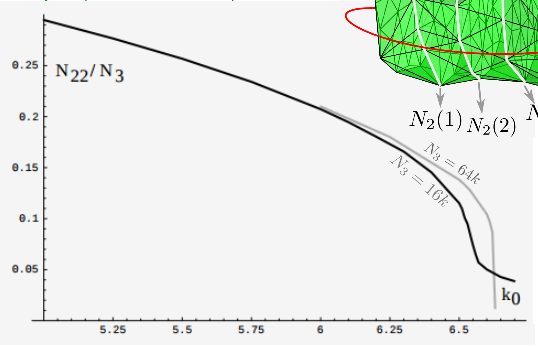

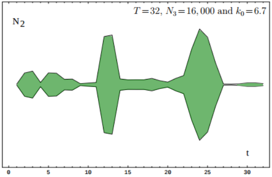

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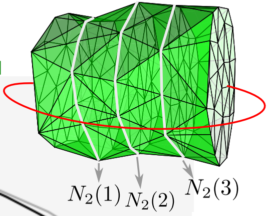
    Strong correlation between slices; condensation!



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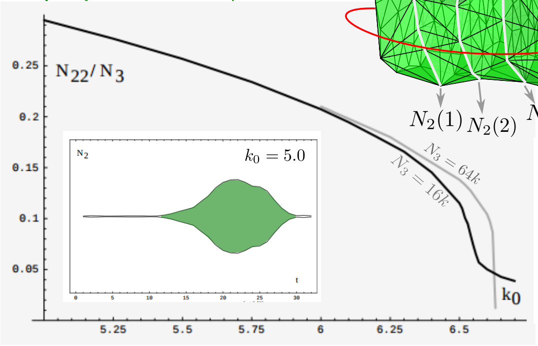

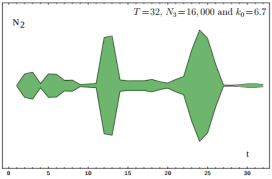

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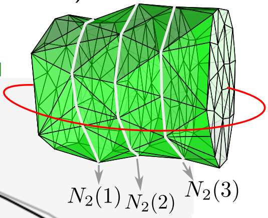
    Strong correlation between slices; condensation!
- Either 1st order phase transition (simplicial triangulations) or no transition (general triangulations).



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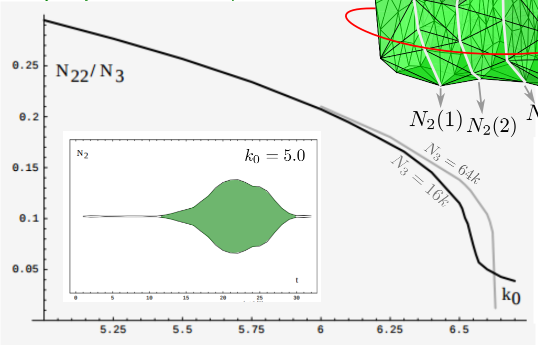
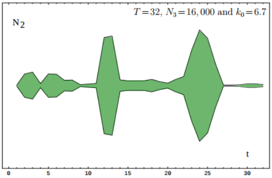
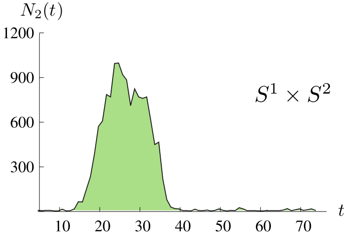




A closer look at the condensation phase

- As $N_3 \to \infty$ the relative fluctuations of $N_2(t')$ w.r.t $\langle N_2(t') \rangle$ decrease to 0.



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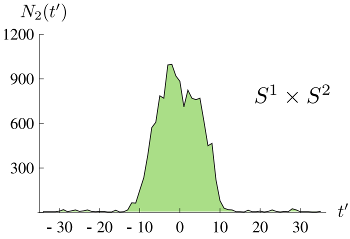

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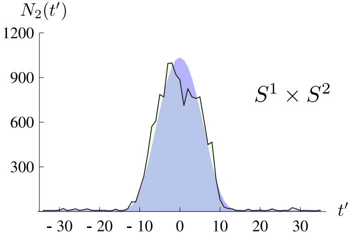

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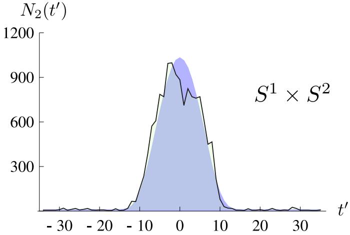

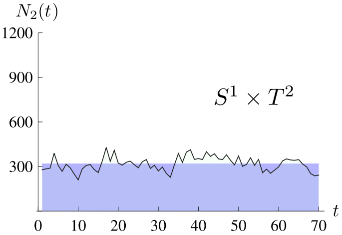

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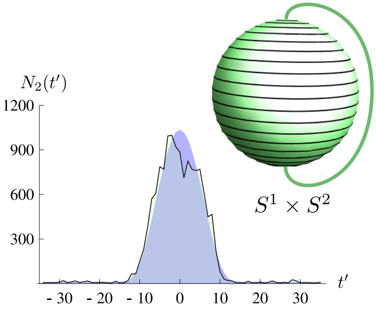

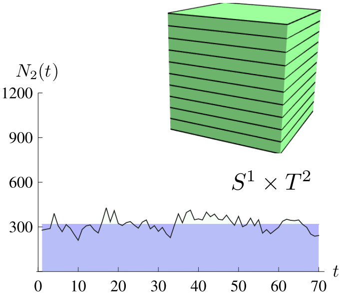

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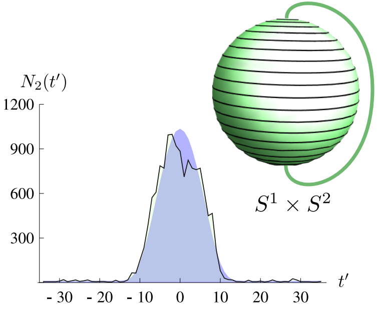
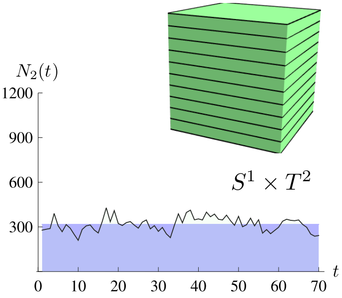
- $\langle N_2(t') \rangle$ accurately matches $a \cdot \cos^2(b \cdot t')$ (which happens to match the volume profile of S^3).
- Spectral dimension $d_s \approx 3$.

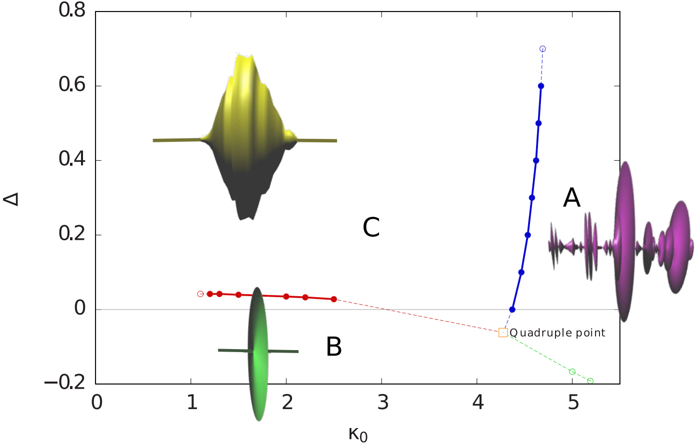




CDT in 4D: the state of the art

- A richer phase diagram in 4D: similar phase C with semi-classical volume profile and $d_s \approx 4$.

[Ambjorn, Coumbe, Gizbert-Studnicki, Goerlich, Jordan, Jurkiewicz, Klitgaard, Loll, ...]



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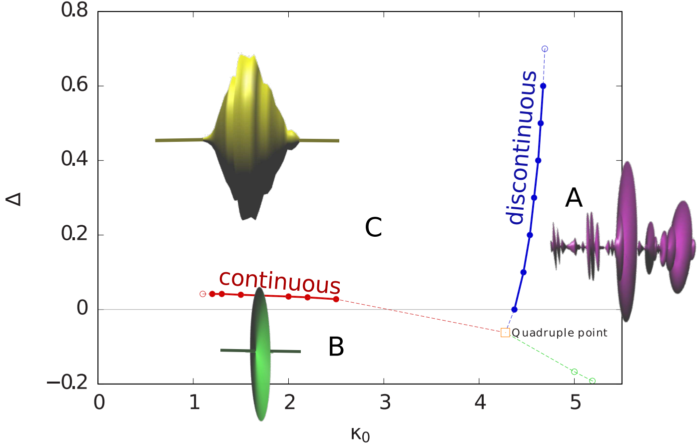

CDT in 4D: the state of the art

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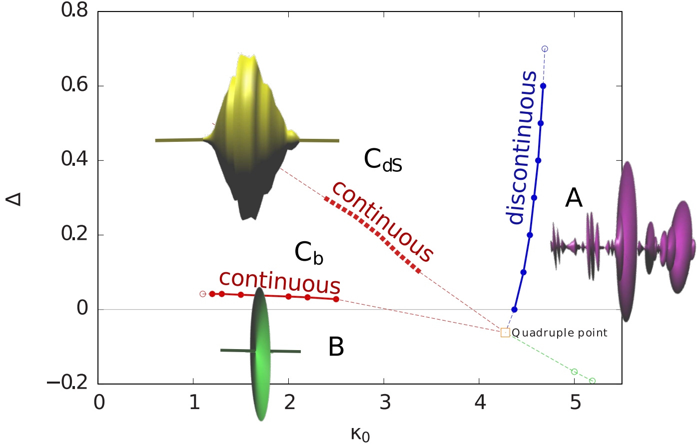
- Now also a continuous phase transition (probably 2nd order)
- Surprisingly another continuous phase transition was recently found.

[Ambjorn, Coumbe, Gizbert-Studnicki, Goerlich, Jordan, Jurkiewicz, Klitgaard, Loll, ...]



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- Continuous phase transitions are essential to model sub-Planckian geometry.
- The possession of a semi-classical thermodynamic limit is a highly non-trivial property in the case of (background-independent) random geometries.