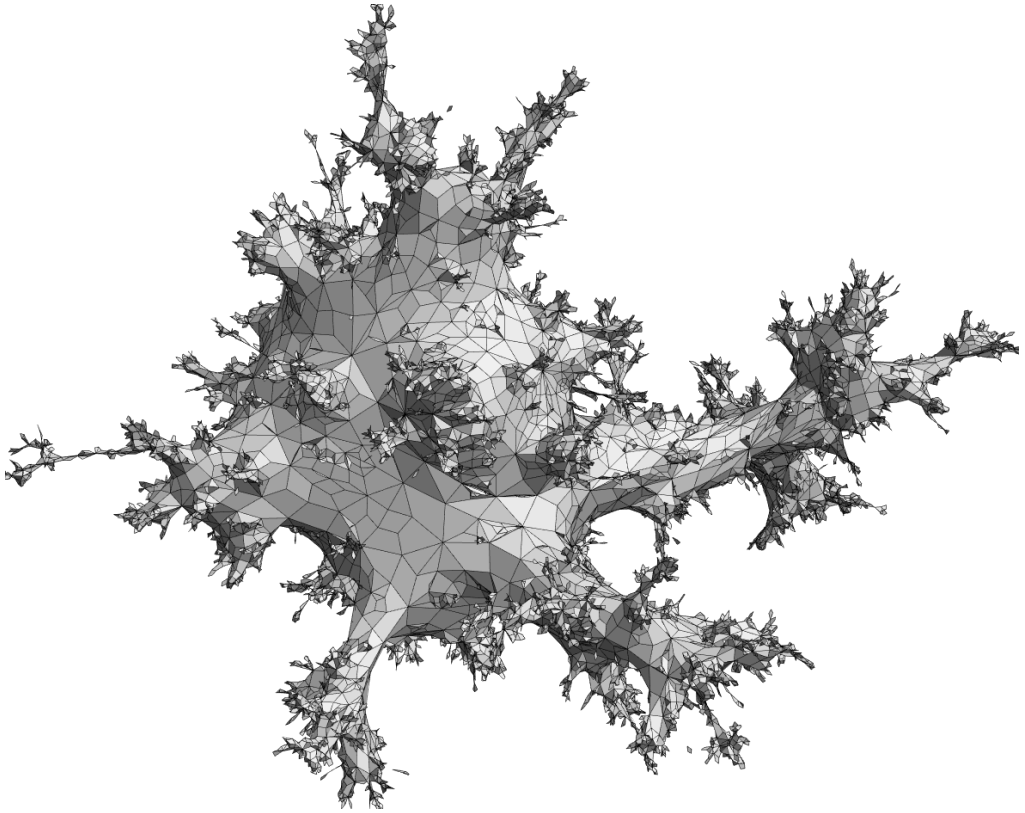
Making Quantum Gravity Computable, 21-06-2017

Monte Carlo methods in Dynamical Triangulations

Part I: 2D random geometry

Timothy Budd



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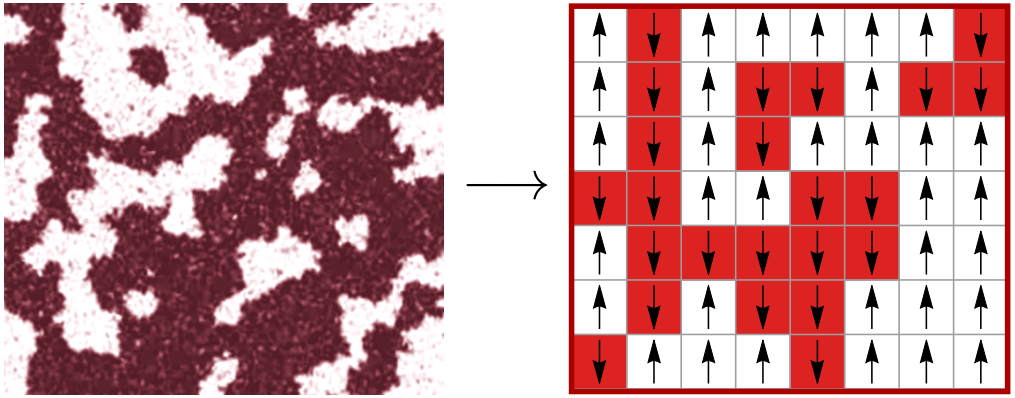
Outline

- Day 1: 2D random geometry
  - Combinatorial representation
  - Markov Chain Monte Carlo (MCMC) methods
  - Matter coupling
  - Observables
- Day 2: Dynamical Triangulations in higher dimensions
  - Quantum gravity
  - Combinatorial representation
  - MCMC methods
  - Phase diagram
  - Causal Dynamical Triangulations
- Tutorials: numerical analysis of various 2D random geometries
  - Measure observables for random geometries (produced by black box)
  - Extract critical exponents.
  - Experiment with (new?) observables.
  - Conclusions will be collected at the end and be discussed.

Discrete geometry?

Discretization in ...

- ...the Ising model: (Barkema's course)



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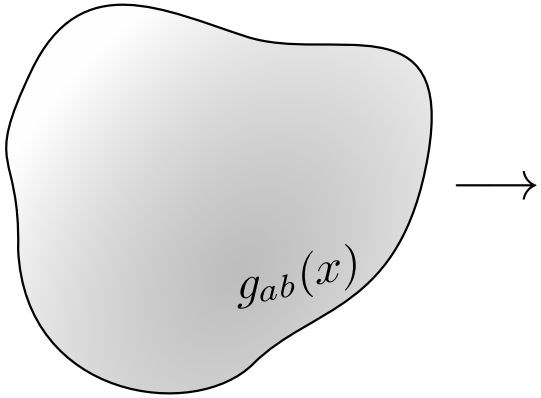

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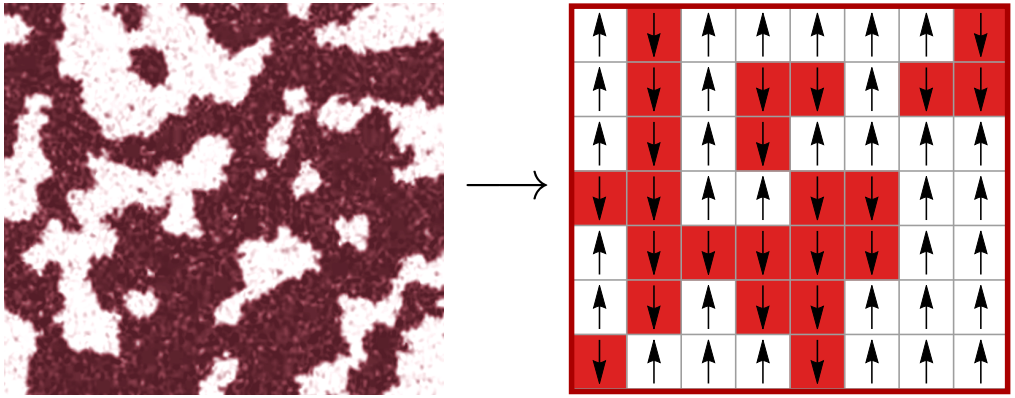

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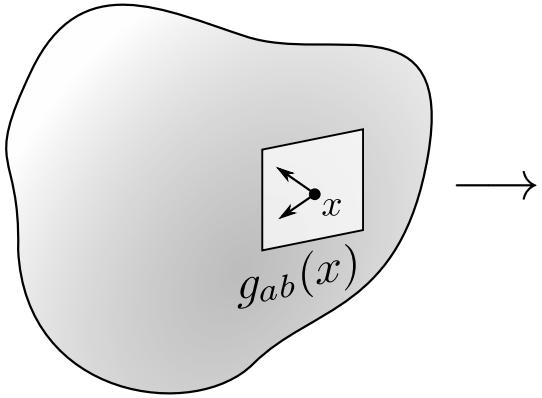

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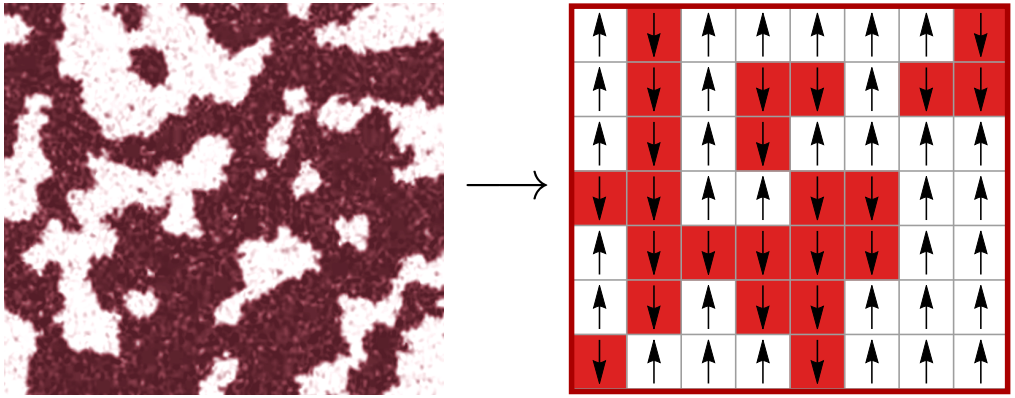

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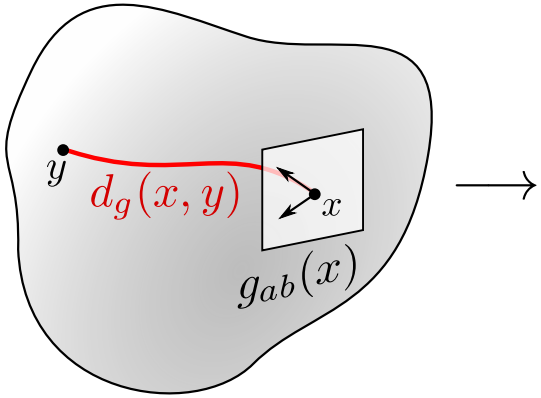

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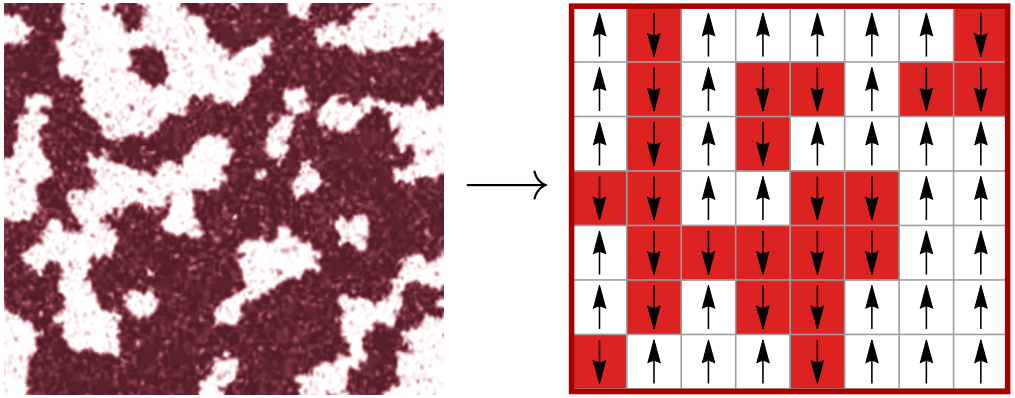

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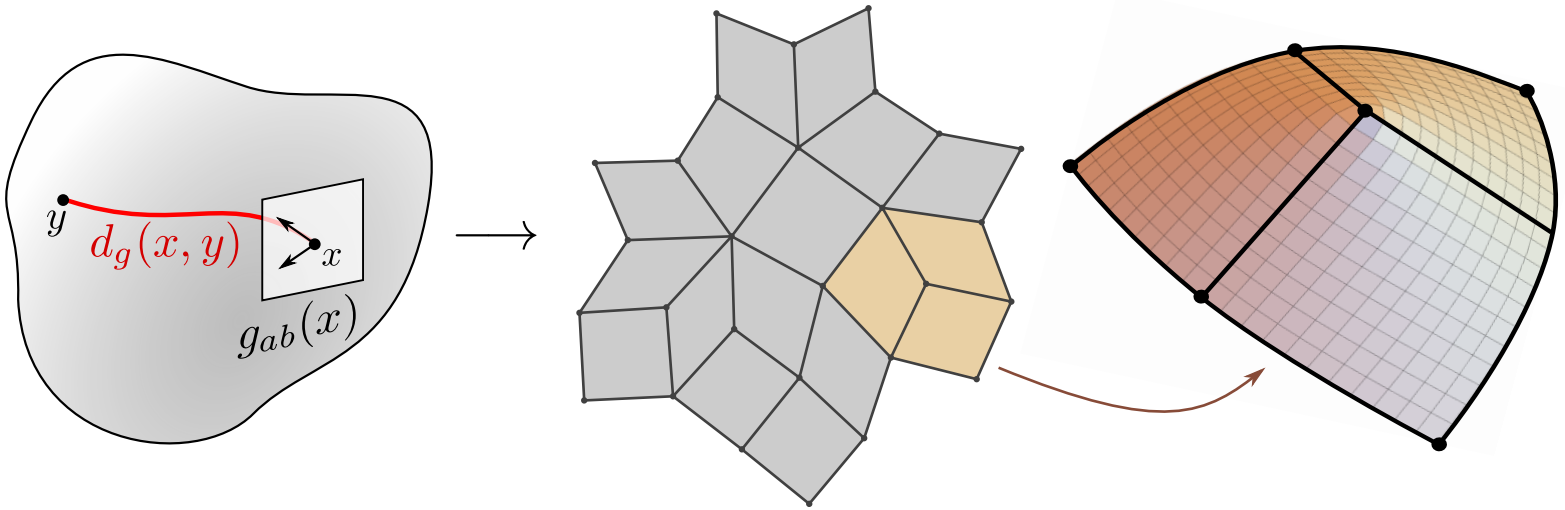

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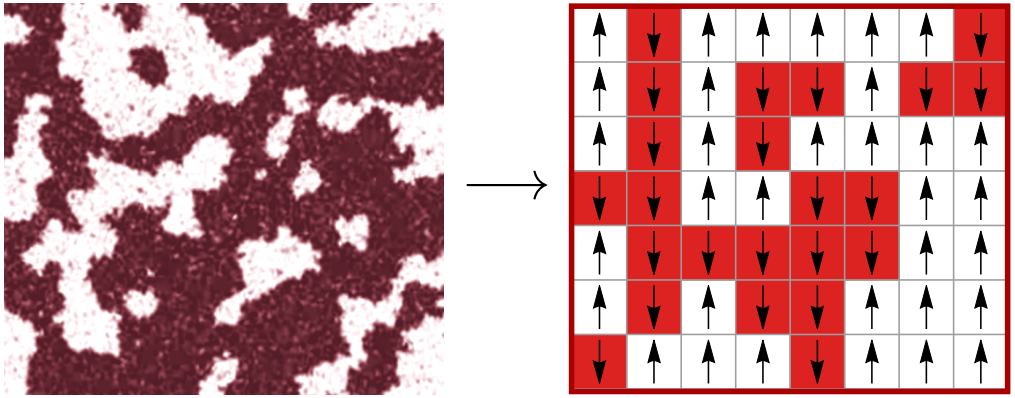

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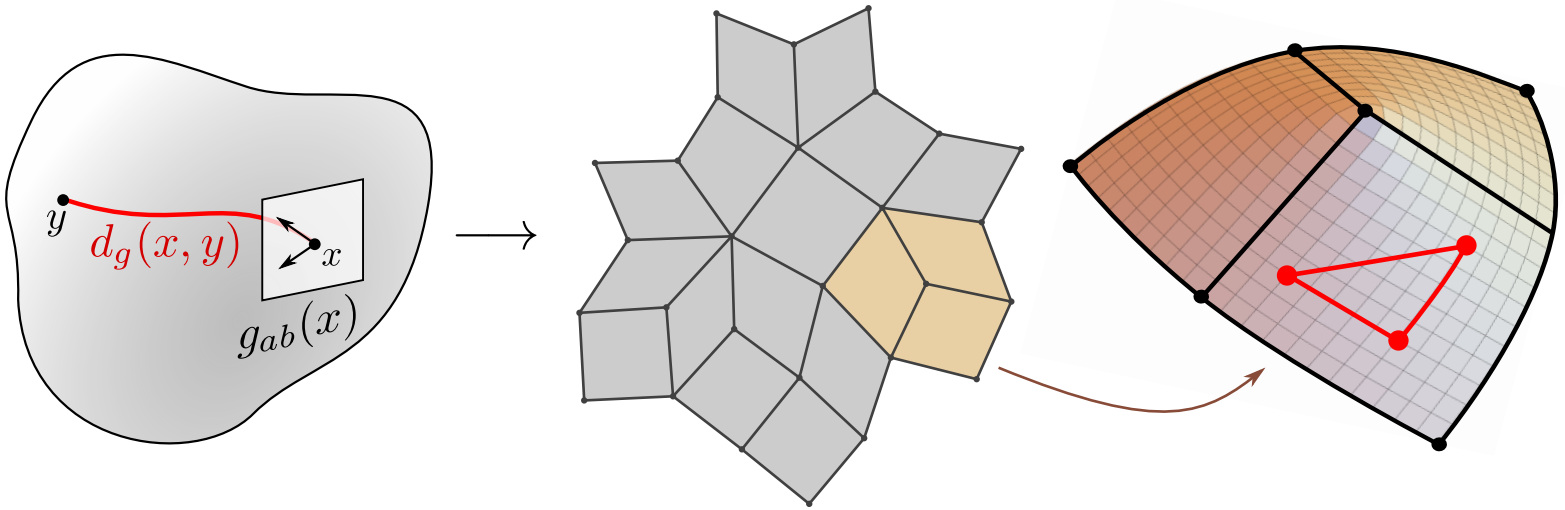

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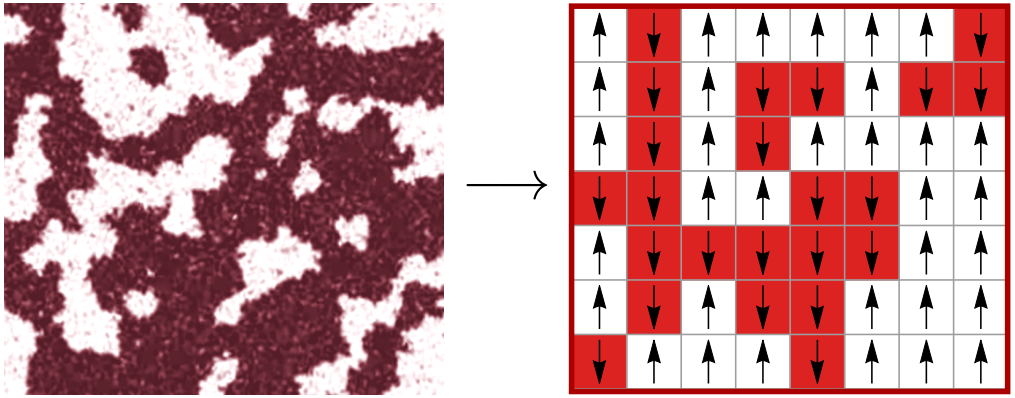

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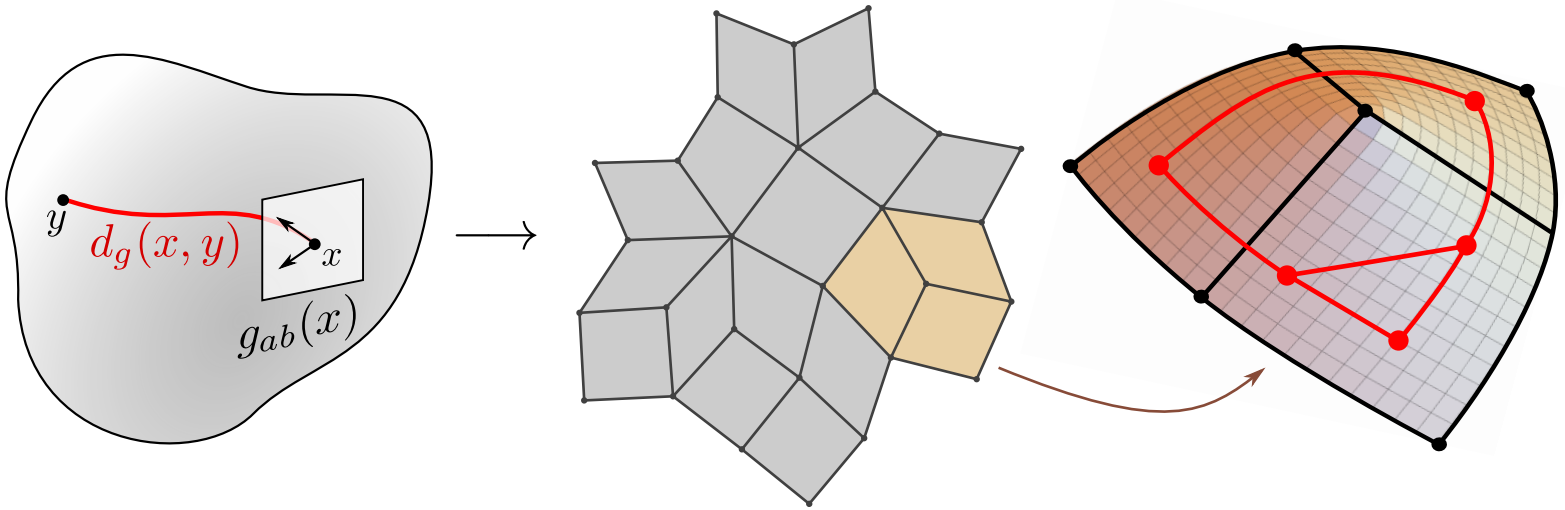

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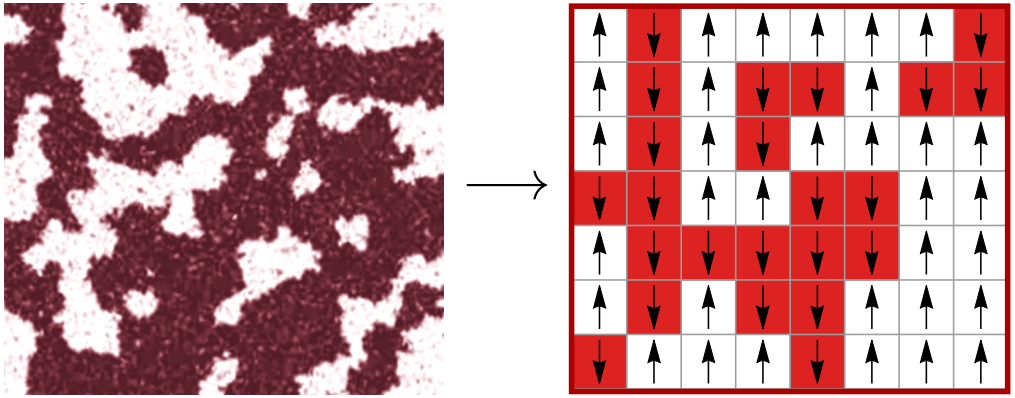

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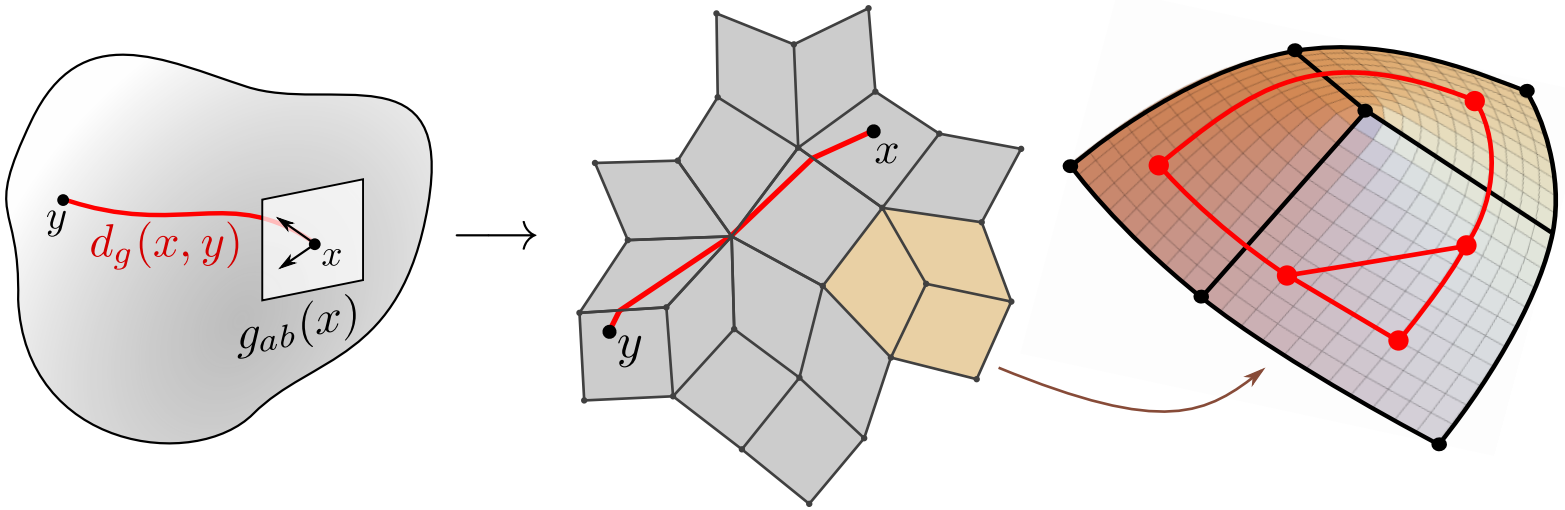

Geometry from polygons

- To change a discrete geometry, one may change . . .
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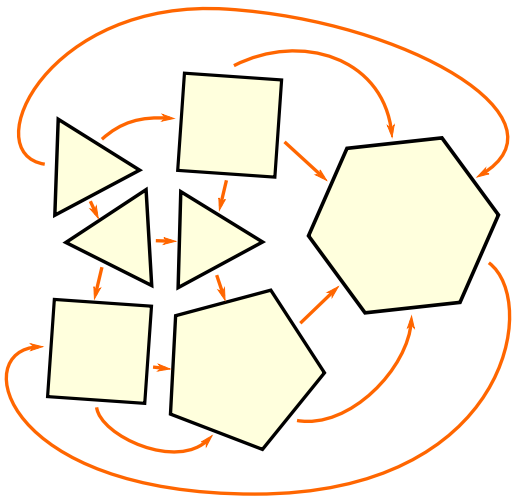
- Fix once and for all the geometry of each polygon of degree k to be that of the regular k-gon in Euclidean space with sides of length 1.
- Then can represent geometry equivalently by
  - a "gluing prescription" on a collection of polygons.
  - a "map": a proper embedding of a graph in a surface;



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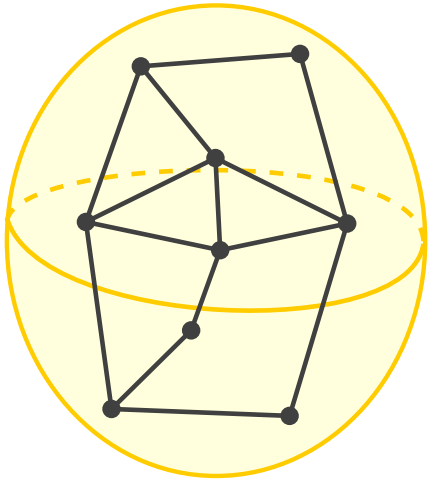
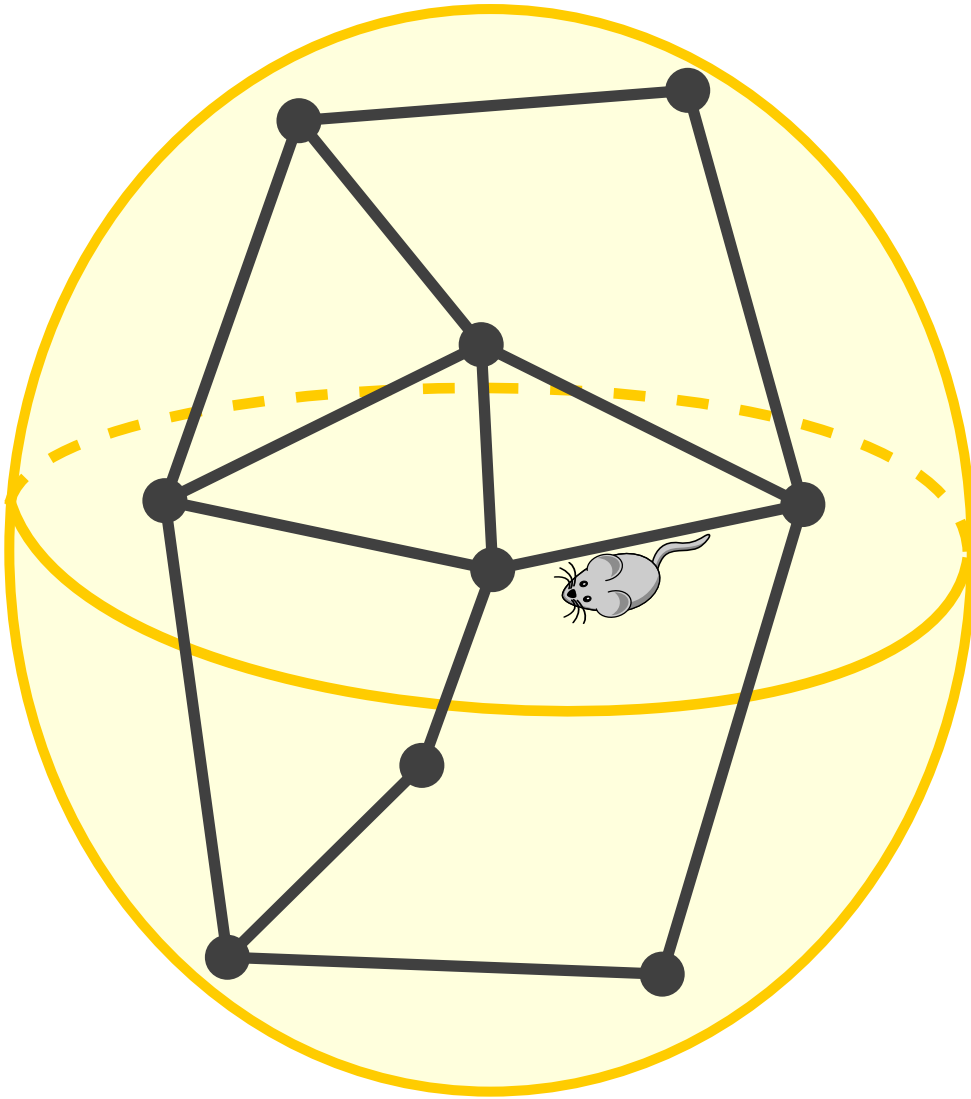


map

Combinatorial description of a map

- Navigate map using "next" and "adjacent".



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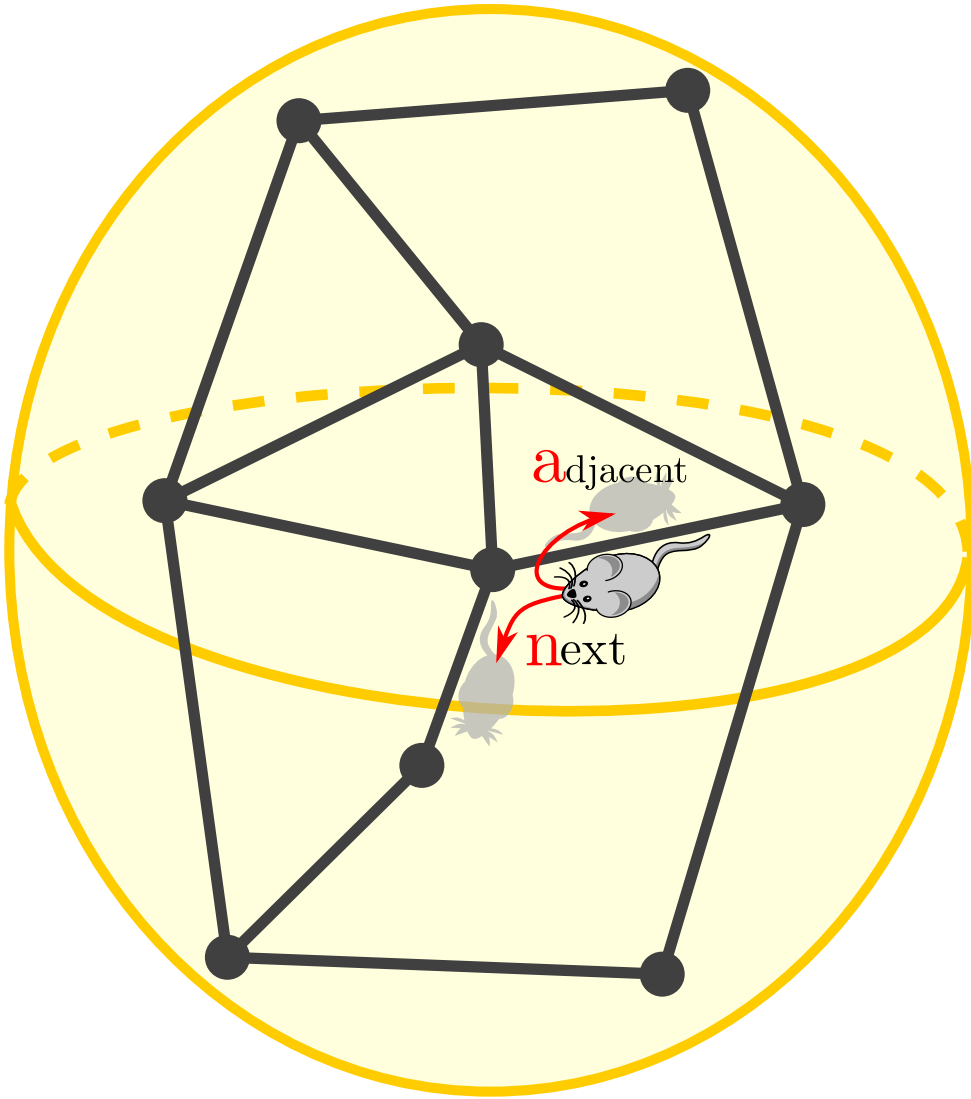

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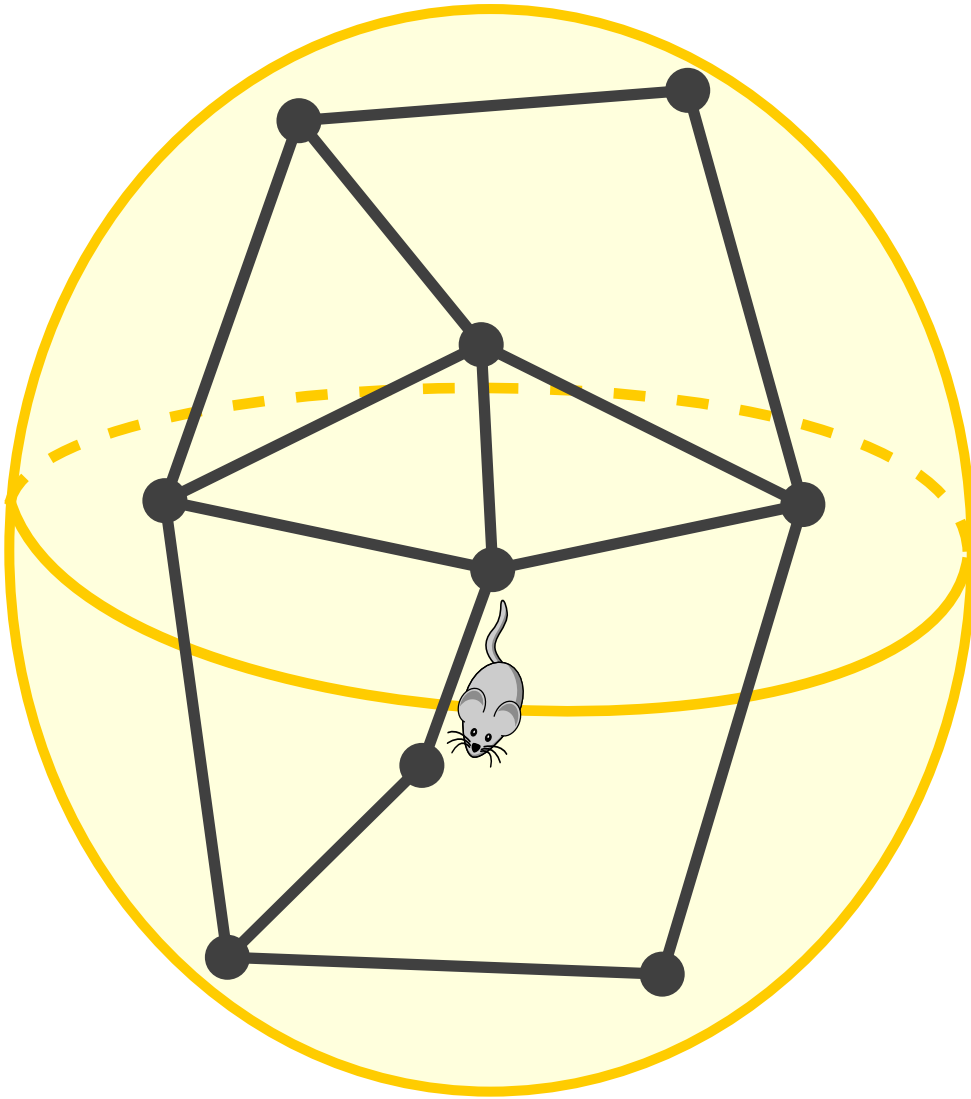

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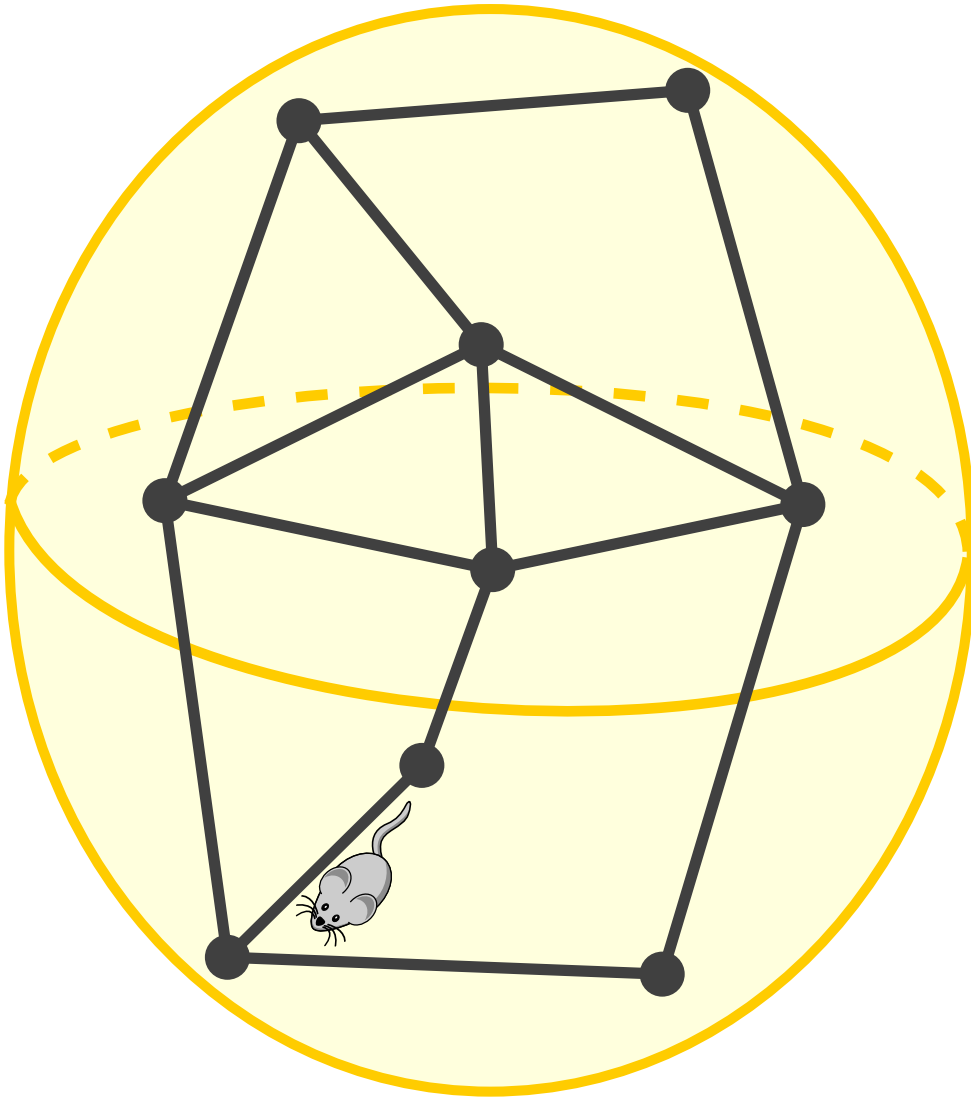

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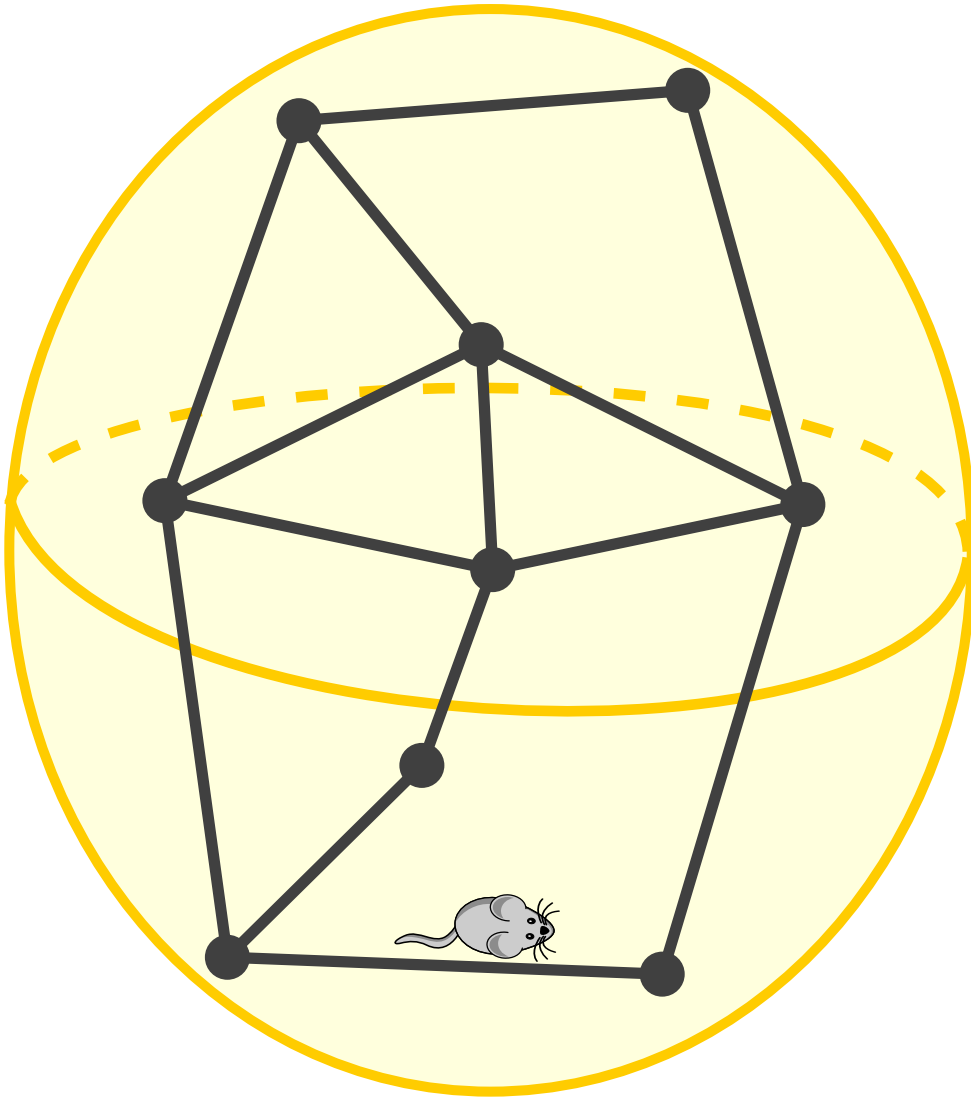

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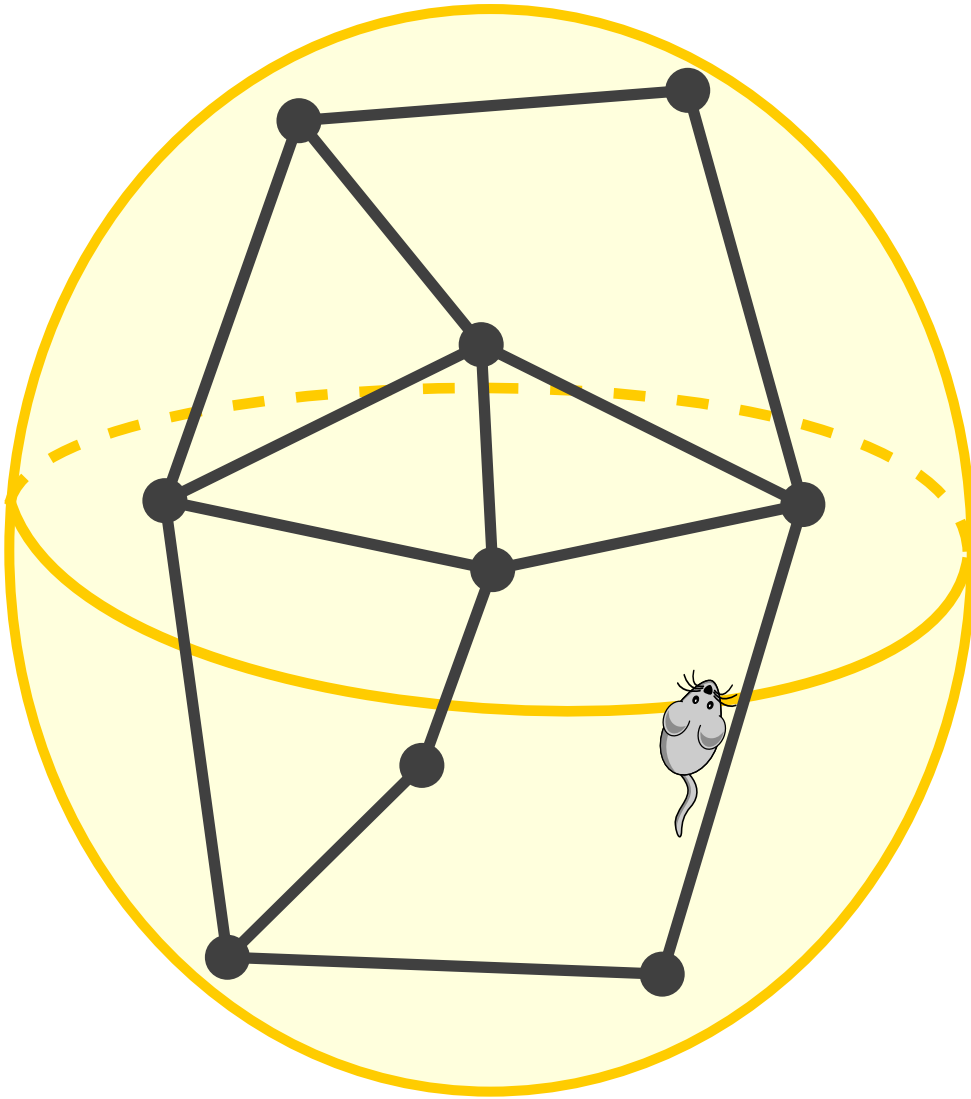

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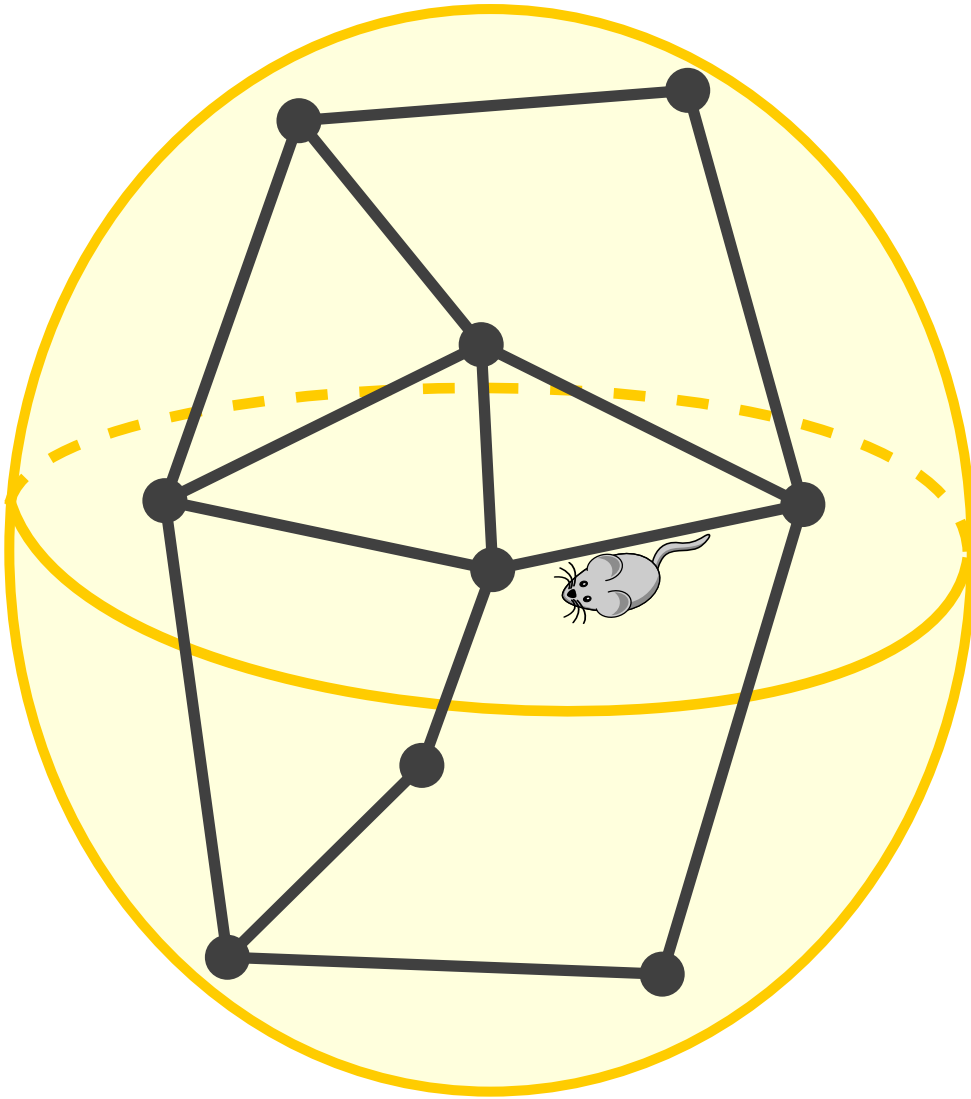

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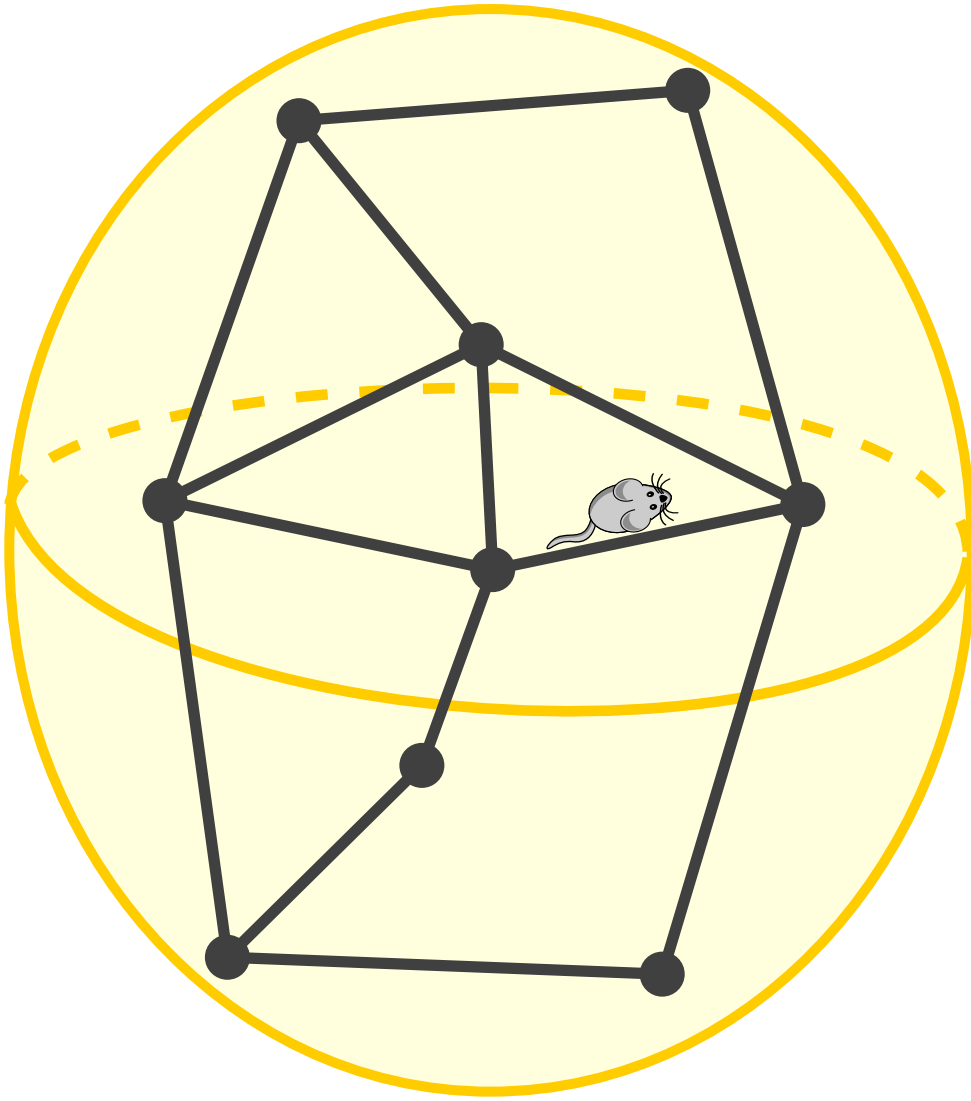

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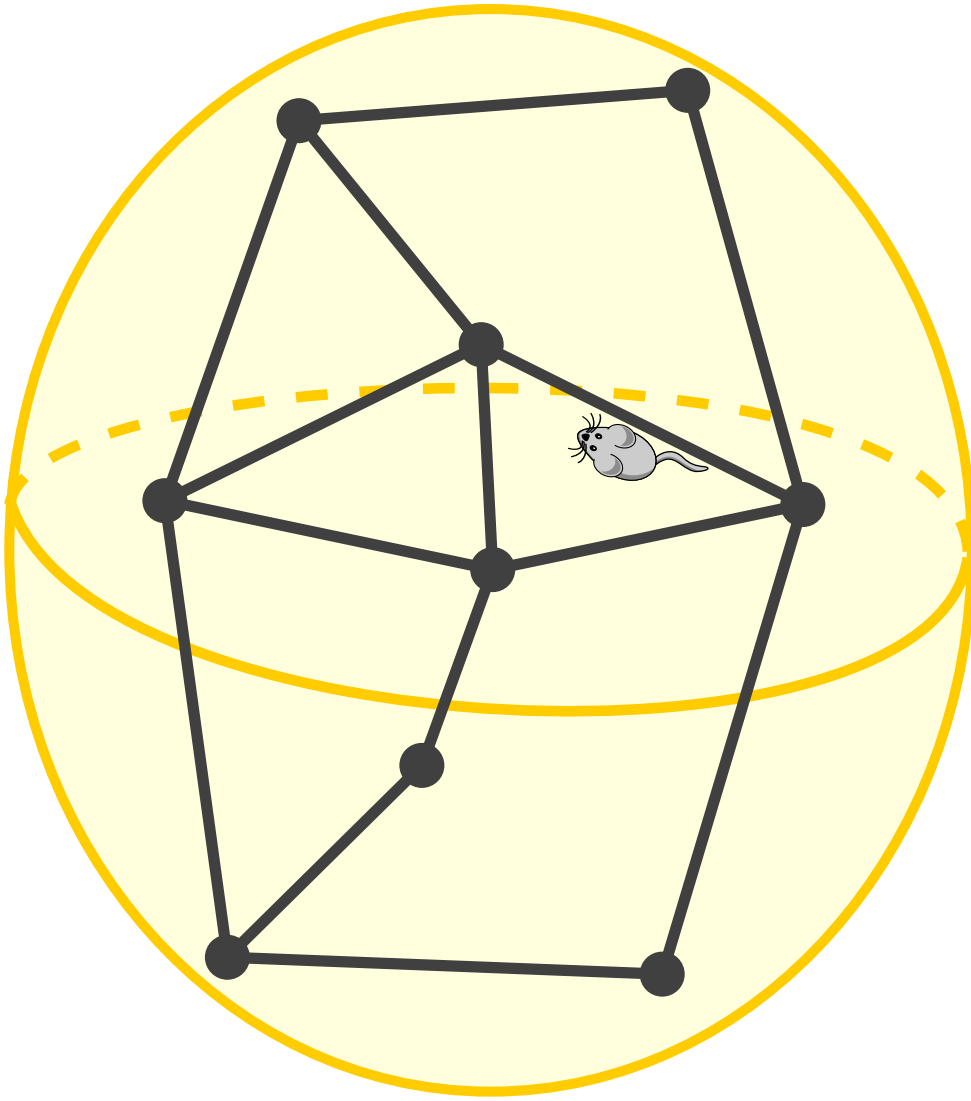

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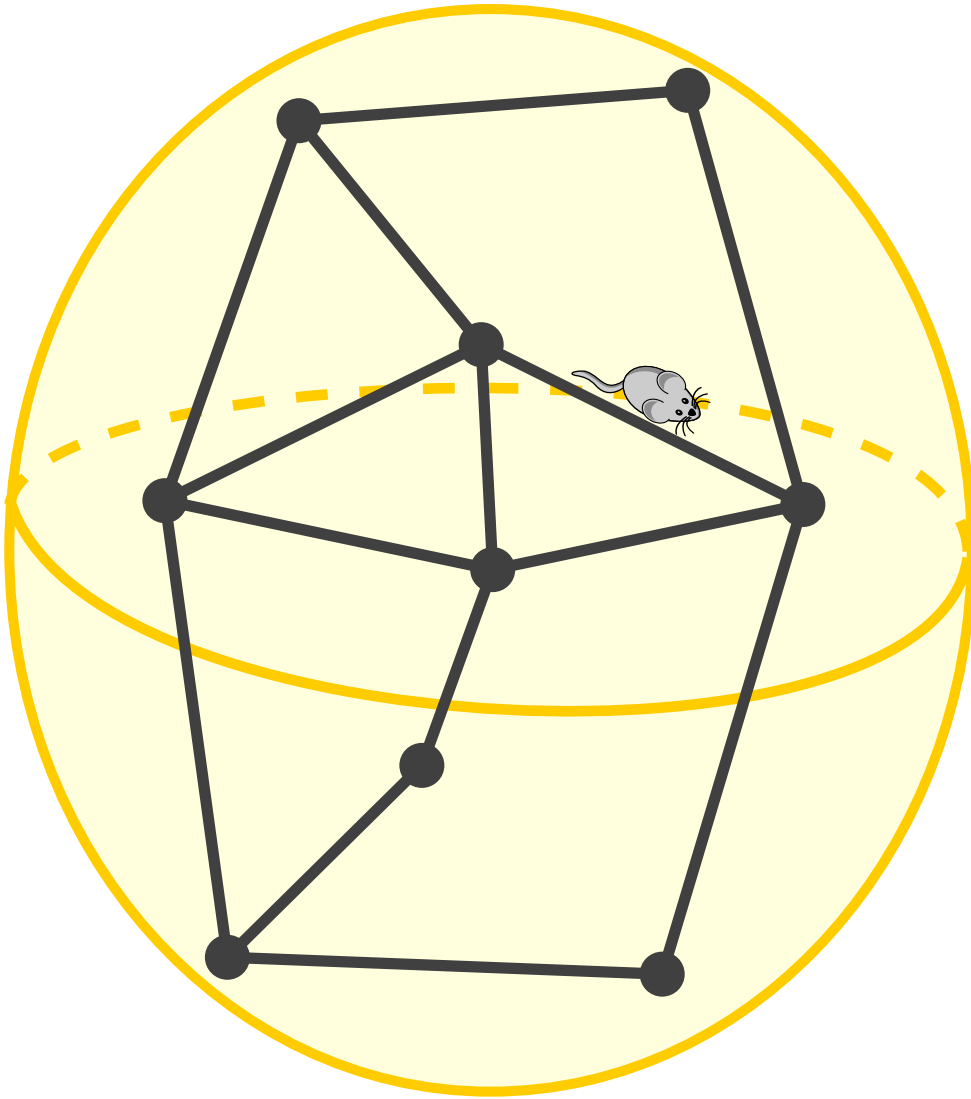

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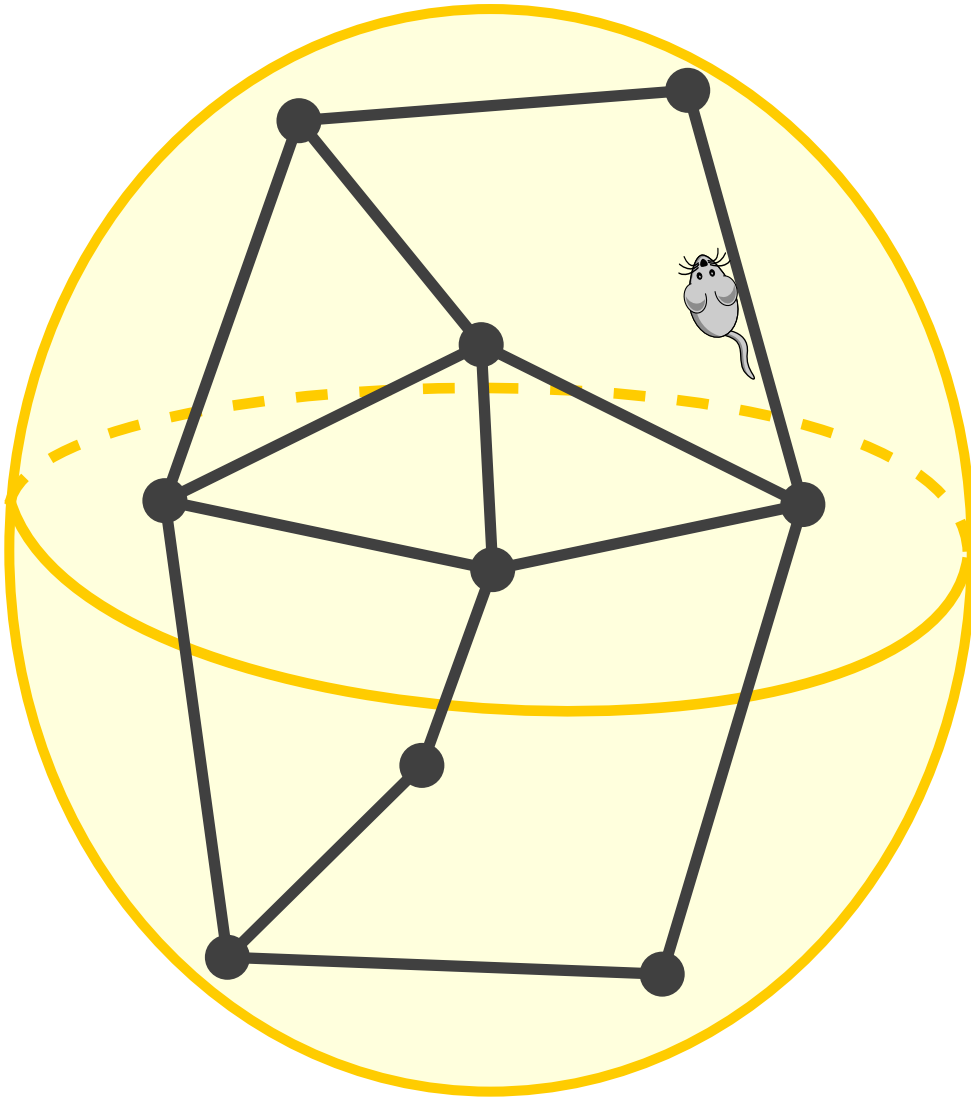

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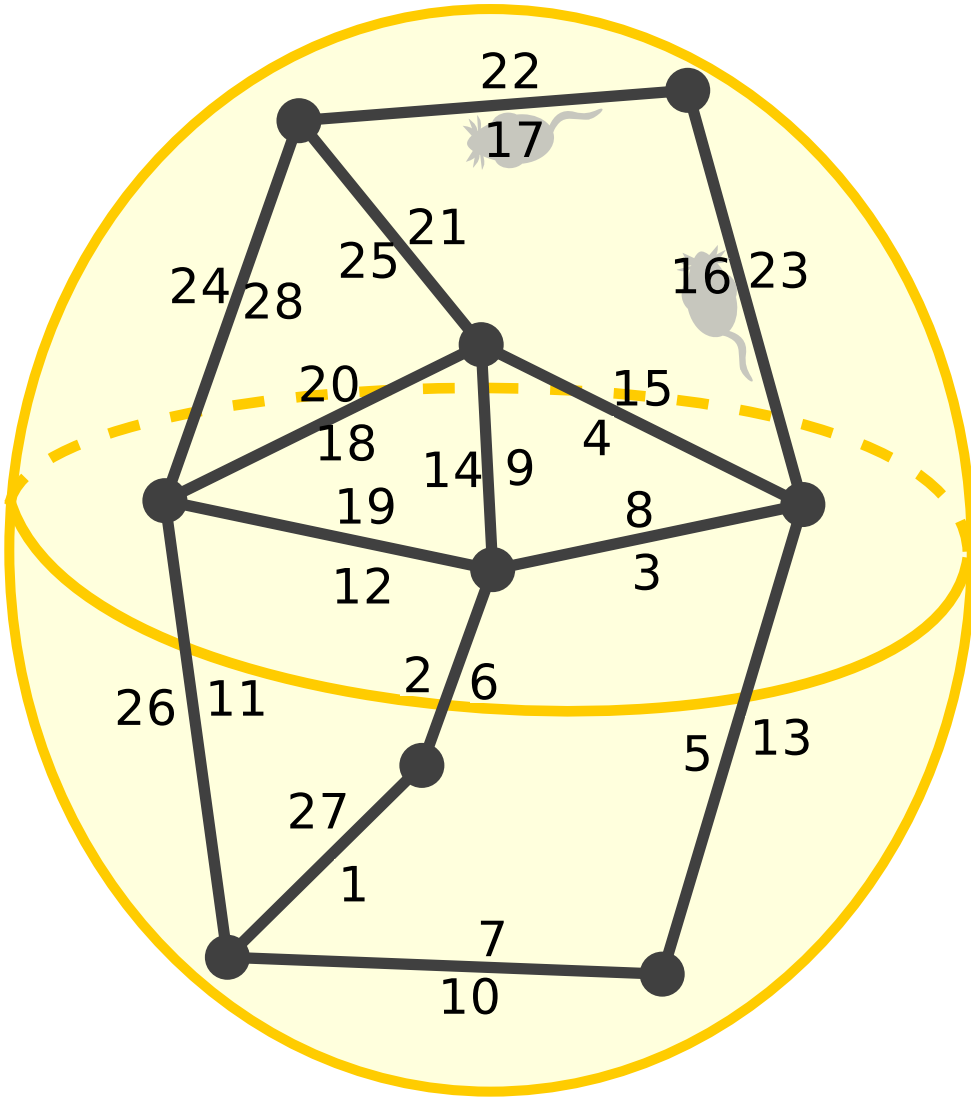

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$$n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 28 \\ 7 & 12 & 6 & 9 & 4 & \cdots & 20 \end{pmatrix}$$

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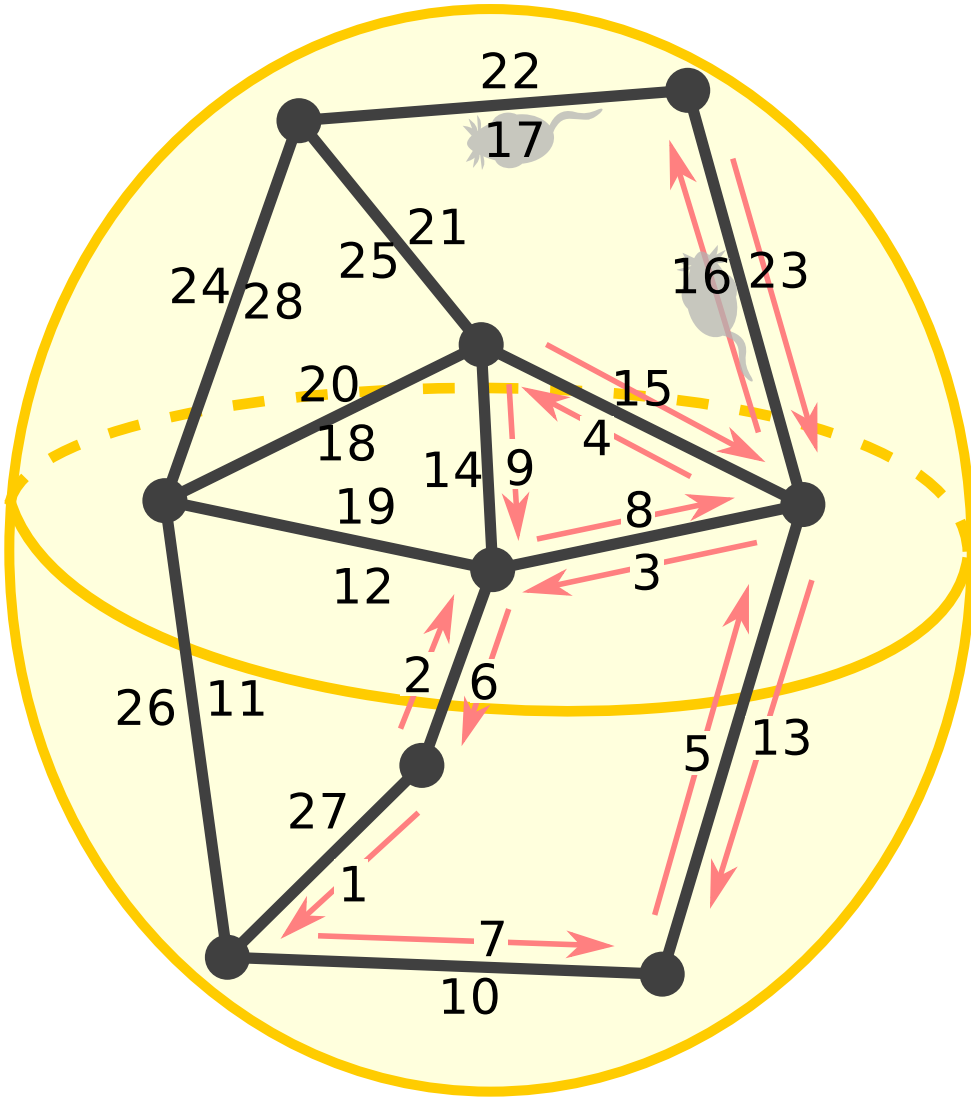

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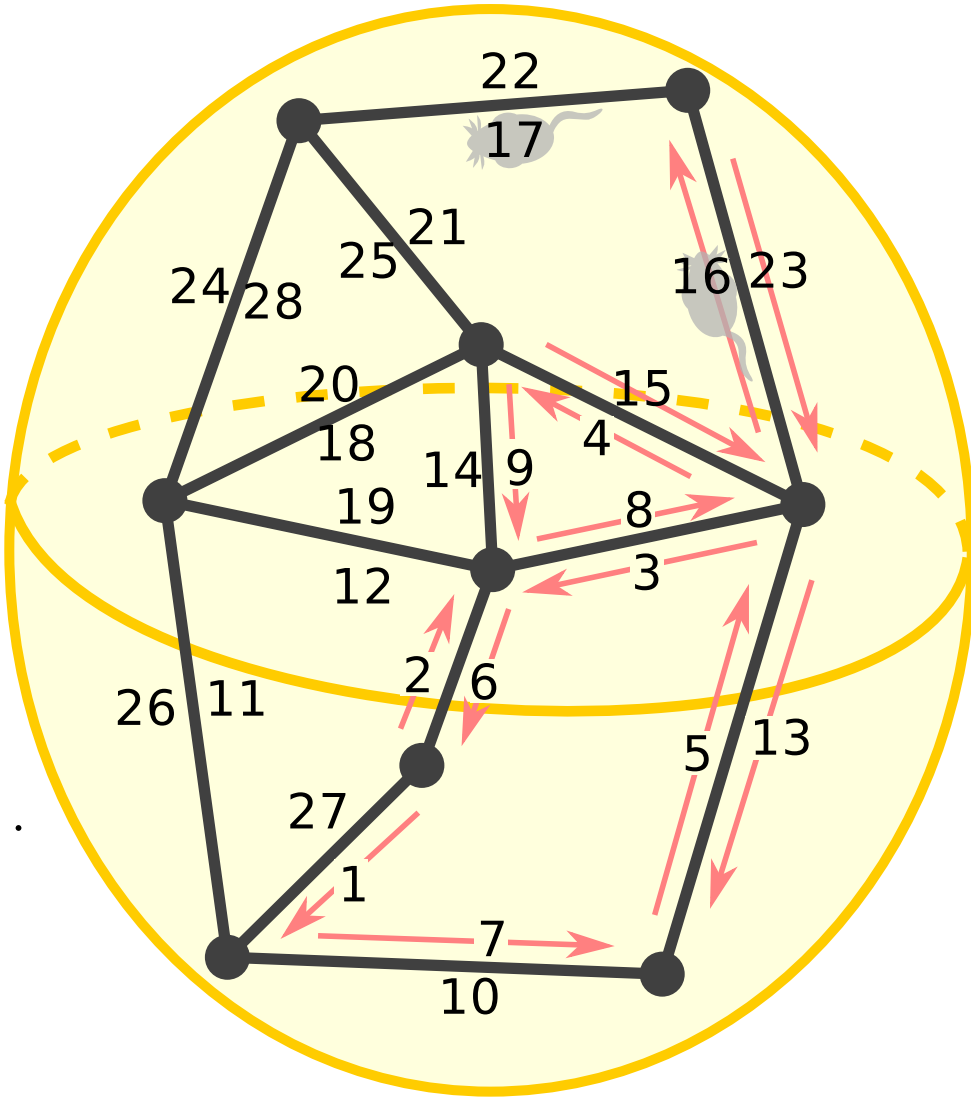

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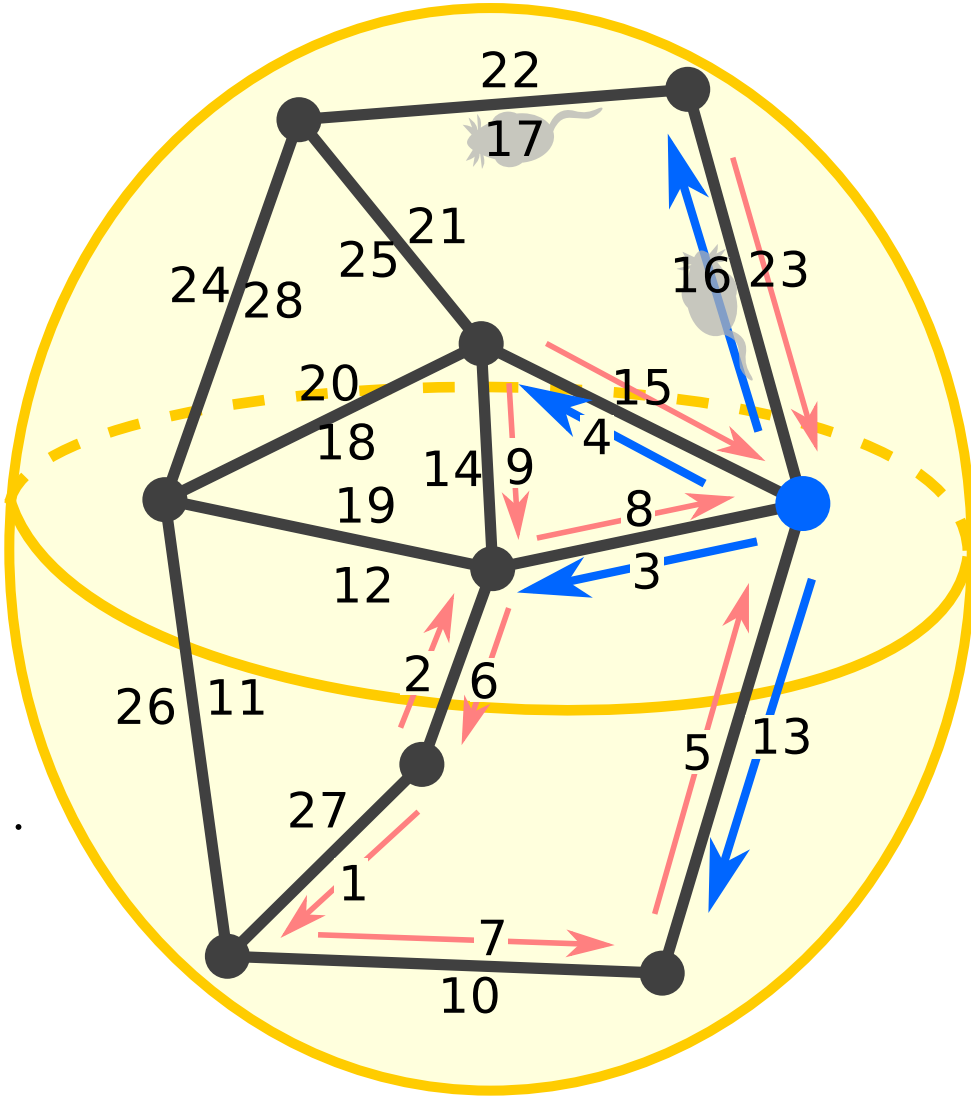
$$n \circ a = (1\,2)(3\,4\,16\,13)(5\,10)\cdots$$



- Any pair (n, a) of permutations on $[2N] := \{1, 2, \ldots, 2N\}$ with $a \circ a = \mathbf{1}$ and $a(x) \neq x$ determines a (labeled) map with N edges, hence a piecewise flat geometry on an (oriented!) surface.

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$$\mathcal{T}_N \equiv \{(n, a) : \text{transitive}, g(n, a) = 0, \text{all cycles of } n \text{ of length } 3\}.$$

- The set $\mathcal{Q}_N$ of *labeled quadrangulations* of S^2 with N edges can be described combinatorially by

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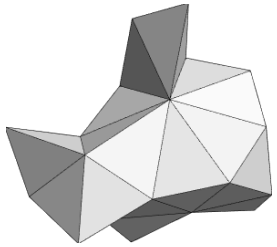
  etc.
- In particular, $|\mathcal{T}_N|, |\mathcal{Q}_N| < ((2N)!)^2 < \infty$.

Random discrete geometries

- The *uniform random* labeled triangulation of S^2 with N edges is an element of $\mathcal{T}_N$ chosen with probability $1/|\mathcal{T}_N|$ each.



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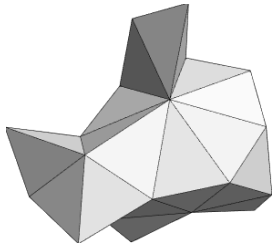

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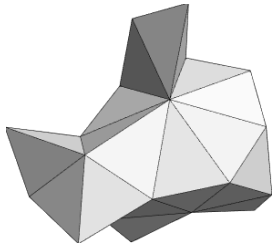

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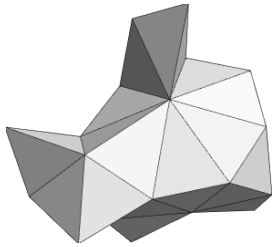

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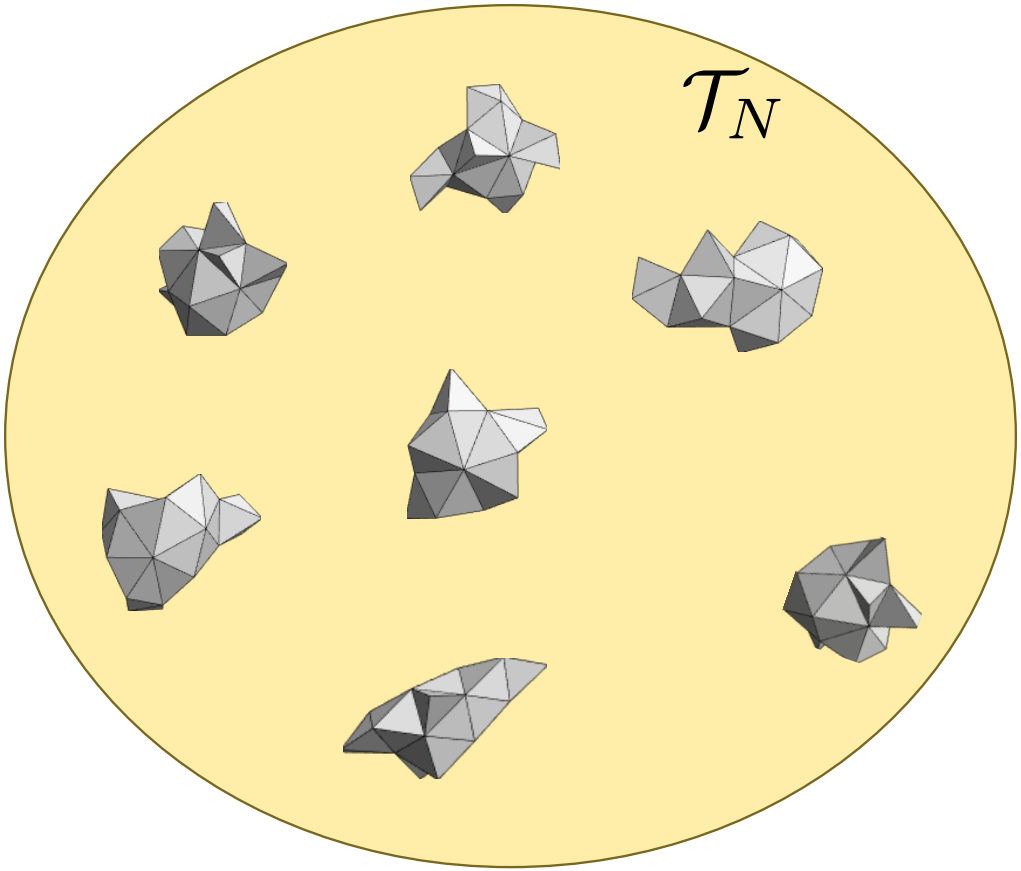
- How to sample from this ensemble? And compute $\langle \mathcal{O} \rangle_N$?
  - The analytic way: combinatorial algorithms; direct random generation.
  - Markov Chain Monte Carlo methods.

A Markov chain on triangulations



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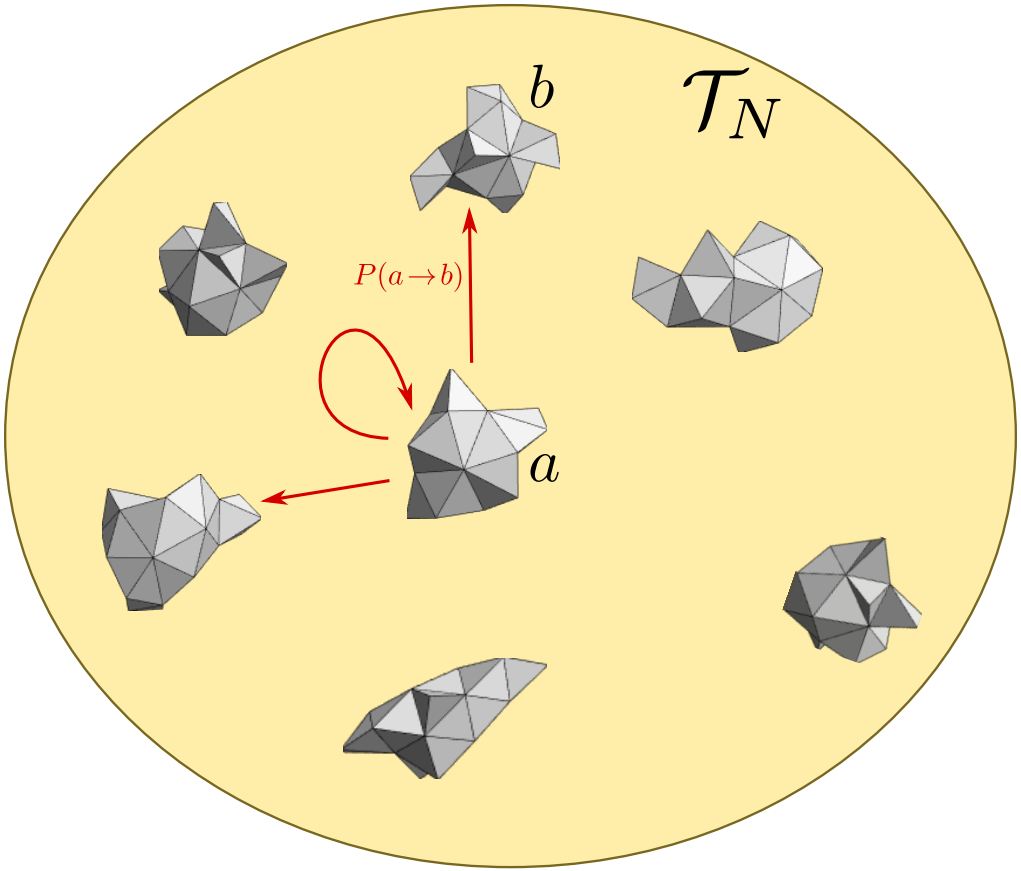

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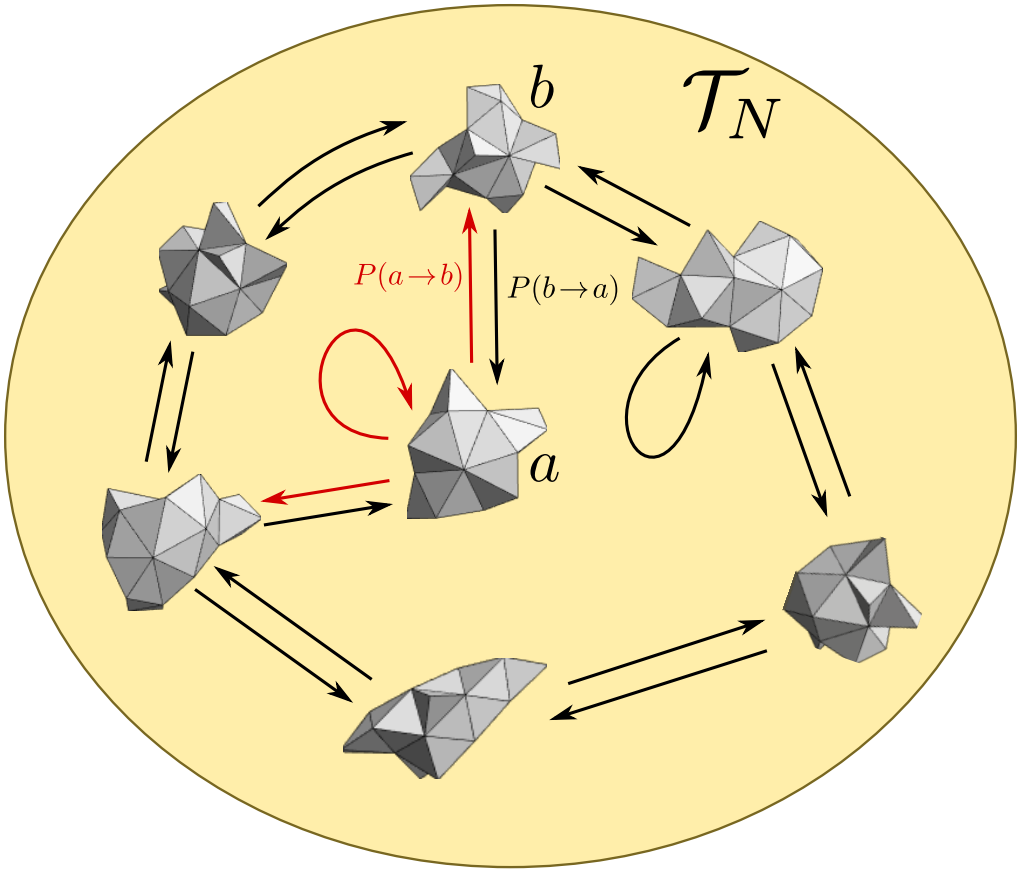
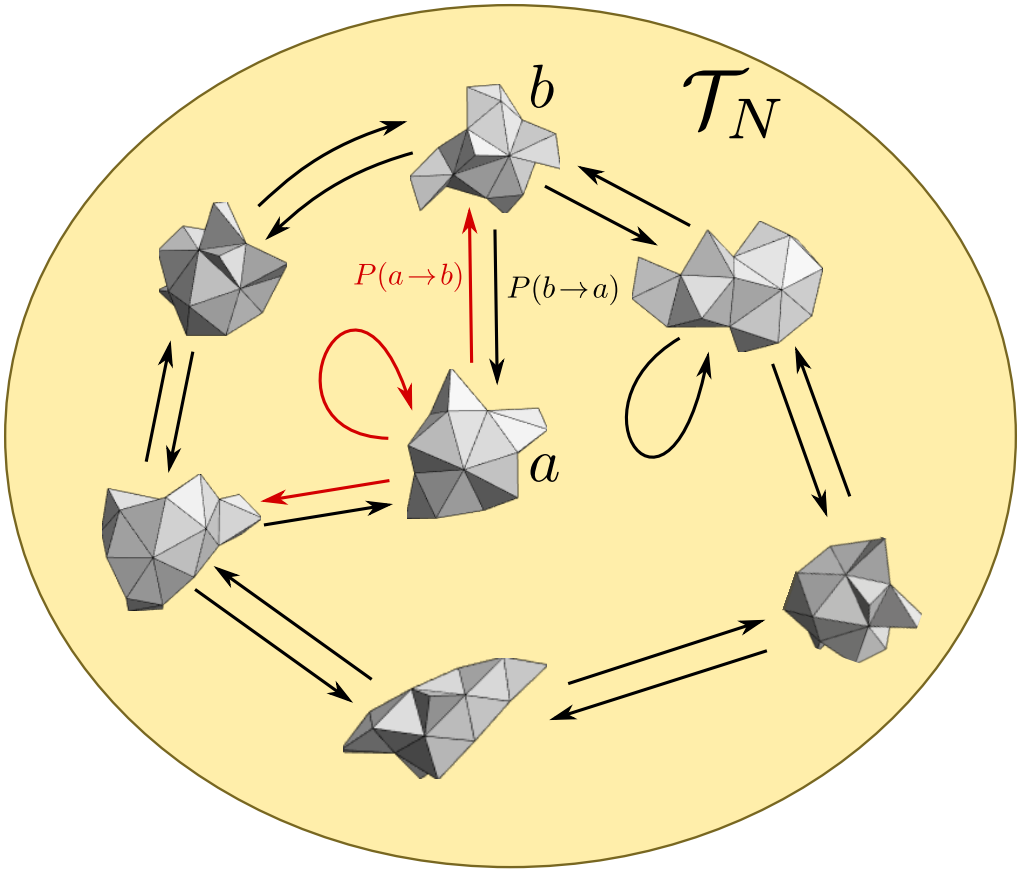


- To obtain a Markov process converging to the uniform distribution on $\mathcal{T}_N$ (from any starting point) it suffices to select an update algorithm that...
  - preserves topology and size of the map;
  - satisfies *Detailed balance*: $P(a \to b) = P(b \to a)$;

A Markov chain on triangulations



- To obtain a Markov process converging to the uniform distribution on $\mathcal{T}_N$ (from any starting point) it suffices to select an update algorithm that...
  - preserves topology and size of the map;
  - satisfies *Detailed balance*: $P(a \to b) = P(b \to a)$;
  - and *Ergodicity*: any state reachable from any other;

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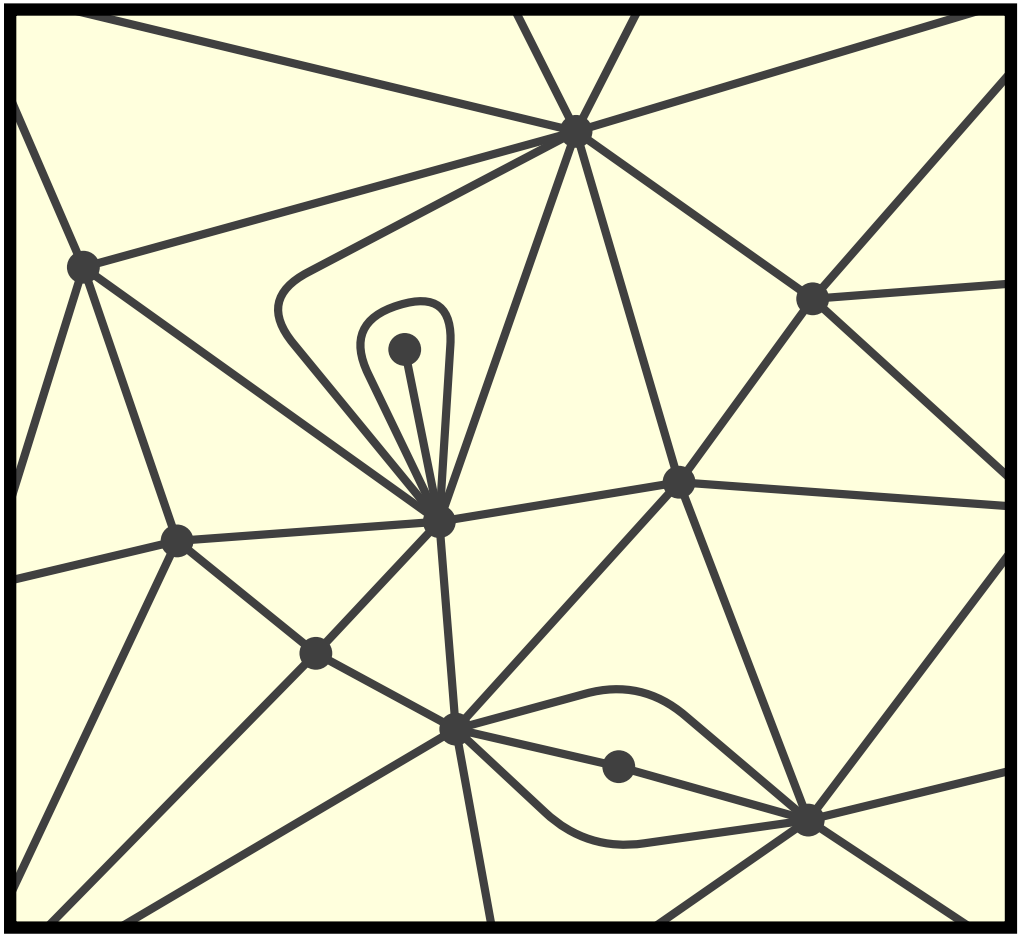

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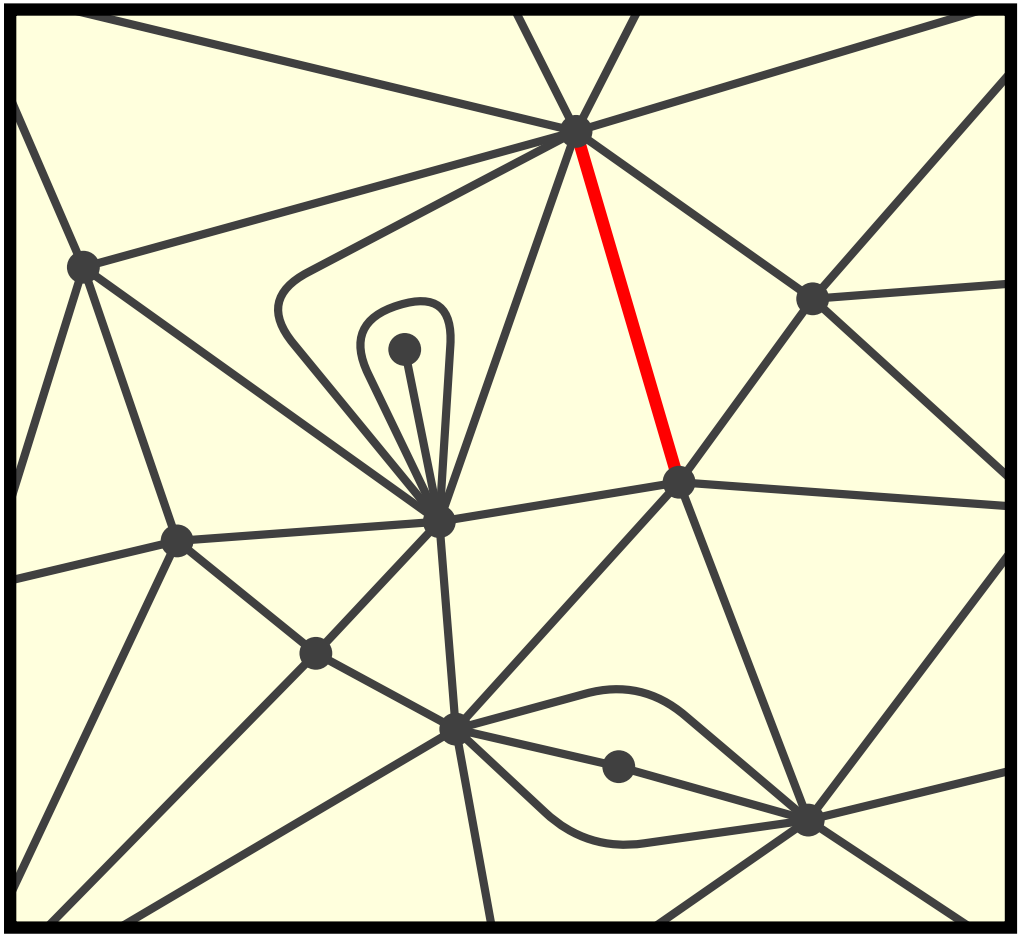

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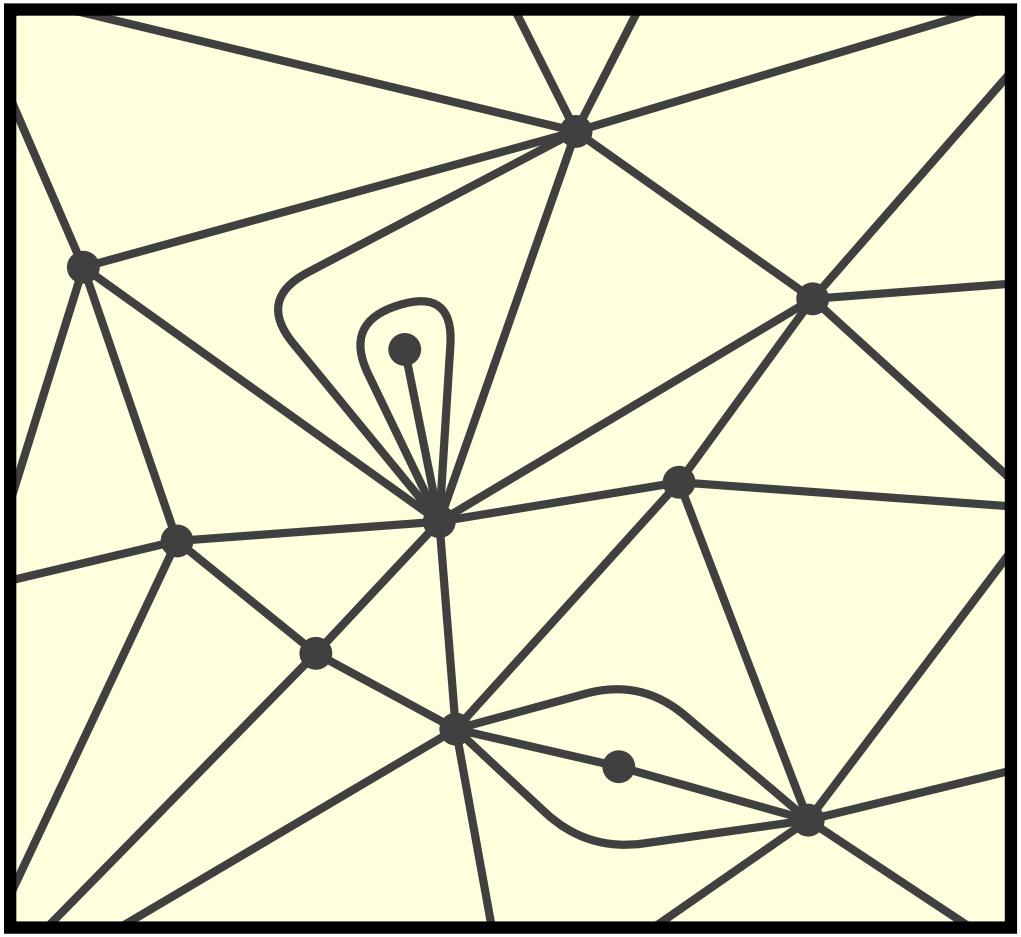

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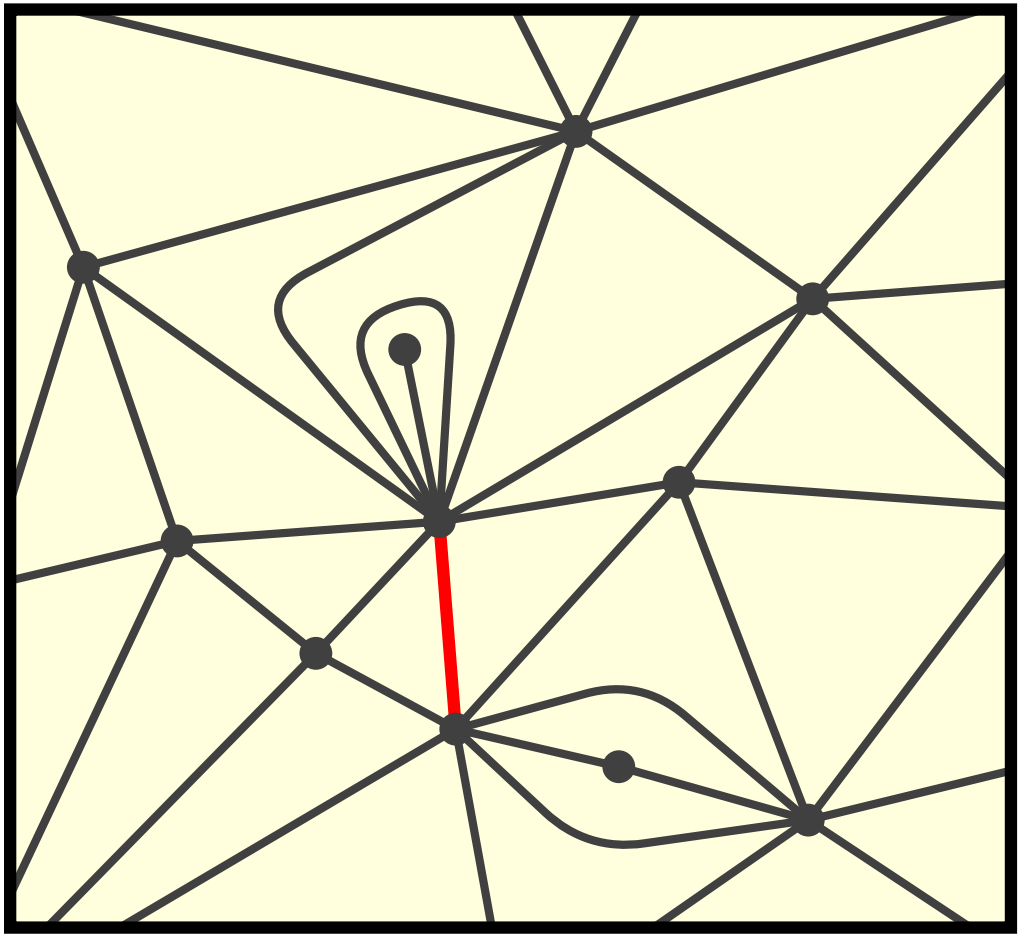

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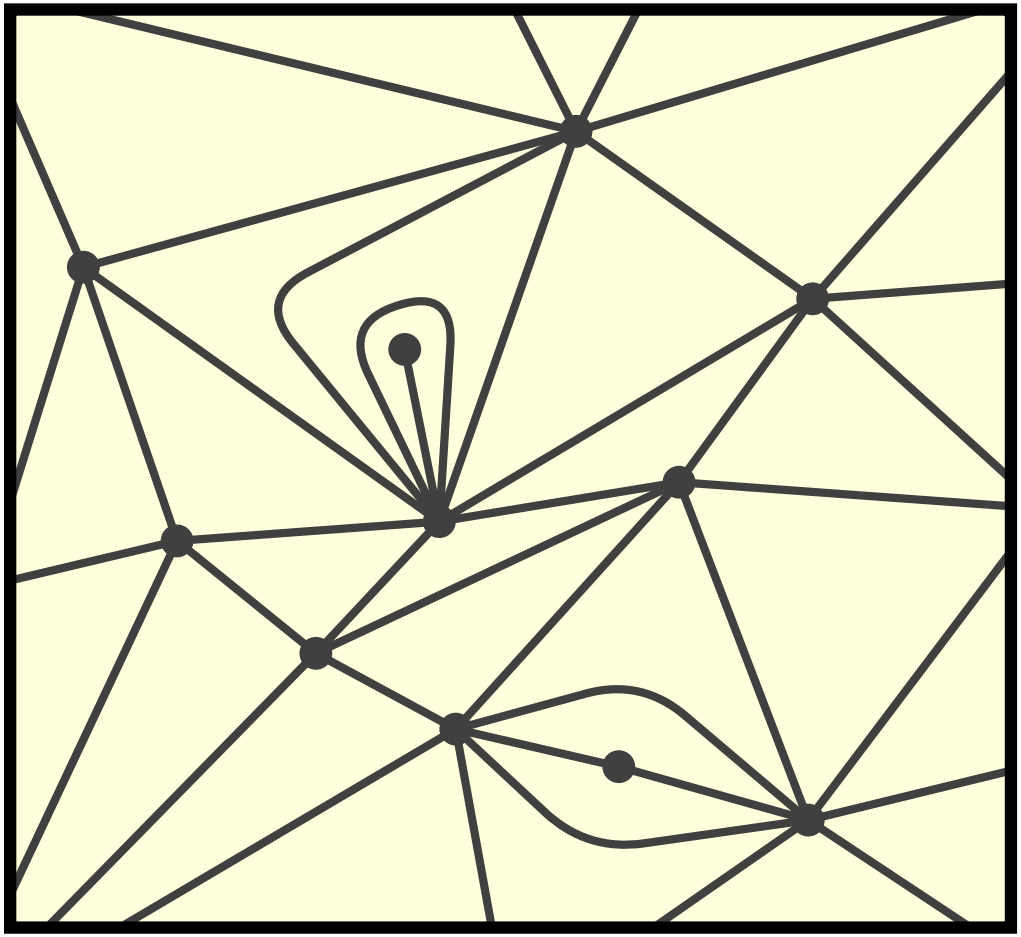

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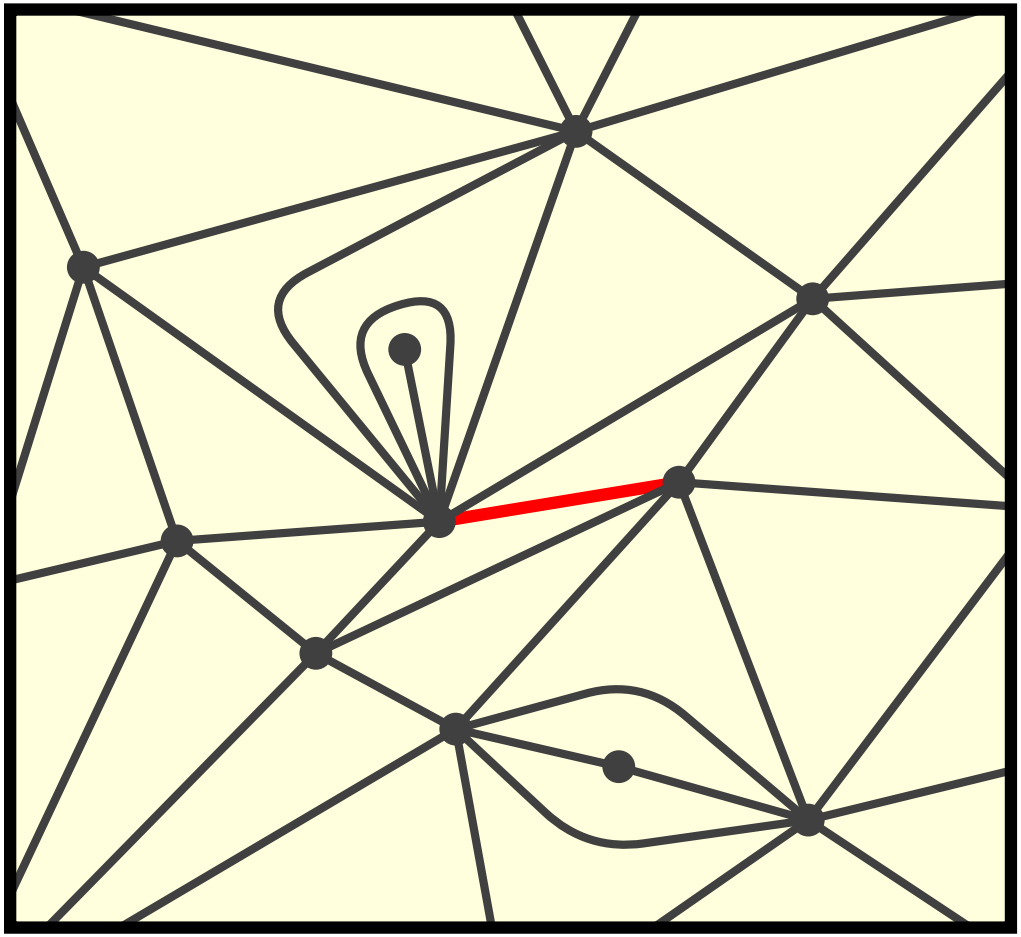

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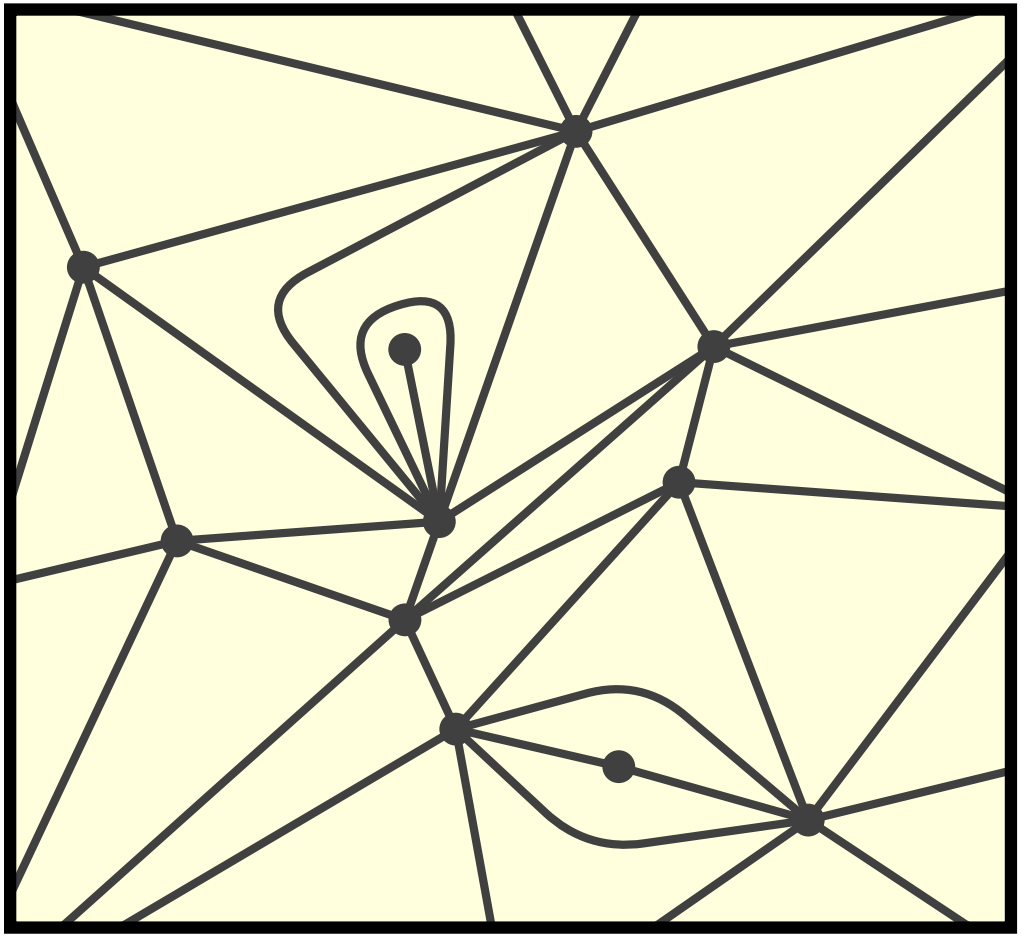

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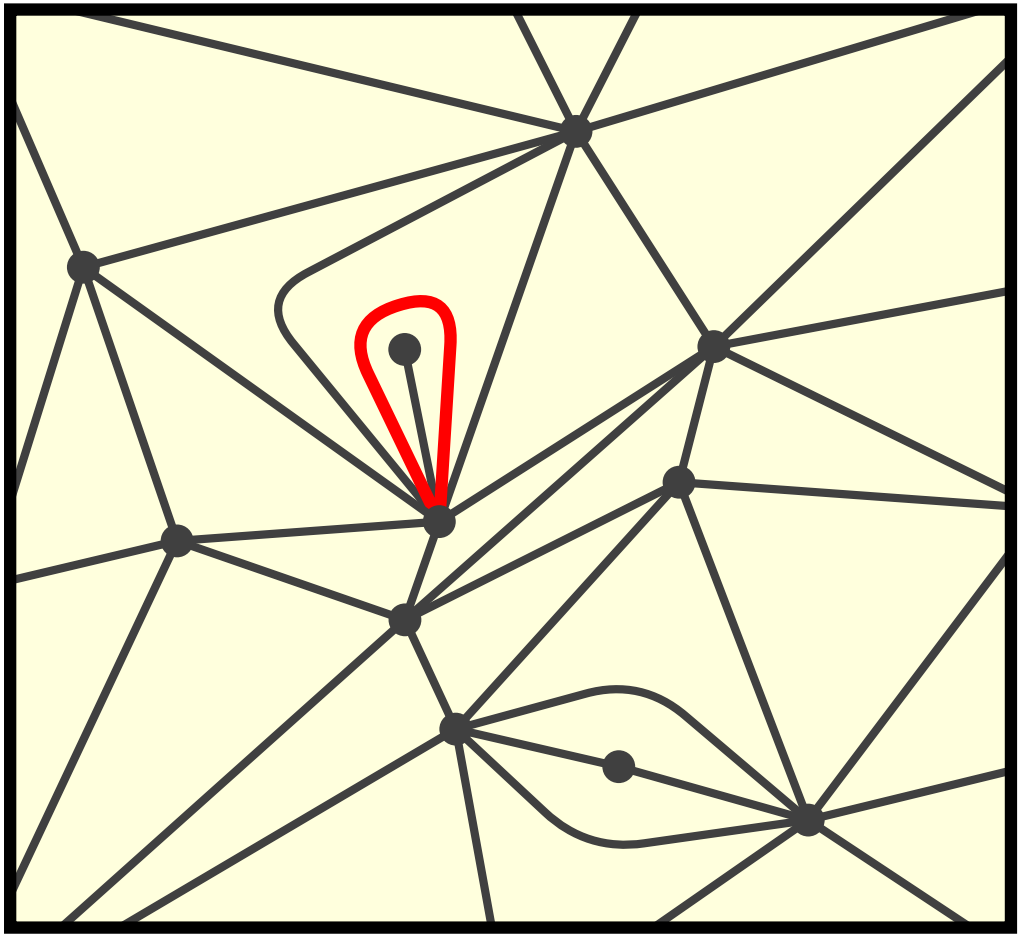

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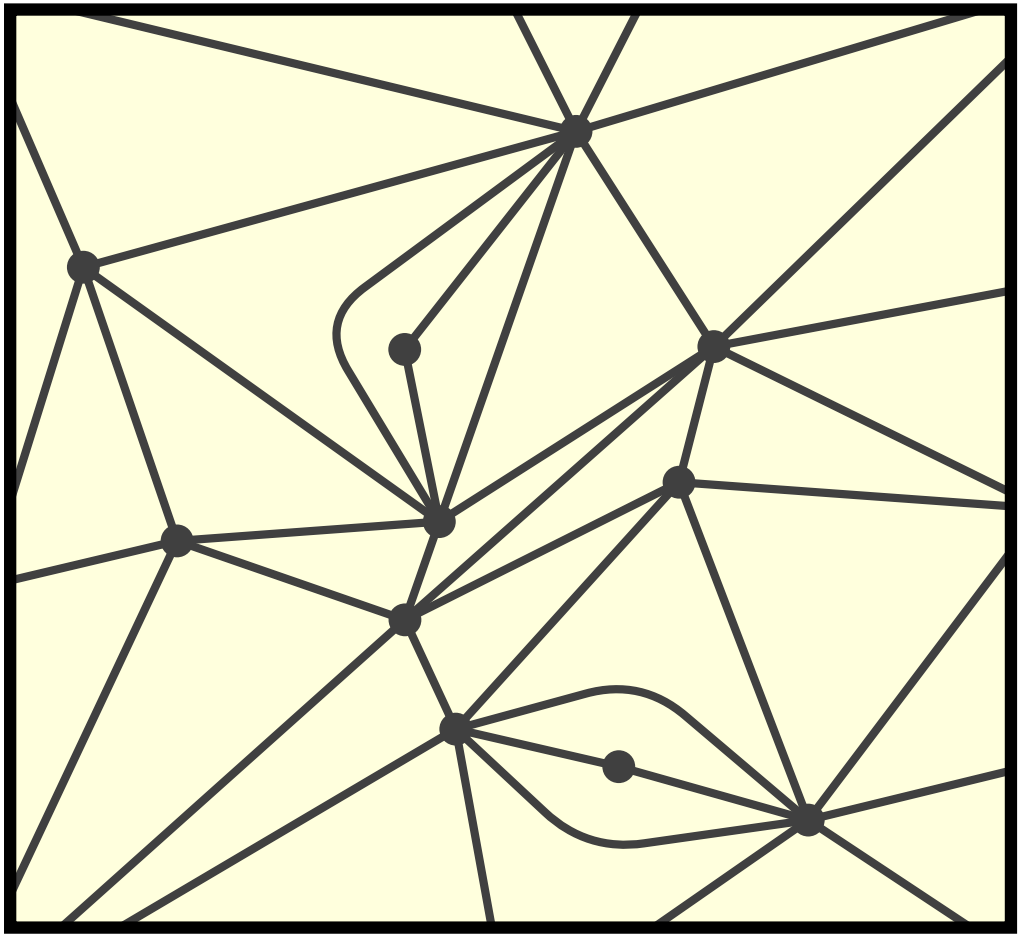

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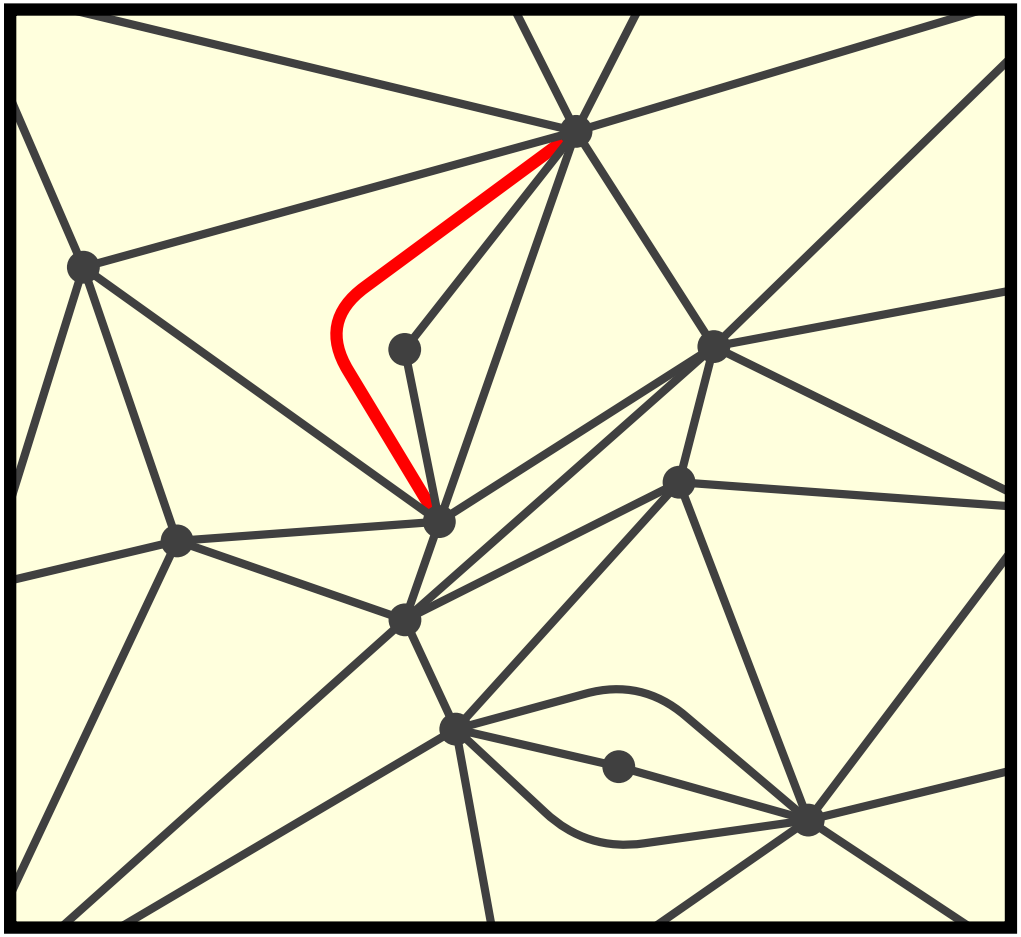

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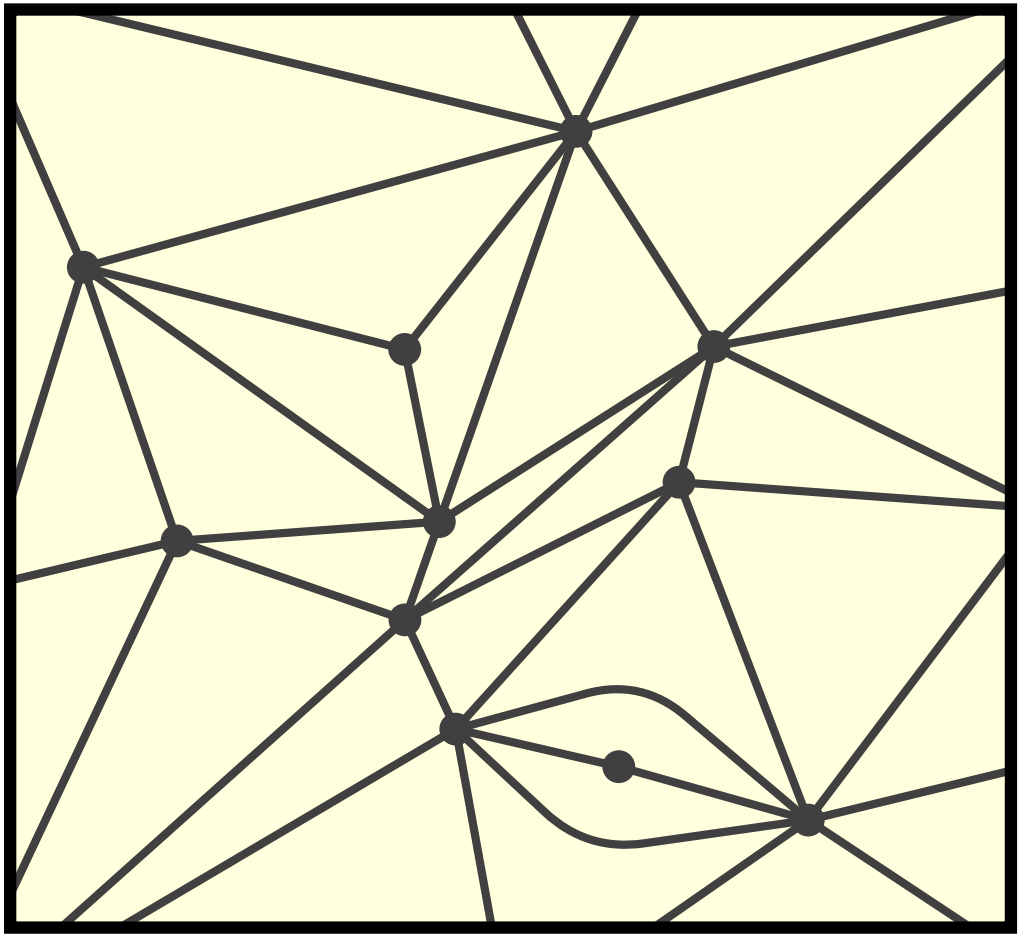

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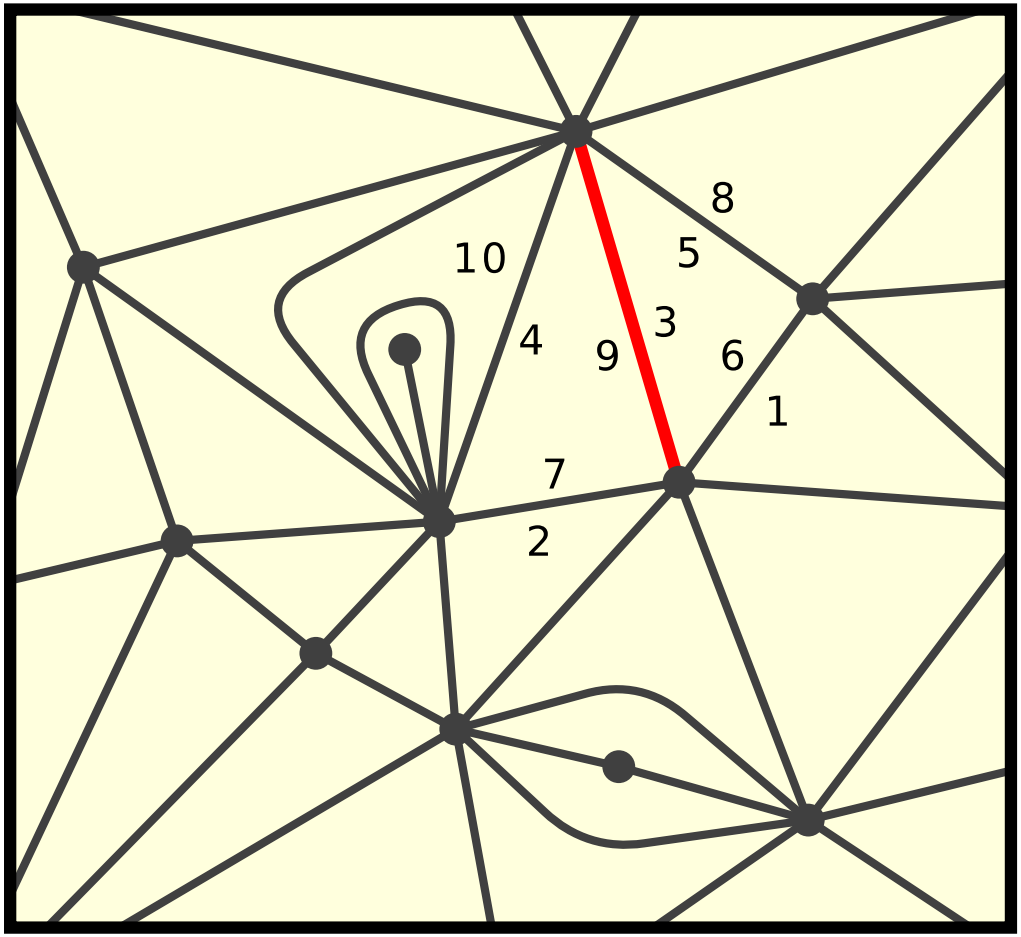

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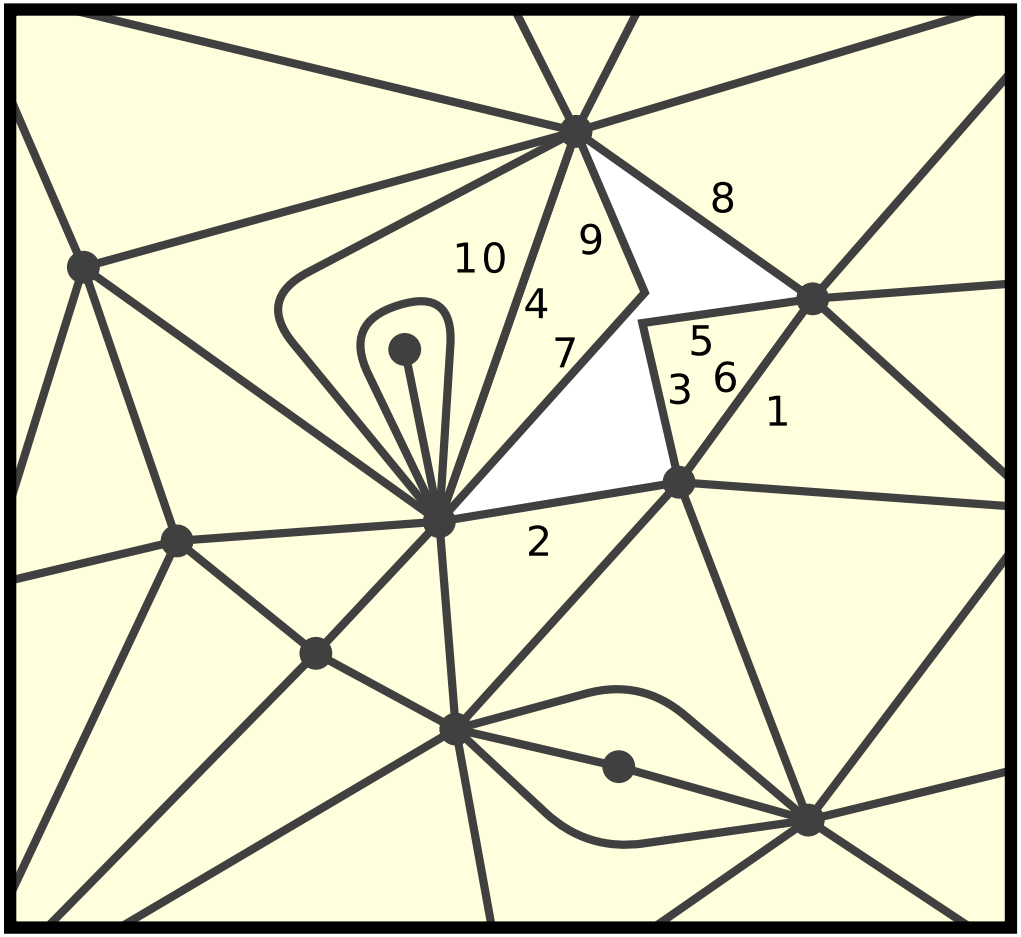

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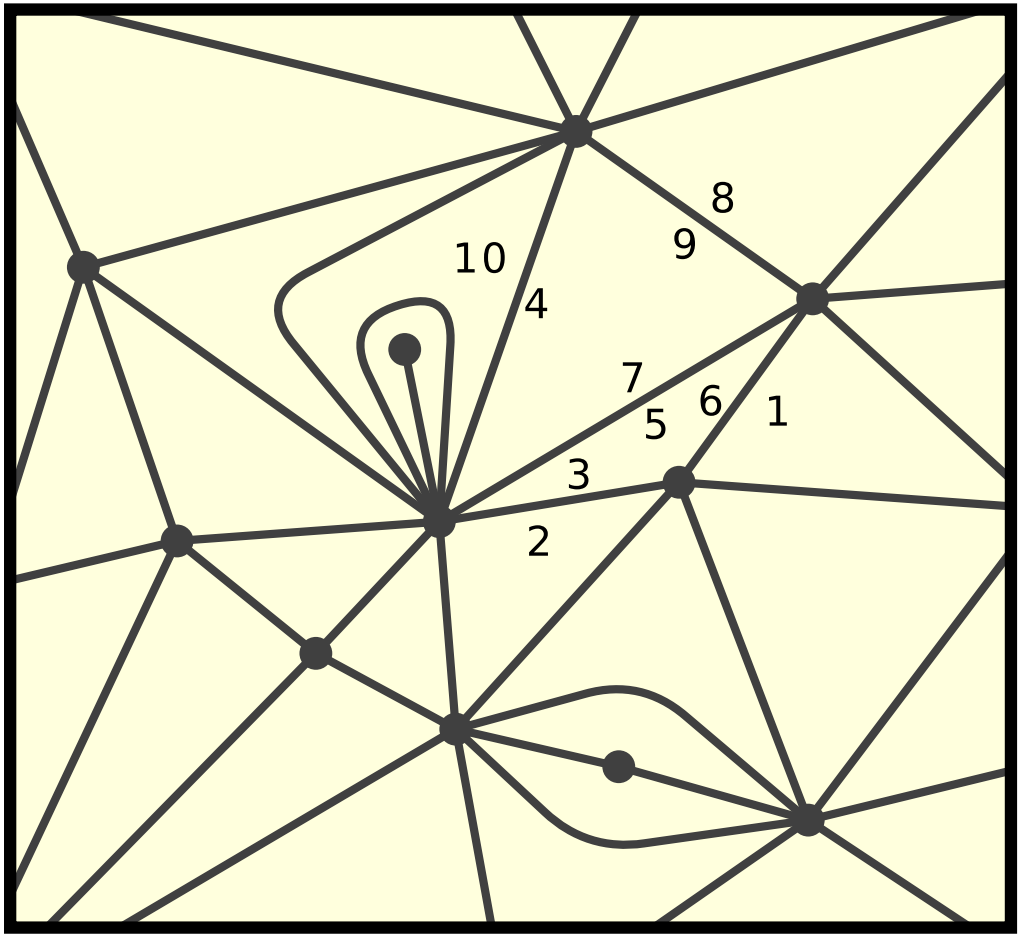

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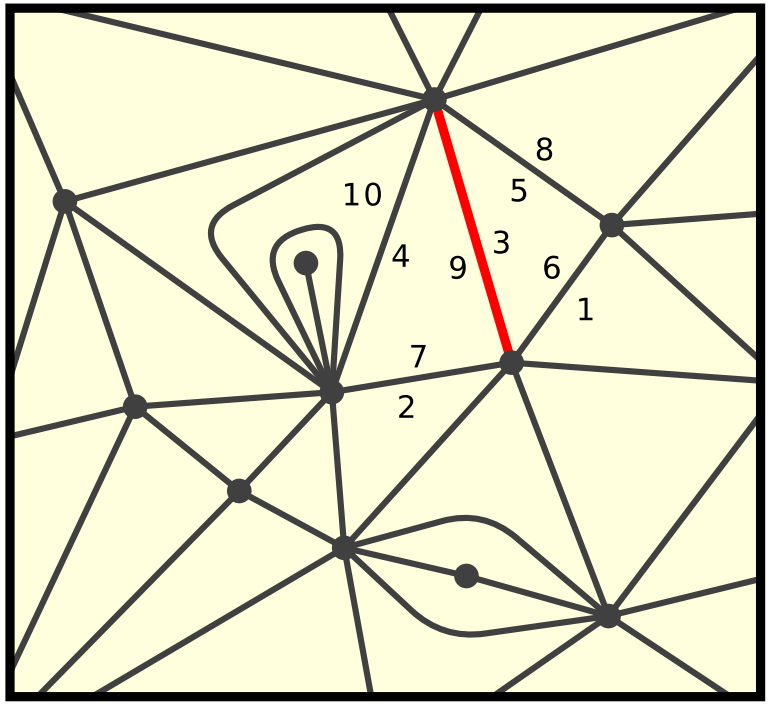

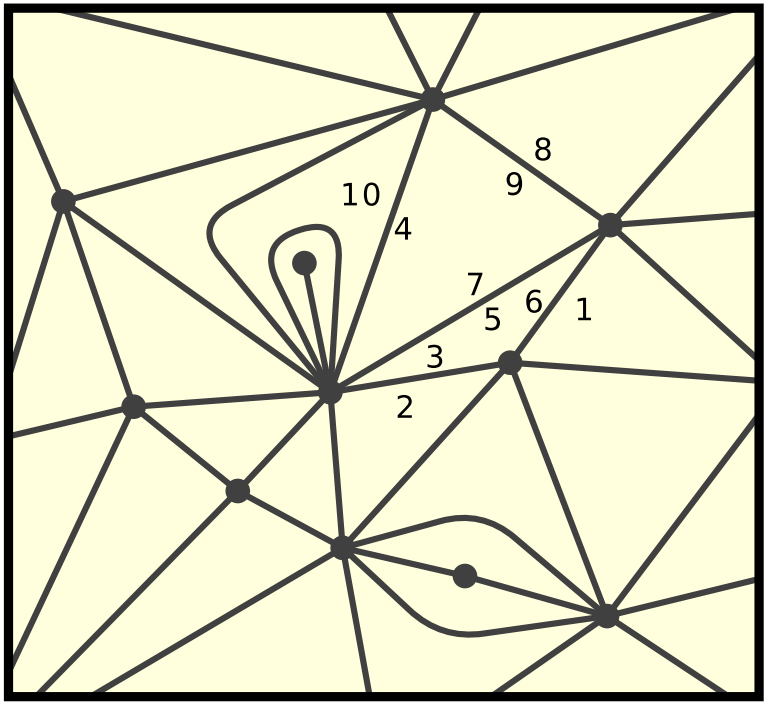

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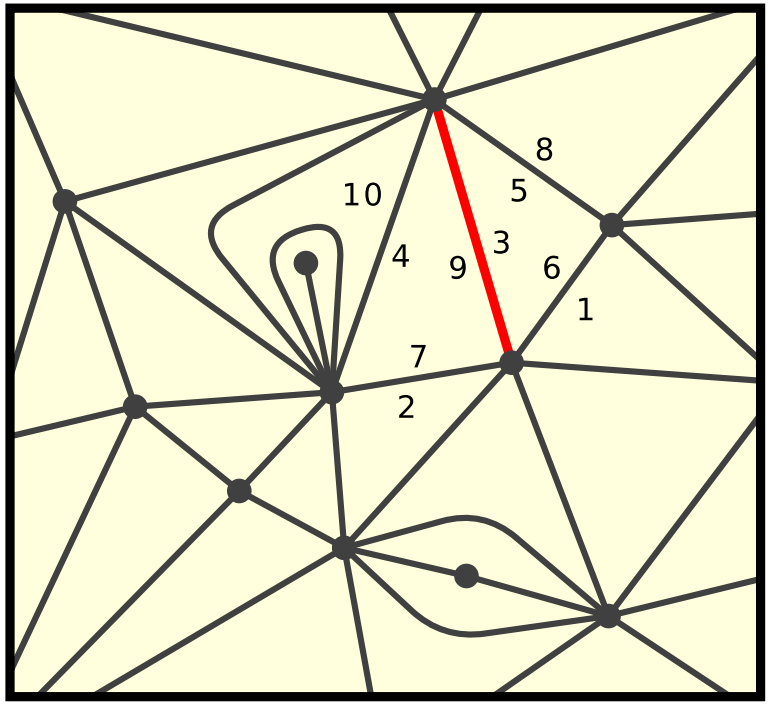

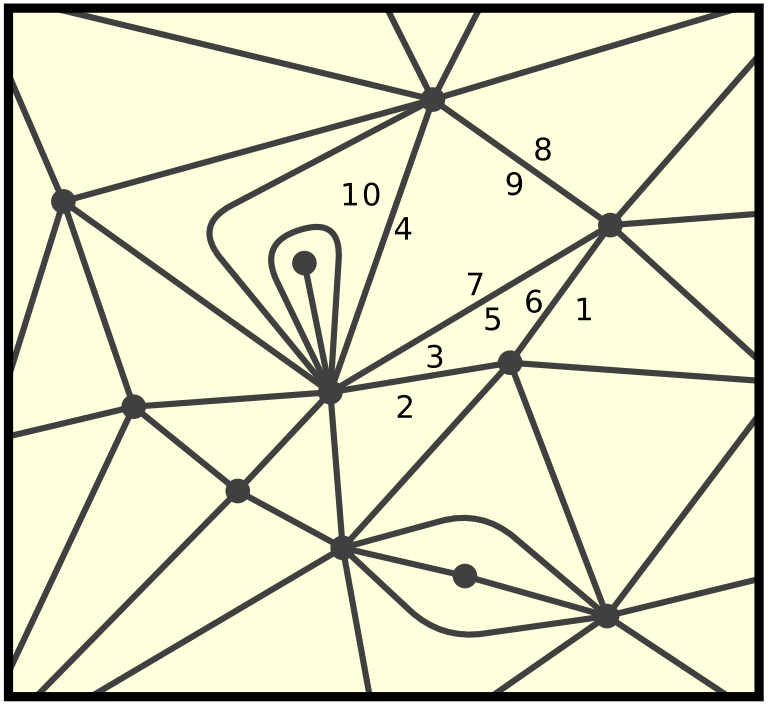

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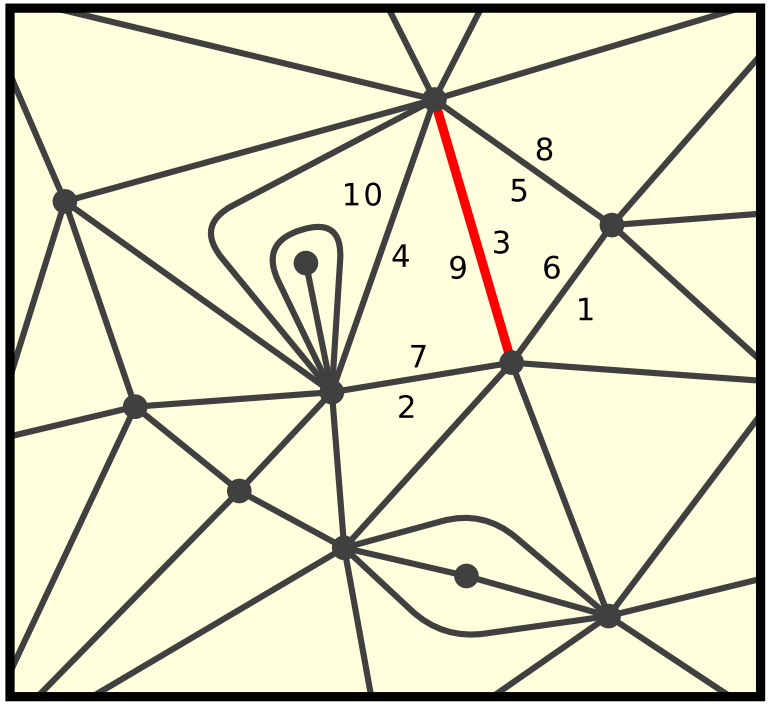

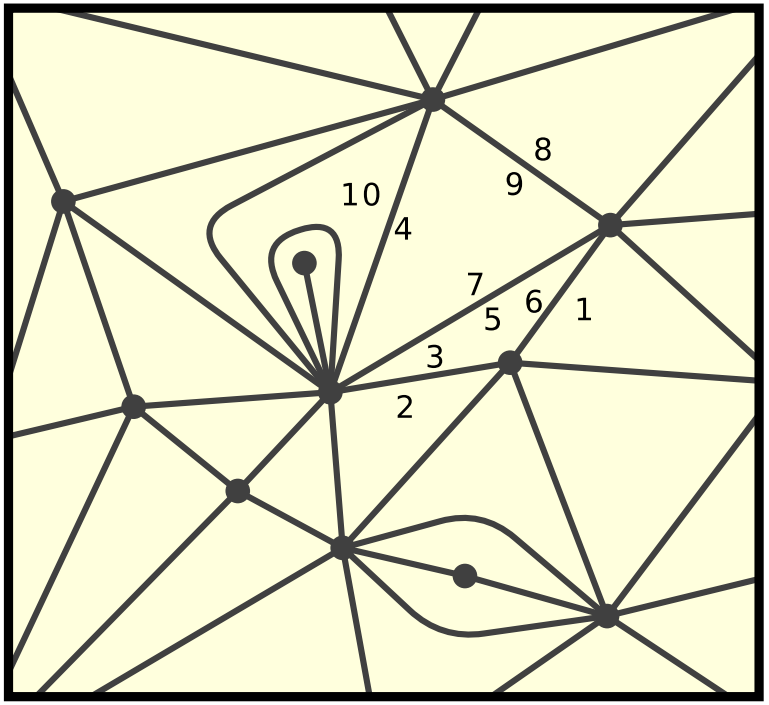

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  [Wagner, '36]
- In practice we don't permute. Why is that OK? Because flipping and permuting commute, and we may require observables $\mathcal{O} : \mathcal{T}_N \to \mathbb{R}$ to be invariant under label permutation.

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- Clearly labeling is useful when representing geometry in the computer.
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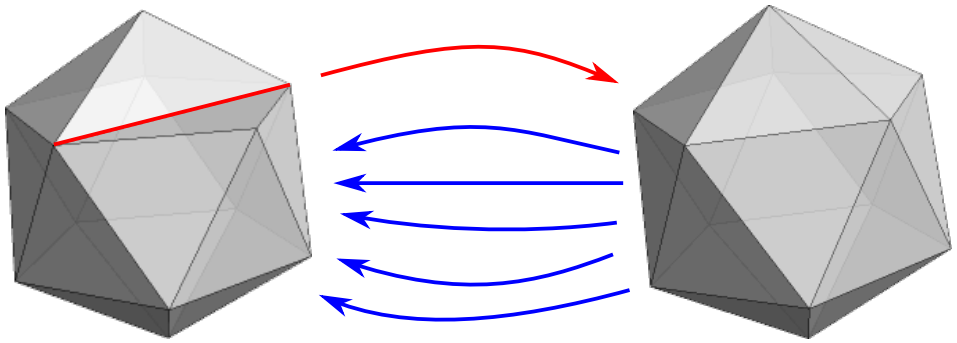
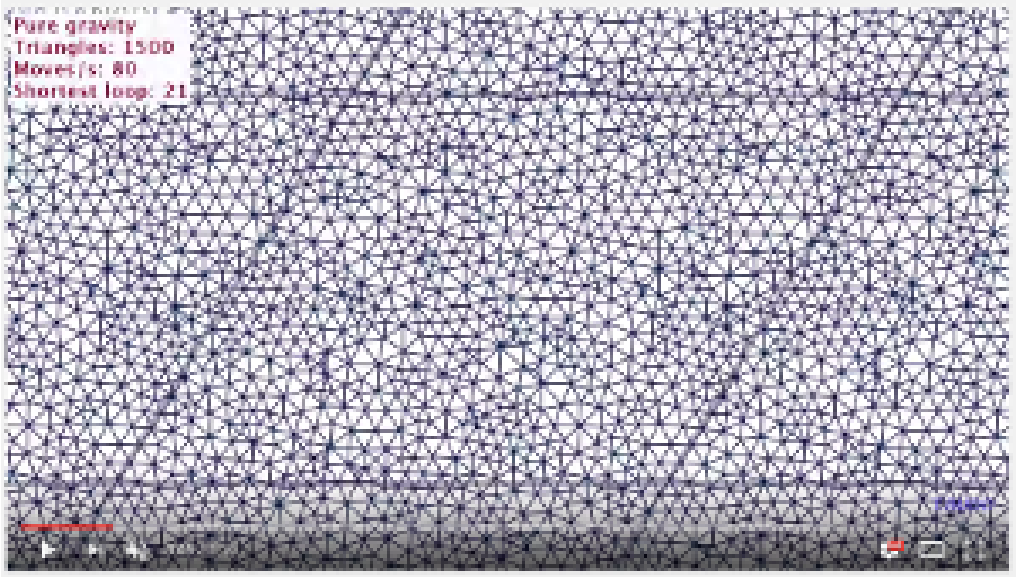


Video of the Markov process on a torus



https://www.youtube.com/watch?v=c3NdgSIe030

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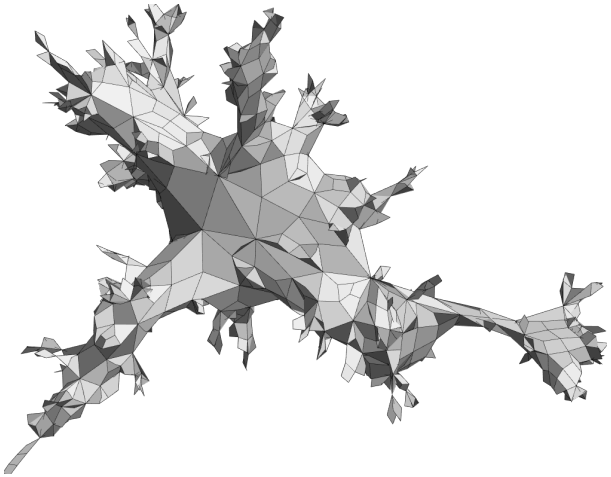
$N = 2000$: 

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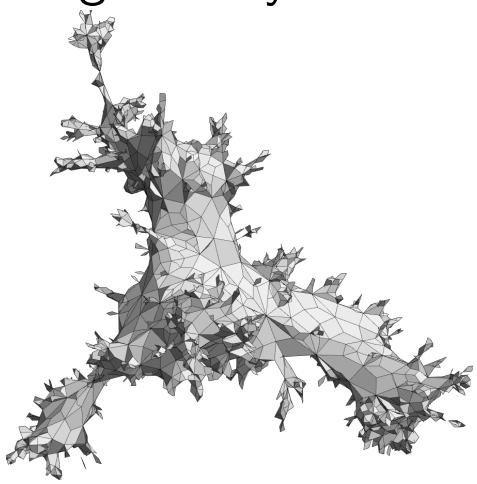
$N = 5000$: 

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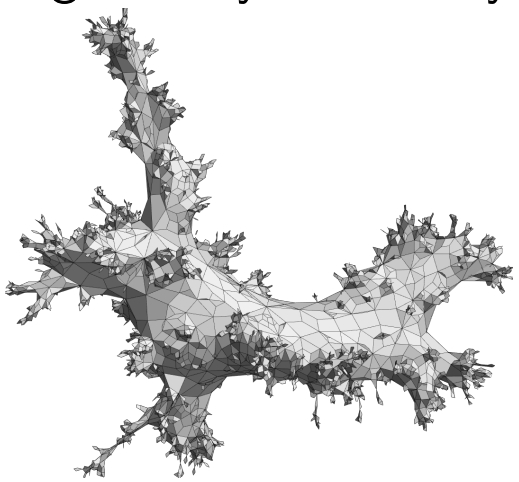

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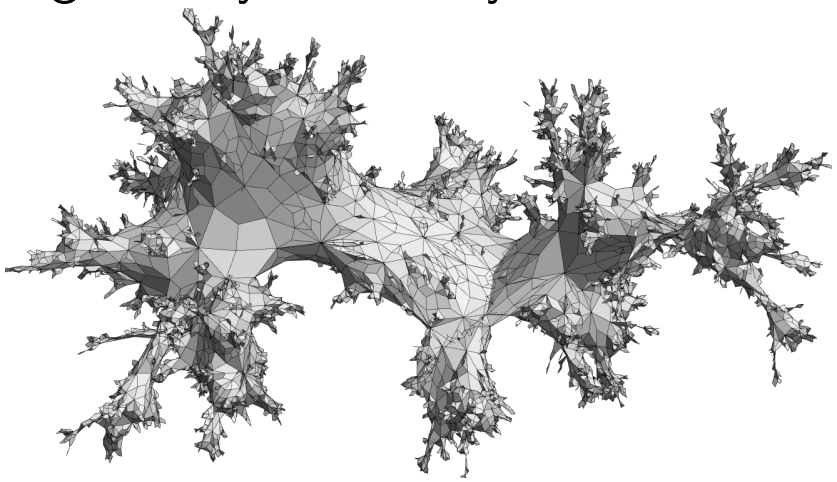

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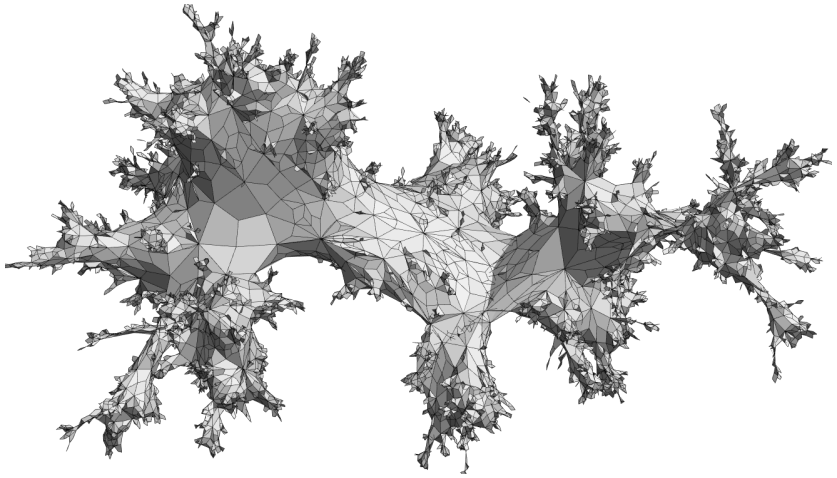

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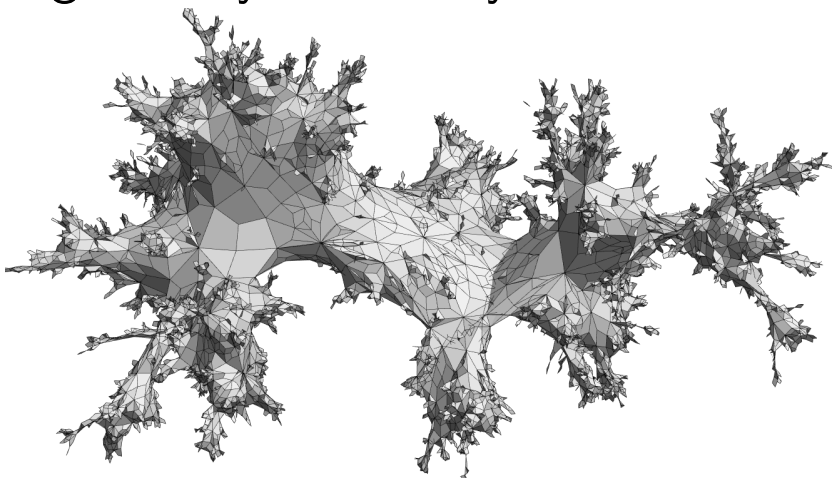


- The large-scale properties are universal as $N \to \infty$: independent of precise ensemble of maps used.
- To determine critical exponents of this universality class: need a family $\mathcal{O}_n$ of observables, with n related to the scale at which the system is probed. Measure $\langle \mathcal{O}_n \rangle_N$ and analyze $N, n \to \infty$ limit.

Matter coupling

- Could introduce a non-trivial action $S[\mathfrak{m}]$ (or energy $\beta E[\mathfrak{m}]$)

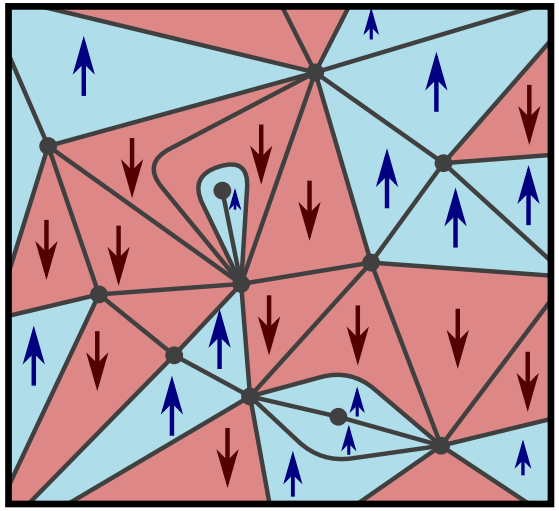
$$Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} 1 \quad \mapsto \quad Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} e^{-S[\mathfrak{m}]}.$$

  Introduce acceptance probability in MCMC to ensure detailed balance.
- However, no local action will change universality class ("phase diagram of pure gravity in 2D is trivial").
- More interesting to couple geometry to matter, e.g. Ising model.

$$Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} \sum_{\text{matter config. } \phi \text{ on } \mathfrak{m}} e^{-S[\mathfrak{m},\phi]}$$

- If the matter is tuned to criticality, the critical exponents of the geometry are affected!
- Widely believed there is a 1-parameter family of universality classes of 2d random geometry, parametrized by total central charge c of coupled matter fields. (no matter: $c = 0$; Ising: $c = 1/2$)

Example: random triangulations + Ising model



$$Z_{N,\mathrm{Ising}} = \sum_{\mathfrak{m}\in\mathcal{T}_N} \sum_{\{s_i\}} e^{\beta J \sum_{\langle ij\rangle} s_i s_j}$$

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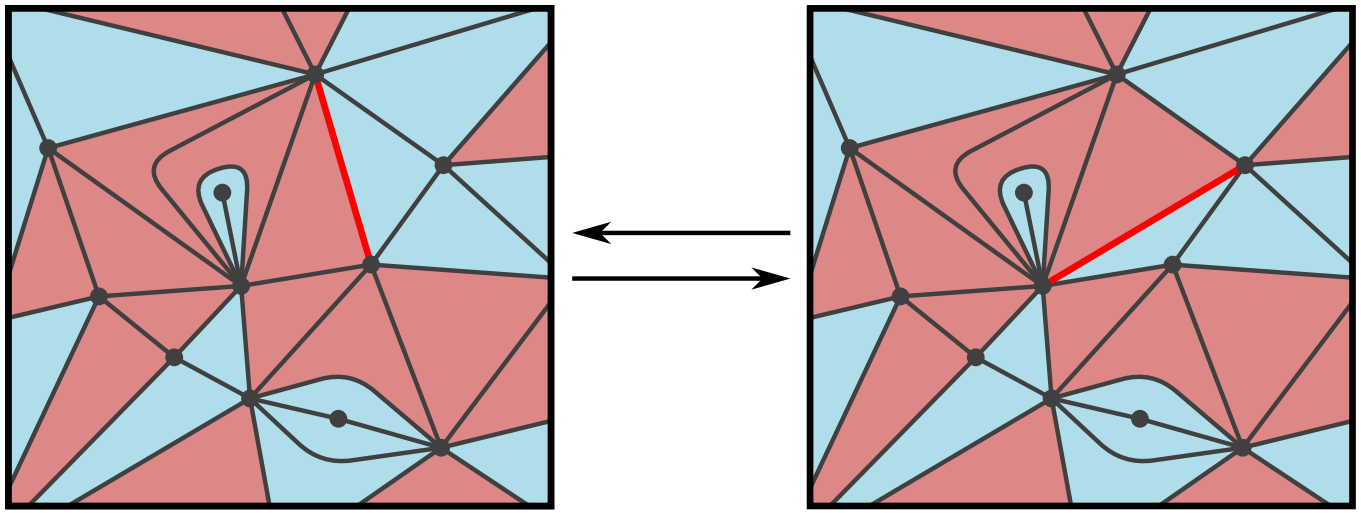

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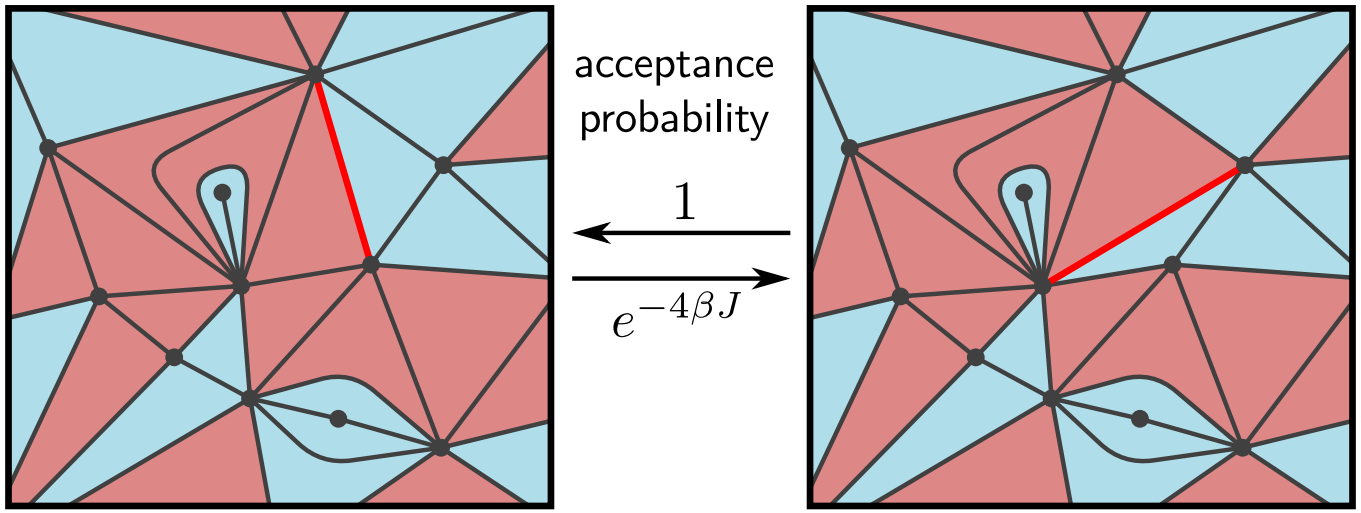

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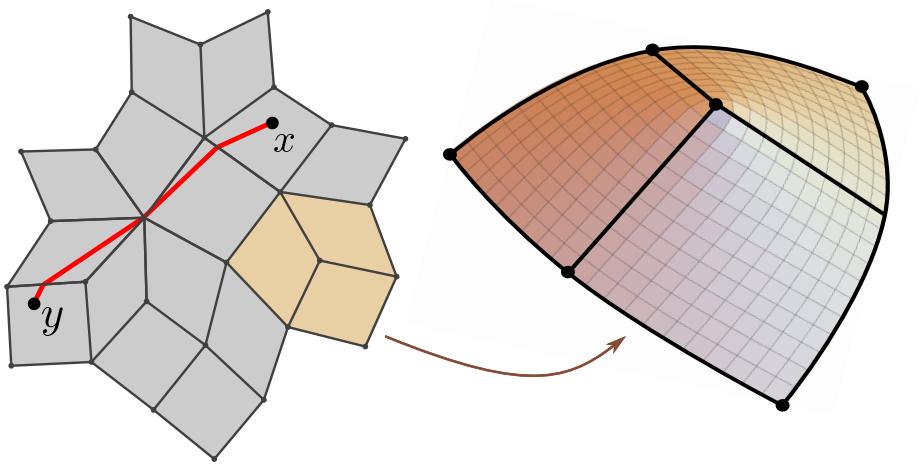
- Introduce Metropolis acceptance probabilities for triangle flip to ensure detailed balance.

Observables: geodesic distances



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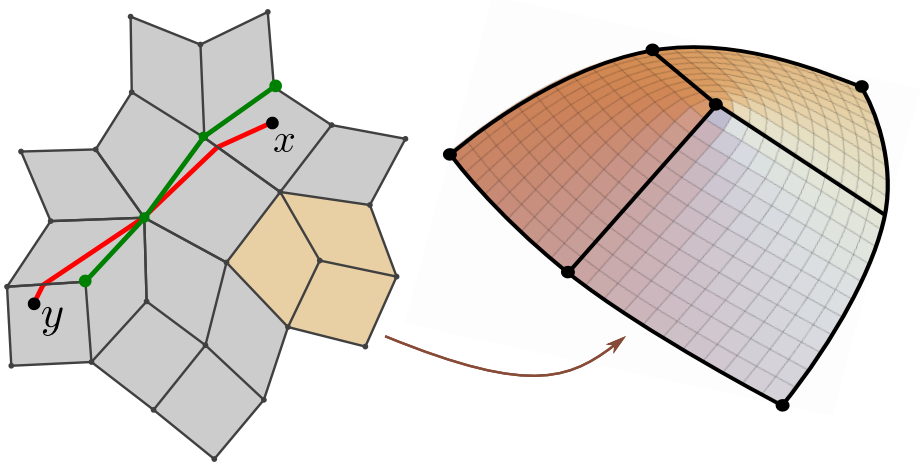

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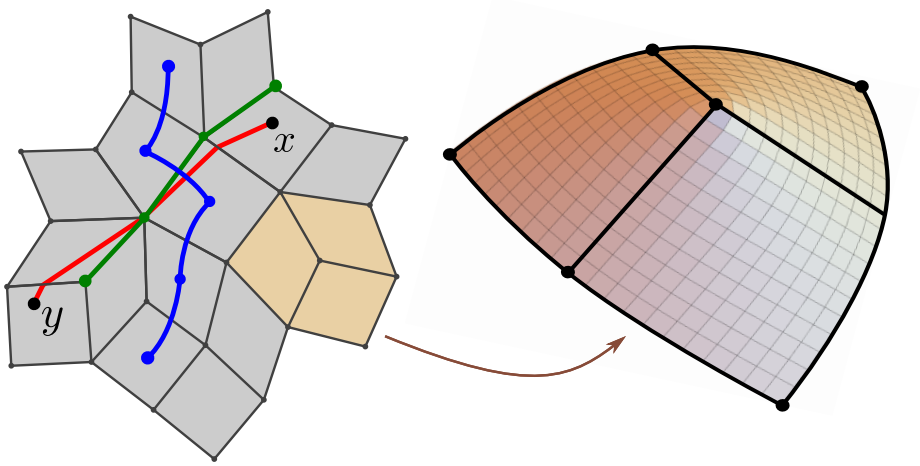

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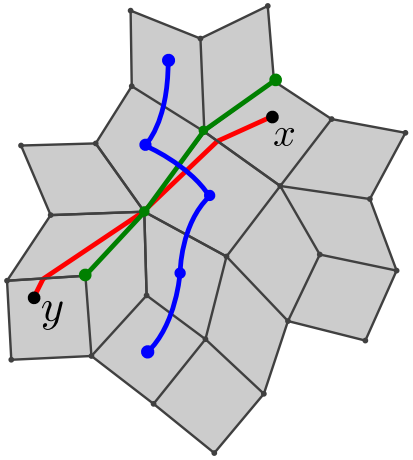

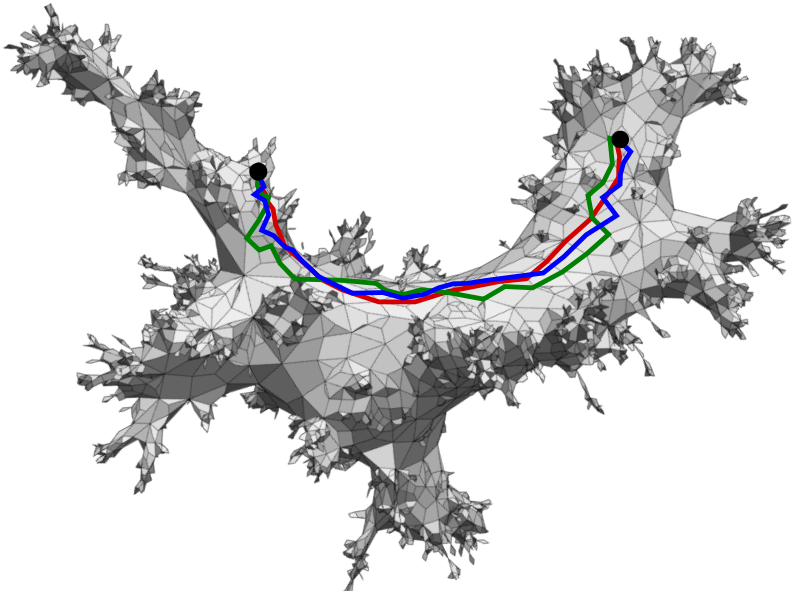

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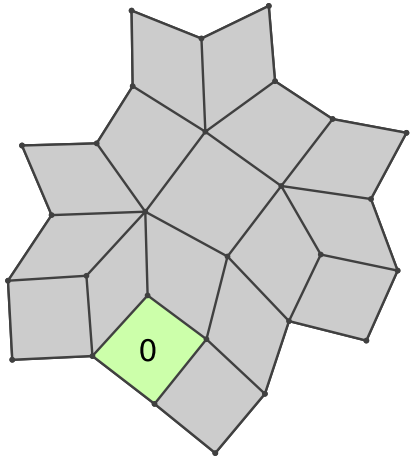

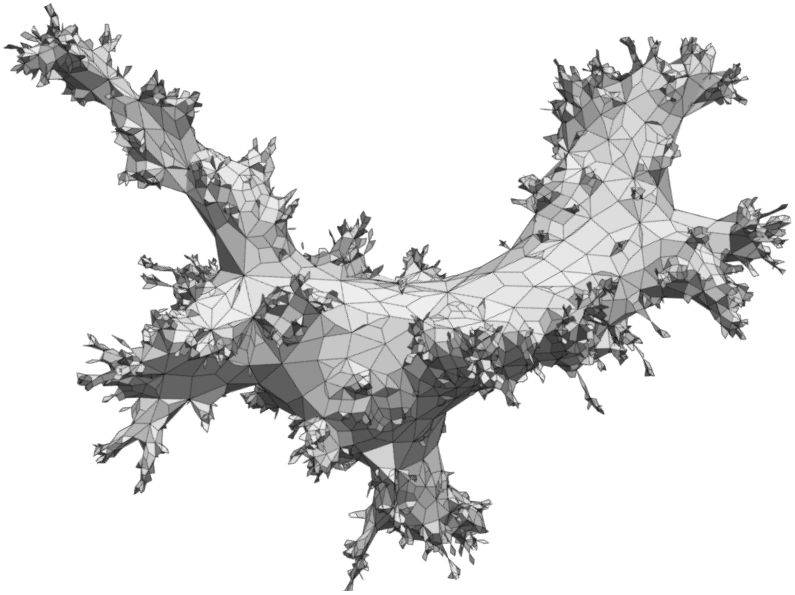

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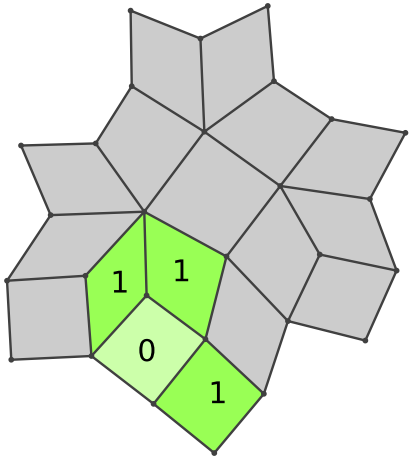

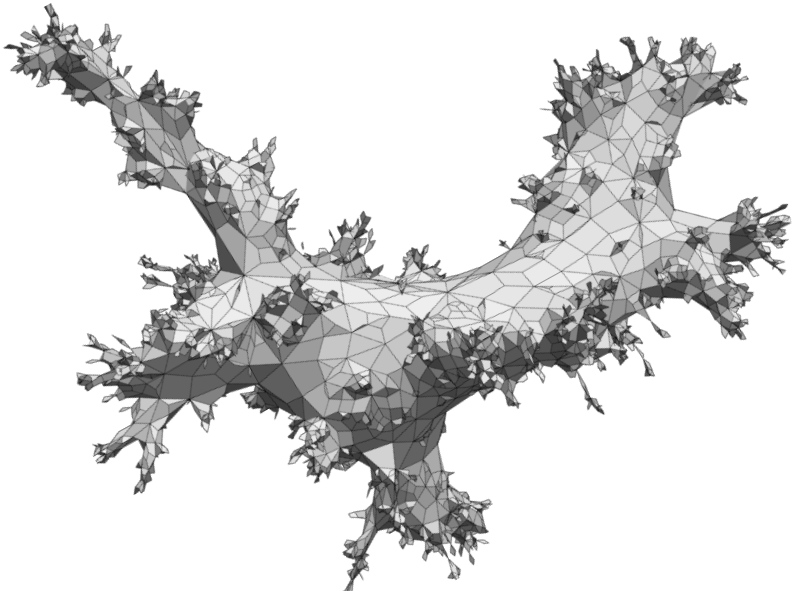

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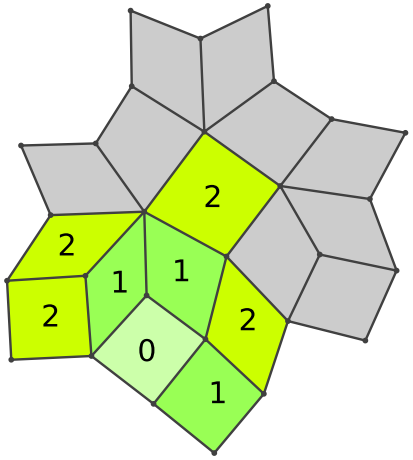

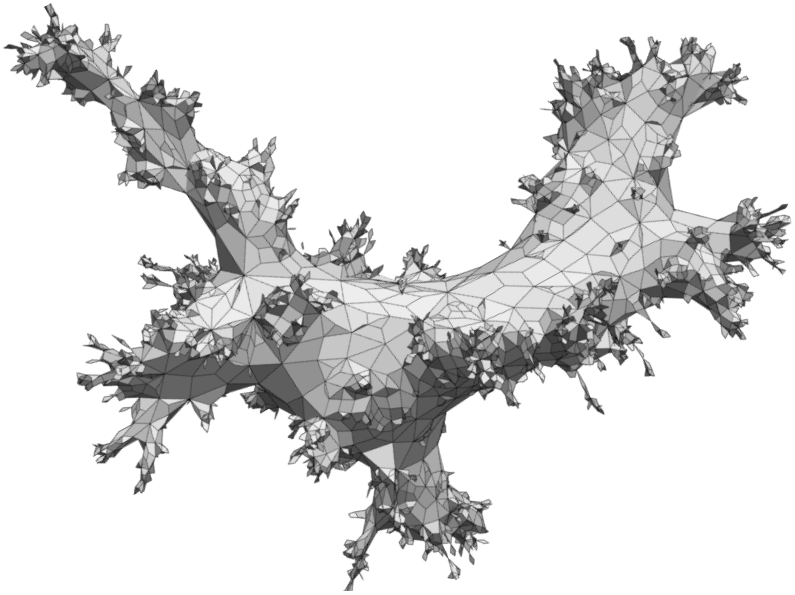

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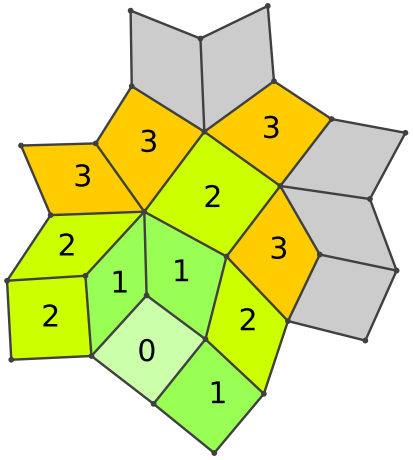

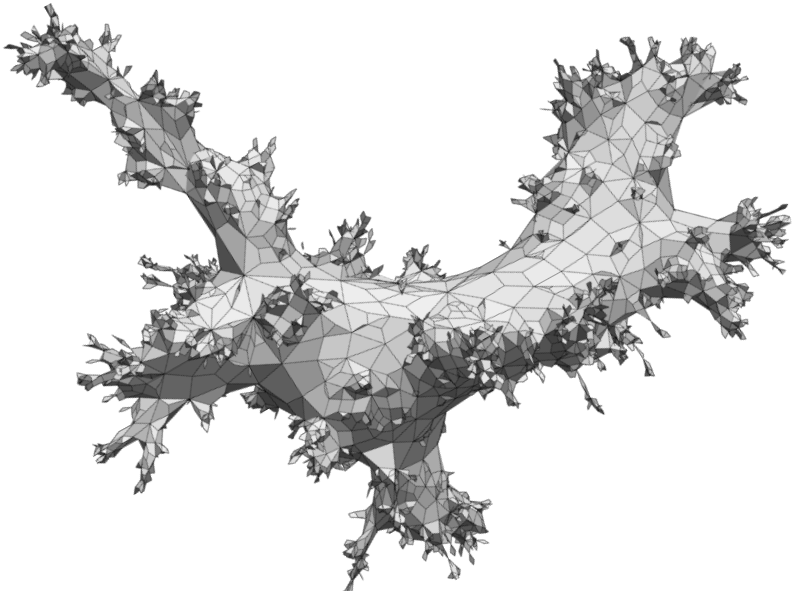

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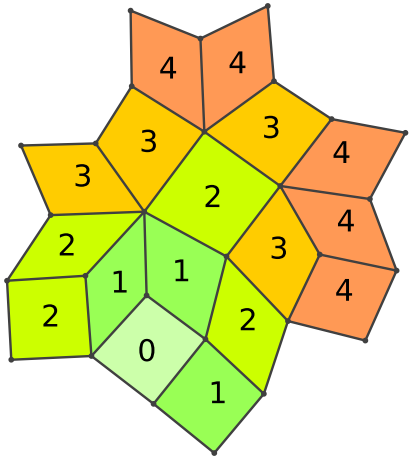

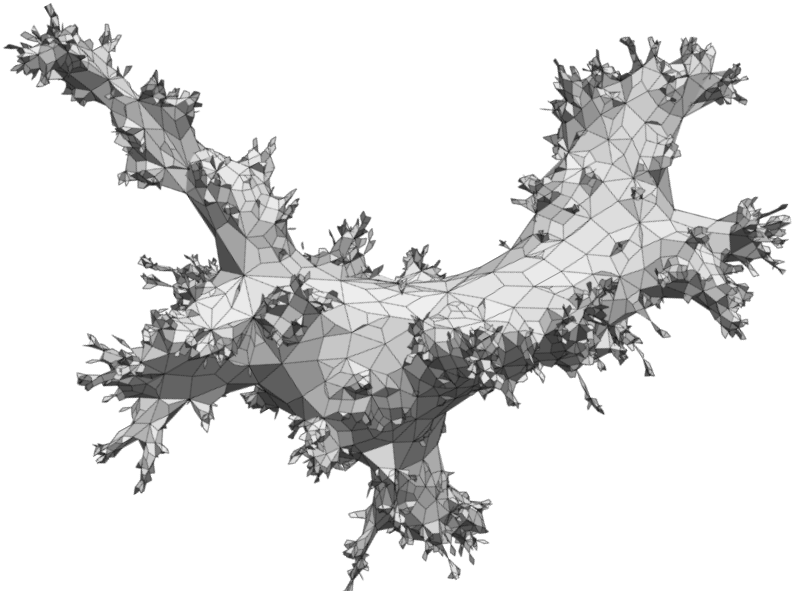

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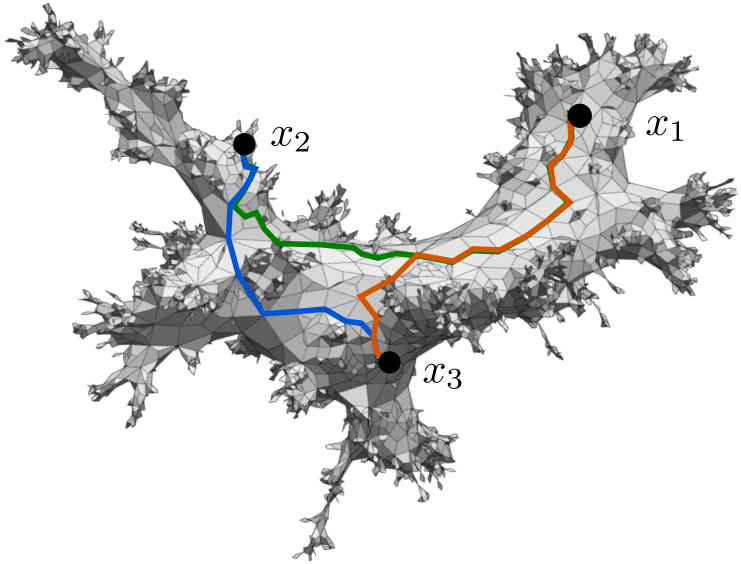

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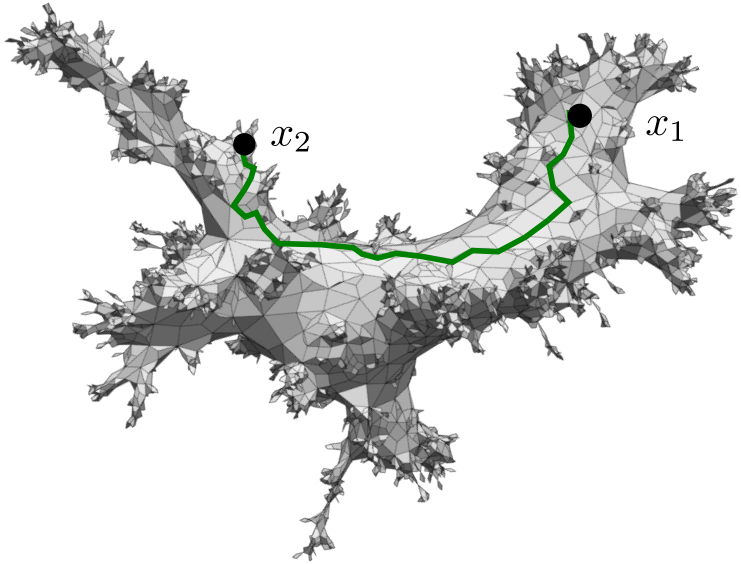

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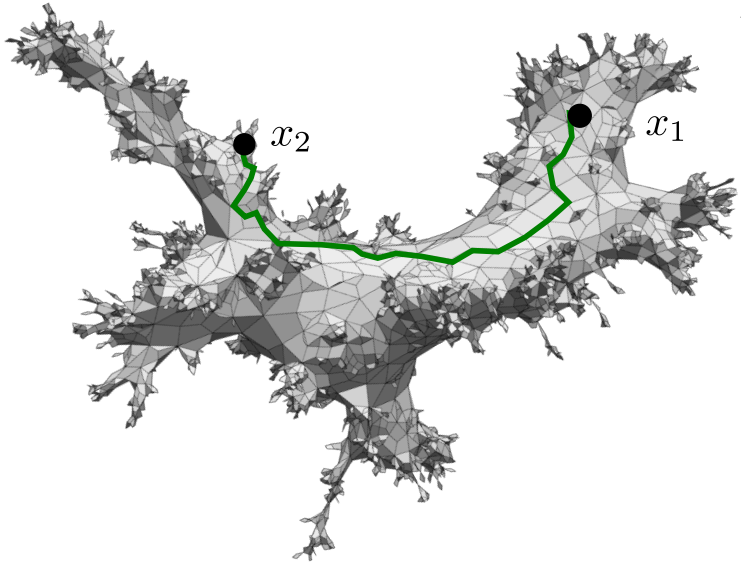

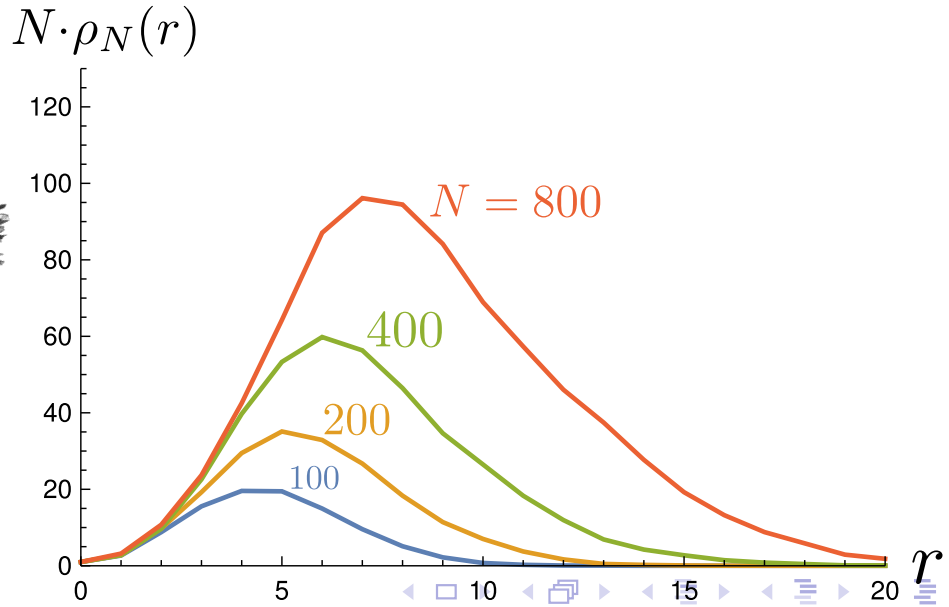

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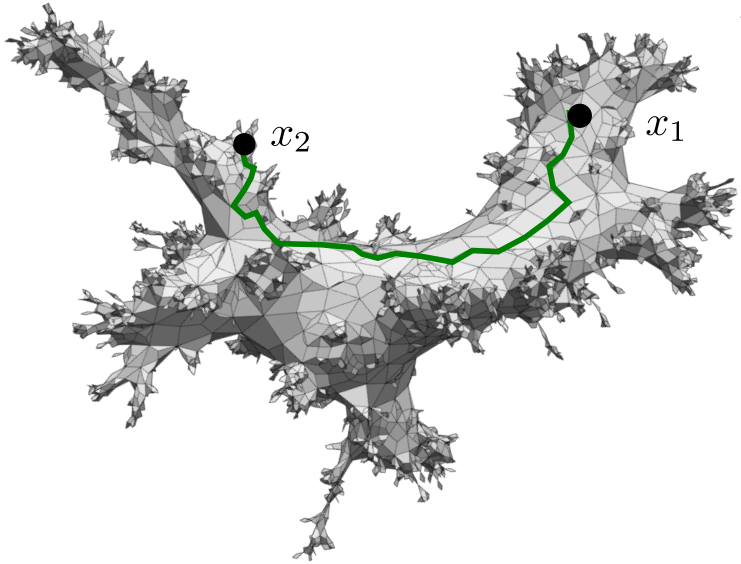
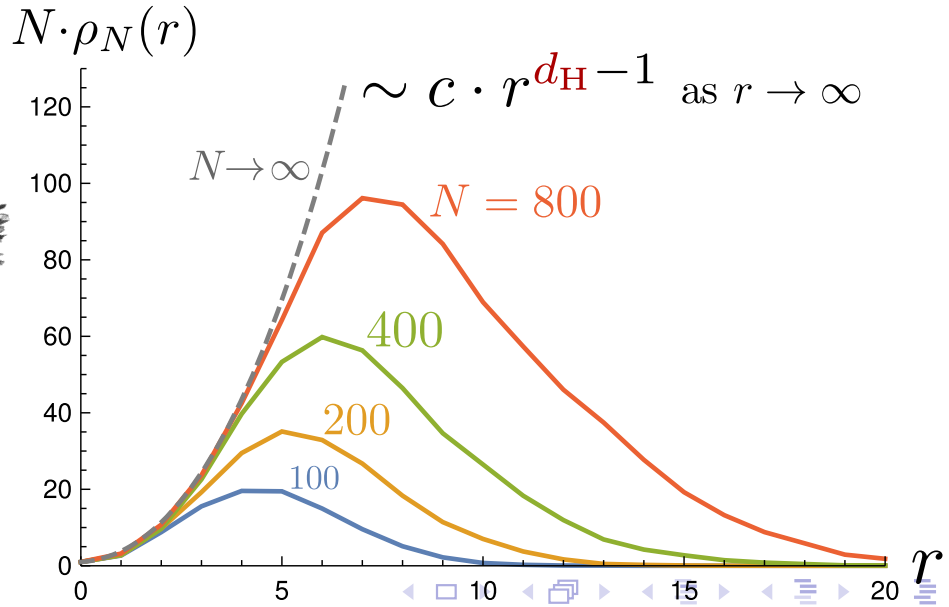
- d_{H} is Hausdorff dimension (which equals d in flat $\mathbb{R}^d$).



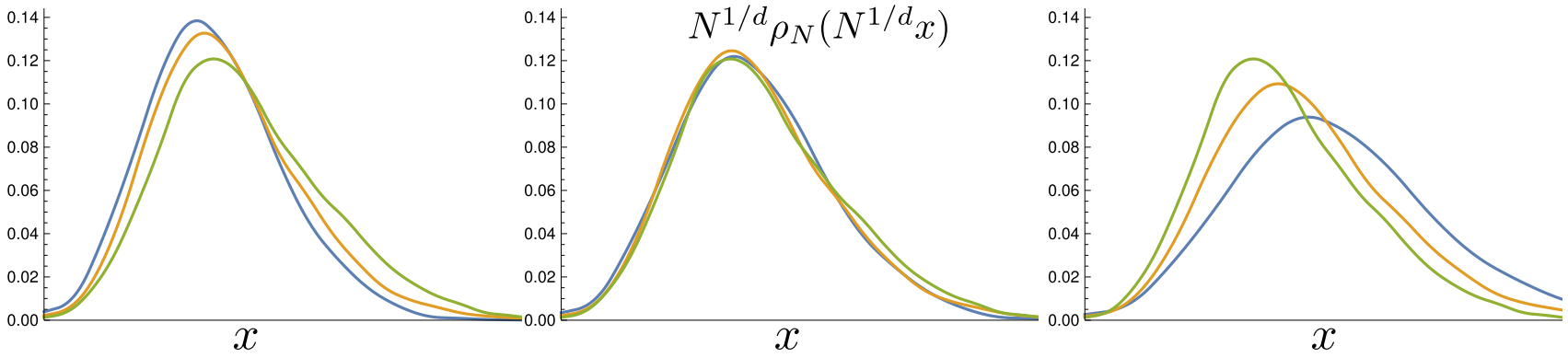


Observables: geodesic distances (continued)

- Instead of taking $N \to \infty$ and then $r \to \infty$, it is usually better to use *finite-size scaling methods*: one expects

$$N^{1/d_{\mathrm{H}}} \rho_N(N^{1/d_{\mathrm{H}}} x) \text{ to converge as } N \to \infty \text{ for any fixed } x \in \mathbb{R}.$$

- Equivalently, we expect the distribution of the distance between two random points to converge as $N \to \infty$ provided we take edge lengths $\sim N^{-1/d_{\mathrm{H}}}$.
- Estimate d_H by "collapsing curves":



Observables: spectral dimension

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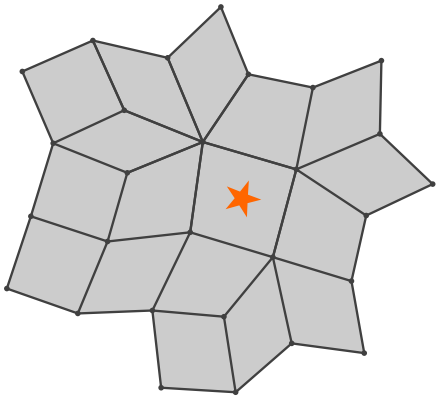

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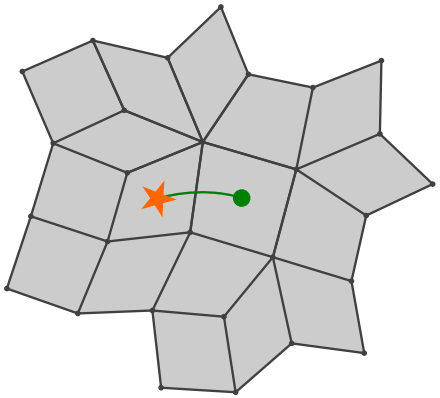

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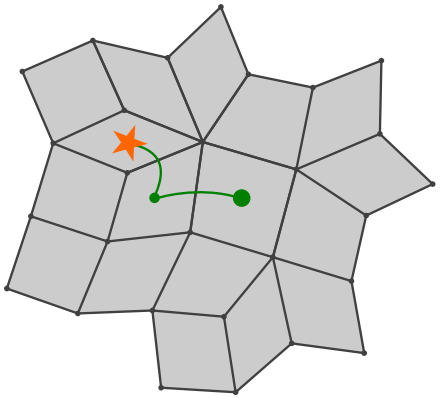

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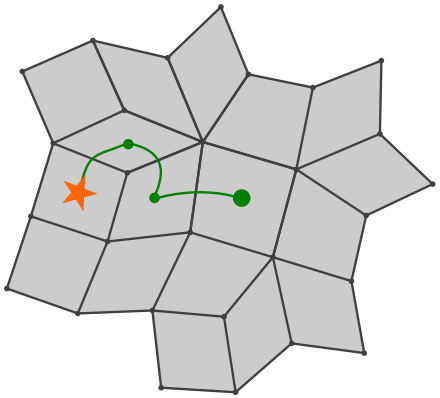

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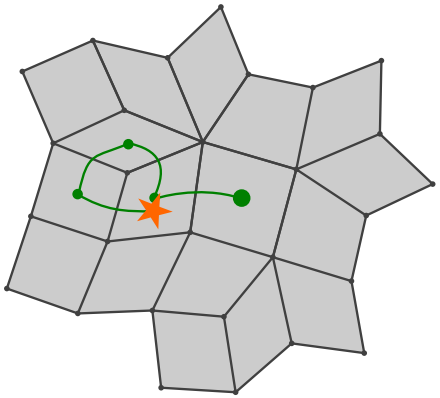

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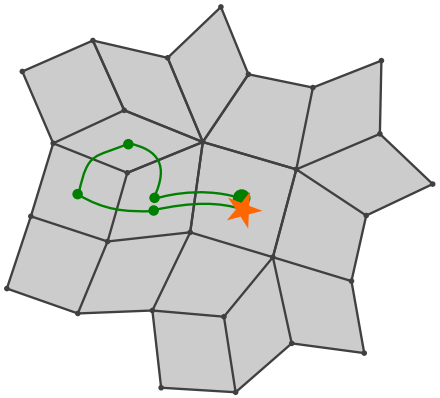

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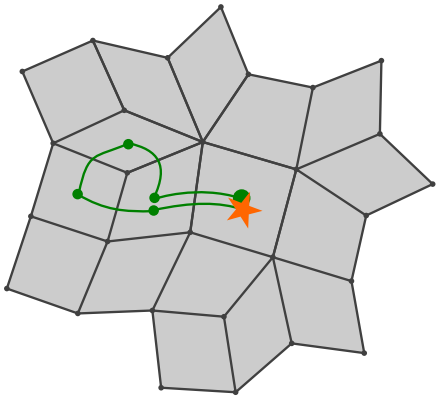

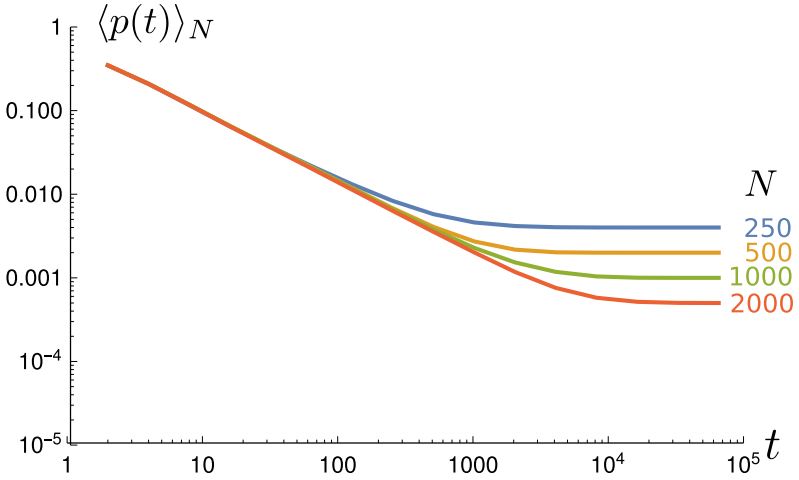

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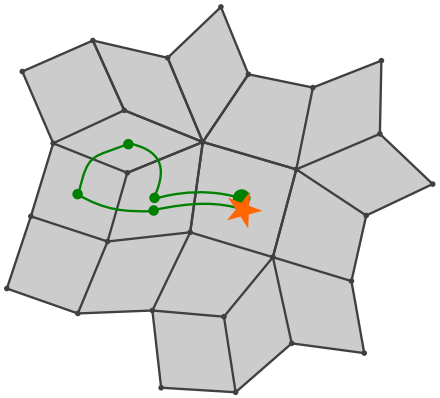
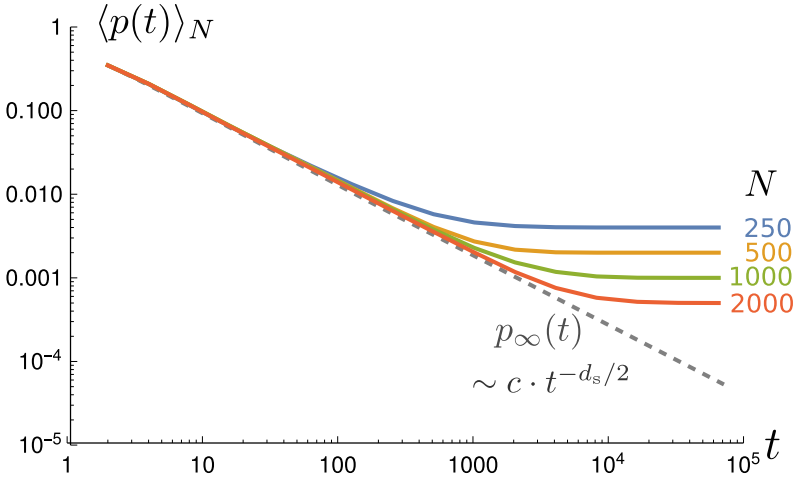
- One expects $\langle p(t;\mathfrak{m})\rangle_N \to p_\infty(t)$ as $N\to\infty$, and $p_\infty(t)\sim t^{-d_s/2}$ as $t\to\infty$.
- d_s is the (annealed) *spectral dimension* (equal to d on $\mathbb{Z}^d$).





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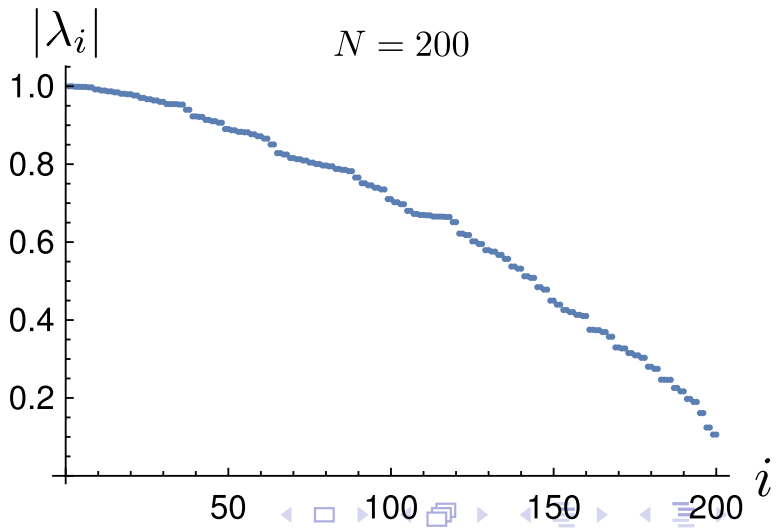

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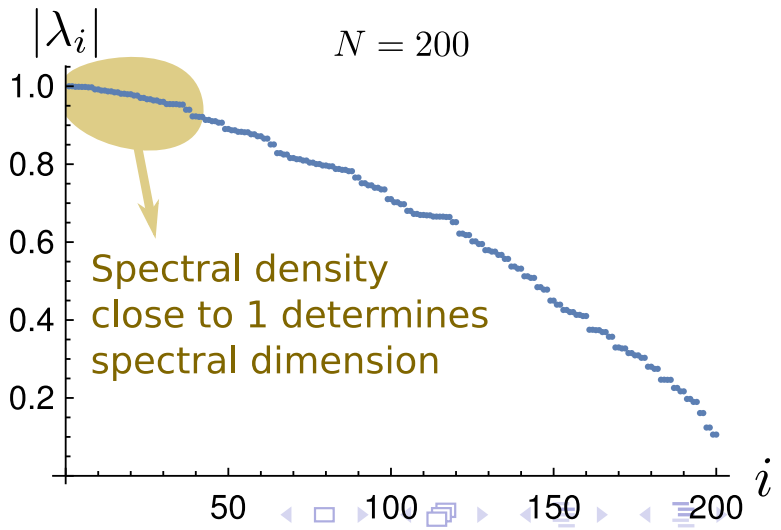

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- Typically expect (unlabeled) partition function to grow like

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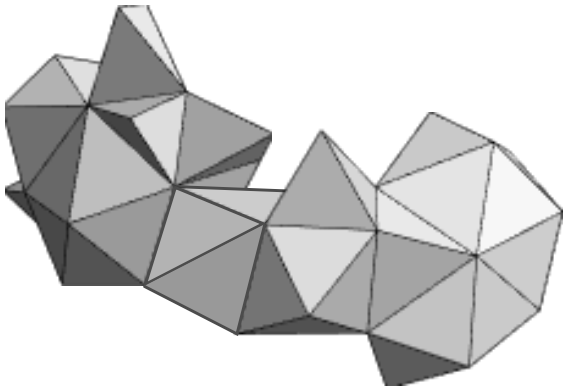

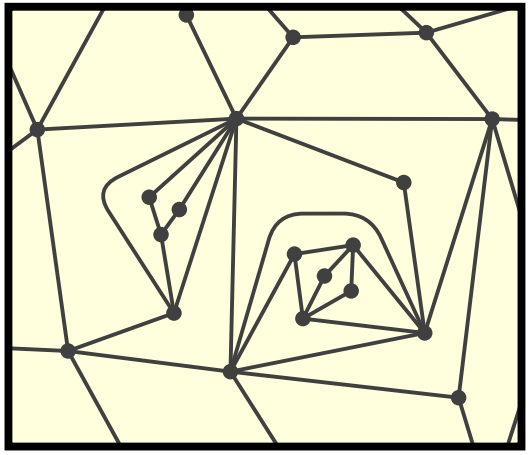

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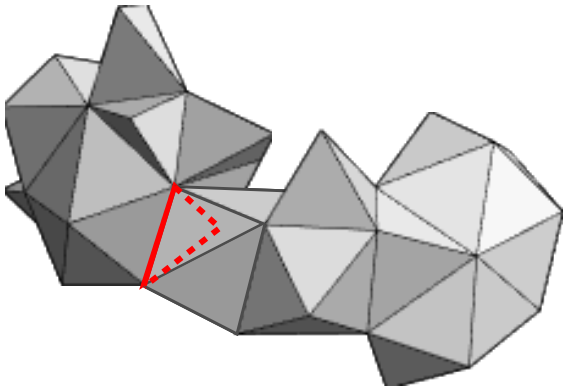

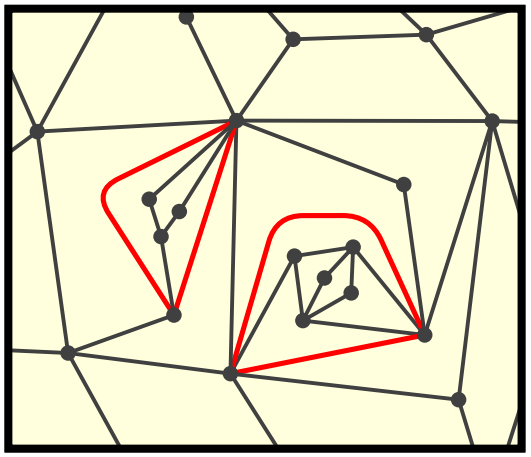

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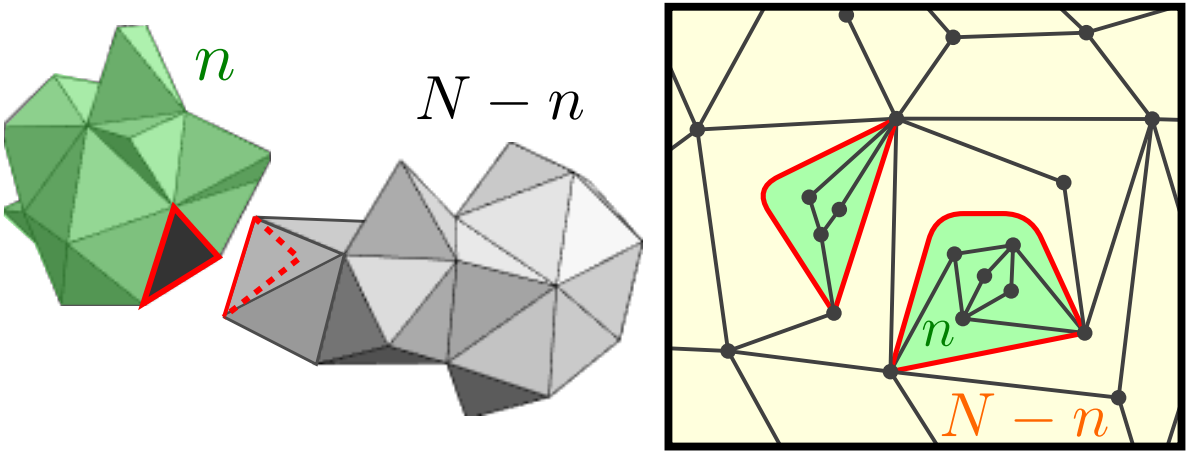

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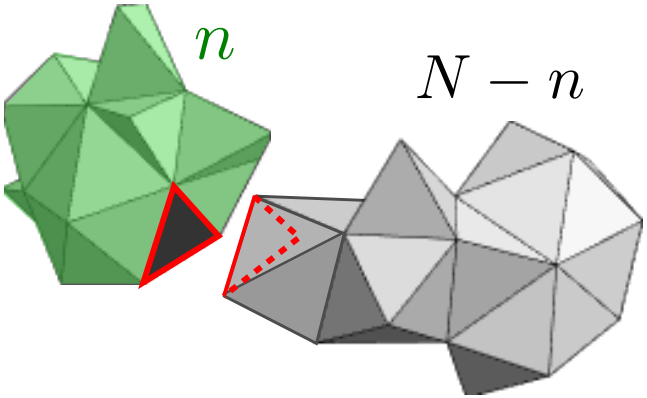
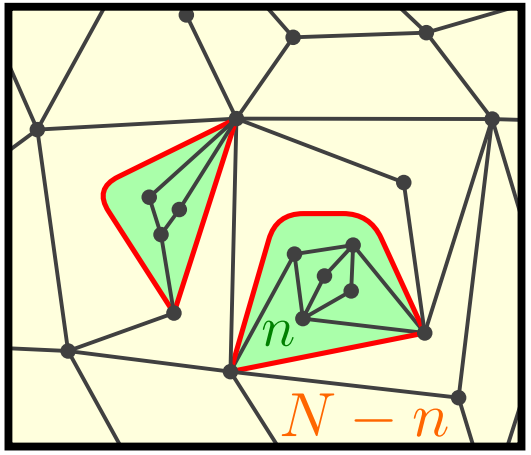
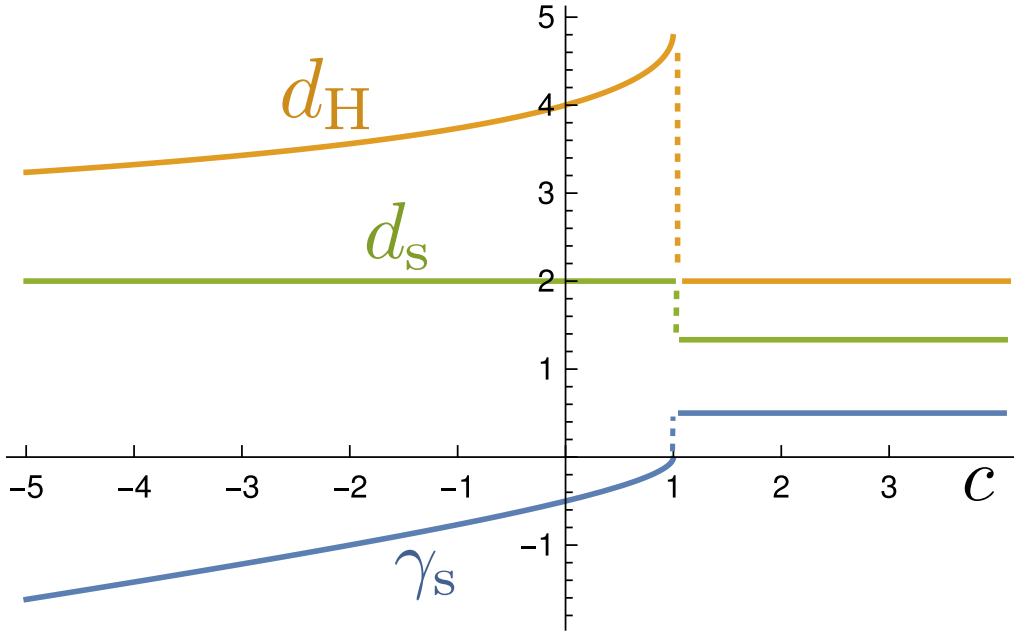
- Histogram of n and plot against $n \cdot (1 - n/N)$.



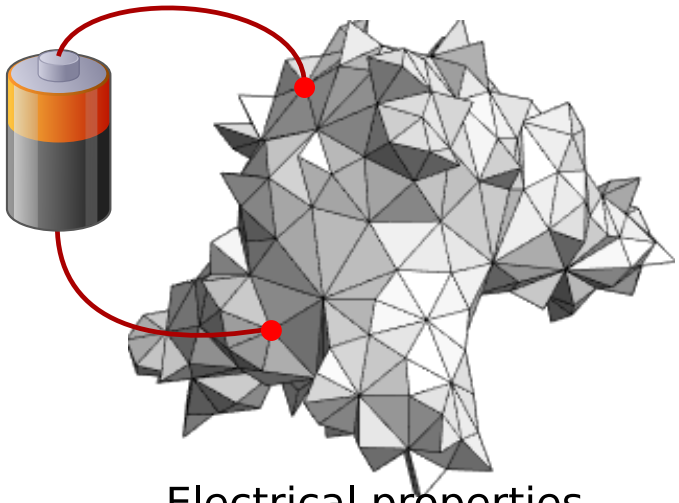


Conjectural relation d_{H}, d_{s}, γ_{s} to central charge c

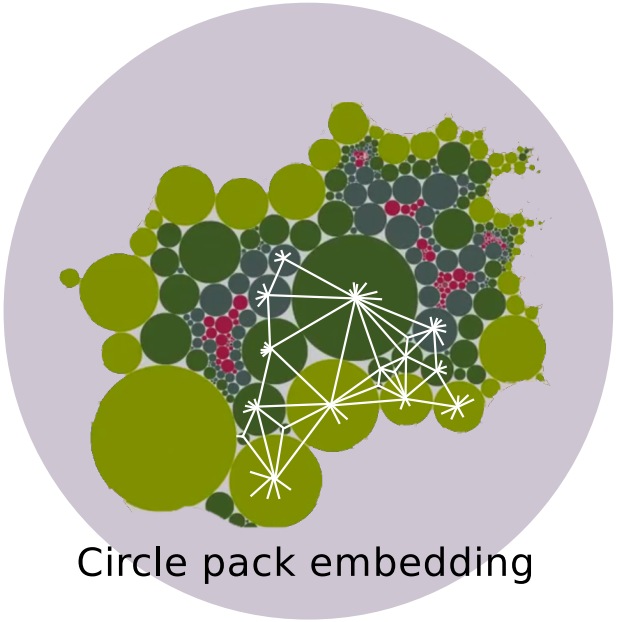
$c \in (-\infty, 1]$	$c > 1$
$\gamma_{\mathrm{s}} = \dfrac{c - 1 - \sqrt{(c-1)(c-25)}}{12}$	$\gamma_{\mathrm{s}} = \dfrac{1}{2}$
$d_{\mathrm{H}} = 2\dfrac{\sqrt{25-c} + \sqrt{49-c}}{\sqrt{25-c} + \sqrt{1-c}}$	$d_{\mathrm{H}} = 2$
$d_{\mathrm{s}} = 2$	$d_{\mathrm{s}} = \dfrac{4}{3}$



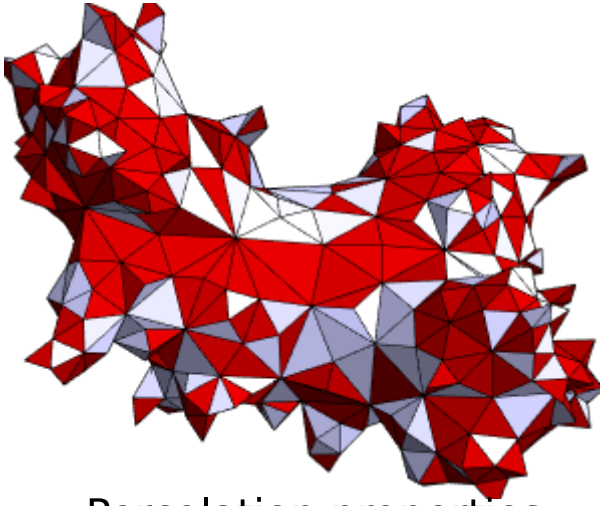
A zoo of observables



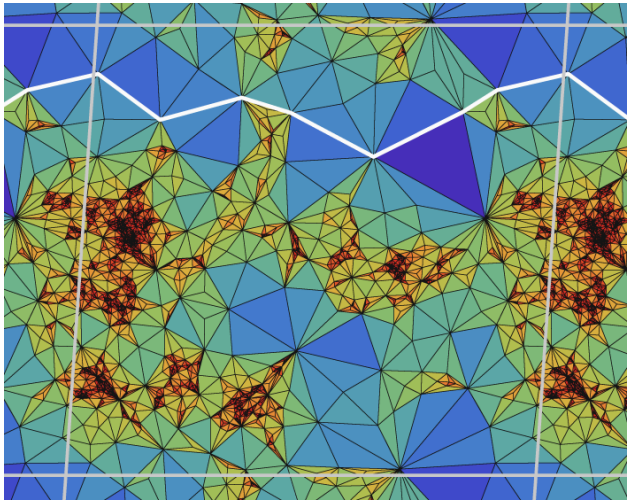
Electrical properties



Circle pack embedding



Percolation properties



Tutte's harmonic embedding

Plan for the tutorial sessions

- Team up (in groups 1, 2, 3).
- Get the program `randgeom` to work on your computer:

  `http://www.nbi.dk/~budd/randgeom/`
- Use the provided Mathematica notebook to get a feel for the random geometries produced by the various models (A, B, C, D).
- Gather data for (at least) one of the observables, and determine estimates for the corresponding critical exponent (d_{H}, d_{s} or γ_{s}) for each of the models.
- Be creative: try to think of a different observable, perform data analysis.
- Towards the end of the 2nd session: send me a brief summary of your results (nice plots?). At the end I will discuss the models and the data, and compare the latter to what we know.

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(All models correspond to a different value of the central charge c. For some models and exponents analytic results are available. One of the models has not been numerically investigated before!)