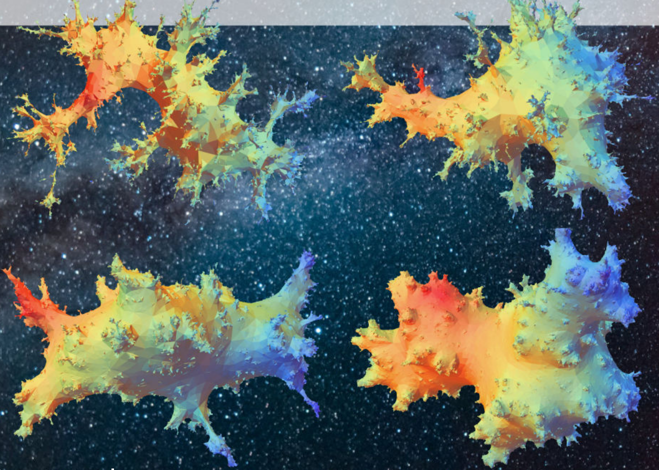
Quantum Gravity and Quantum Geometry @ Nijmegen, 31-10-2019

Trees and Fractal Dimensions in 2D Quantum Gravity

Timothy Budd

based on joint work with Jerome Barkley, arXiv:1908.09469



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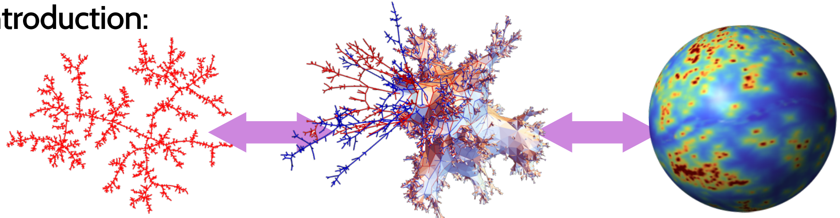
Radboud University

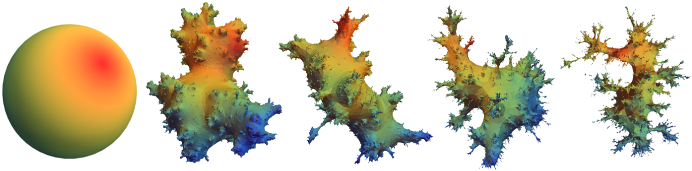
Outline

Introduction:

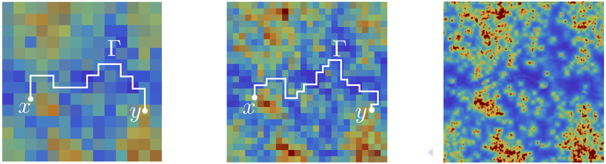


trees | random surfaces | Liouville Quantum Gravity

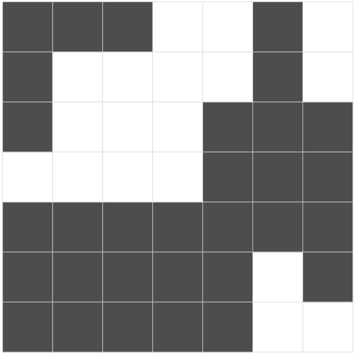
Fractal dimensions in random surfaces:

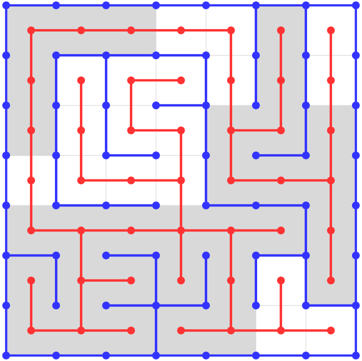


Fractal dimensions in Liouville Quantum Gravity:



Statistical systems and trees?





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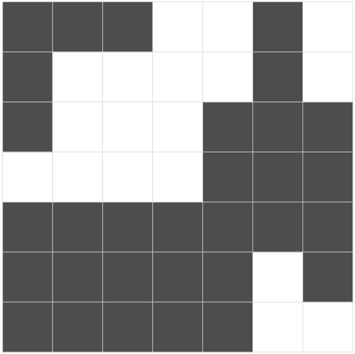

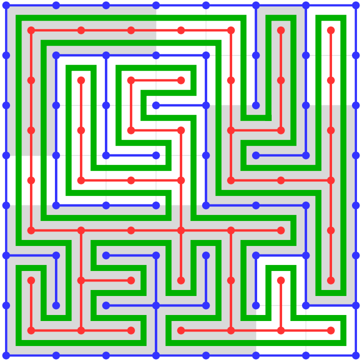

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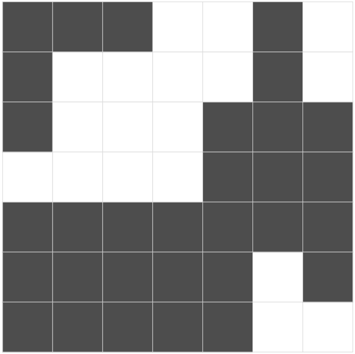

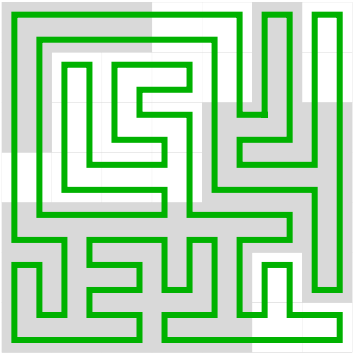

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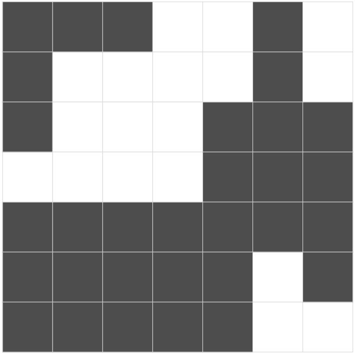

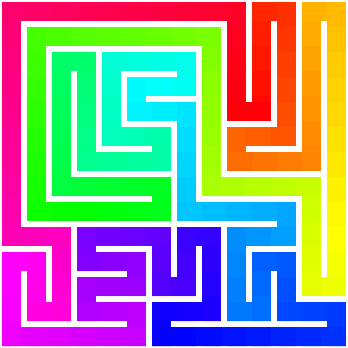

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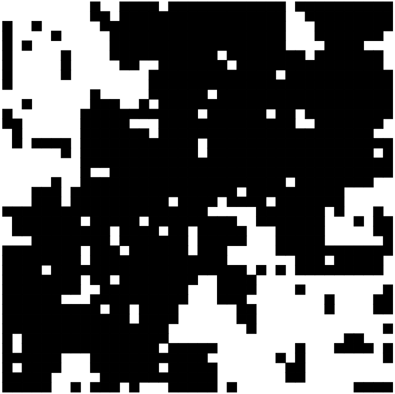

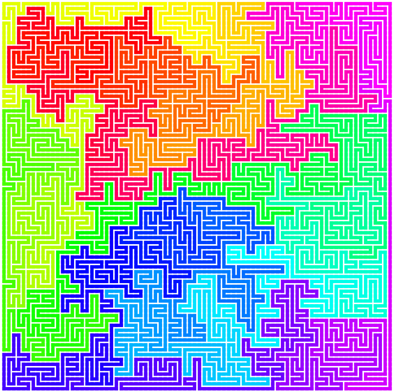

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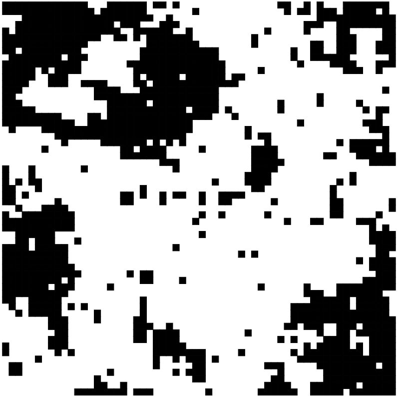

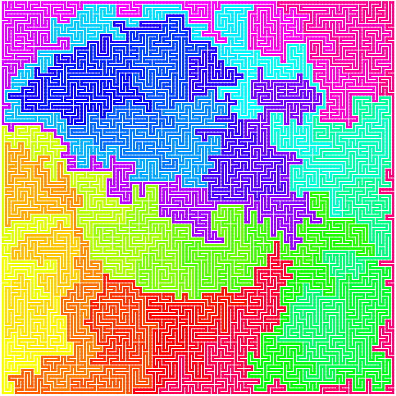

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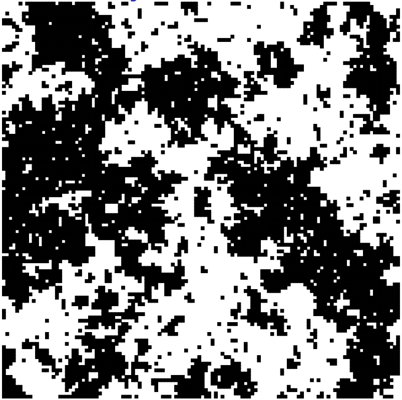

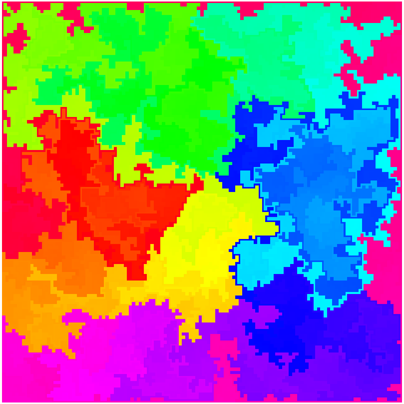

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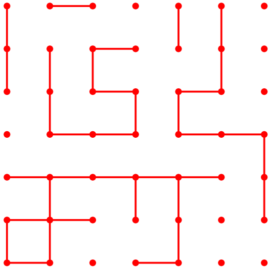

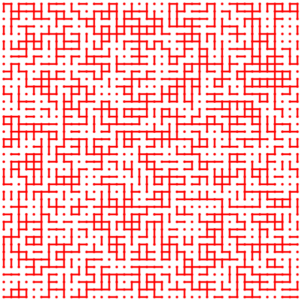

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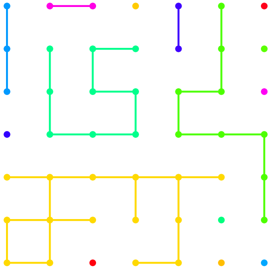

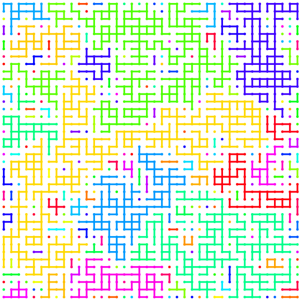

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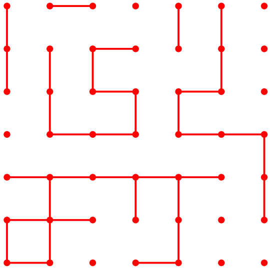

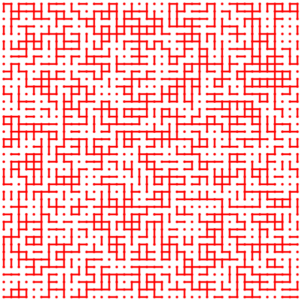

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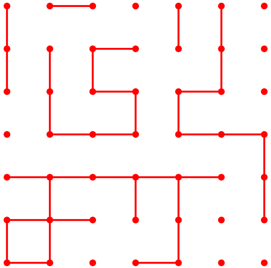

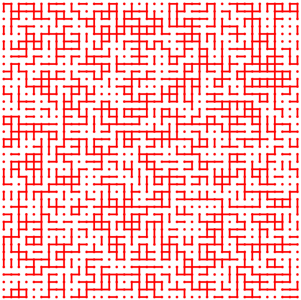

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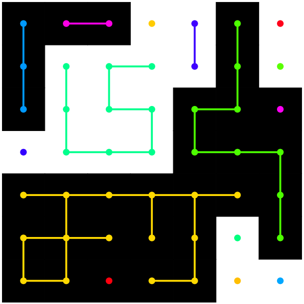

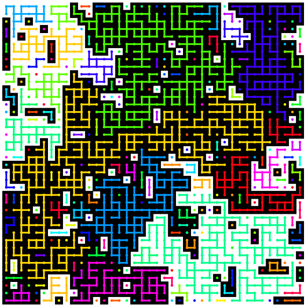

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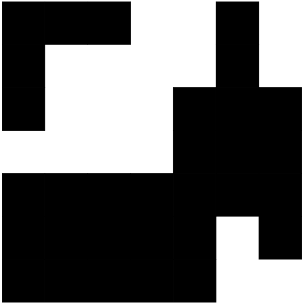

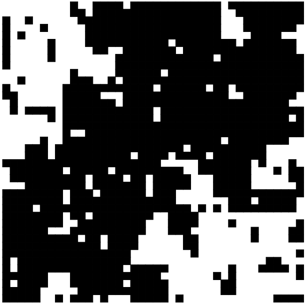

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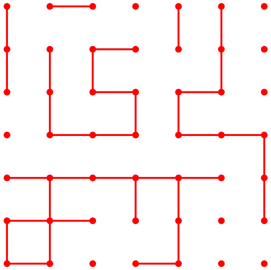

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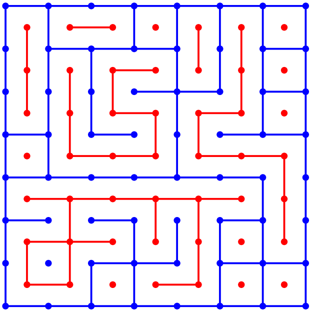

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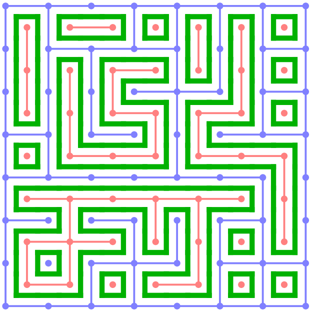

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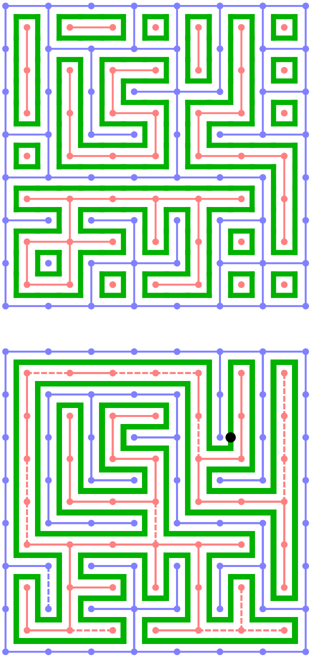

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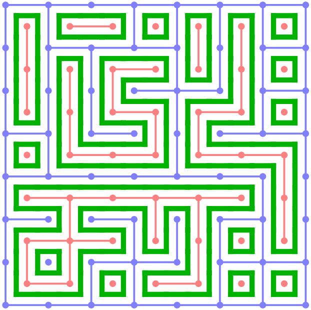

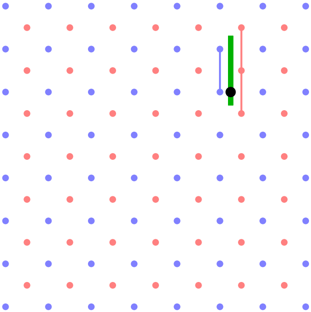

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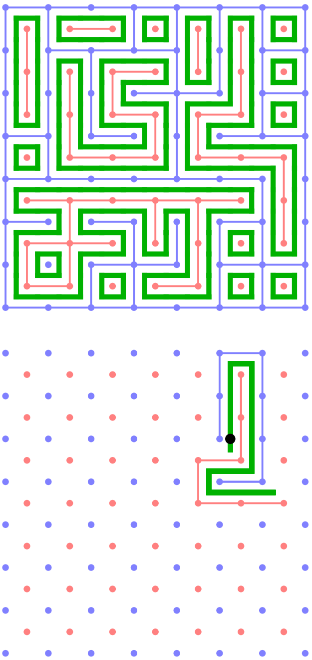

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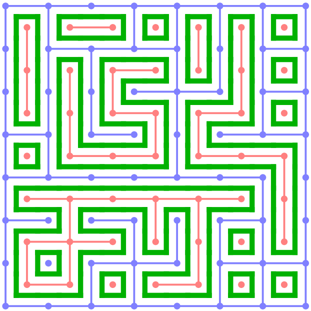

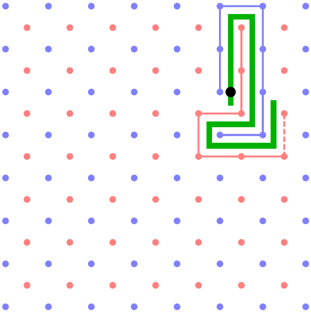

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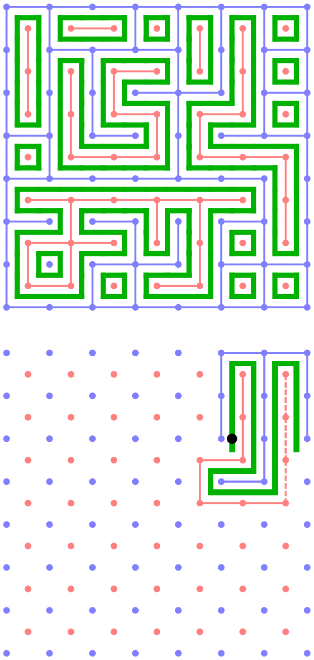

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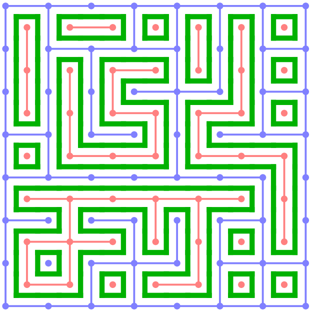

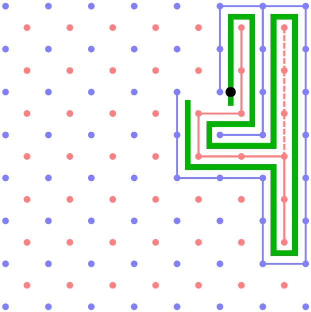

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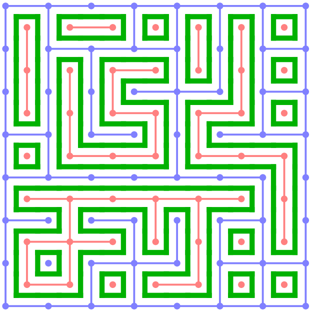

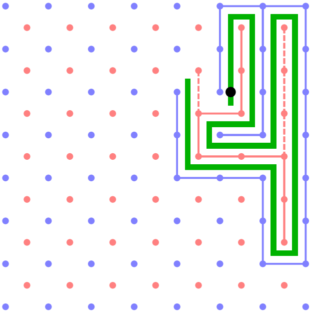

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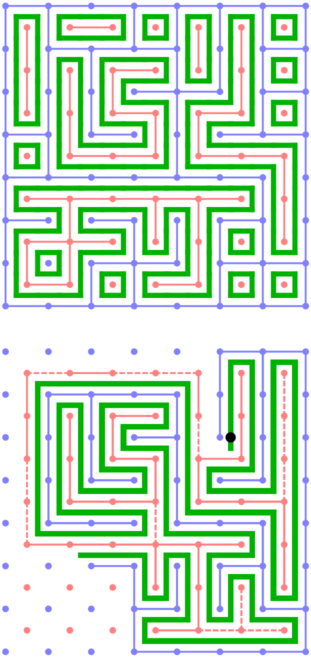

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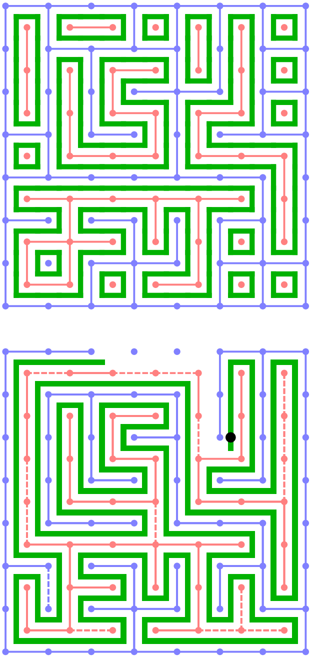

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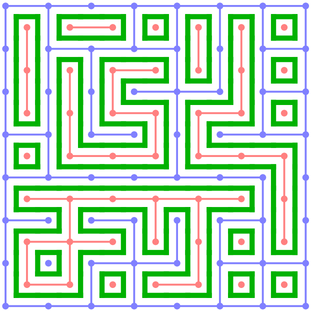

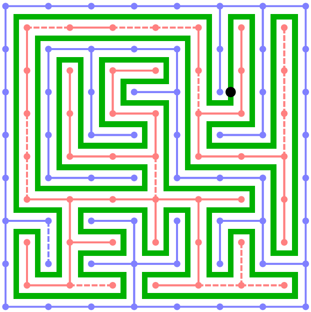

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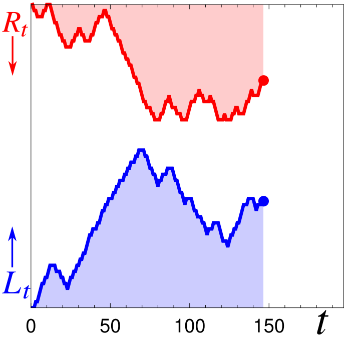

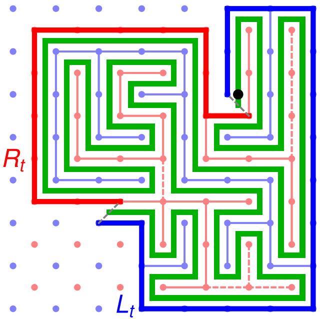

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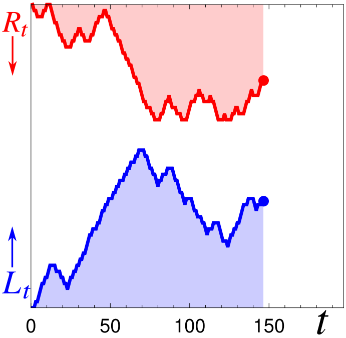

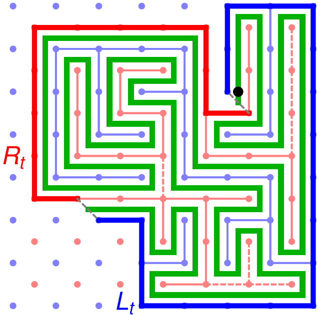

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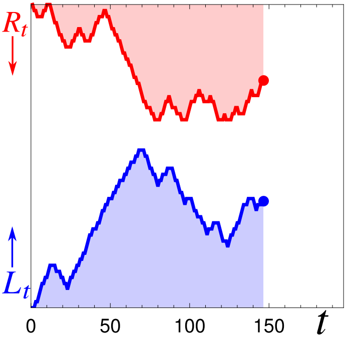

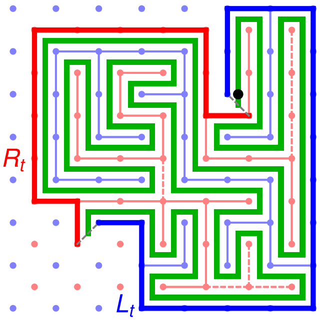

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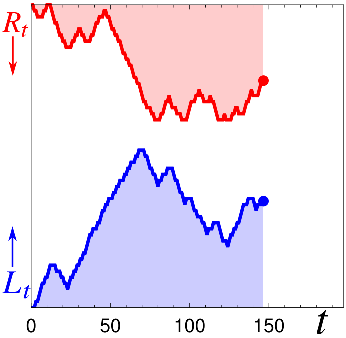

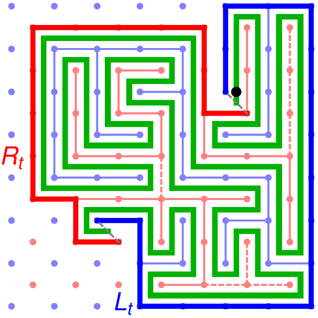

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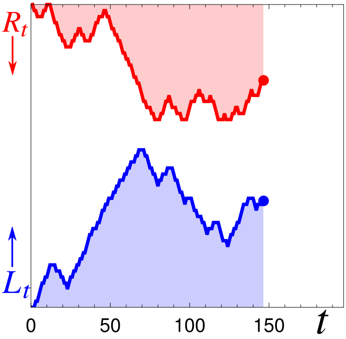

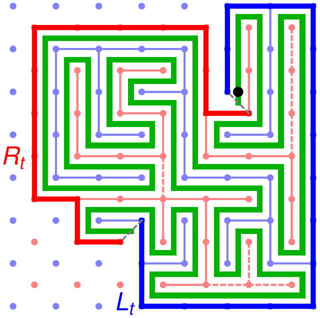

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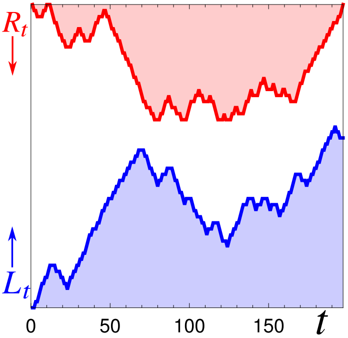

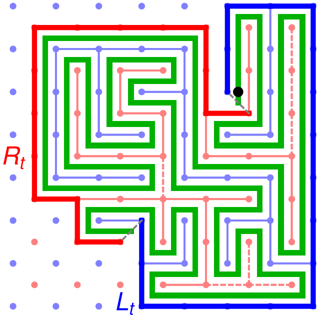

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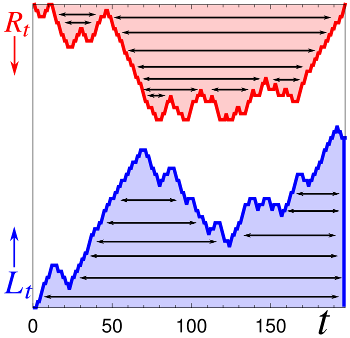

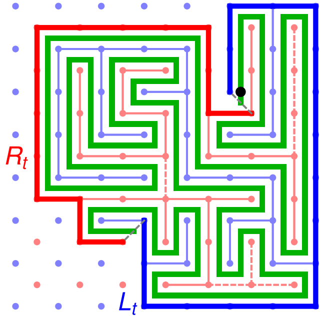

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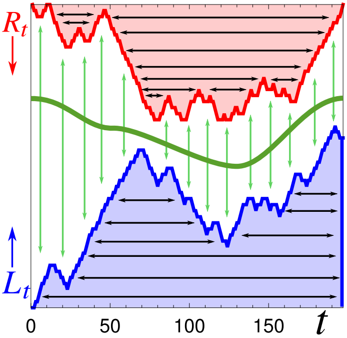

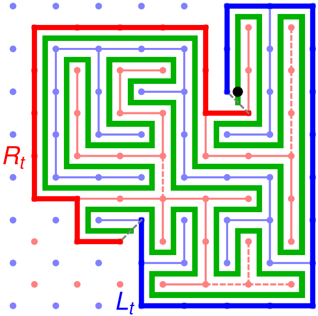

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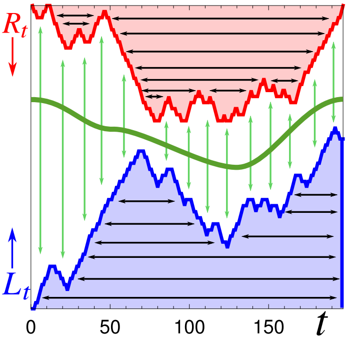

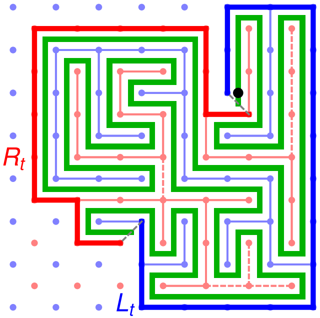

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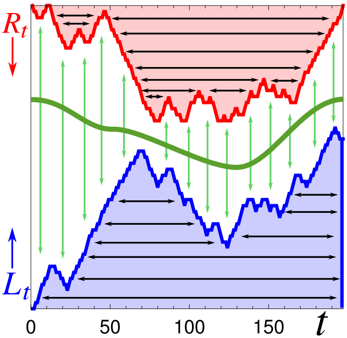
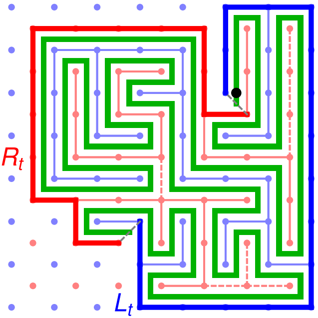


No, ...

...only when coupled to gravity (in continuum limit)!



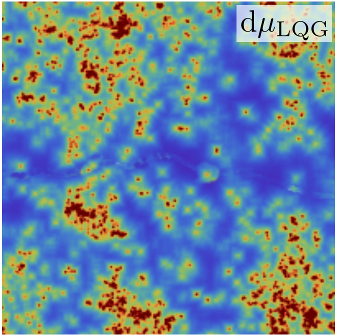
Liouville Quantum Gravity

- CFT of the Liouville field $\phi : \mathbb{C} \to \mathbb{R}$

$$\gamma \in [0,2], \quad g_{ab} = e^{\gamma\phi}\hat{g}_{ab}$$

$$S[\phi] = \frac{1}{4\pi}\int \mathrm{d}^2 z \sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q_\gamma \hat{R}\phi + \lambda e^{\gamma\phi})$$

[Polyakov, Knizhnik, Zamolodchikov, David, ...]



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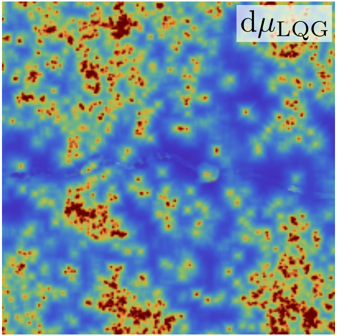

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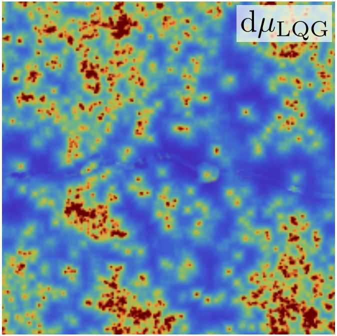
[Duplantier, Sheffield, David, Rhodes, Vargas, . . .]



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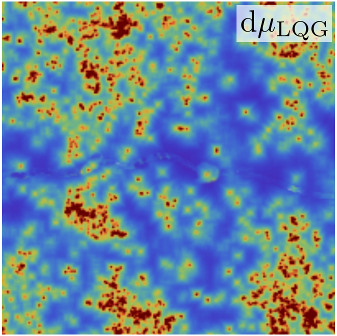
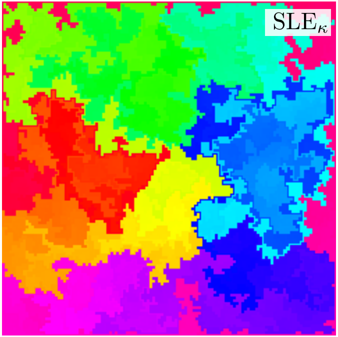
- $\mathrm{d}\mu_{\rm LQG}$ is conformally invariant, and intimately related to SLE_κ with

$$\kappa = \frac{16}{\gamma^2}, \quad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$$





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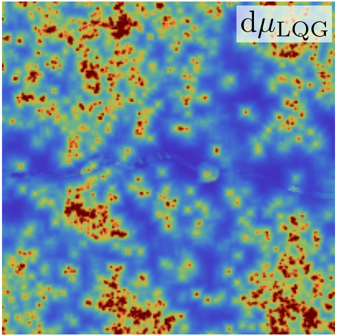

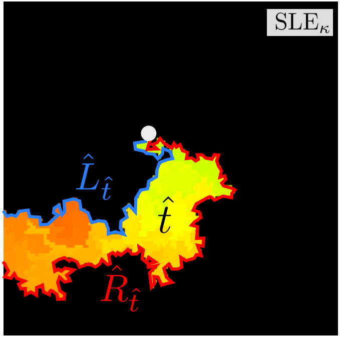

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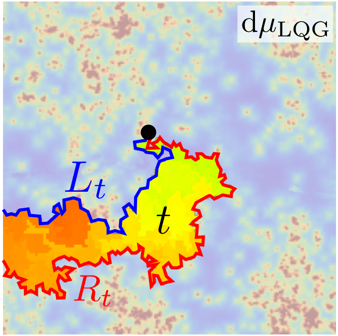
- The law of $(\hat{L}_{\hat{t}}, \hat{R}_{\hat{t}})$ is complicated, but reparametrizing in terms of $g_{ab} = e^{\gamma\phi}\hat{g}_{ab}$,

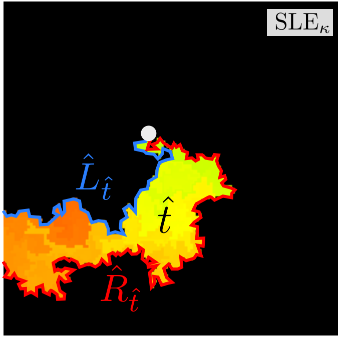
$$\begin{pmatrix} R_t \\ L_t \end{pmatrix} \overset{law}{=} \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \vec{X}_t, \quad \alpha = \frac{\pi}{8}(\gamma^2 - 2),$$

with $\vec{X}_t$ a standard 2D Brownian motion!

[Sheffield, Duplantier, Miller, Gwynne, . . .]

$$\mathrm{SLE}_{\kappa=16/\gamma^2} + \mathrm{LQG}_\gamma \;=\; \text{2D Brownian motion}_\alpha$$





Liouville Quantum Gravity

$$\gamma \in [0,2], \quad g_{ab} = e^{\gamma\phi}\hat{g}_{ab}$$

$$S[\phi] = \frac{1}{4\pi}\int \mathrm{d}^2 z \sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q_\gamma \hat{R}\phi + \lambda e^{\gamma\phi})$$

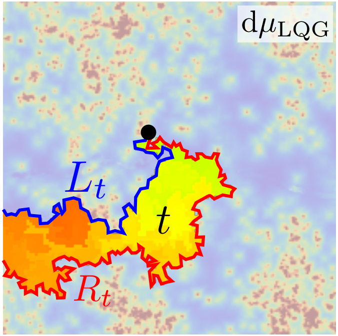
- SLE_κ can be parametrized by contour length functions $(\hat{L}_{\hat{t}}, \hat{R}_{\hat{t}})$ as function of area $\hat{t}$ w.r.t. Euclidean metric $\hat{g}_{ab}$.
- The law of $(\hat{L}_{\hat{t}}, \hat{R}_{\hat{t}})$ is complicated, but reparametrizing in terms of $g_{ab} = e^{\gamma\phi}\hat{g}_{ab}$,

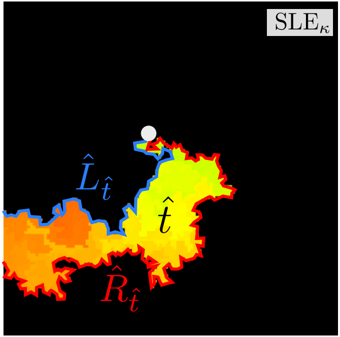
$$\begin{pmatrix} R_t \\ L_t \end{pmatrix} \overset{law}{=} \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \vec{X}_t, \quad \alpha = \frac{\pi}{8}(\gamma^2 - 2),$$

with $\vec{X}_t$ a standard 2D Brownian motion!

[Sheffield, Duplantier, Miller, Gwynne, ...]

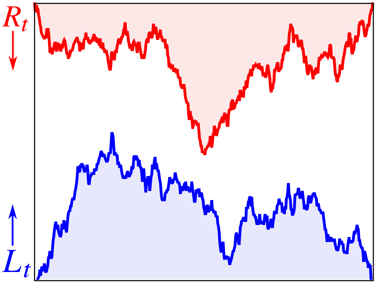
$$\underbrace{\text{SLE}_{\kappa=16/\gamma^2}}_{\text{matter}} + \underbrace{\text{LQG}_\gamma}_{\text{gravity}} = \underbrace{\text{2D Brownian motion}_\alpha}_{\text{pair of trees}}$$





From trees back to 2D quantum gravity

$$\begin{pmatrix} R_t \\ L_t \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \vec{X}_t$$



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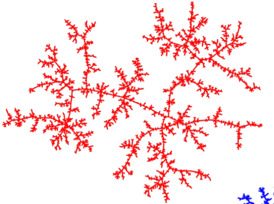

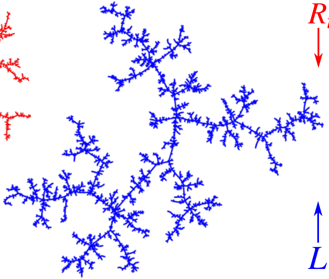

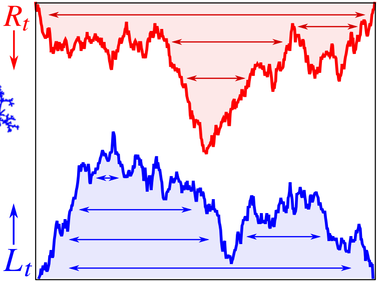

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From trees back to 2D quantum gravity

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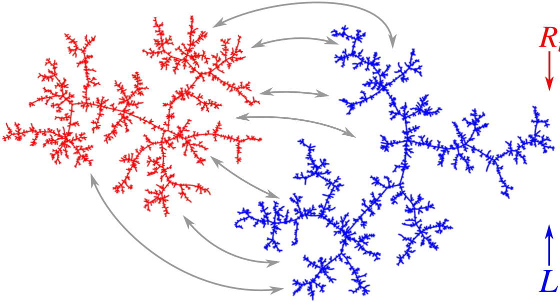
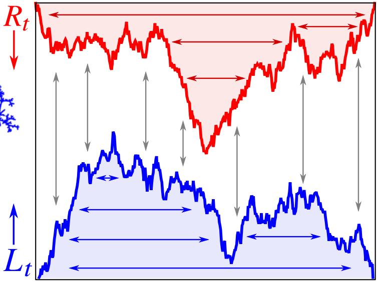


"Mating of trees"

[Duplantier, Miller, Sheffield, '14]



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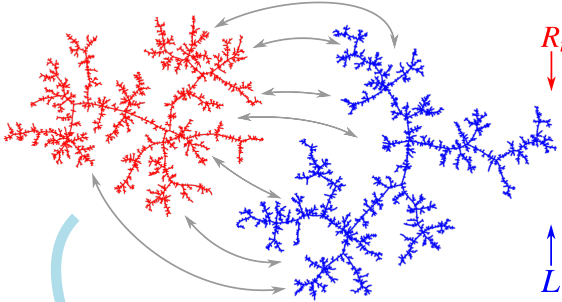
From trees back to 2D quantum gravity



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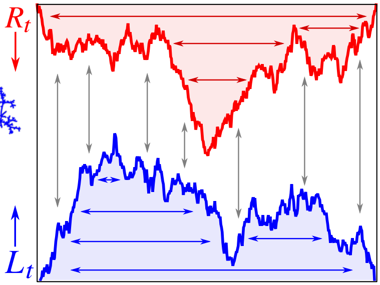

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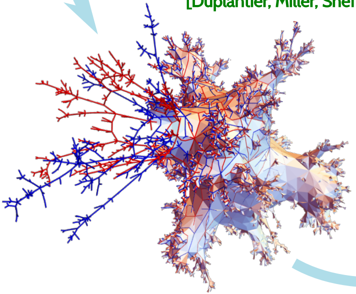
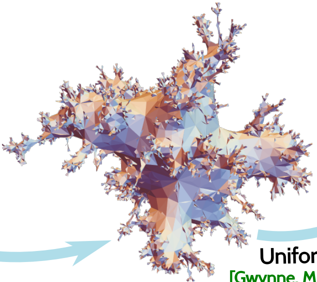
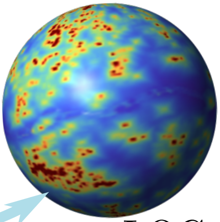




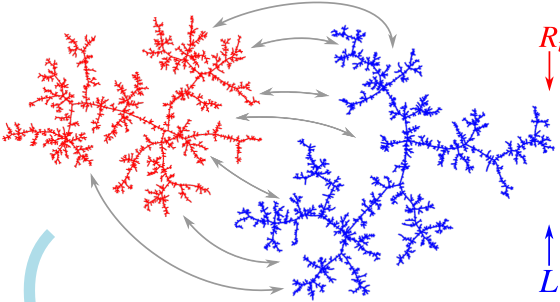
Uniformization

[Gwynne, Miller, Sheffield, '17]



LQG_γ

From trees back to 2D quantum gravity



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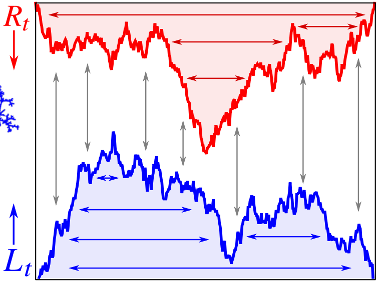

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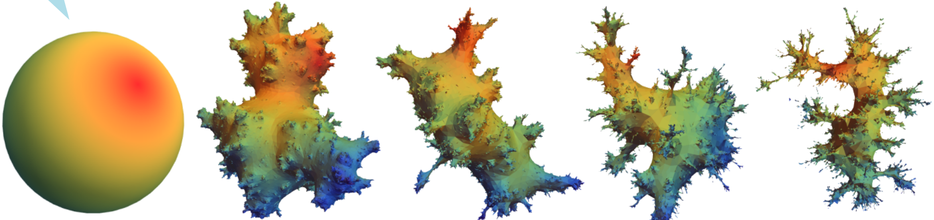
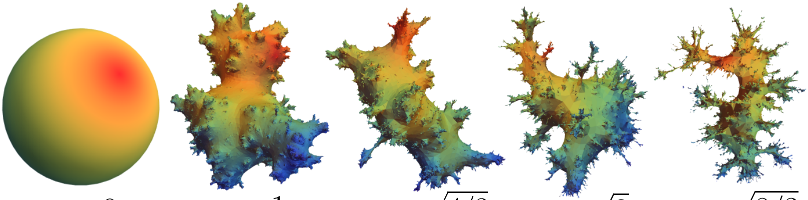


$\gamma = 0$ $\gamma = 1$ $\gamma = \sqrt{4/3}$ $\gamma = \sqrt{2}$ $\gamma = \sqrt{8/3}$



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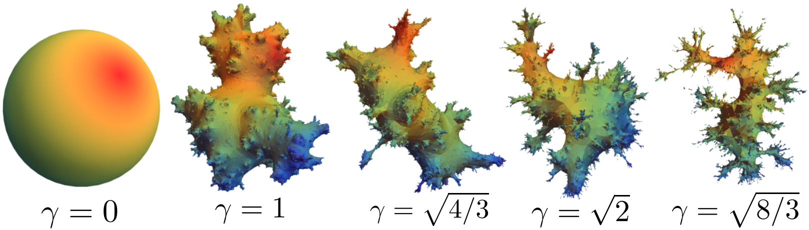
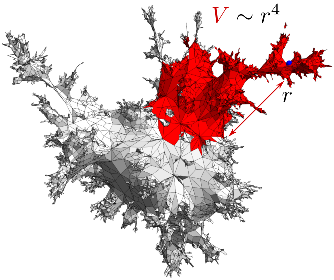
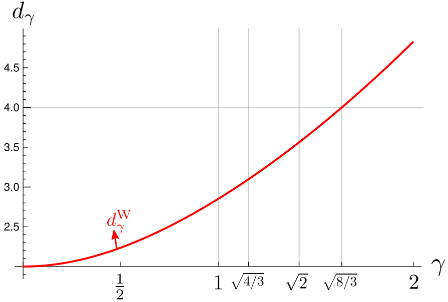
- 1-parameter family of scale-invariant geometries: UV fixed points of the renormalization group flow of 2D quantum gravity.



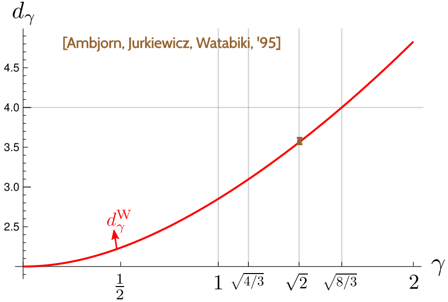
- 1-parameter family of scale-invariant geometries: UV fixed points of the renormalization group flow of 2D quantum gravity.

- Fractal properties? Hausdorff dimension d_γ?
  - "Classical gravity": $d_{\gamma=0} = 2$
  - "Pure gravity": $d_{\gamma=\sqrt{8/3}} = 4$ [Ambjørn, Watabiki, '95] [Schaeffer, Chassaing, Le Gall, Miermont, ...].
  - "Gravity coupled to matter": $d_\gamma = ?$





- Watabiki's conjecture [Watabiki, '93]: $d_\gamma^W = 1 + \frac{\gamma^2}{4} + \sqrt{(1 + \frac{\gamma^2}{4})^2 + \gamma^2}$
- Derivation not without issues, but agrees very well with numerical data...



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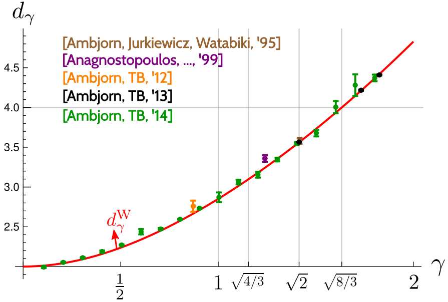

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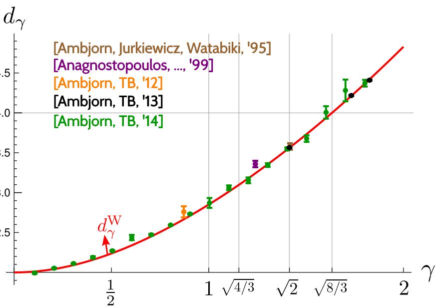

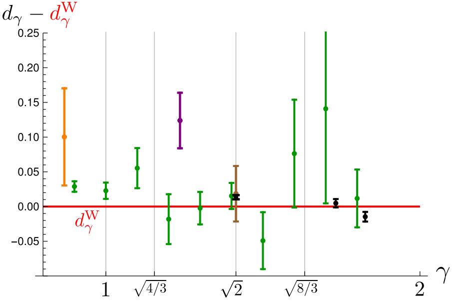

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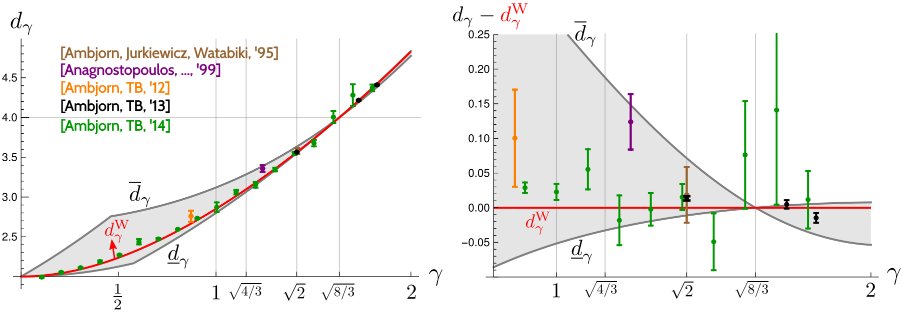

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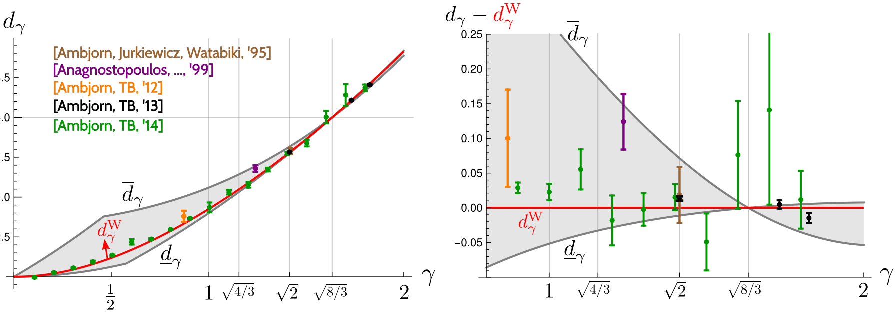
- ...and with recent rigorous bounds. [Ding, Gwynne, Pfeffer, Ang, '18-'19]



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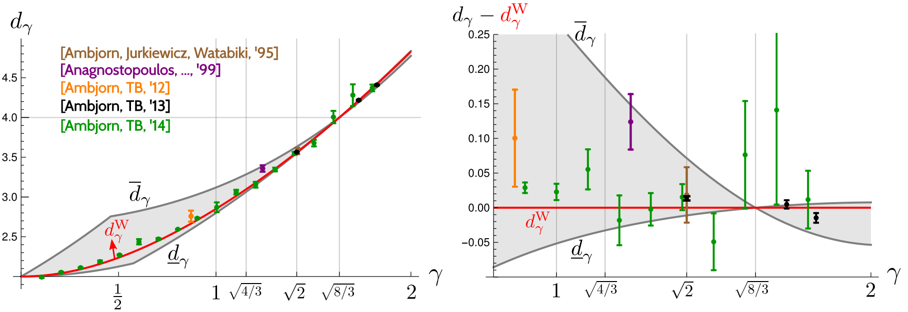
[Ding, Goswami, '16] $$d_\gamma \geq 2 + C\frac{\gamma^{4/3}}{\log \gamma^{-1}}, \quad \text{while} \quad d_\gamma^W = 2 + O(\gamma^2).$$



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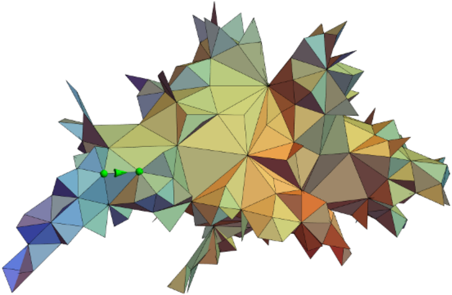
- Use tree encoding to obtain higher accuracy for $\gamma < \sqrt{8/3}$!

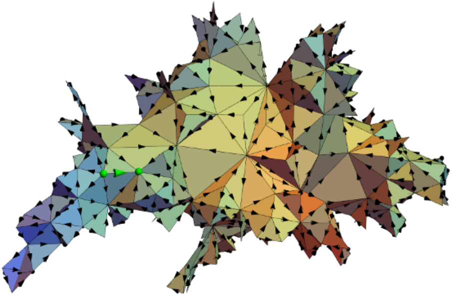
Bipolar-oriented triangulations

- Consider triangulation of S^2 with an oriented root edge.



Bipolar-oriented triangulations

- Consider triangulation of S^2 with an oriented root edge.
- Bipolar orientation: assignment of directions that is acyclic and has no sources or sinks except at root.



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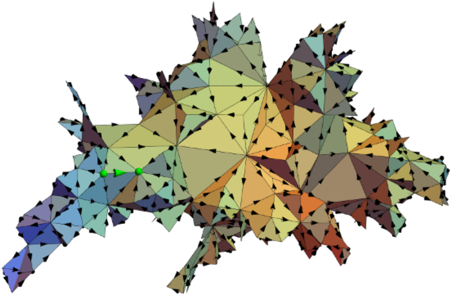

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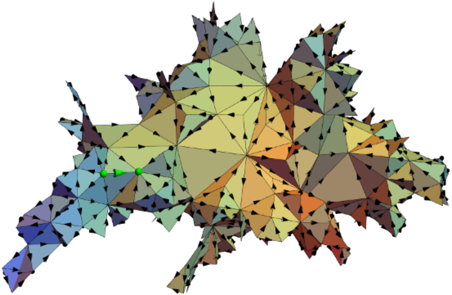

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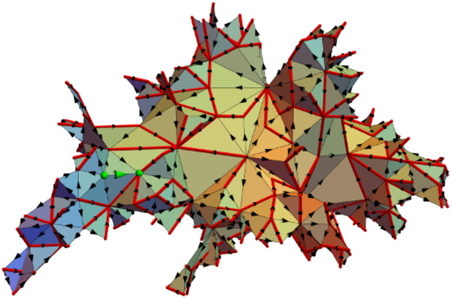
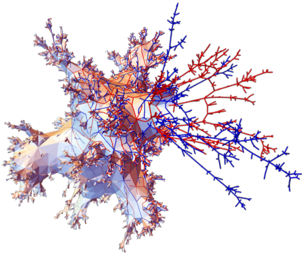
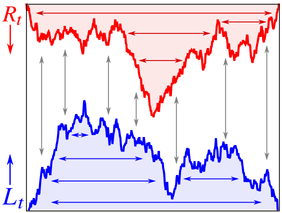
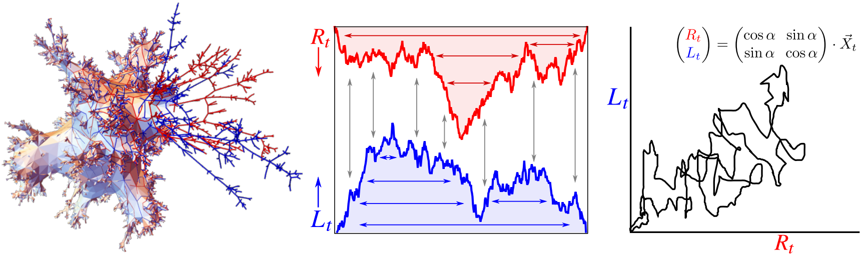





R_t
L_t



R_t
L_t
L_t
R_t
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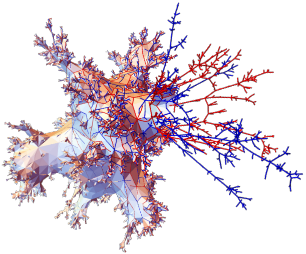
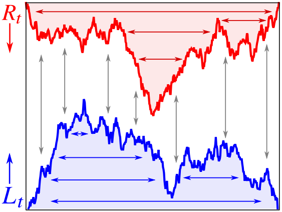
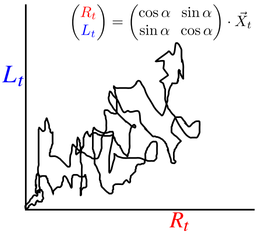
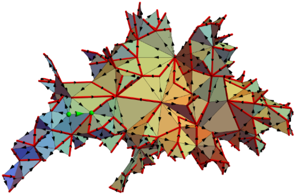


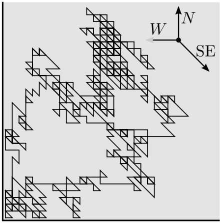


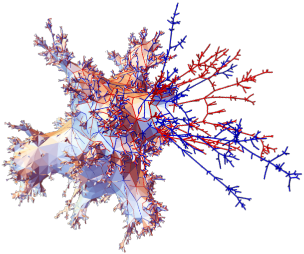


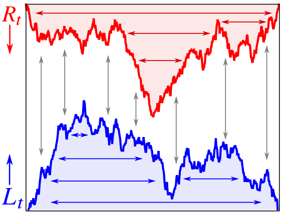


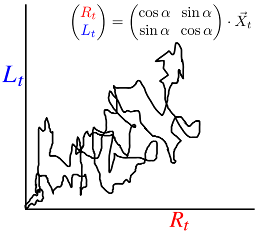
Bijection

[Kenyon, Miller, Sheffield, Wilson, '15]






R_t
L_t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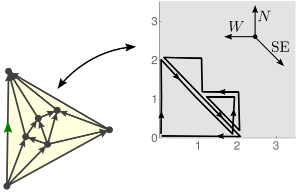
L_t
R_t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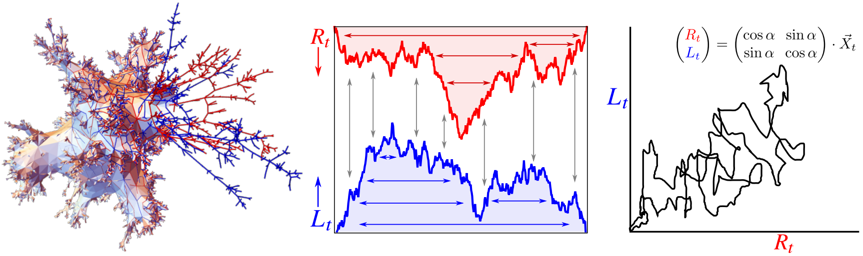

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L_t
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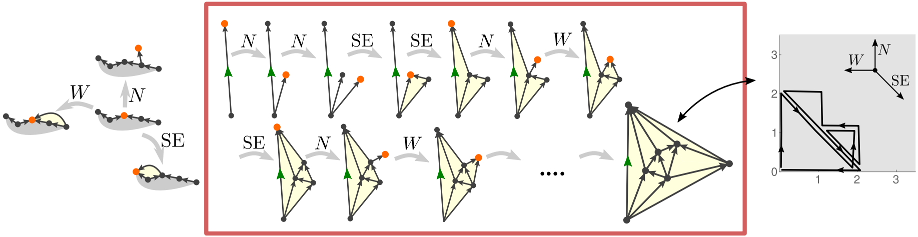
L_t
R_t



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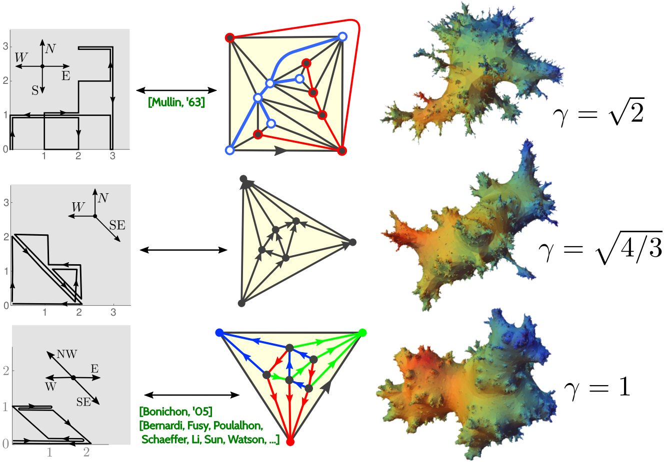
Spanning-tree-decorated quadrangulations

Bipolar-oriented triangulations

Schnyder-wood-decorated triangulations



- Similar encoding known for several other models.

Spanning-tree-decorated quadrangulations

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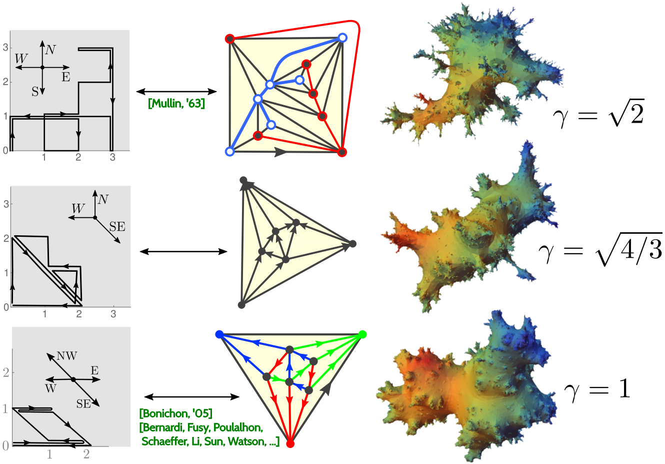
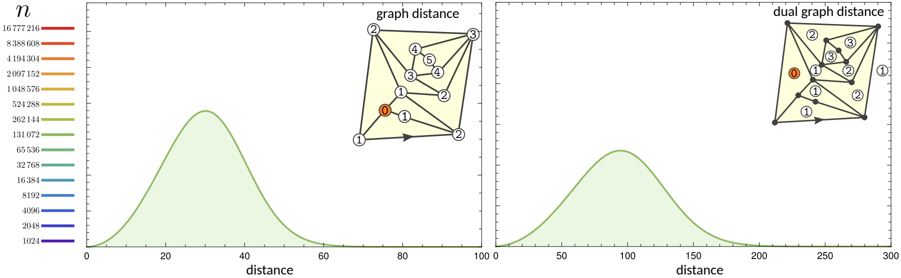
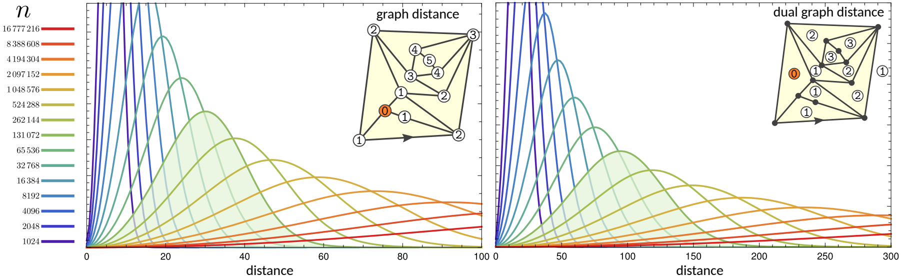


- Similar encoding known for several other models.
- Lattice walks can be sampled in linear time: much more efficient than MCMC methods.

Finite-size scaling analysis of distances [Barkley, TB, '19]



Finite-size scaling analysis of distances [Barkley, TB, '19]



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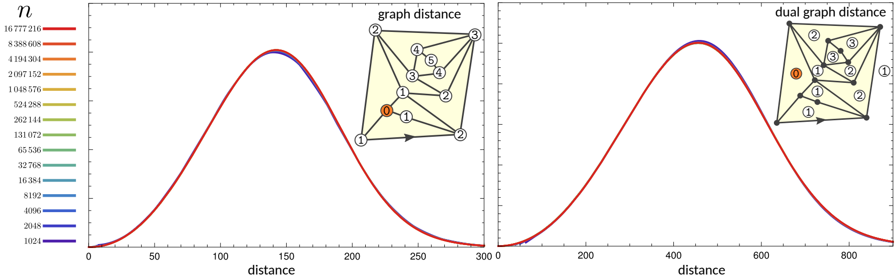

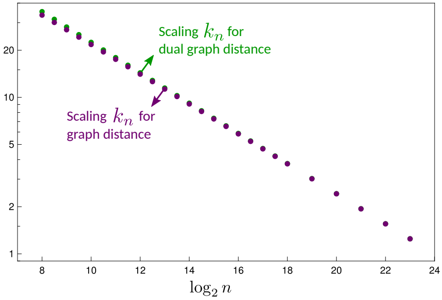

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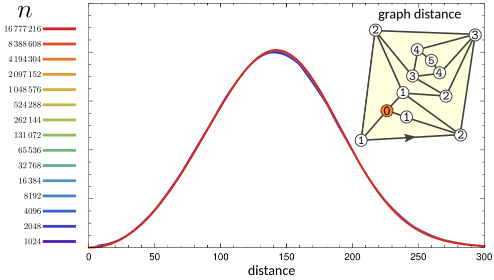

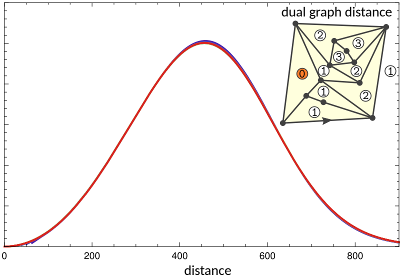

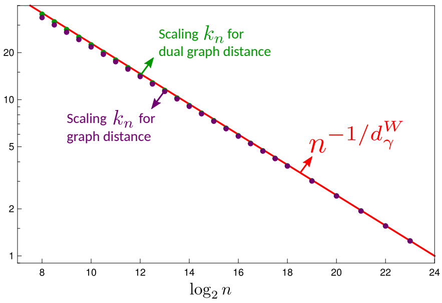

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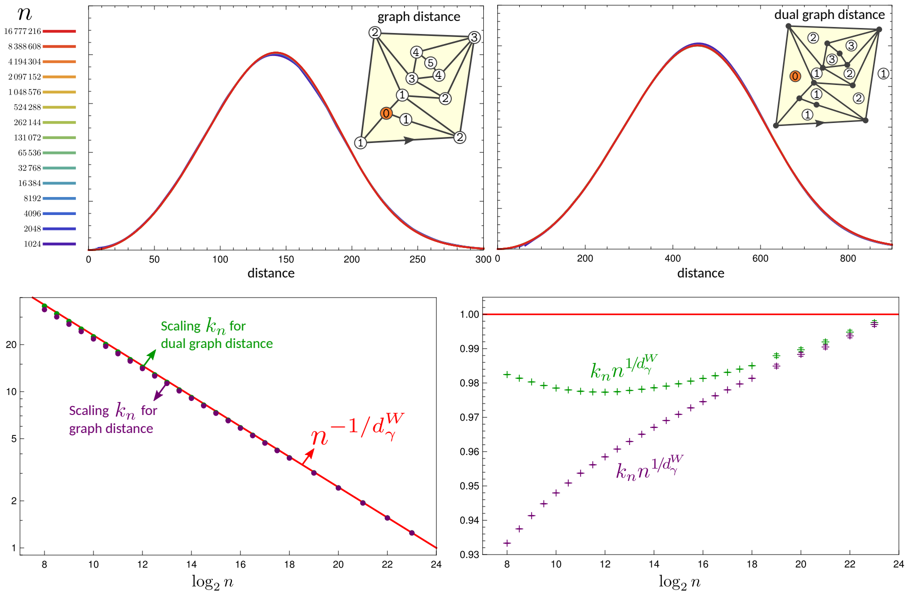

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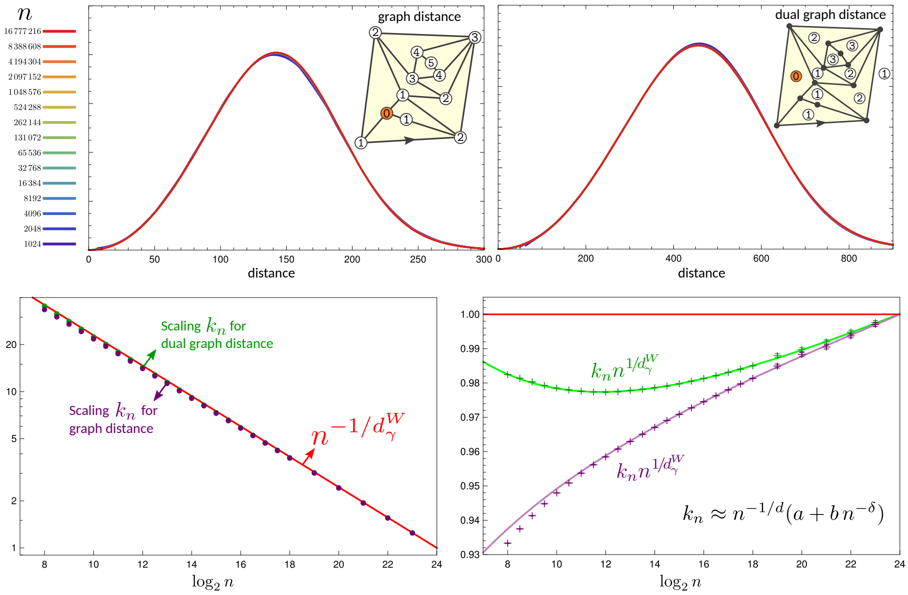

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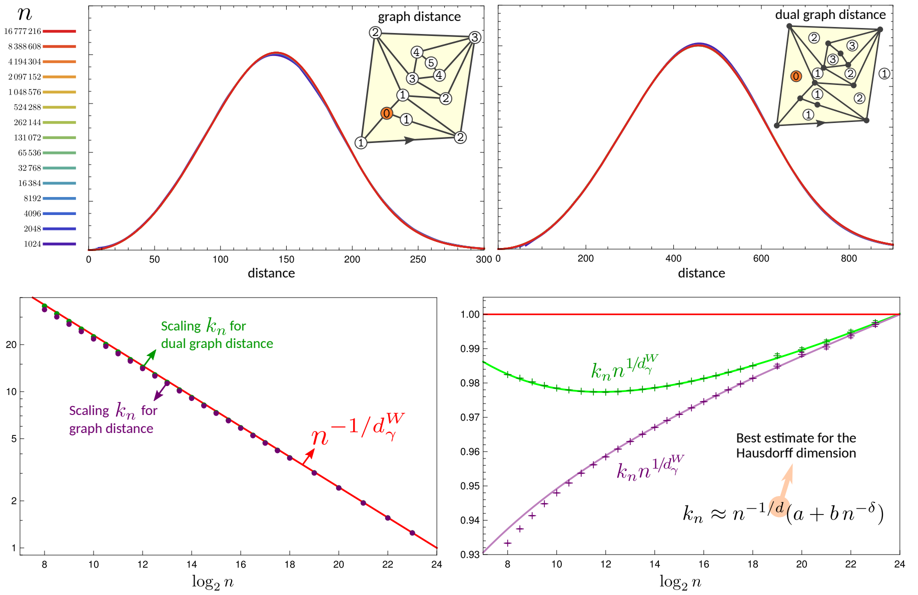
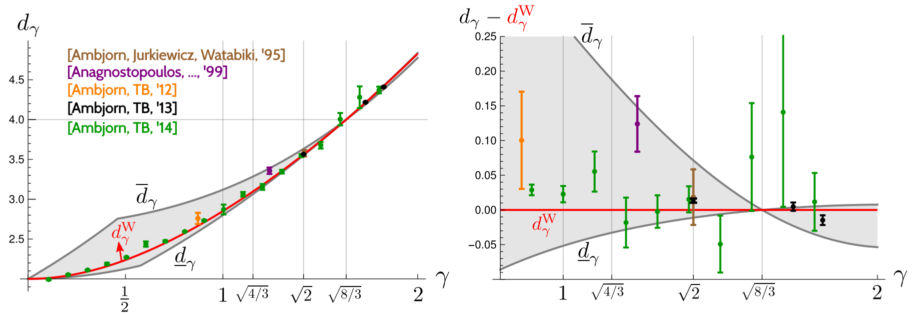


New Hausdorff dimension estimates [Barkley, TB, '19]



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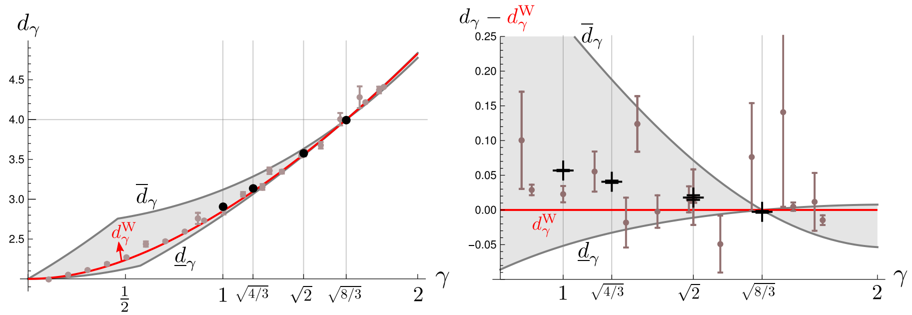

- Significant deviation ($> 20\sigma$) from $d^W_\gamma = 1 + \frac{\gamma^2}{4} + \sqrt{(1+\frac{\gamma^2}{4})^2 + \gamma^2}$.

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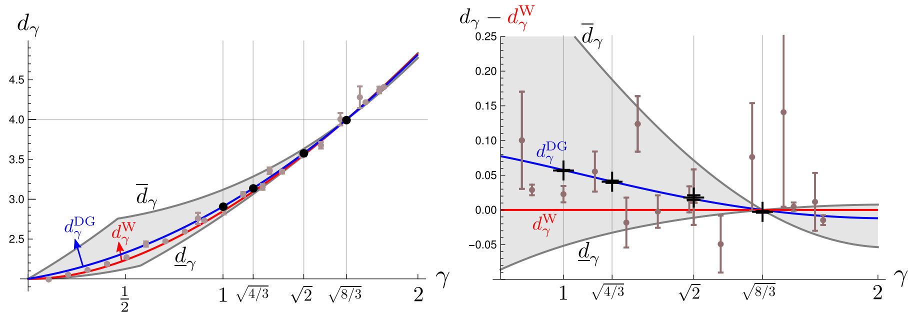
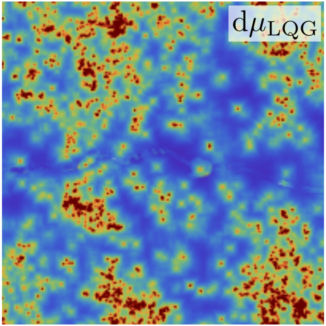


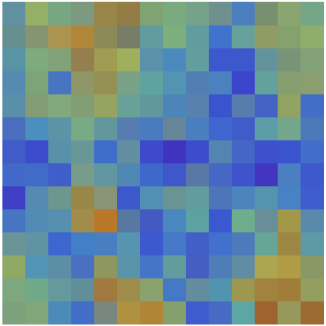
- Significant deviation ($> 20\sigma$) from $d_\gamma^W = 1 + \frac{\gamma^2}{4} + \sqrt{(1 + \frac{\gamma^2}{4})^2 + \gamma^2}$.
- Perfectly consistent with $d_\gamma^{\mathrm{DG}} = 2 + \frac{\gamma^2}{2} + \frac{\gamma}{\sqrt{6}}$. [Ding, Gwynne, '18]

Approaching from Liouville side

- Volume measure in LQG_γ:
  $\mathrm{d}\mu_{\mathrm{LQG}} \approx \sqrt{g}\,\mathrm{d}^2 z, \quad g_{ab} = e^{\gamma\phi}\hat{g}_{ab}.$

$$S[\phi] = \frac{1}{4\pi}\int \mathrm{d}^2 z \sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q_\gamma \hat{R}\phi + \lambda e^{\gamma\phi})$$





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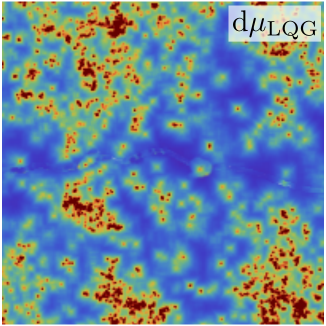

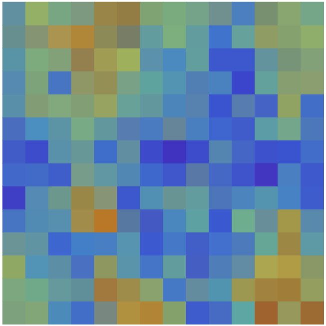

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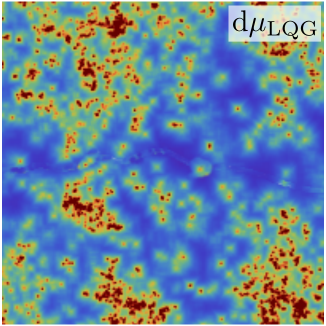

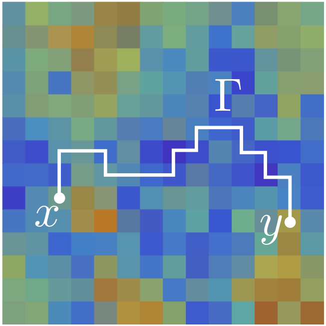

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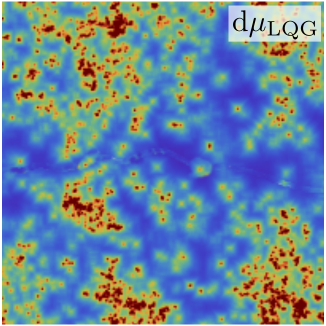

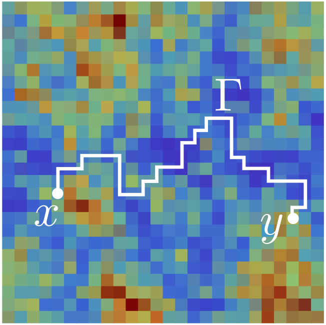

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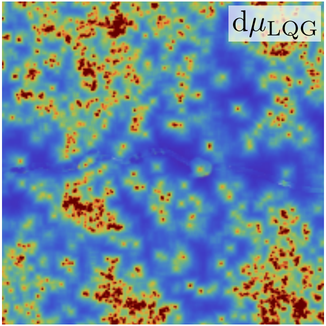
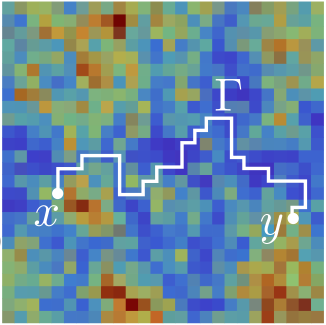
- Describes distances in LQG_γ only if
  - $\xi = \gamma/d_\gamma$ (fractal dimension)
  - $\lambda = 1 - \frac{2}{d_\gamma} - \frac{\gamma^2}{2d_\gamma}$ (conformal invariance)

[Ding, Gwynne, Ang, . . .]





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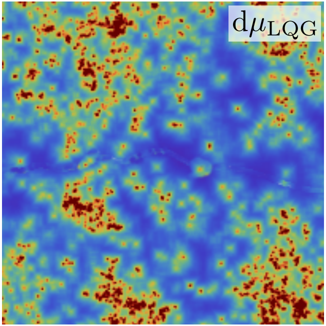
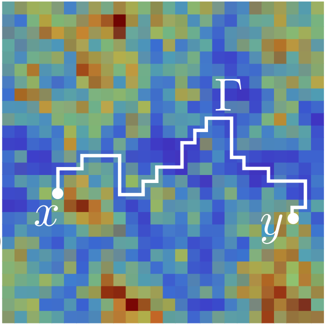
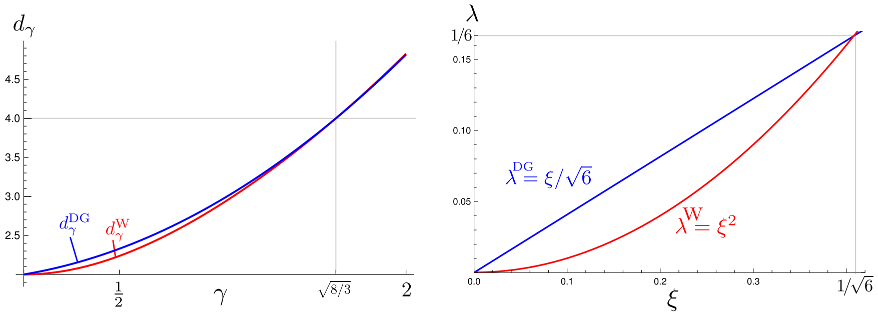
  [Ding, Gwynne, Ang, ...]
- Given pair (ξ, λ) can solve for (γ, d_γ)!

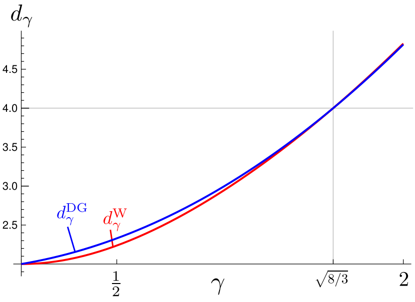


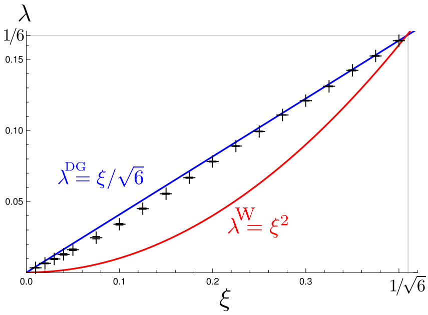


Results from finite-size scaling in LQG_γ [Barkley, TB, '19]



Results from finite-size scaling in LQG$_\gamma$ [Barkley, TB, '19]





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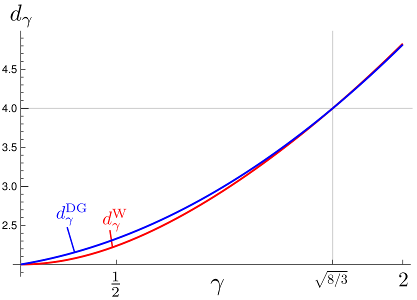

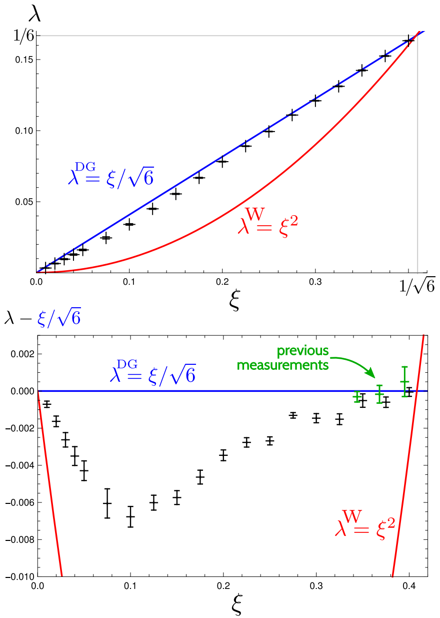

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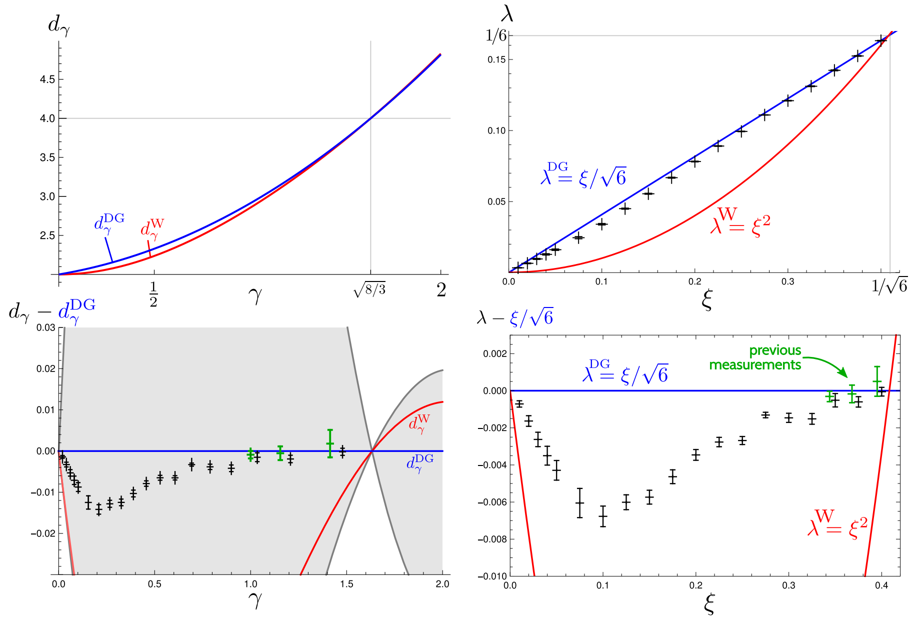

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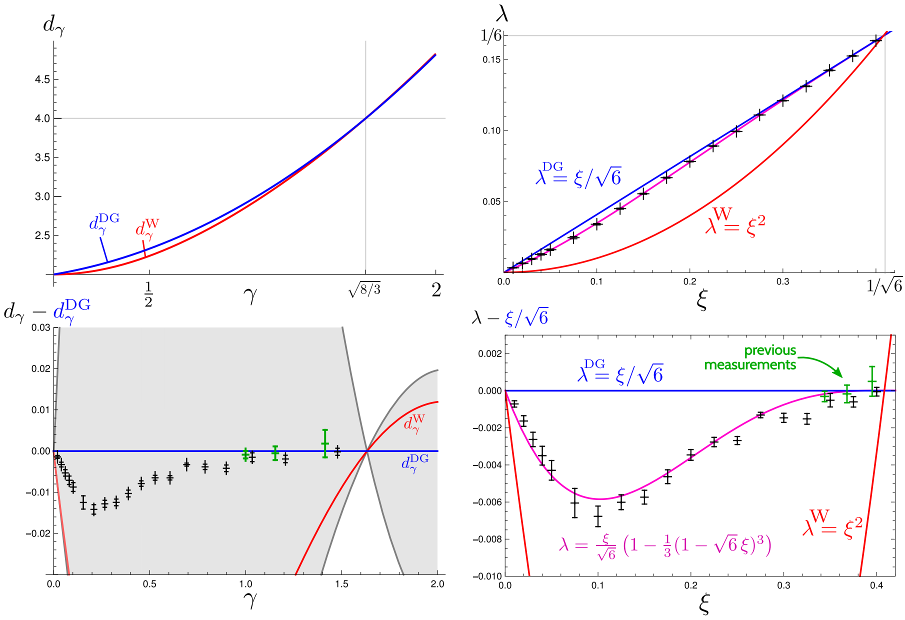

Wrapping up

- After 25 years Watabiki's conjecture is dead.
  Long live Ding-Gwynne's, ... but for how long?

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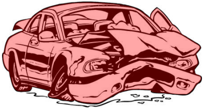

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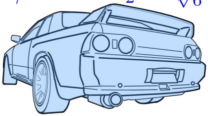
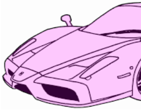


$d_\gamma =$



- Building (quantum) geometries from scale-invariant trees is a very recent and fruitful perspective on 2D quantum gravity. How about higher dimensions?