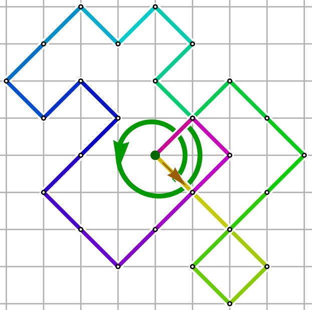
Journées de combinatoire de Bordeaux, LaBRI
January 25th, 2017

Winding of walks on the square lattice

Timothy Budd



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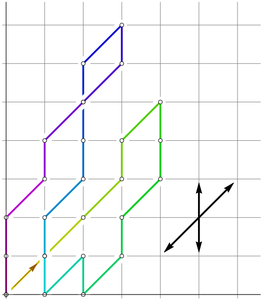
Introduction: Gessel sequence

- In 2001 Ira Gessel conjectured the number of walks with $2n$ steps $\in \{N, S, SW, NE\}$ in the quadrant starting and ending at 0 to be

$$16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = 2, 11, 85, 782, \ldots$$



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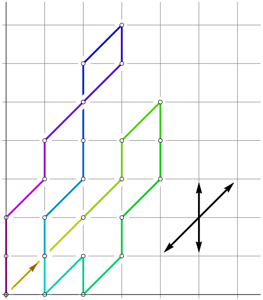

- Proving this turned out to be a notoriously difficult problem, but by now we have...
  - ...a computer-aided proof. [Kauers, Koutschan, Zeilberger, '08]
  - ...a human (complex-analytic) proof. [Bostan, Kurkova, Raschel, '13]
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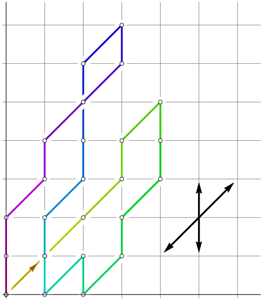

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- As we will see, counting walks by winding angle provides a natural alternative route.

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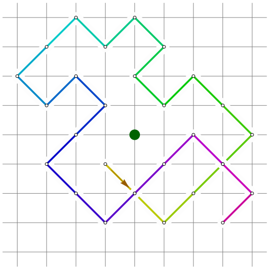
- To a walk w on $\mathbb{Z}^2$ avoiding 0 we can naturally associate a winding angle

$$\theta_w := \sum_{i=1}^{|w|} \measuredangle(w_{i-1}, 0, w_i).$$



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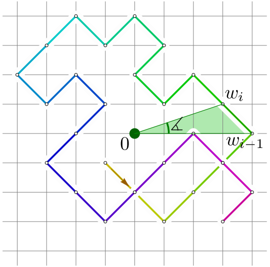

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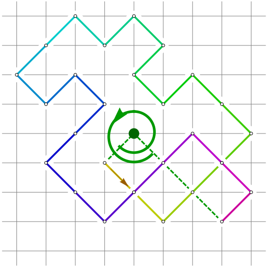

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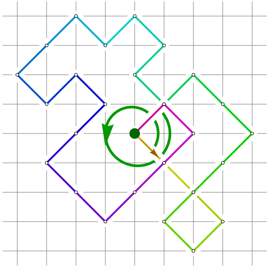

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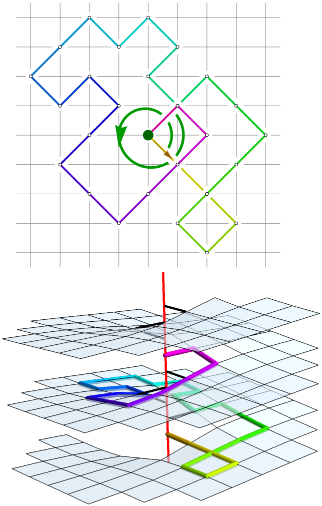

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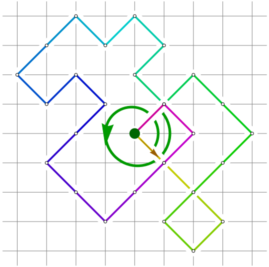
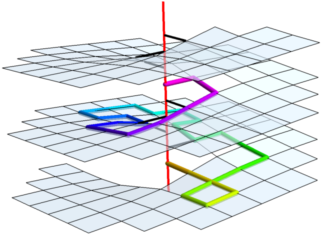
- Extends unambiguously to excursions from the origin.
- Natural interpretation as walks in the universal cover of $\mathbb{Z}^2 \setminus \{0\}$.
- Main goal today is to determine the GF for simple excursions from origin

$$F(t,b) := \sum_w t^{|w|} e^{ib\,\theta_w}$$

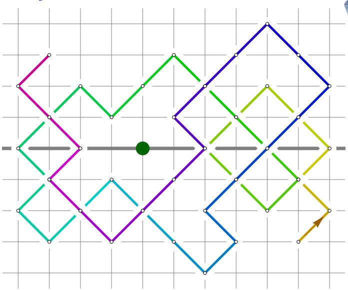
$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \dots$$





Decomposing into walks on the slit plane

- The general idea: decompose into a sequence of walks on the slit plane.



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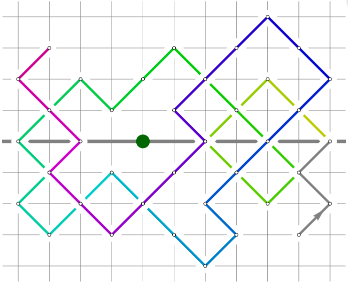

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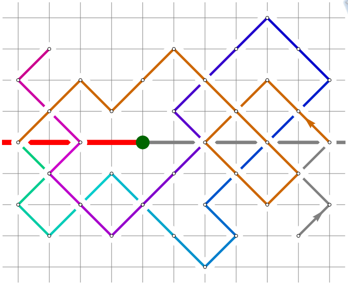

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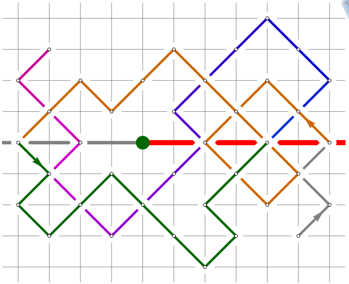

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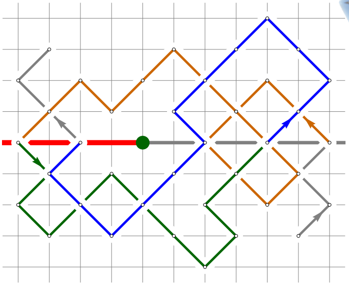

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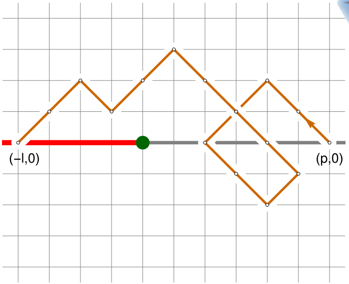
- Denote by $H^{(p,l)}(t)$ the GF for walks $(p,0) \to (-l,0)$ that hit the slit from above (counted by t^{length}).



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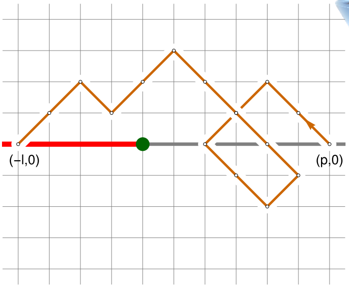

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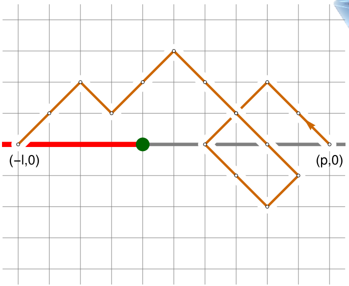

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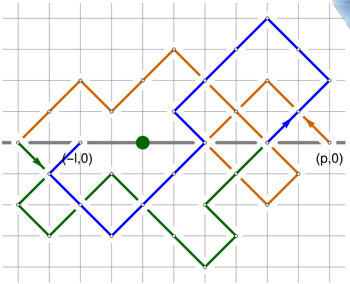

- $H^{(l,p)} = \frac{l}{p} H^{(p,l)}$, so introduce symmetric "matrix" $\mathcal{H} := \left(\sqrt{\frac{l}{p}} H^{(p,l)}\right)_{p,l \geq 1}$
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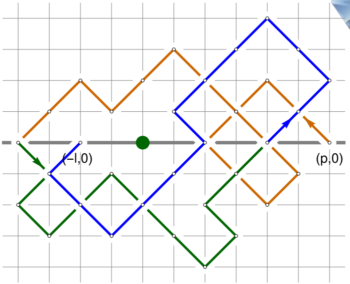

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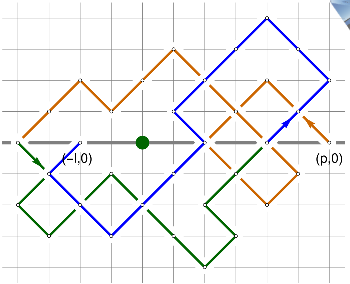

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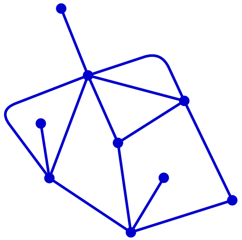
- Planar map = a multigraph properly embedded in the plane up to homeomorphism. Take it to be rooted on the outer face.



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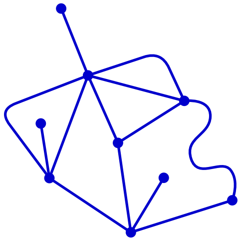

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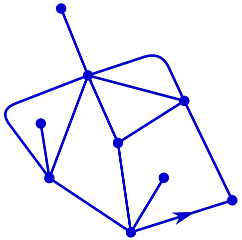

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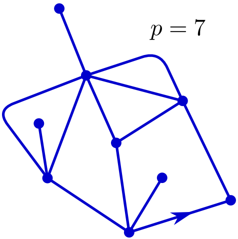

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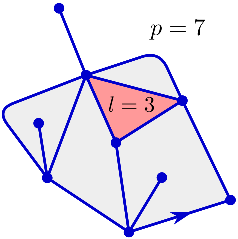

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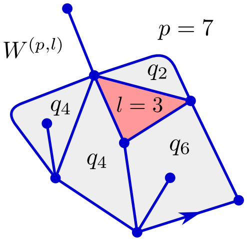

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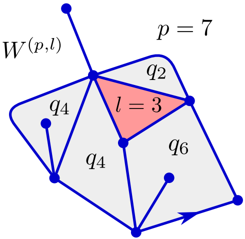
- For quasi-bipartite maps ($q_1 = q_3 = \cdots = 0$) it takes a universal form (see e.g. [Collet, Fusy, '12])

$$W^{(p,l)} = \frac{1}{l}\frac{2}{p+l}\alpha(l)\alpha(p)\left(\frac{\rho_{\mathbf{q}}}{4}\right)^{(p+l)/2} \qquad \alpha(p) := \frac{p!}{\lfloor \frac{p}{2} \rfloor ! \lfloor \frac{p-1}{2} \rfloor !}$$



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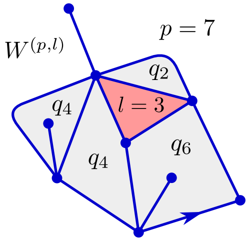
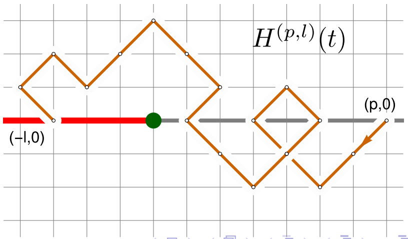
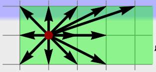
- Remarkably $H^{(p,l)}(t) = W^{(p,l)}\Big|_{\rho_{\mathbf{q}} \to \rho(t) := \frac{1-\sqrt{1-16t^2}}{8t^2}-1}$





A bijective explanation

Proposition

For any step set $\mathfrak{S} \subset$  *, there exists a bijection*

$$\Phi^{(p,l)} : \{\mathfrak{S}\text{-}\textit{walks } (p,0) \to (-l,0) \textit{ hitting slit from above}\}$$

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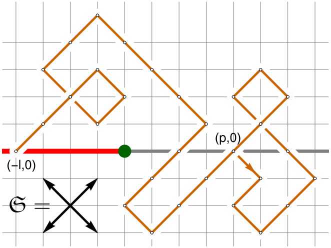

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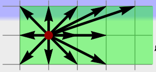
For any step set $\mathfrak{S} \subset$  *, there exists a bijection*

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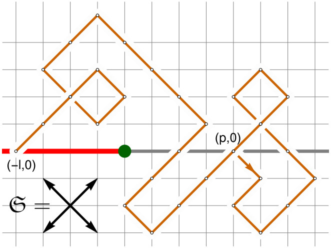

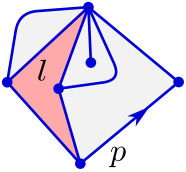

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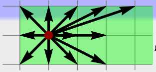
For any step set $\mathfrak{S} \subset$ *, there exists a bijection*

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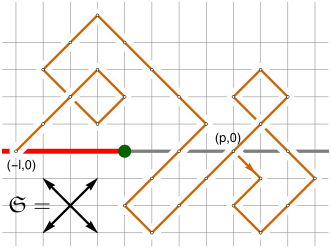
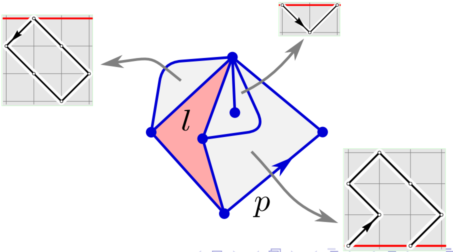
  - for each face (except root or marked) of degree k an excursion $(0,0) \to (k-2,0)$ above or below x-axis.





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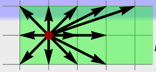
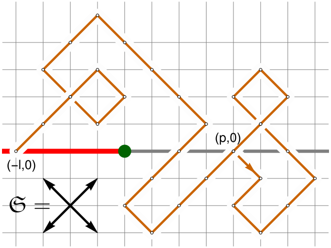
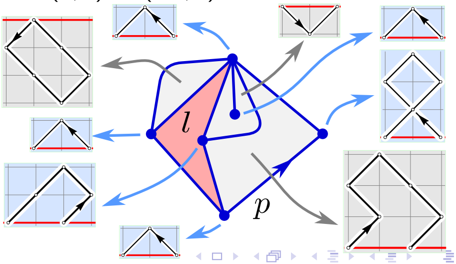
For any step set $\mathfrak{S} \subset$ , *there exists a bijection*

$$\Phi^{(p,l)} : \{\mathfrak{S}\text{-}walks\ (p,0) \to (-l,0)\ hitting\ slit\ from\ above\}$$

$$\longrightarrow \left\{ \begin{array}{l} \text{“}\mathfrak{S}\text{-walk-decorated maps” with root face degree } p \\ \text{and marked face degree } l \end{array} \right\}$$

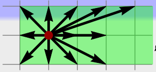
- A $\mathfrak{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree k an excursion $(0,0) \to (k-2,0)$ above or below x-axis.
  - for each vertex an excursion $(0,0) \to (-2,0)$ above x-axis





A bijective explanation

Proposition

For any step set $\mathfrak{S} \subset$  *, there exists a bijection*

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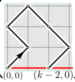
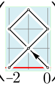
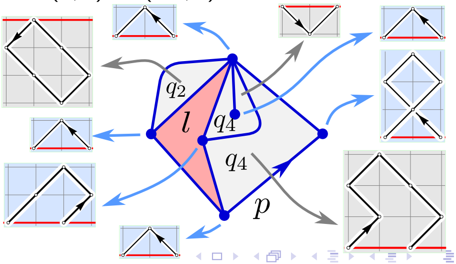
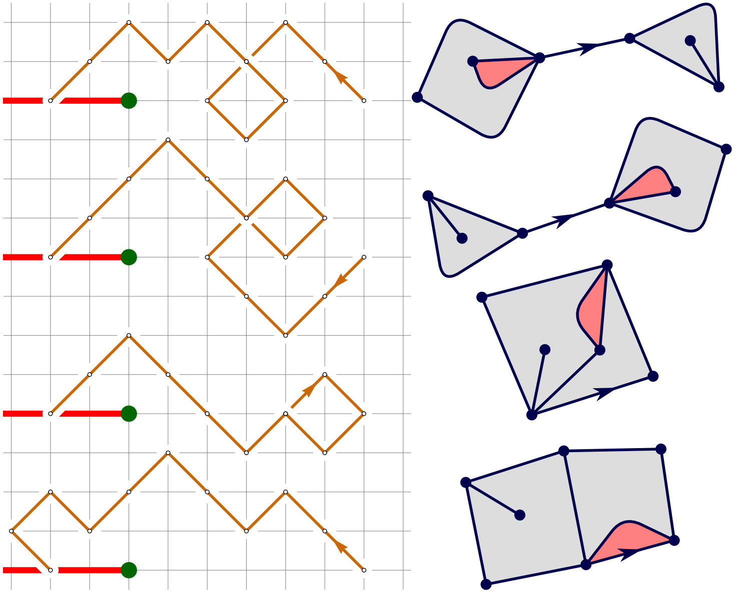
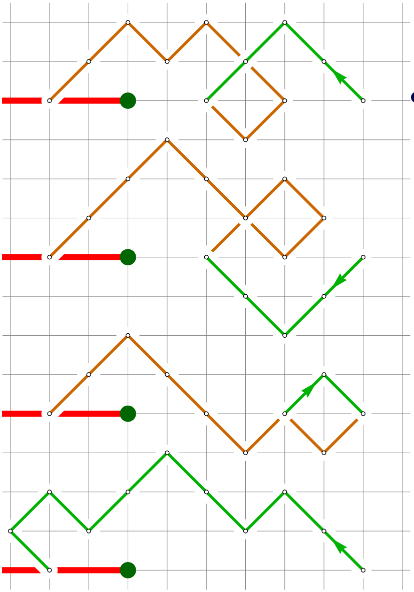
- Substituting in $W^{(p,l)}(q_i)$ the GFs

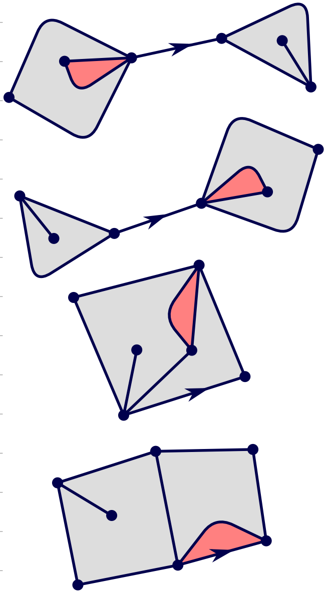
$q_k \to \left(\right.$  $\left. \right) \cdot \left(\right.$  $\left. \right)^{\frac{k-2}{2}}$

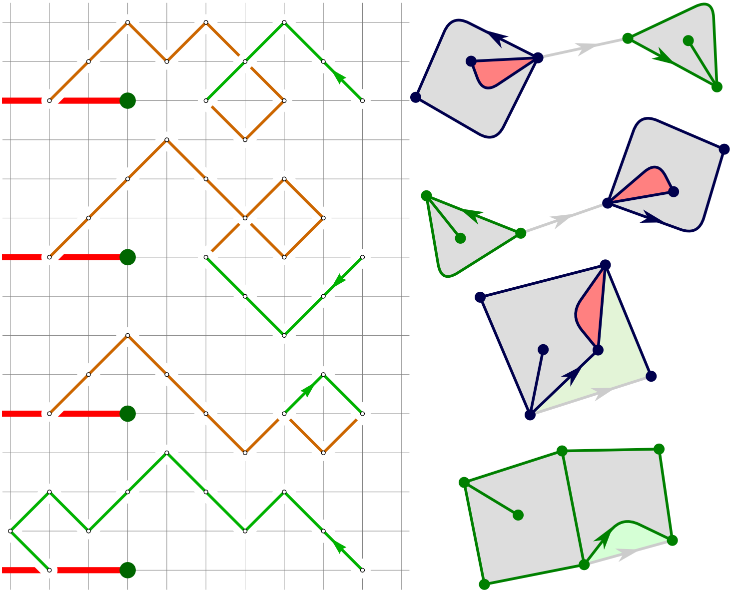
leads to $H^{(p,l)}(t)$.



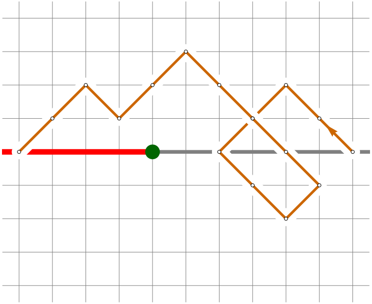


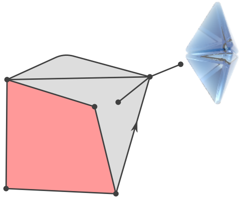






From walks to (rigid) loop-decorated maps





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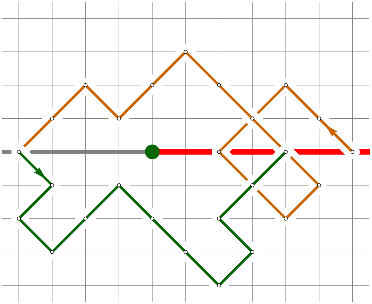

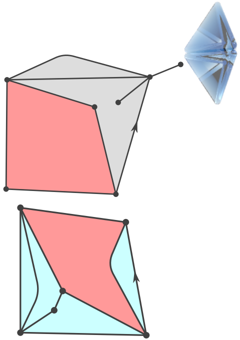

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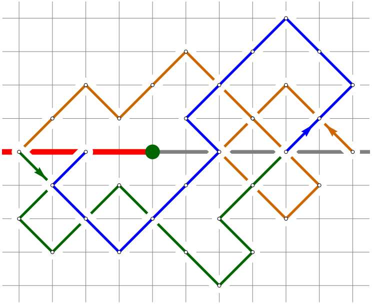

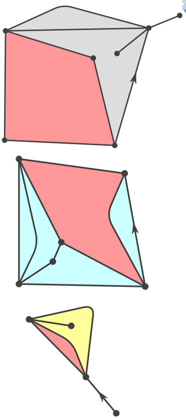

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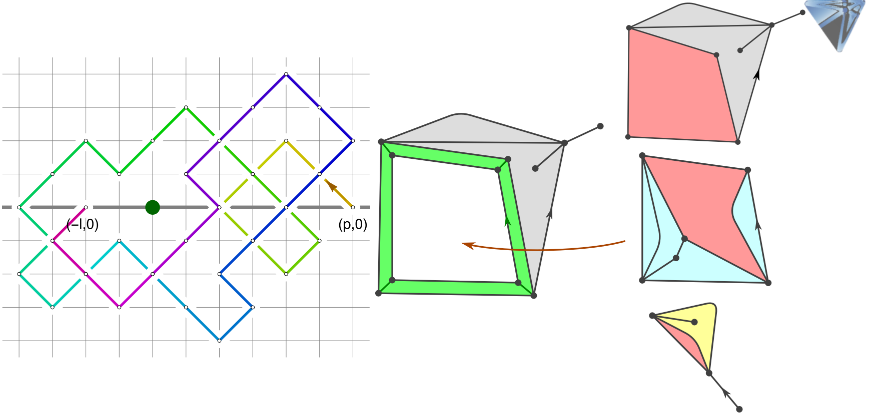

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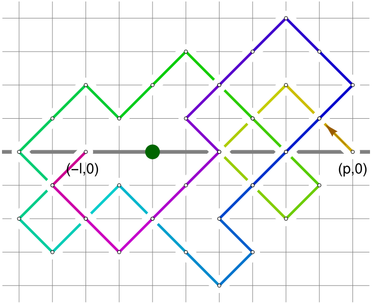

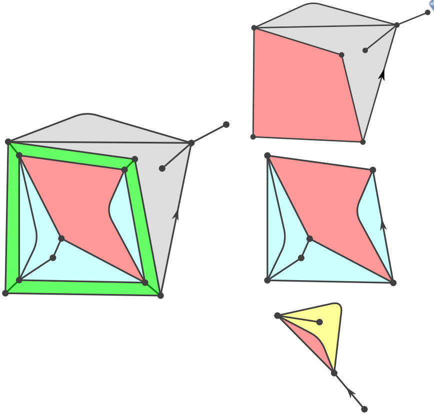

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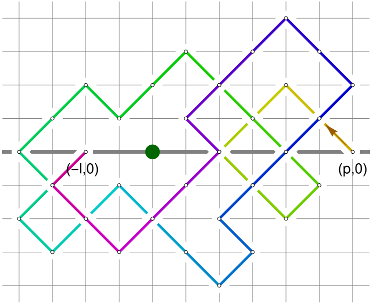

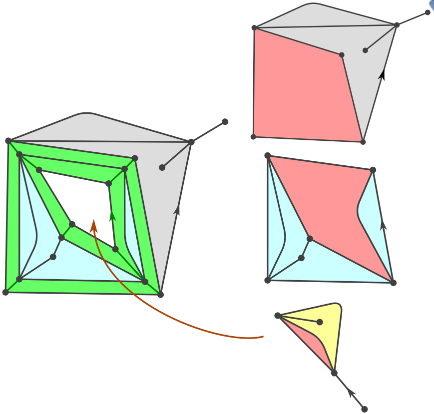

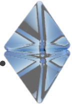

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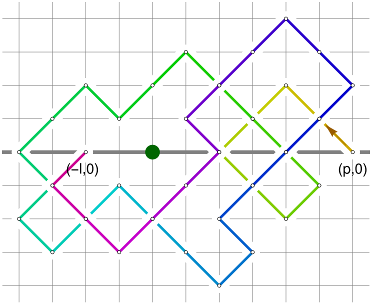

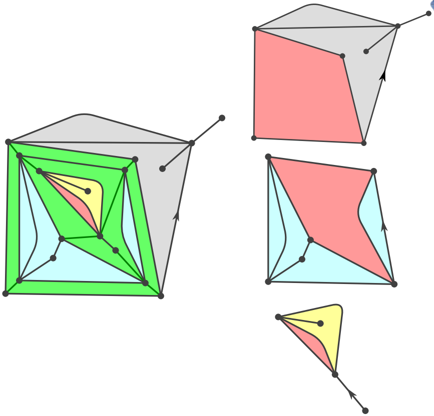

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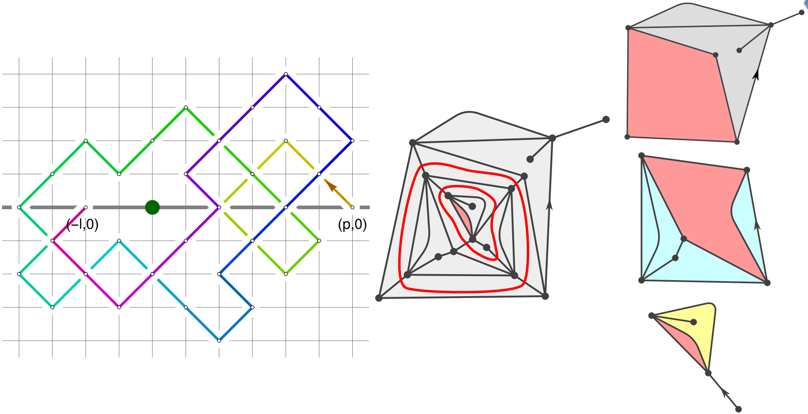

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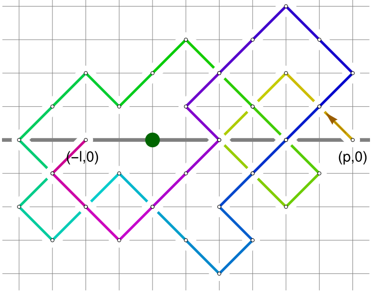

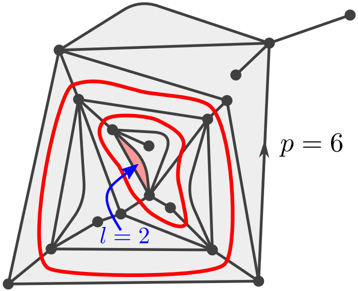

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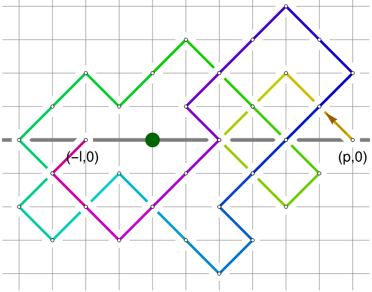

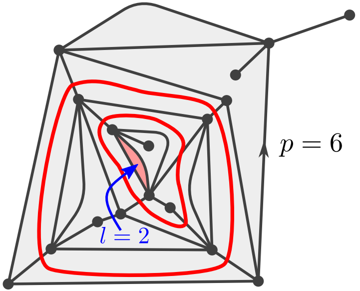

- Recall

$$\sum_{\text{such walks}} t^{|w|} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2\cos(\pi b))^N \left(\mathcal{H}^N\right)_{pl} = \sqrt{\frac{p}{l}} \left(\frac{2\cos(\pi b)\mathcal{H}}{I - 2\cos(\pi b)\mathcal{H}} \right)_{pl}$$

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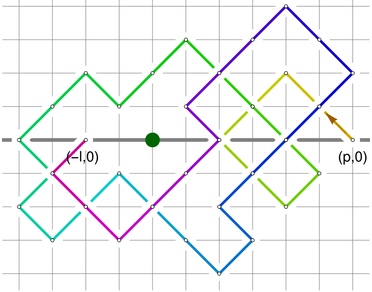

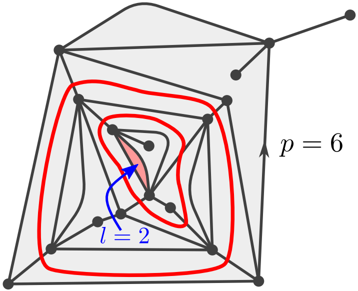

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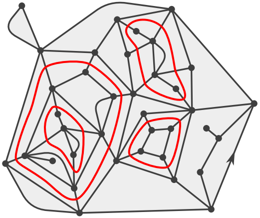
- Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees p, l carrying a weight

$$(2\cos(\pi b))^{\#\text{loops}+1} \prod_{\text{regular faces}} q_{\text{degree}}$$

Planar maps coupled to a rigid $O(n)$ loop model

- Rigid $O(n)$ model: a planar map + disjoint loops, that intersect solely quadrangles through opposite sides. Enumerated with

  $$\text{weight} \quad n^{\#\text{loops}} g^{\#\text{loop faces}} \prod_{\text{regular faces}} q_{\text{degree}}$$



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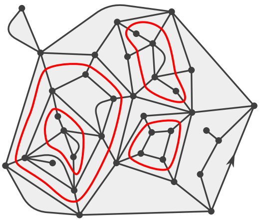

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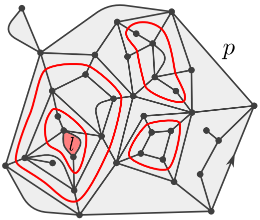

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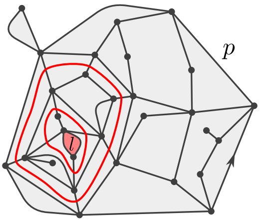

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- Recently in [Borot, Bouttier, Duplantier, '16] (for triangulations) exact statistics for the nesting of loops was obtained, i.e. distribution of # loops surrounding a marked vertex/face.
- Importantly: the form of the GF $\mathcal{G}^{(p,l)}(n, g, \mathbf{q})$ is universal and is not affected by suppressing loops that do not surround the marked face.

- We know that (with $n = 2\cos(\pi b)$ and appropriate g,$\mathbf{q}$)

$$\sqrt{\frac{p}{l}}\left(\frac{\mathcal{H}}{I - n\mathcal{H}}\right)_{pl} = \qquad \mathcal{G}^{(p,l)}(n, g, \mathbf{q})$$

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- Adapting GF from [Borot, Bouttier, Duplantier, '16] and computing a series expansion:

$$= 4 \sum_{m=1}^{\infty} \frac{1}{q^m + q^{-m} - n} \frac{\cos(2\pi m\, v(x_2)) \; x_1 \frac{\partial}{\partial x_1} \cos(2\pi m\, v(x_1))}{m(q^{-m} - q^m)}$$

where $q = q(4t) = t^2 + 8t^4 + \cdots$ is the nome of modulus $4t$ and

$$v(x) := \mathrm{cd}^{-1}(-x/\sqrt{\rho}, \rho)/(4K(\rho)), \quad \rho(t) = \frac{1 - \sqrt{1 - 16t^2}}{8t^2} - 1$$

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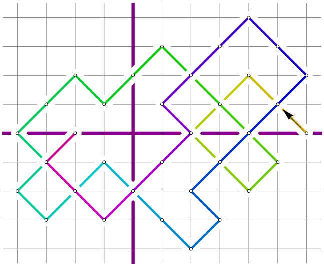
Proposition (Diagonalization of $\mathcal{H}$)

$\mathcal{H} = U^T \cdot \Lambda_q \cdot U$ in the sense of operators on $\ell^2(\mathbb{R})$ with

$$\Lambda_q = diag\left(\frac{1}{q^m + q^{-m}} \right)_{m\geq 1}, \quad U_{mp} = \sqrt{\frac{4p}{m(q^{-m} - q^m)}} [x^p] \cos(2\pi m\, v(x))$$

Refinement: increase winding angle resolution

- Up to now: decomposed walk into sequence of walks on slit plane, each numerated by $\sqrt{\frac{p}{l}}\mathcal{H}_{pl}$.



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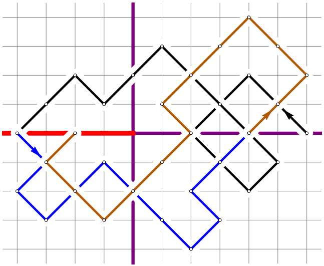

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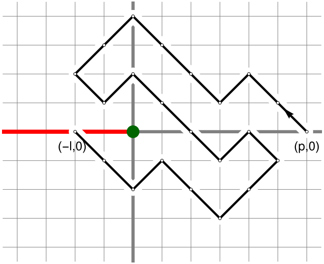

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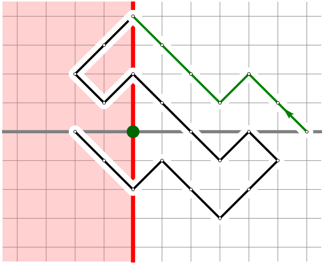

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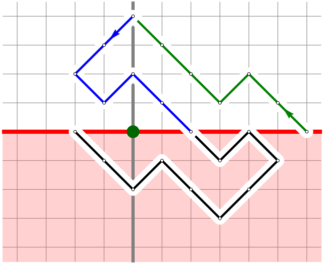

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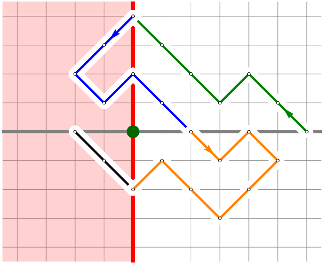

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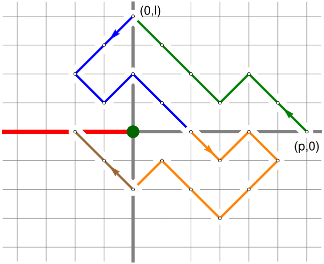
$$2\mathcal{H} = (2\mathcal{J})(\mathcal{J}+\mathcal{J}\cdot 2\mathcal{H}), \quad \mathcal{J} = \sqrt{\frac{4\mathcal{H}}{I+2\mathcal{H}}}$$



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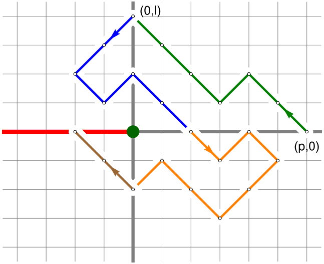

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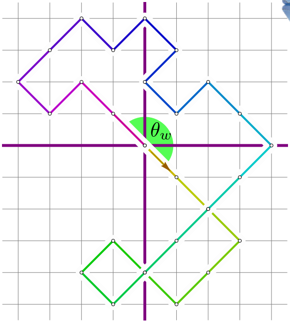
- Hence $\mathcal{J}$ has same eigenmodes as $\mathcal{H}$ but eigenvalues are $\frac{1}{q^{m/2}+q^{-m/2}}$ instead of $\frac{1}{q^m+q^{-m}}$. Such operations $q \to \sqrt{q}$ on elliptic functions are well-known as "Landen transformations".

Winding angle of excursions

- Wish to enumerate excursions from origin by length and winding angle:

$$F(t,b) := \sum_{w} t^{|w|} e^{ib\,\theta_w}$$

$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \ldots$$



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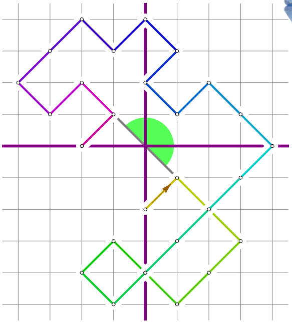

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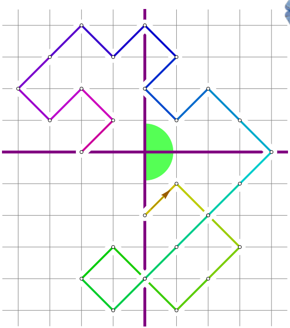

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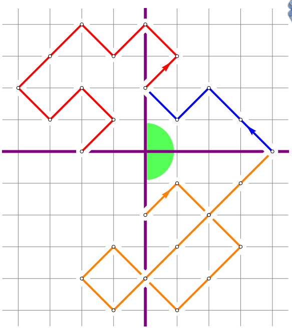

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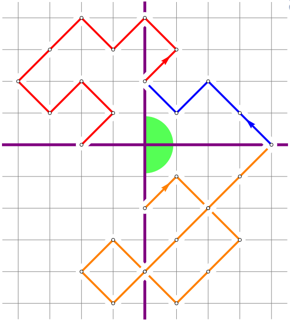

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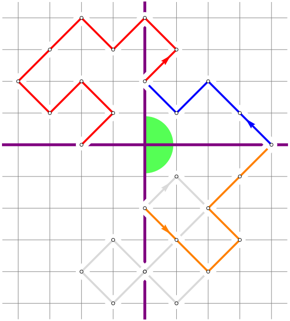

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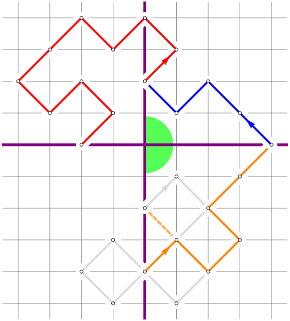

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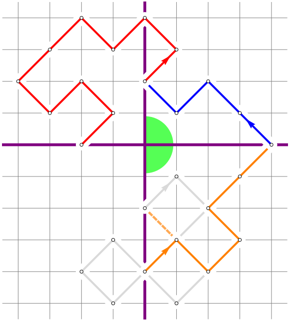

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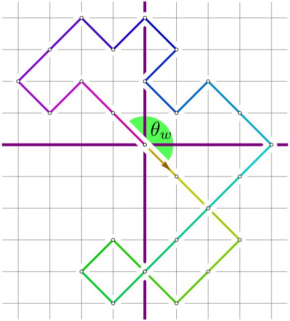

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Theorem (Excursions in the $\frac{n\pi}{4}$-cone.)

For any set of integers $-n < m-n < p < m < n$ the generating function $F_{n,m,p}(t)$ for excursions from the origin with winding angle $\frac{p\pi}{2}$ staying strictly inside angular region $(\frac{p+m-n}{4}\pi, \frac{p+m}{4}\pi)$ is given by

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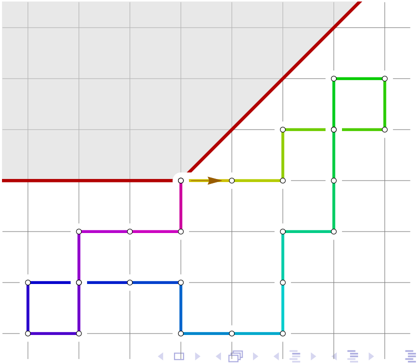

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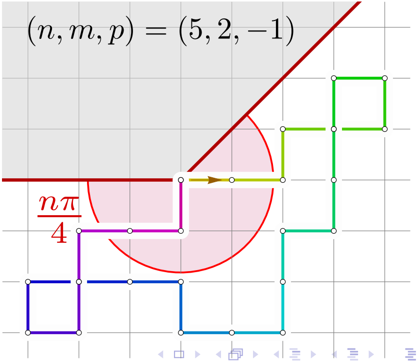

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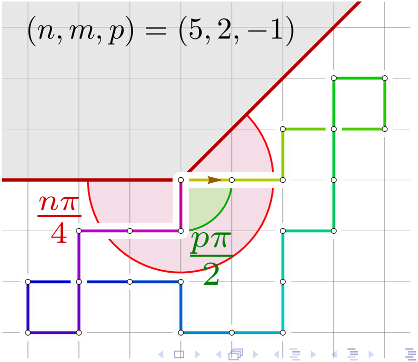

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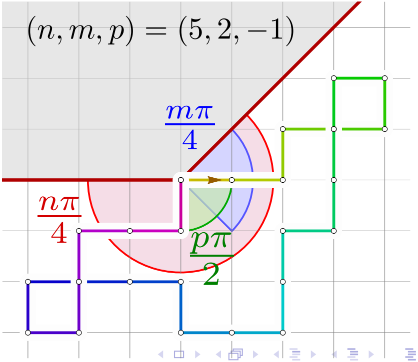

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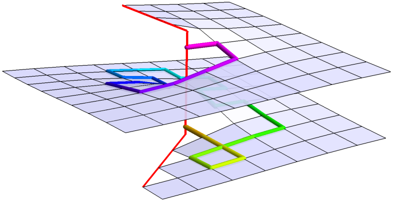

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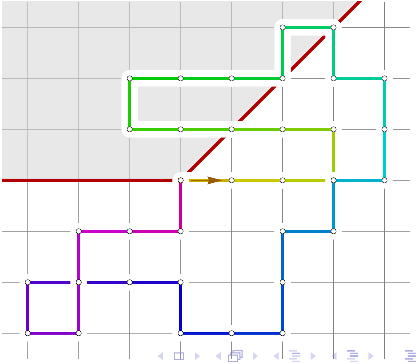

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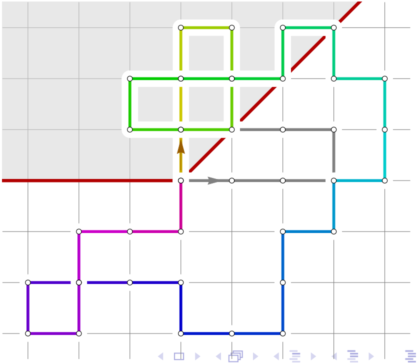

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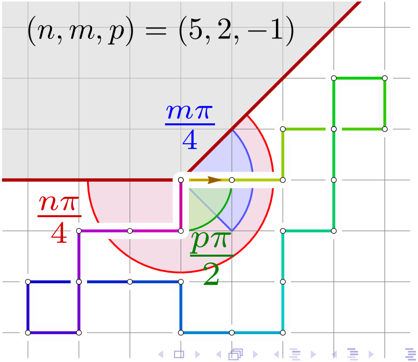

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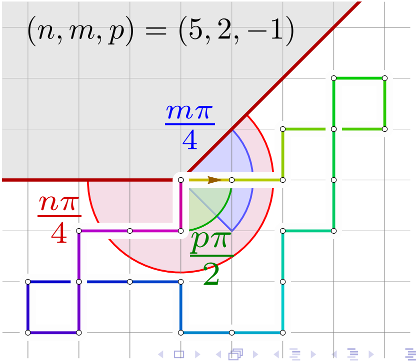

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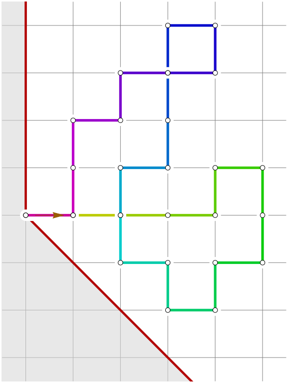
- Special case: $(n, m, p) = (3, 2, 0)$



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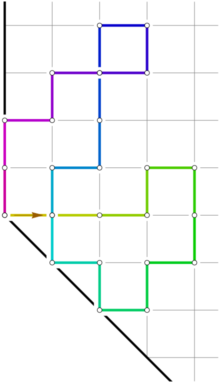

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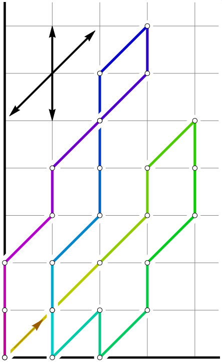

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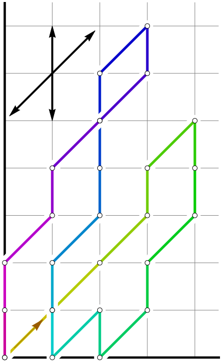

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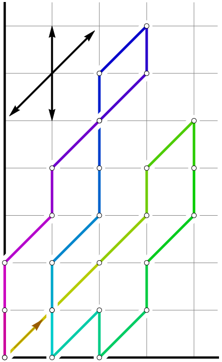

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  which is an algebraic series.
- Can reproduce the known formula

$$\sum_{n=0}^{\infty} t^{2n}\, 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = \frac{1}{2t^2}\left[{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{6}; \tfrac{2}{3}; (4t)^2\right) - 1\right].$$

  by checking that both solve same algebraic equation.

Further questions

- Generating functions for walks with full control on the endpoint?
- Other walks with small steps?
- Finally, here is an interpretation of the nome q as function of the elliptic modulus k. Why is it so simple?

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Thanks for your attention!

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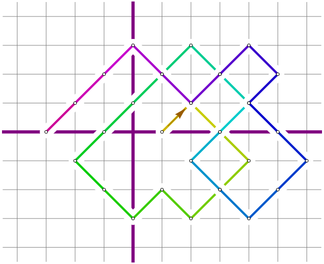
- Recall $\sqrt{\frac{p}{l}}((2\mathcal{H})^N)_{pl}$ enumerates walks $(p,0) \to (\pm l, 0)$ that alternate between half-axes N times.



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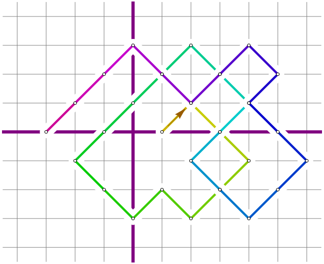

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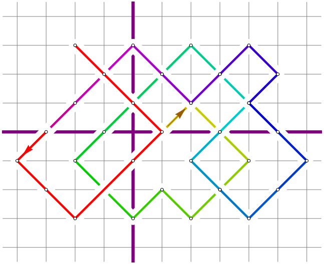

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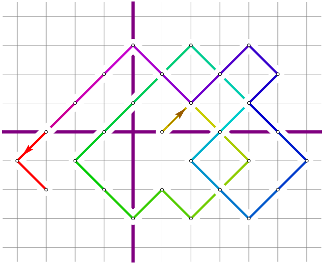

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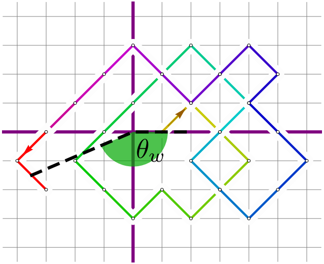

$$\sum_w t^{|w|} e^{i\pi b(\lfloor \frac{\theta_w}{\pi} \rfloor + \frac{1}{2})} = \frac{4t\cos(\pi b/2)}{1-4t}\sum_{N\geq 0}\cos^N(\pi b)\sum_{l\geq 1}\frac{1}{\sqrt{l}}((2\mathcal{H})^N - (2\mathcal{H})^{N+1})_{1,l}$$

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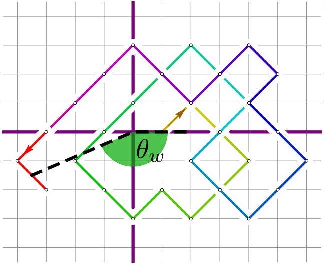
- Recall $\sqrt{\frac{1}{l}}((2\mathcal{H})^N)_{1l}$ enumerates walks $(1,0) \to (\pm l, 0)$ that alternate between half-axes N times.
- Then $\frac{1}{1-4t}\sum_{l\geq 1}\frac{1}{\sqrt{l}}((2\mathcal{H})^N - (2\mathcal{H})^{N+1})_{1,l}$ enumerates all walks alternating exactly N times.

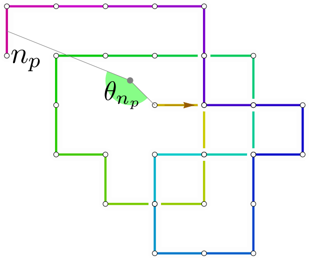


$$\sum_w t^{|w|} e^{i\pi b(\lfloor \frac{\theta_w}{\pi}\rfloor + \frac{1}{2})} = \frac{4t\cos(\pi b/2)}{1-4t}\sum_{N\geq 0}\cos^N(\pi b)\sum_{l\geq 1}\frac{1}{\sqrt{l}}((2\mathcal{H})^N - (2\mathcal{H})^{N+1})_{1,l}$$

$$= \frac{1}{1-4t}\frac{\pi}{2K(4t)}\sum_{k=-\infty}^{\infty}\frac{2e^{i\pi b(k+\frac{1}{2})}}{q^{k+\frac{1}{2}} + q^{-k-\frac{1}{2}}} = \frac{\operatorname{cn}(b\,K(4t), 4t)}{1-4t}$$

Another application: hyperbolic secant law

- Recall $\sqrt{\frac{1}{l}}((2\mathcal{H})^N)_{1l}$ enumerates walks $(1,0) \to (\pm l, 0)$ that alternate between half-axes N times.
- Then $\frac{1}{1-4t}\sum_{l\geq 1}\frac{1}{\sqrt{l}}((2\mathcal{H})^N - (2\mathcal{H})^{N+1})_{1,l}$ enumerates all walks alternating exactly N times.



$$\sum_w t^{|w|} e^{i\pi b(\lfloor \frac{\theta_w}{\pi} \rfloor + \frac{1}{2})} = \frac{4t\cos(\pi b/2)}{1-4t}\sum_{N\geq 0}\cos^N(\pi b)\sum_{l\geq 1}\frac{1}{\sqrt{l}}((2\mathcal{H})^N - (2\mathcal{H})^{N+1})_{1,l}$$

$$= \frac{1}{1-4t}\frac{\pi}{2K(4t)}\sum_{k=-\infty}^{\infty}\frac{2e^{i\pi b(k+\frac{1}{2})}}{q^{k+\frac{1}{2}} + q^{-k-\frac{1}{2}}} = \frac{\mathrm{cn}(b\,K(4t), 4t)}{1-4t}$$

Theorem (Winding angle of SRW on $\mathbb{Z}^2$ around $(-\frac{1}{2}, \frac{1}{2})$)

If $n_p \geq 1$ is a geometric RV with parameter $0 < p < 1$ then

$$\mathbb{P}\left[k\pi < \theta_{n_p} < (k+1)\pi\right] = \frac{\mathrm{sech}(\pi(k+\frac{1}{2})T)}{\sum_{k\in\mathbb{Z}}\mathrm{sech}(\pi(k+\frac{1}{2})T)}, \quad T = \frac{K(\sqrt{1-p^2})}{K(p)}$$