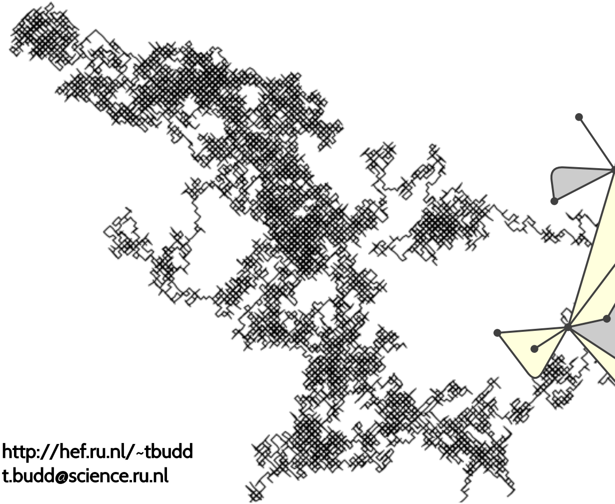
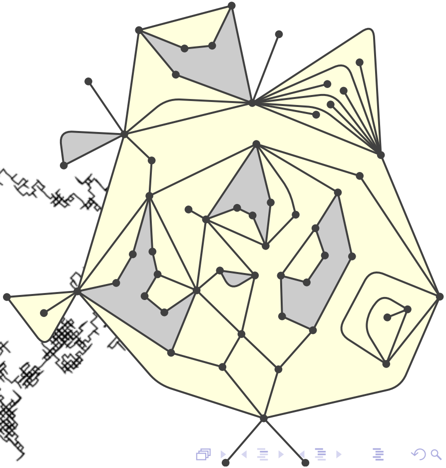
Random Geometry follow up, Isaac Newton Institute, Cambridge, 17 July 2018

Lattice Walks & Peeling of Planar Maps

Timothy Budd





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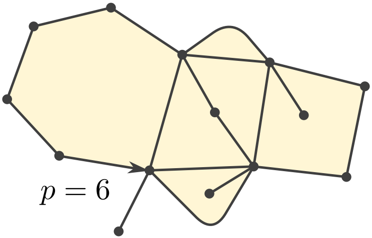
Plan

- Review (Miermont's Lecture)
  - Boltzmann planar maps
  - Peeling exploration
- Relation between random walks on $\mathbb{Z}^2$ and Boltzmann planar maps
- Rigid $O(n)$ loop model on planar maps
  - Peeling exploration
  - Nesting of loops vs. winding of random walks
  - Coding the $O(2)$ model via lattice walks

Reminder: Boltzmann planar maps

- $\mathcal{M}_p = \{$rooted, bipartite planar maps of perimeter $2p\}$.



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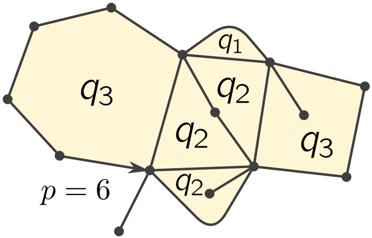

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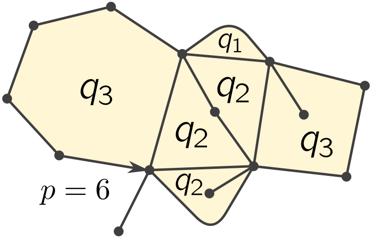

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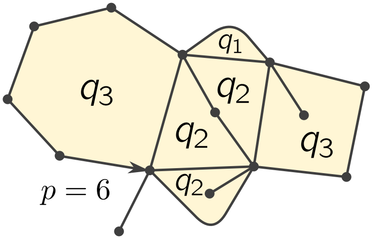
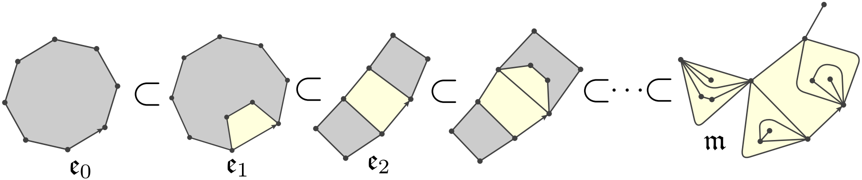
- If $\mathbf{q}$ is admissible then $w_{\mathbf{q}}(\,\cdot\,|\mathcal{M}_p)$ defines the **q**-Boltzmann planar map $\mathfrak{m}^{(p)}$ of perimeter $2p$.



Reminder: peeling exploration [Watabiki, Angel, Curien, Le Gall, TB, ...]

- Describe an exploration of $\mathfrak{m}$ by a sequence $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{m}$ of submaps containing holes (the unexplored regions).



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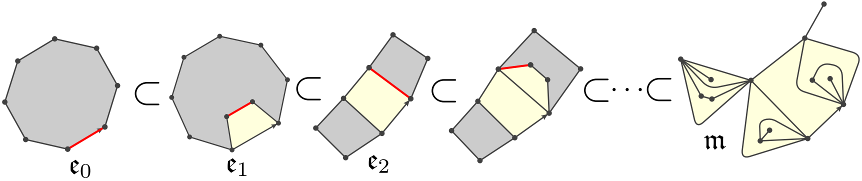
- Fix a peeling algorithm to decide across which edge to explore next.



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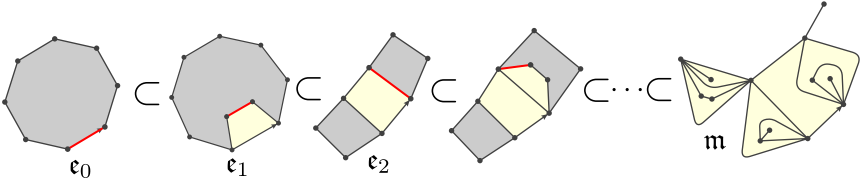
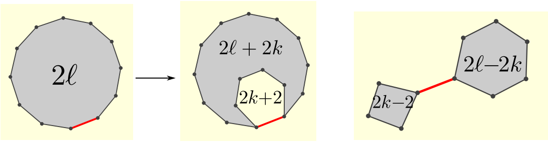


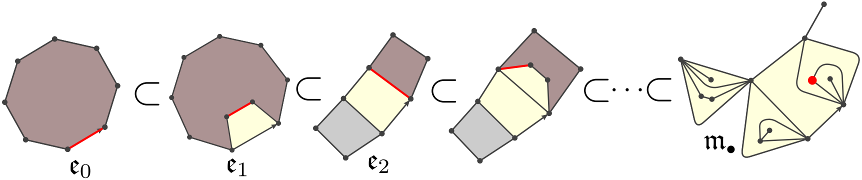
- For a $\mathbf{q}$-Boltzmann planar map $\mathfrak{m} = \mathfrak{m}^{(p)}$, $(\mathfrak{e}_i)$ is a Markov process with transition probabilities



Transition probability: $\dfrac{q_{k+1} W^{(l+k)}}{W^{(l)}}$ $\dfrac{W^{(k-1)} W^{(l-k)}}{W^{(l)}}$

Reminder: targeted peeling exploration

- If the map $\mathfrak{m}_\bullet$ has a marked vertex, one may track the hole containing the vertex.



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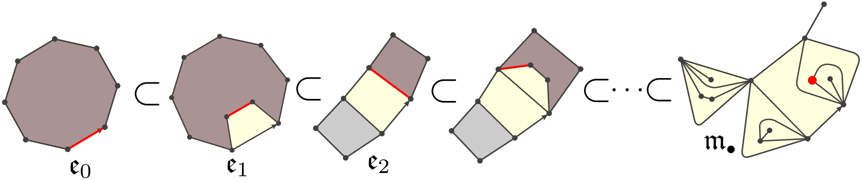
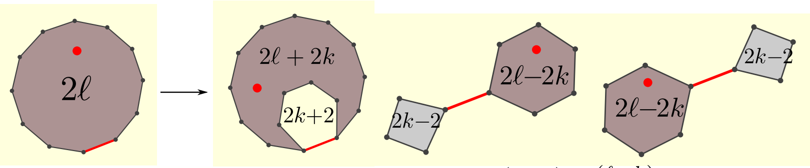
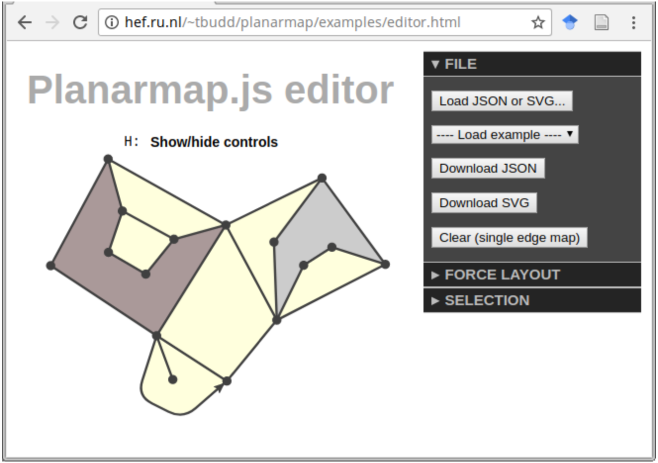


- For a pointed $\mathbf{q}$-Boltzmann planar map $\mathfrak{m}_\bullet^{(p)}$



Transition probability: $\dfrac{q_{k+1} W_\bullet^{(\ell+k)}}{W_\bullet^{(\ell)}}$ $\qquad 2\dfrac{W^{(k-1)} W_\bullet^{(\ell-k)}}{W_\bullet^{(\ell)}}$

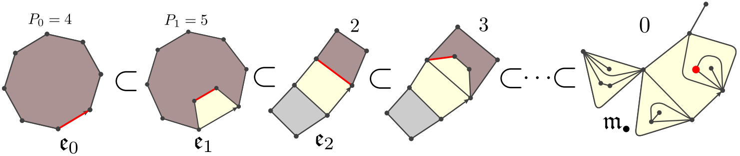
Planar map editor: try for yourself!

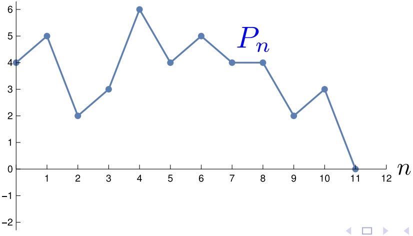


http://hef.ru.nl/~tbudd/planarmap/examples/editor.html

Reminder: perimeter process

- The perimeter process (P_n) tracks the half-perimeter of the hole containing the marked vertex.





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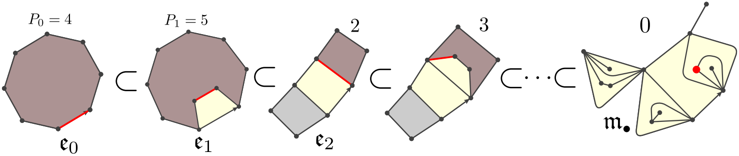
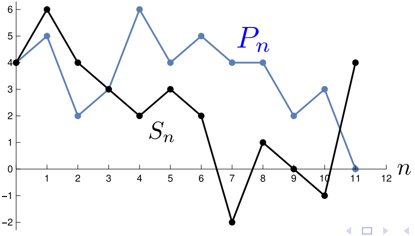
- The perimeter process (P_n) tracks the half-perimeter of the hole containing the marked vertex.



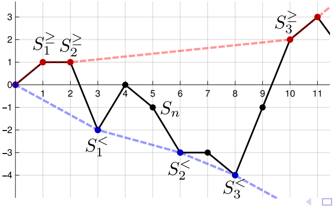
- If $\mathbf{q}$ admissible, (P_n) has the law of a random walk (S_n) with distribution $\nu_\mathbf{q}$ conditioned to hit $\mathbb{Z}_{\leq 0}$ at 0:

$$p(\ell, \ell+k) = \frac{h^{\downarrow}(\ell+k)}{h^{\downarrow}(\ell)} \nu_\mathbf{q}(k), \quad h^{\downarrow}(\ell) = 4^{-\ell} \binom{2\ell}{\ell}, \quad \nu_\mathbf{q}(k) = \begin{cases} q_{k+1} \, (4R_\mathbf{q})^k \\ 2W^{(-k-1)} \, (4R_\mathbf{q})^k \end{cases}$$



Wiener-Hopf factorization

- Denote by $(S_i^<)$ the strict descending ladder process of (S_n) and by $(S_i^{\geq})$ the weak ascending ladder process.



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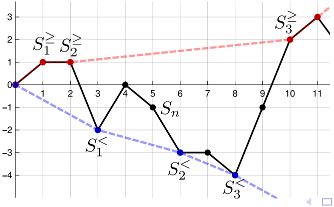
$$1 - \mathbb{E}e^{i\theta S_1} = (1 - \mathbb{E}e^{i\theta S_1^{<}})(1 - \mathbb{E}e^{i\theta S_1^{\geq}})$$



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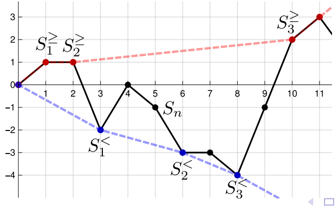

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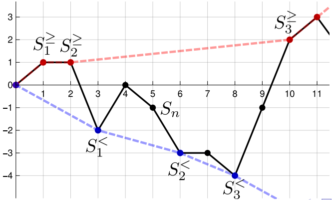
Theorem (TB, '15)

The map $\mathbf{q} \to \nu_\mathbf{q}$ is a bijection between admissible $\mathbf{q}$ and probability distributions on $\mathbb{Z}$ for which $(S_i^<) \overset{(\mathrm{d})}{=} (T_i)$.



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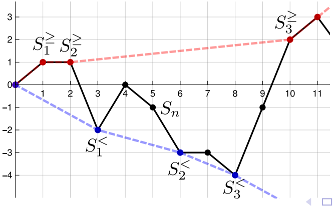
Moreover, $\mathbf{q}$ is critical $\iff$ (S_n) oscillates $\iff$ $(S_i^\geq)$ non-defective.



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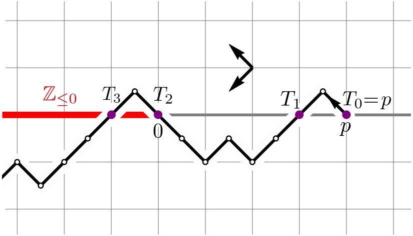

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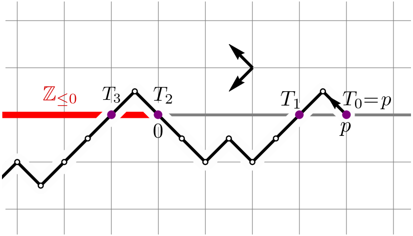

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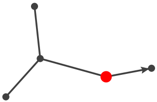

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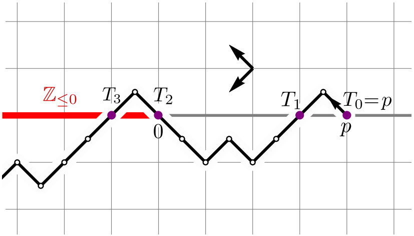
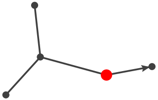
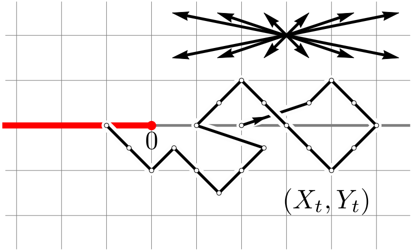




- If $\mathbf{q} = 0$, then $(S_i) = (S_i^<) \overset{(d)}{=} (T_i)$.
- One can get random walks (S_i) for certain $\mathbf{q} \neq 0$ by looking at axis intersections of more general lattice walks on $\frac{1}{2}\mathbb{Z}^2$.

- Consider a 2d random walk (X_t, Y_t) s.t. X_t has i.i.d. increments in $\mathbb{Z} + \frac{1}{2}$ and Y_t is an independent simple RW on $\frac{1}{2}\mathbb{Z}$.



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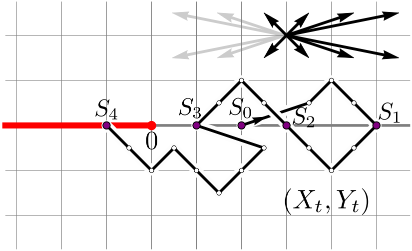

Proposition

The law of the sequence of axis intersections of (X_t, Y_t) is equal to that of (S_i) for some admissible $\mathbf{q}$ iff $X_{t+1} - X_t \geq -\frac{1}{2}$ and $(X_t) \not\to \infty$.

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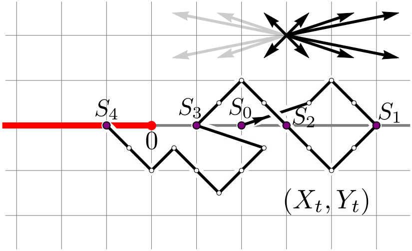

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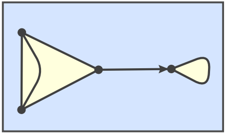
- Proof sketch: Inspired by [Bousquet-Mélou, Schaeffer, '02]
  - Axis intersections of (X_t, Y_t) are equal in law to $(X_{2T_i})_i$.
  - "Subordination by (T_i) commutes with Wiener-Hopf factorization".

$$1 - \mathbb{E}e^{i\theta X_{2T_1}} = \sqrt{1 - \mathbb{E}e^{i\theta X_2}} = \sqrt{(1 - \mathbb{E}e^{i\theta X_2^<})(1 - \mathbb{E}e^{i\theta X_2^\geq})} = \sqrt{1 - \mathbb{E}e^{i\theta X_2^<}}\sqrt{1 - \mathbb{E}e^{i\theta X_2^\geq}}$$

  - Thus statement holds iff (X_t) has descending ladder process $X_{2t}^< = t$.

Combinatorial explanation? Compare fragmentations

- Consider the fragmentation induced by the peeling process of a planar map (in the more general non-bipartite setting).



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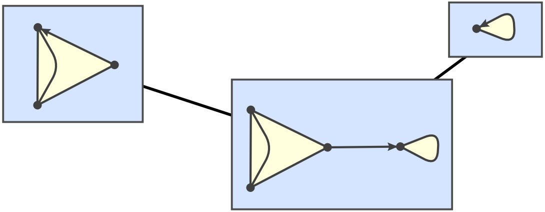

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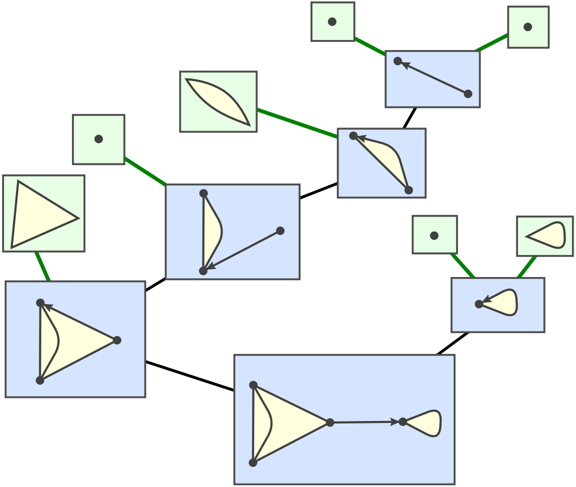

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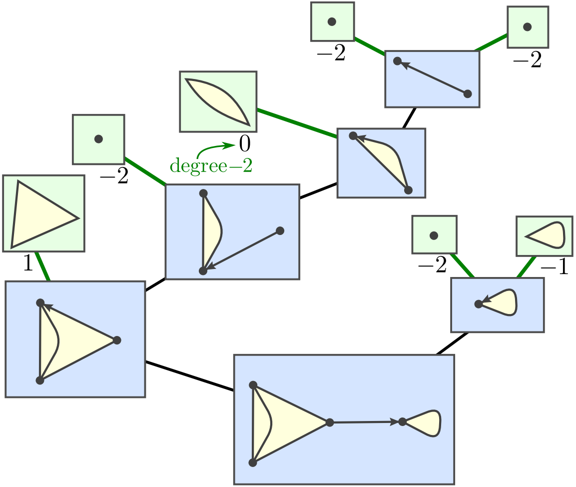

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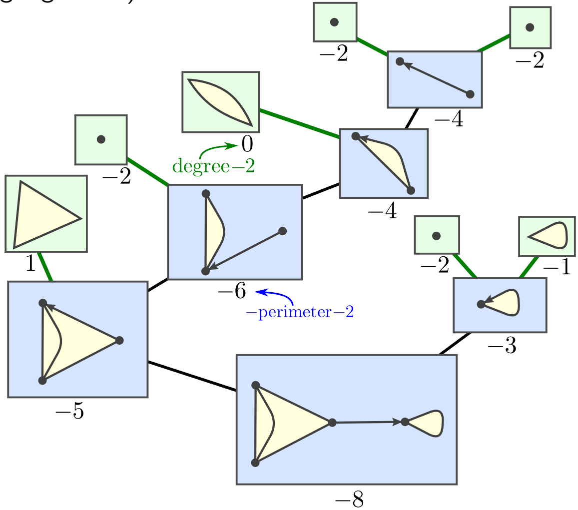
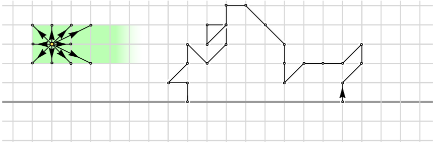
- Consider the fragmentation induced by the peeling process of a planar map (in the more general non-bipartite setting).
- The labeled tree unique characterizes the planar map (for fixed peeling algorithm).



Combinatorial explanation? Compare fragmentations

- Consider the fragmentation of an excursion in the upper-half plane.



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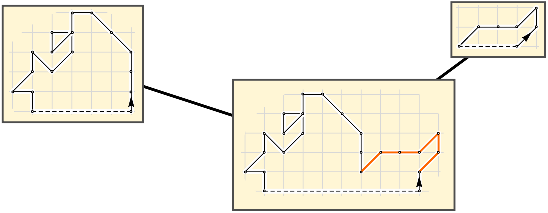

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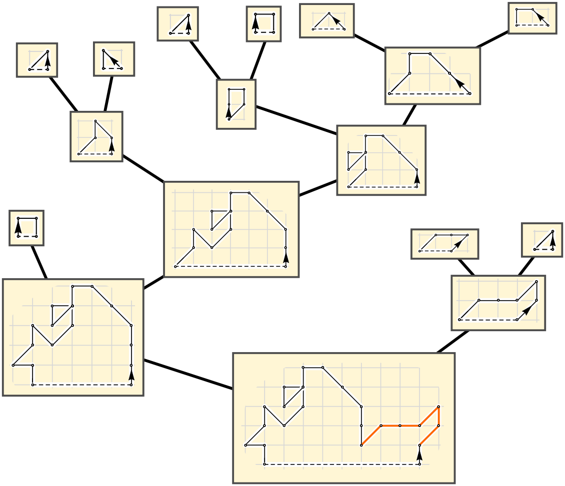

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- Consider the fragmentation of an excursion in the upper-half plane.
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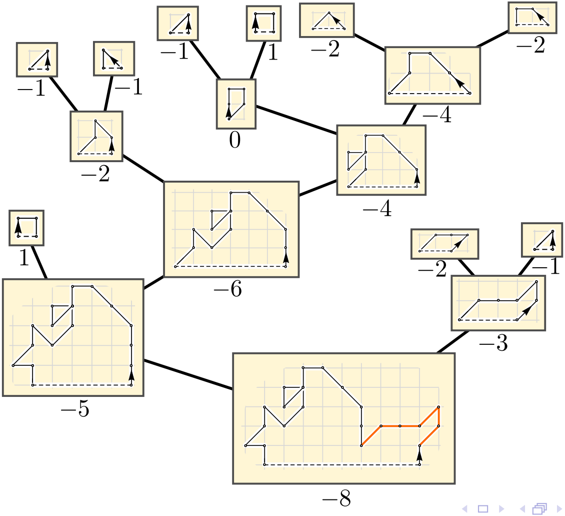

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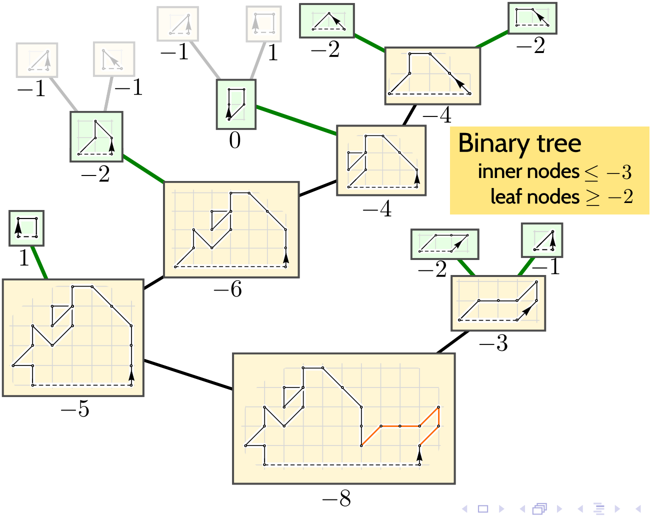
- Label de fragments by their extent.
- Determine the maximal subtree with labels ≤ -3 on inner nodes.



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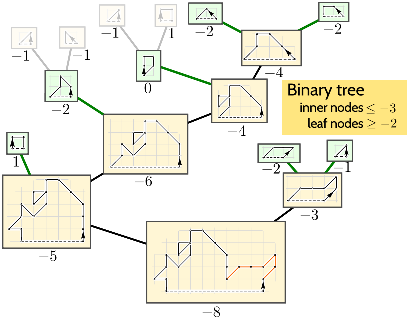
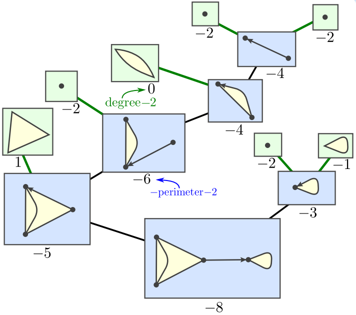
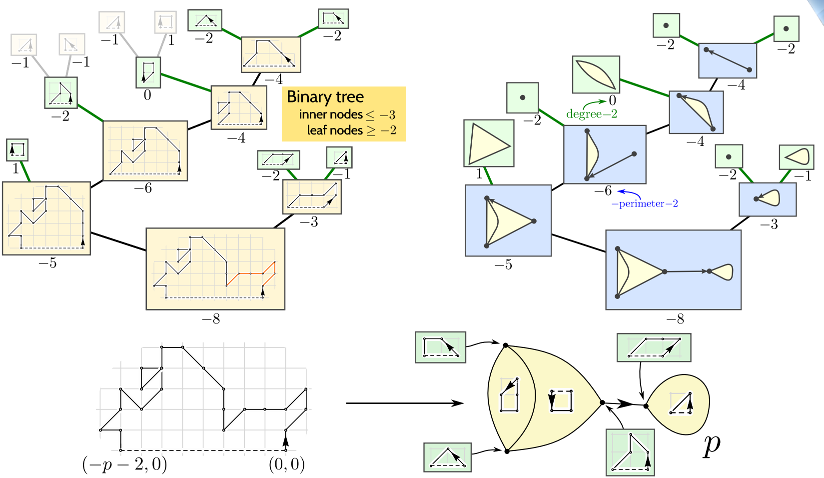
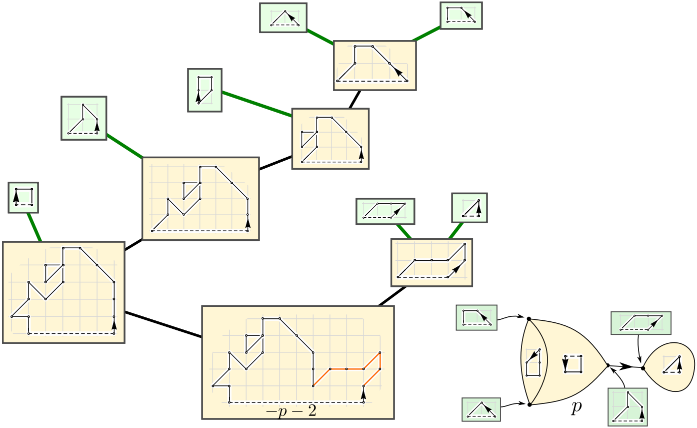


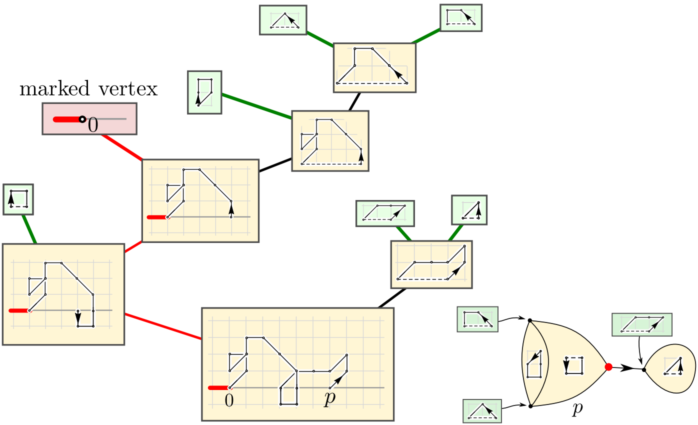


Combinatorial explanation? Compare fragmentations

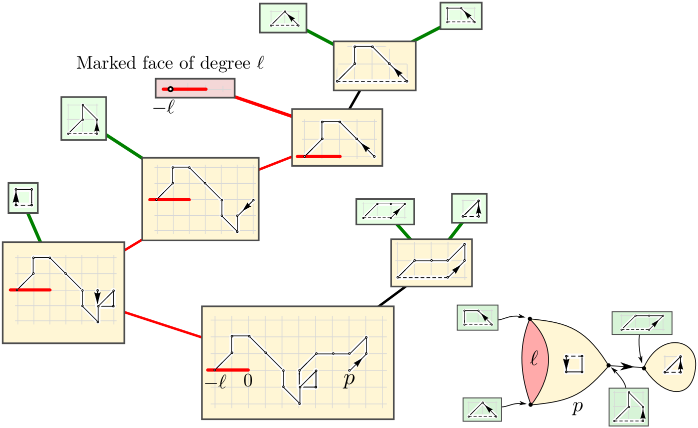


- Matching the trees determines a bijection between ↑-excursions of extent $-p-2$ and maps of perimeter p decorated with:
  - an ↑-excursion of extent -2 for each vertex;
  - an ↕-excursion of extent $k-2$ for each face of degree k.


$-p-2$
p



- The bijection extends to walks on the slit plane and decorated planar maps with a marked vertex or marked face.



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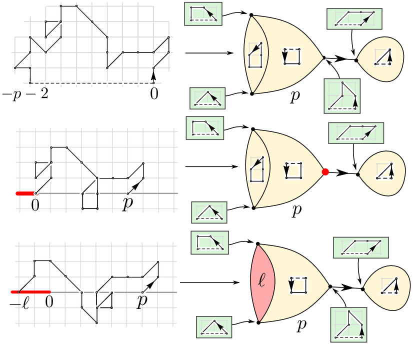
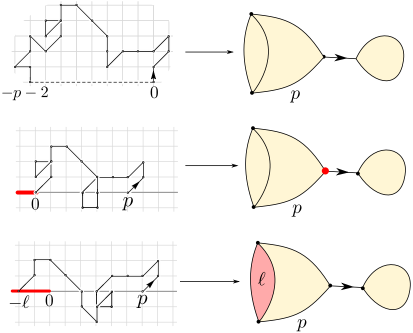


- Taking the image of a random walk (X_t, Y_t) and forgetting the decoration yields a $\mathbf{q}$-Boltzmann planar map, with $\nu_{\mathbf{q}}$ the axis-return distribution of the walk.



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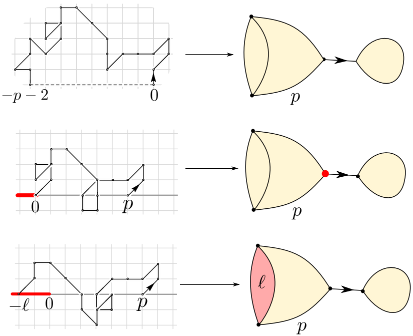

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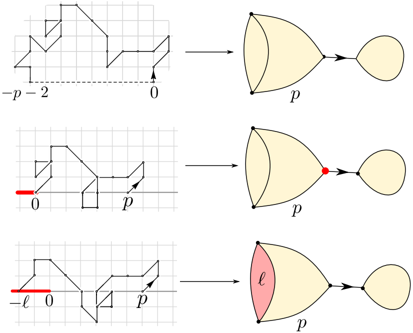

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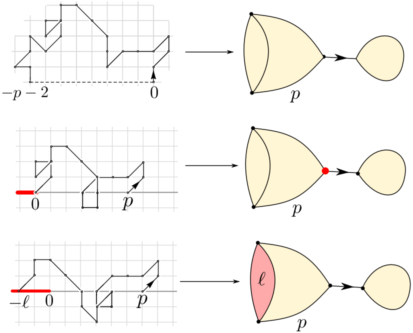

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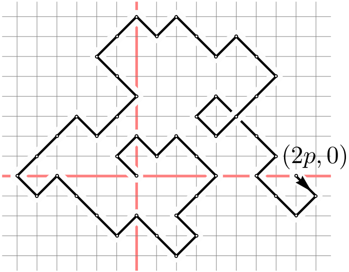
- $\mathbf{q}$ is critical iff (X_t) has no drift.
- If (X_t) in dom. of attr. of an α-stable process for $\alpha \in (1, 2]$, then (S_t) is in dom. of attr. of an $\frac{\alpha}{2}$-stable process with Lévy measure

$$\frac{\cos a\pi}{x^a}\mathbf{1}_{x>0}\mathrm{d}x + \frac{1}{|x|^a}\mathbf{1}_{x<0}\mathrm{d}x, \qquad a = 1 + \tfrac{\alpha}{2} \in (\tfrac{3}{2}, 2].$$

A glimpse of loops



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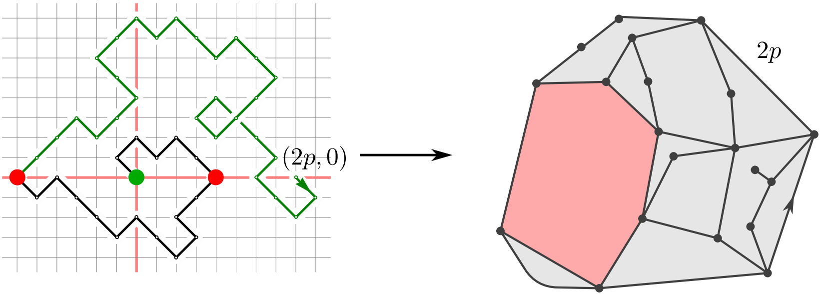

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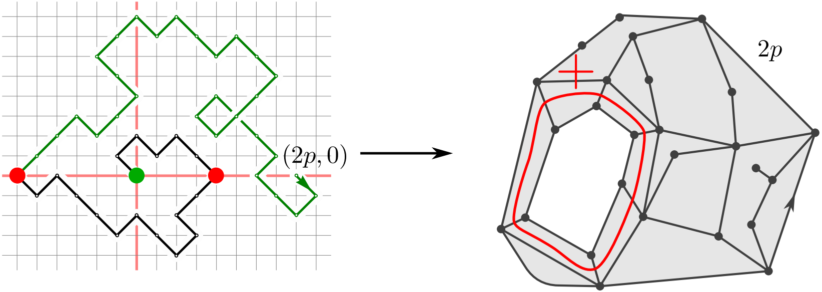

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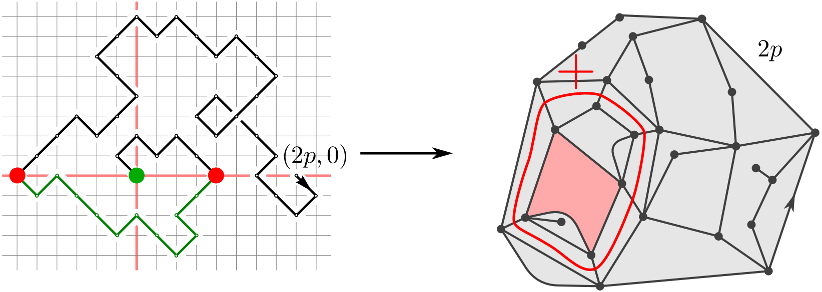

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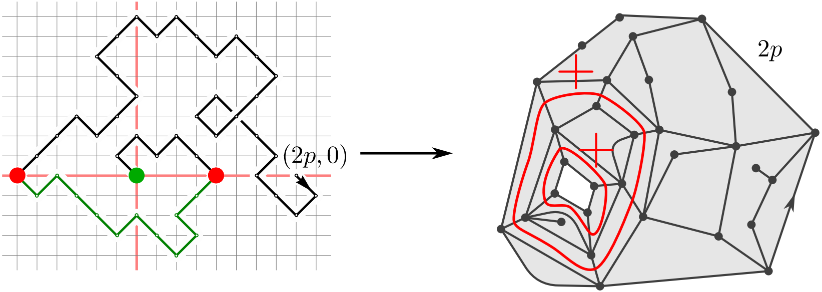

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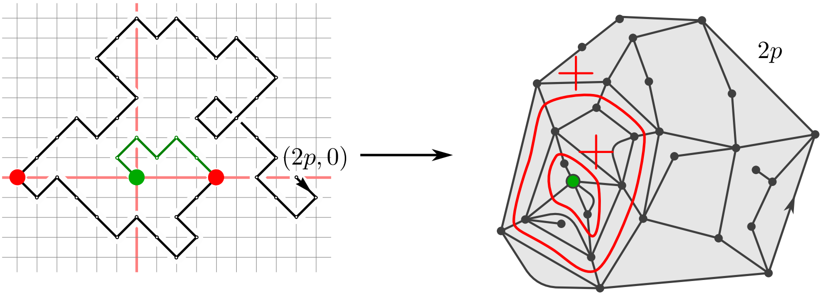
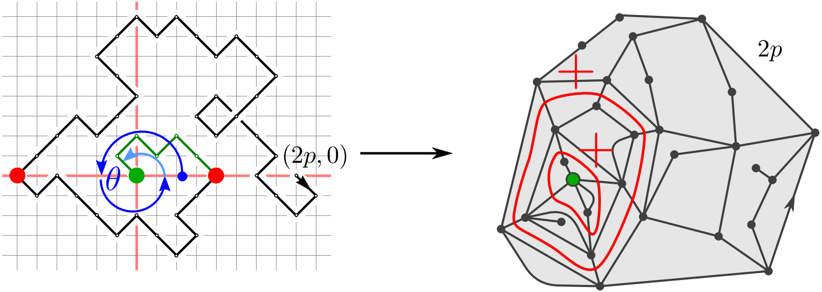


- A simple diagonal random walk $(p,0) \to (0,0)$ is mapped to a $\mathbf{q}$-Boltzmann planar map with signed, nested loops with distribution

$$\propto g^{\#\square} \prod_{\text{reg. faces } f} q_{\frac{\deg(f)}{2}}$$

for some g and $\mathbf{q}$ as before.

A glimpse of loops



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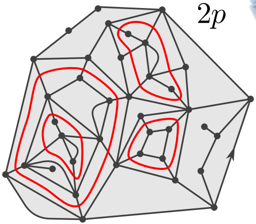
  for some g and **q** as before.
- The winding angle θ of the walk (ignoring the last bit) is $\sum_{\text{loops}} \pm\pi$.

Reminder: $O(n)$ loop model

[Stanley, Domany, Mukamel, Nienhuis, Kostov, Eynard, Zinn-Justin, Kristjansen . . . , 70's–90's]

- Let $\hat{\mathcal{M}}_p$ be the set of loop-decorated maps of boundary $2p$.



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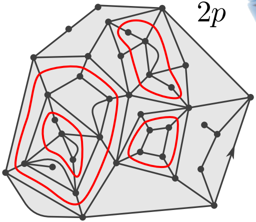

- Let $\hat{\mathcal{M}}_p$ be the set of loop-decorated maps of boundary $2p$.
- The rigid $O(n)$ loop model corresponds to the measure $w_{n,g,\mathbf{q}}(\,\cdot\,|\hat{\mathcal{M}}_p)$, where

  [Borot, Bouttier, Guitter, '11]

$w_{n,g,\mathbf{q}}(\mathfrak{m}) = n^{\#}$  $g^{\#}$  $\displaystyle\prod_{\text{reg. faces } f} q_{\frac{\deg(f)}{2}}$

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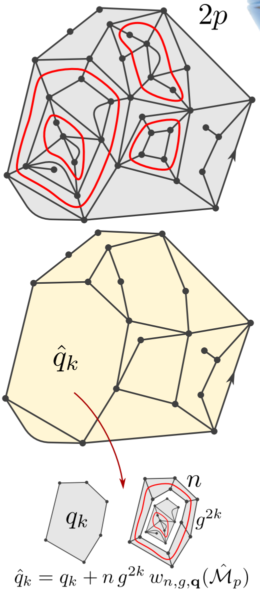

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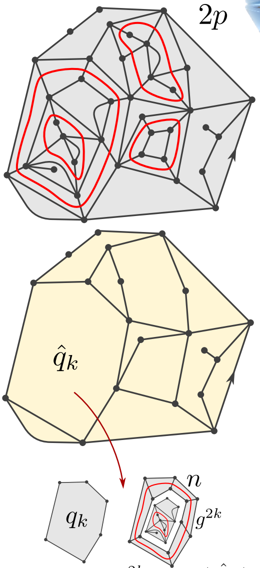

$$\hat{q}_k = q_k + n\, g^{2k}\, w_{n,g,\mathbf{q}}(\hat{\mathcal{M}}_p)$$

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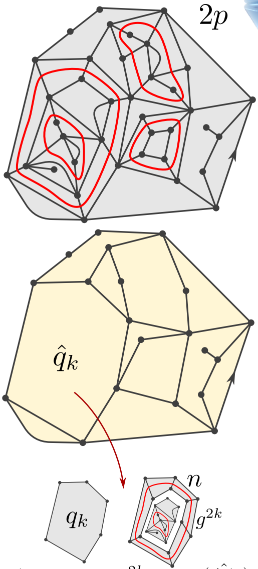
- For $n \in (0, 2]$ the non-generic scaling limits are conjecturally related to $\mathrm{LQG}_\gamma + \mathrm{CLE}_\kappa$, $n = -2\cos(4\pi/\kappa)$
  - Dense phase: $\kappa \in [4, 8)$, $\gamma = \sqrt{16/\kappa}$
  - Dilute phase: $\kappa \in (2, 4]$, $\gamma = \sqrt{\kappa}$



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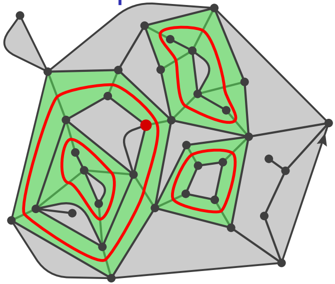
Targeted peeling exploration with loops



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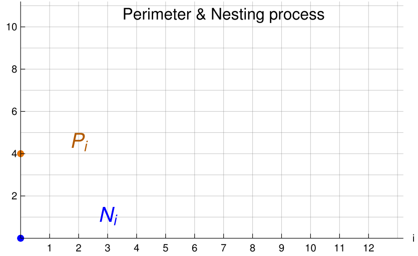

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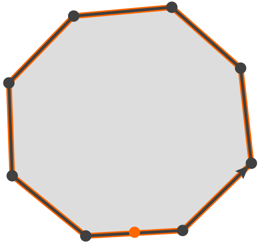

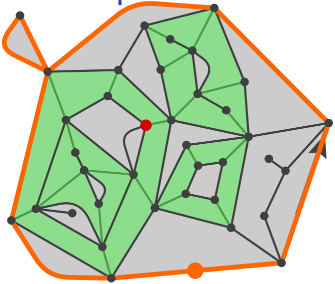

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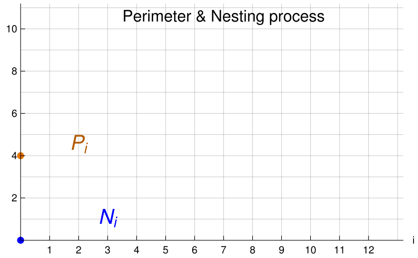

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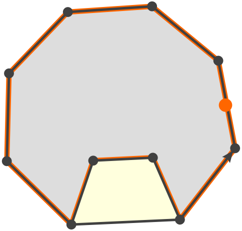

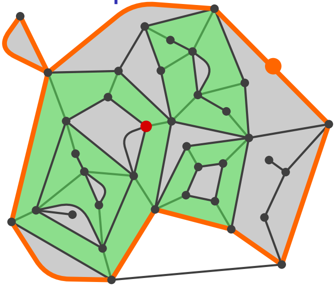

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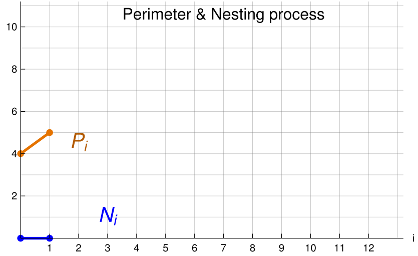

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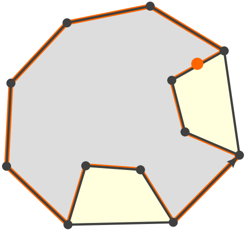

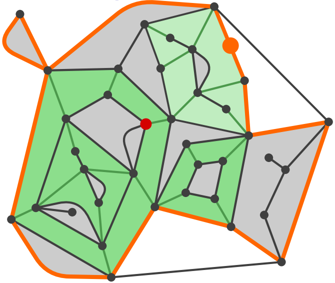

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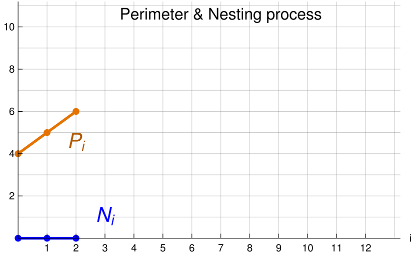

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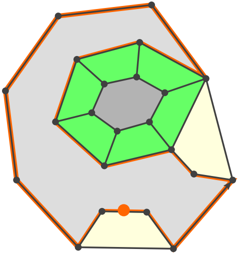

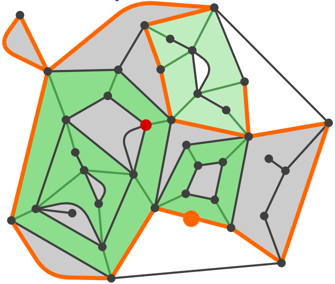

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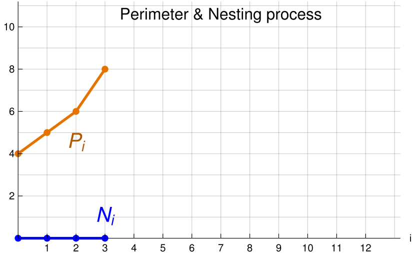

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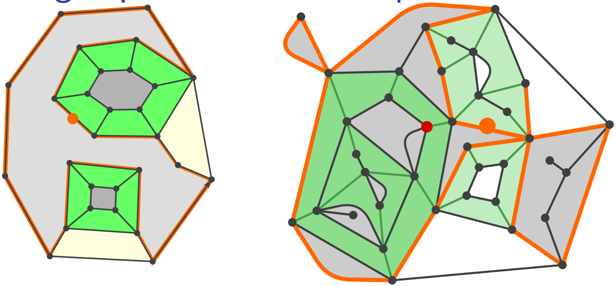

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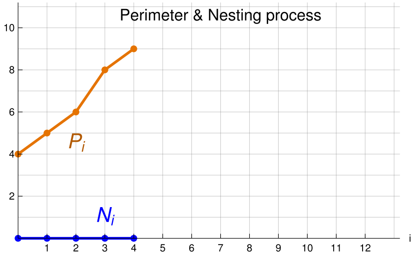

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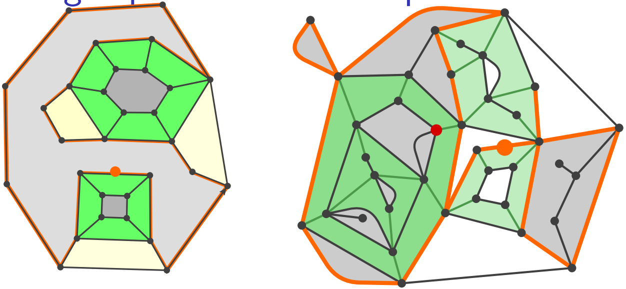

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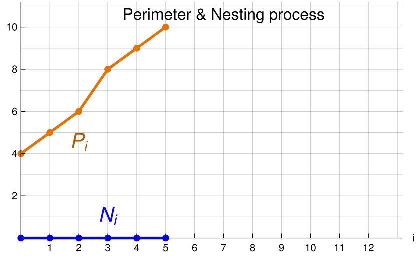

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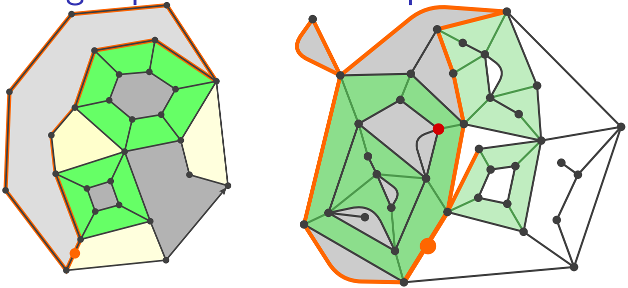

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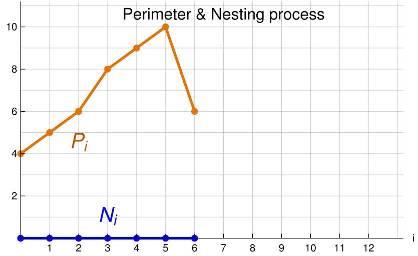

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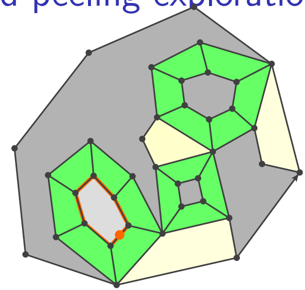

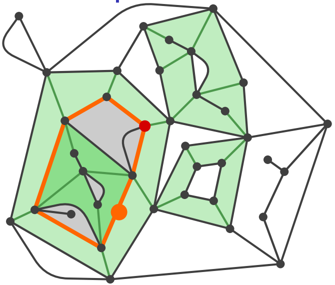

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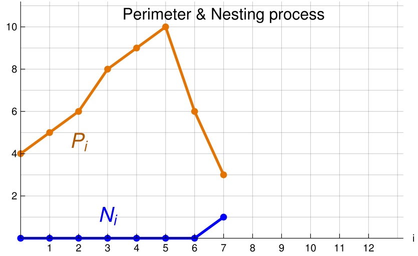

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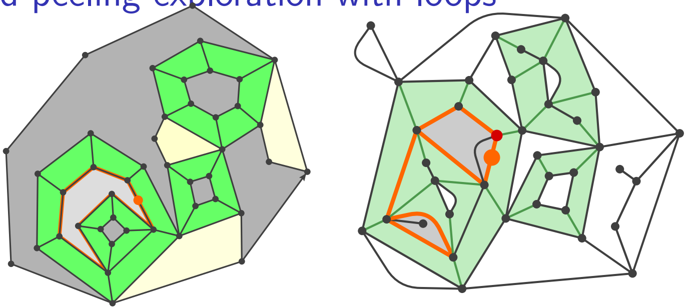

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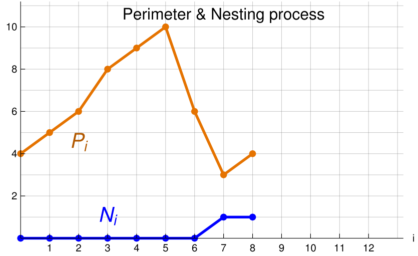

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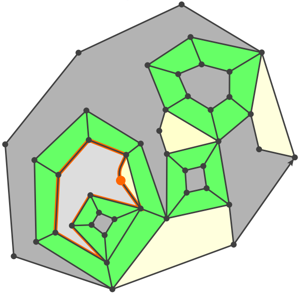

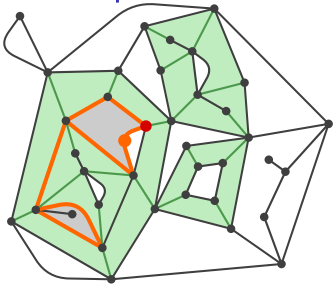

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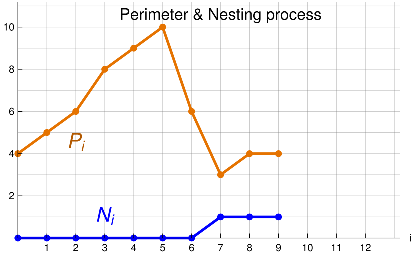
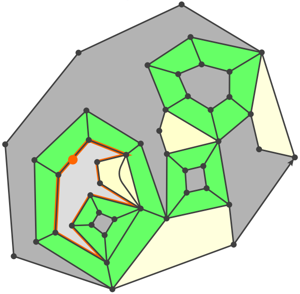
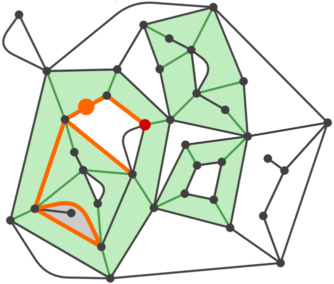


Targeted peeling exploration with loops





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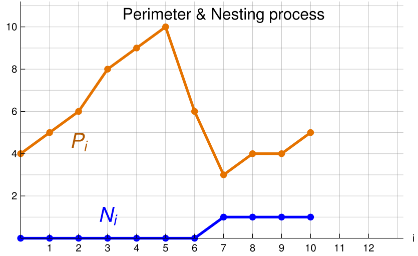

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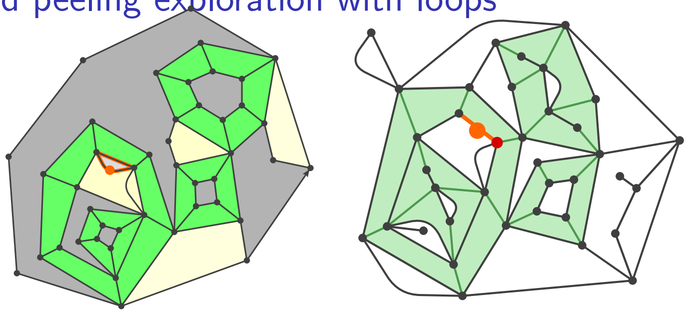

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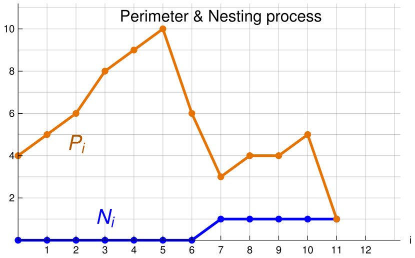

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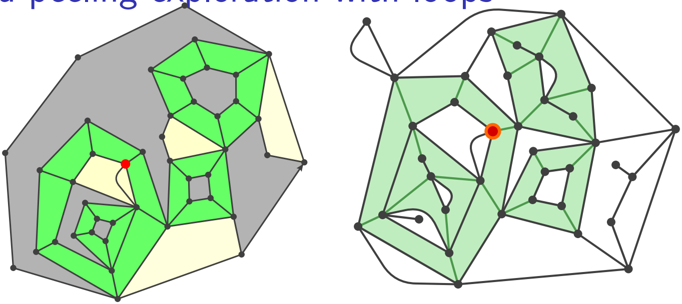

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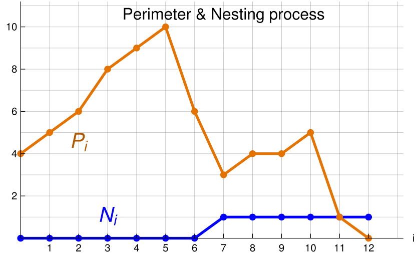
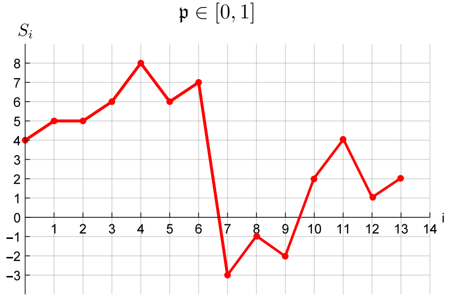


Ricocheted random walk [TB,'18+]

- Let (S_i) be the random walk with law $\nu_{\hat{\mathbf{q}}}$.



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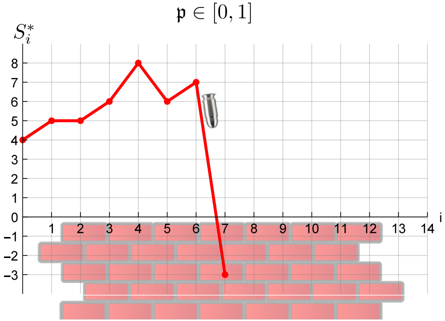

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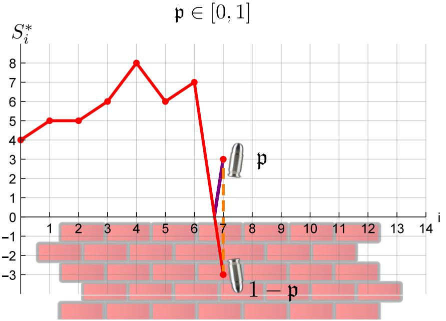

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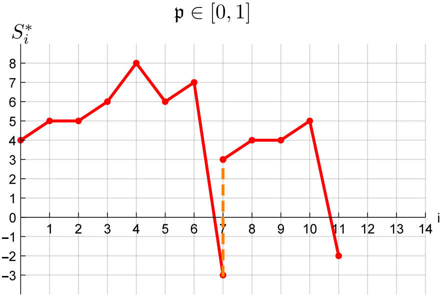

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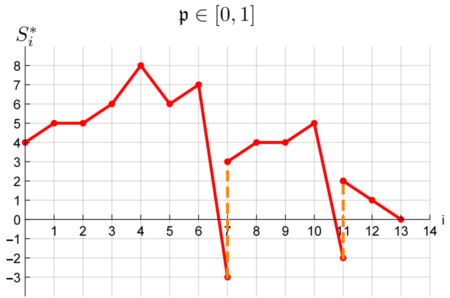

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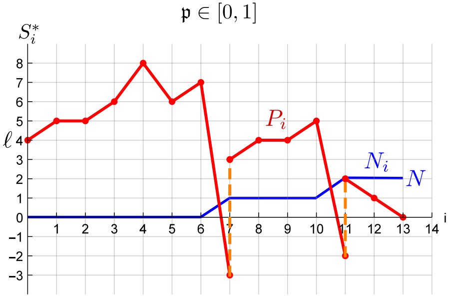

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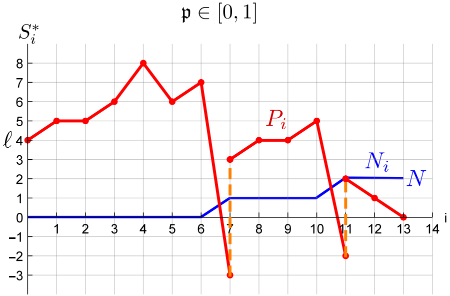

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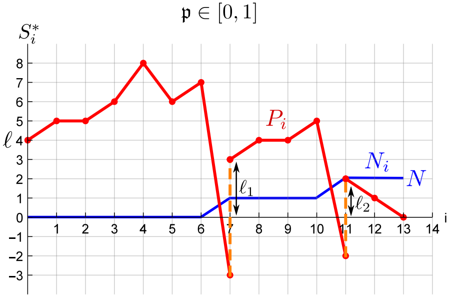
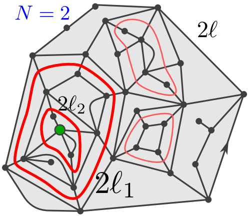
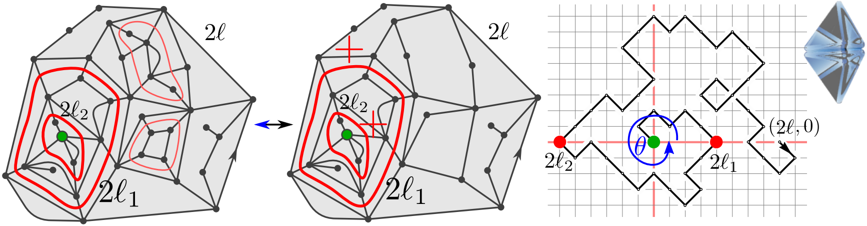
- The law of nested loop lengths $(\ell_j)_{j=1}^{N}$ is independent of $\hat{\mathbf{q}}$!





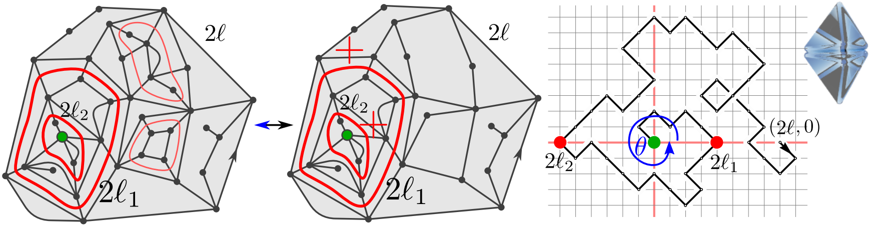


Theorem (TB,'18+)

Let $n = 2$ and $(n, g, \mathbf{q})$ non-generic critical and $N^{(\ell)}$ the # nested loops in the corresponding pointed map of boundary 2ℓ. Let $\theta^{(\ell)}$ be the winding angle of a random walk started at $(2\ell, 0)$. Then

$$\mathbb{E}[z^{N^{(\ell)}}] = \mathbb{E}[e^{ib\,\theta^{(\ell)}}] = \frac{1}{1+\cos \pi b}[x^{2\ell}]\left(\frac{1-x}{1+x}\right)^{b}, \quad b = \tfrac{1}{\pi}\arccos z.$$

For $n \in (0, 2)$ this distribution is simply tilted by $(\frac{n}{2})^{N^{(\ell)}}$.



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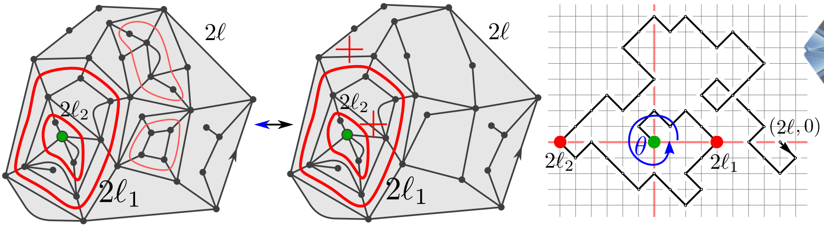

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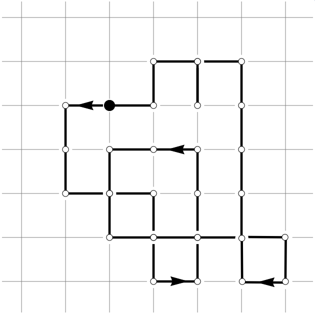
- Inspired by this many more exact statistics of the winding of simple random walks can be obtained [TB, '17]

Byproduct: winding field of a random loop

- Consider a uniform loop of length 2ℓ on $\mathbb{Z}^2$.



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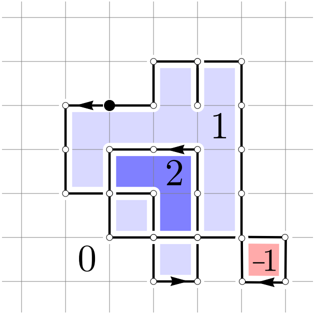

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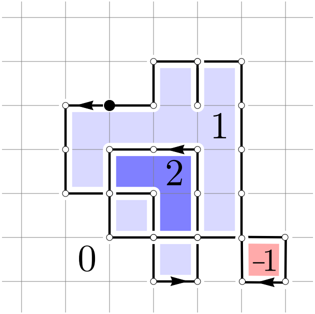

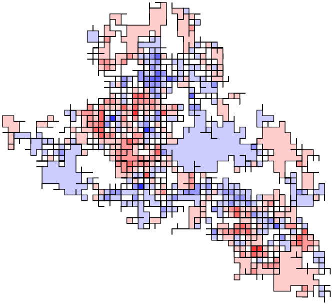

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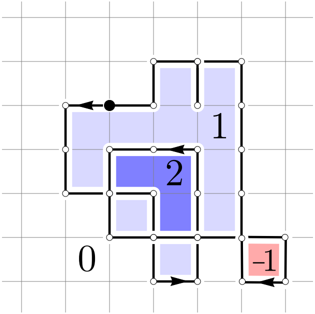

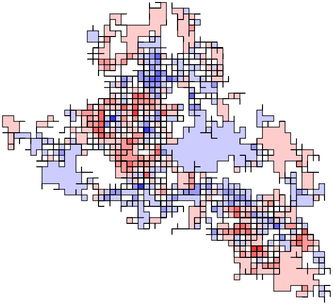

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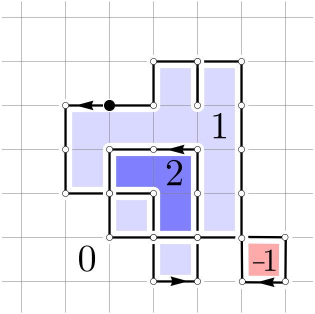

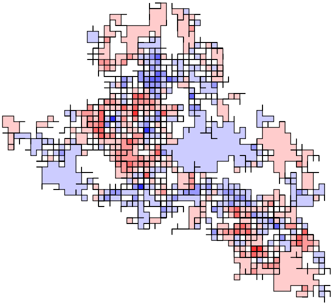

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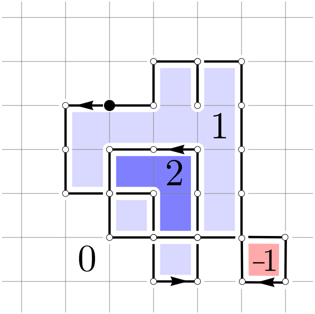

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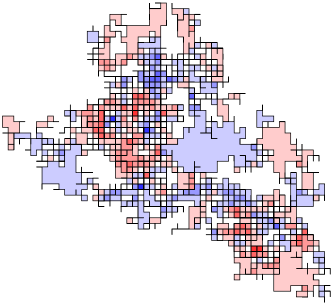
- Reproduces result of the 2d Brownian bridge as $\ell \to \infty$.

  [Garban, Trujillo-Ferreras,'06]



Mating of trees and convergence to LQG+SLE

- Miller's lecture: If you have a random map with a statistical model coded (à la mating of trees) by a random walk on $\mathbb{Z}^2$ with independent increments, then strong coupling with mated-CRT maps allows one to import results from $\text{LQG}_\gamma + \text{SLE}_\kappa$. [Gwynne, Holden, Sun, Miller, Sheffield]

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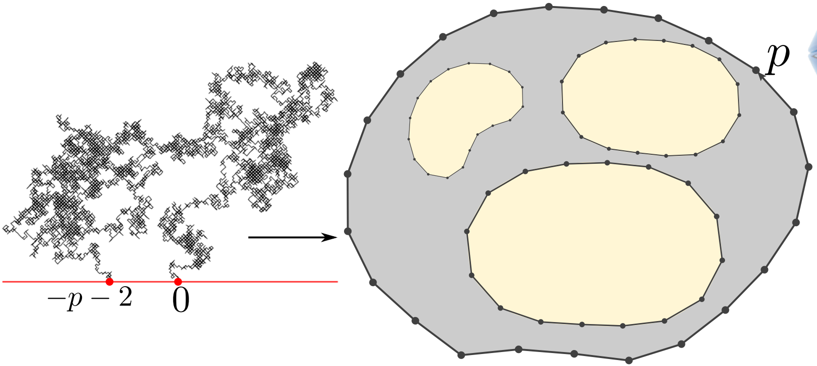
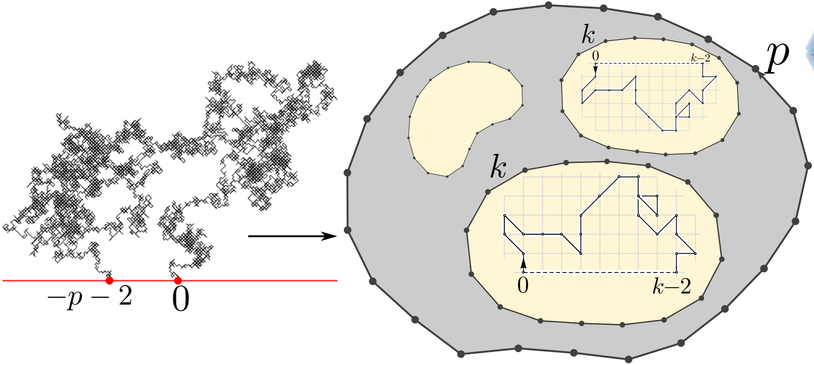
  - $\text{LQG}_2 + \text{CLE}_4$: $O(2)$ loop model-decorated maps $\leftrightarrow$ simple random walk on $\mathbb{Z}^2$???



- An ↑-excursion of extent $-p-2$ encodes a planar map with boundary length p with decoration.



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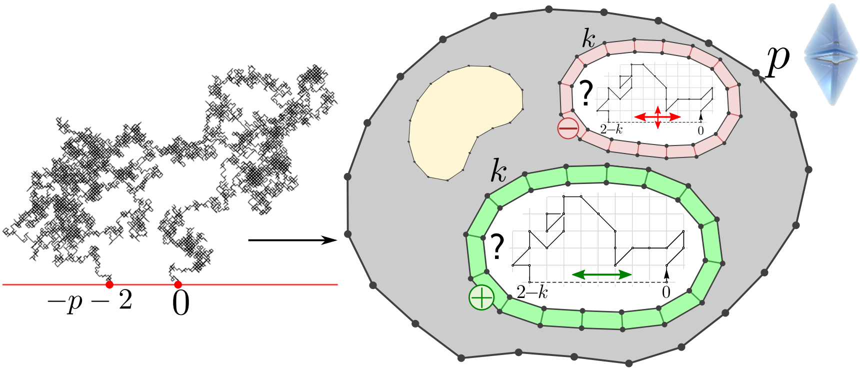

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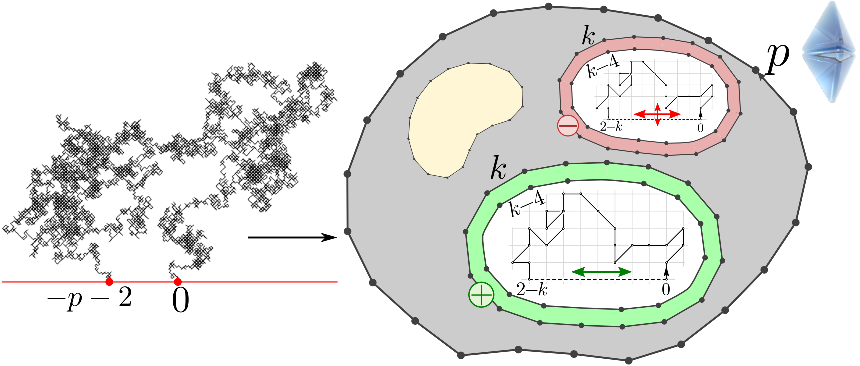

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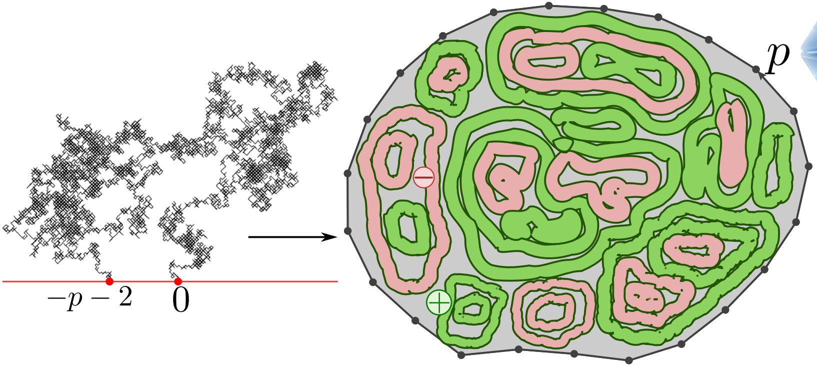

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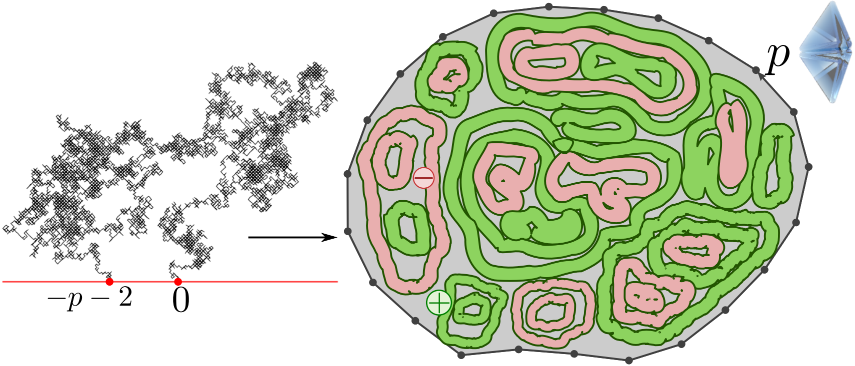

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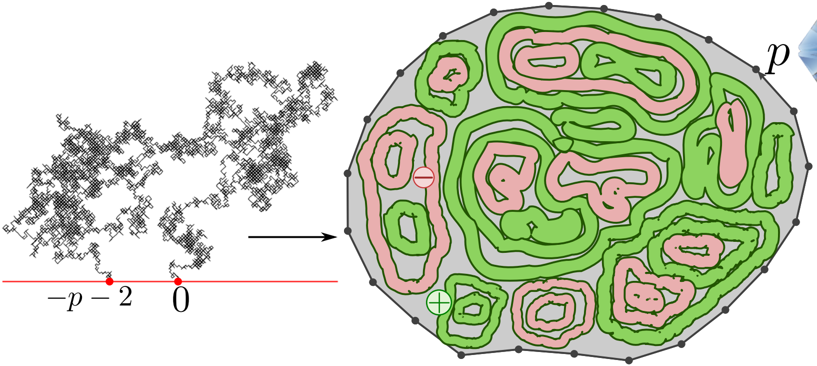

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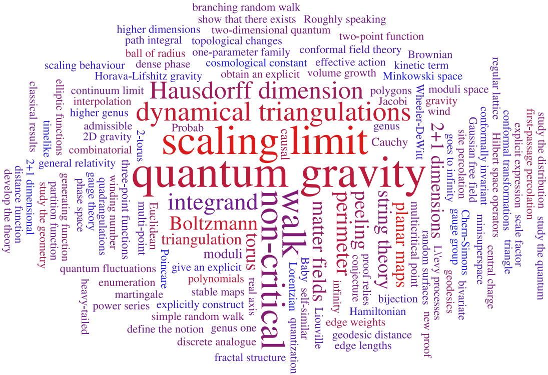
- Homework*: extend to $O(n)$, $n \in (0,2)$.

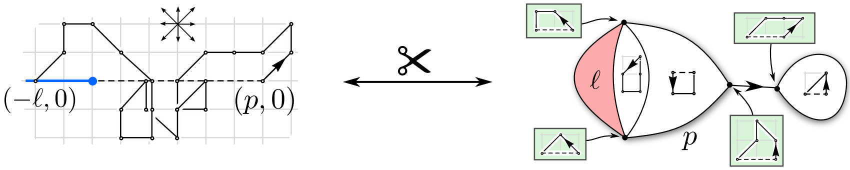
Thank you!



(My life according to `https://scimeter.org`)

Backup slides

Walks on slit plane encode maps



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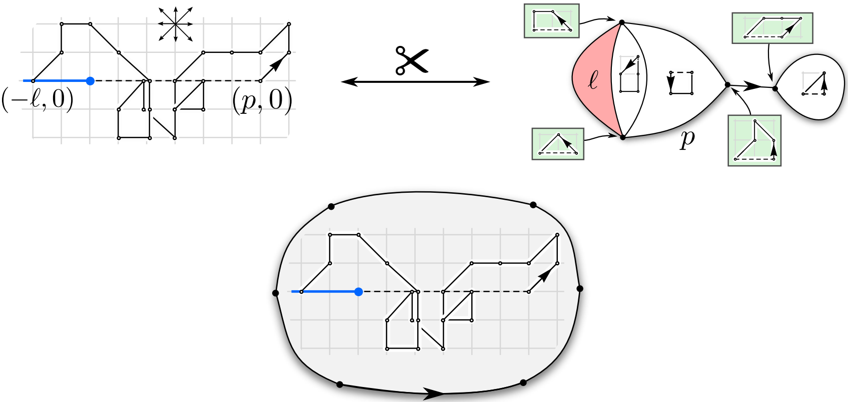

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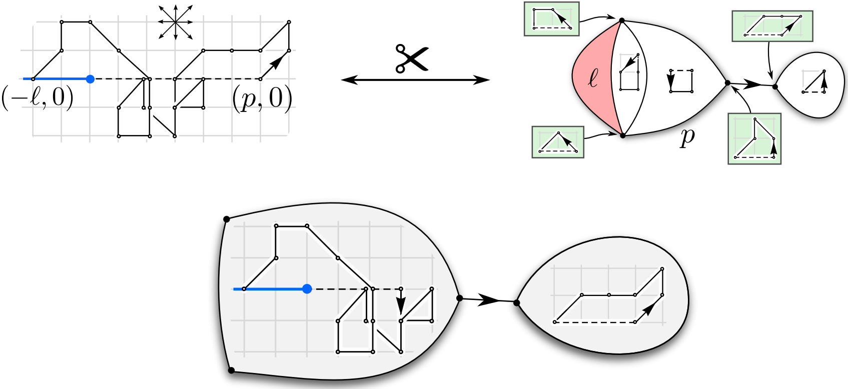

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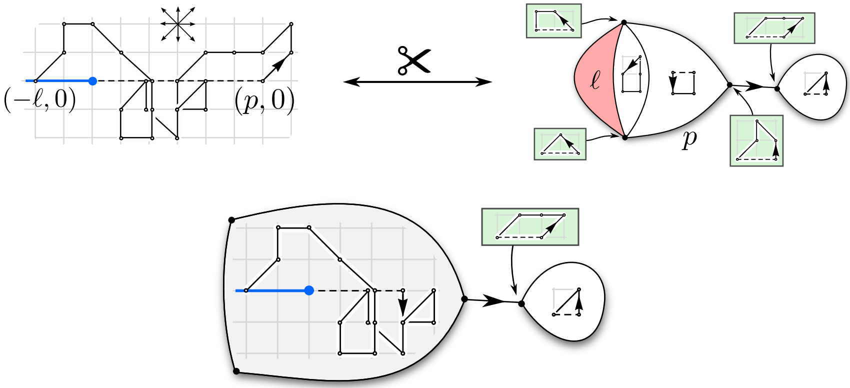

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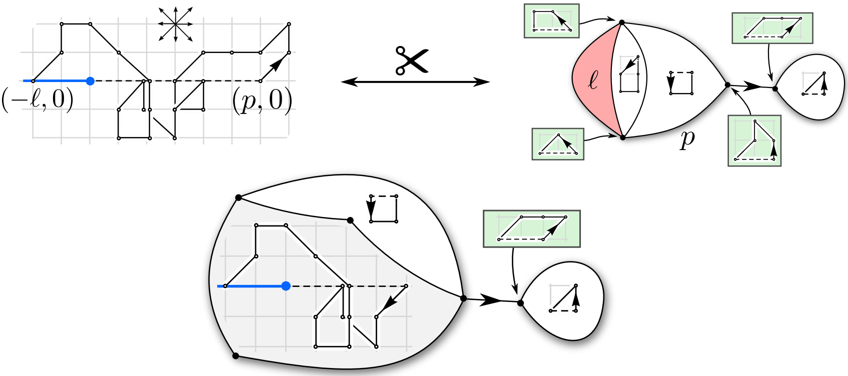

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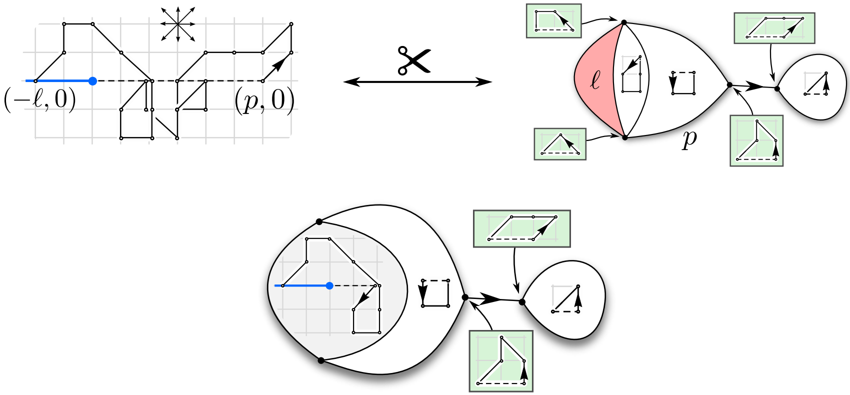

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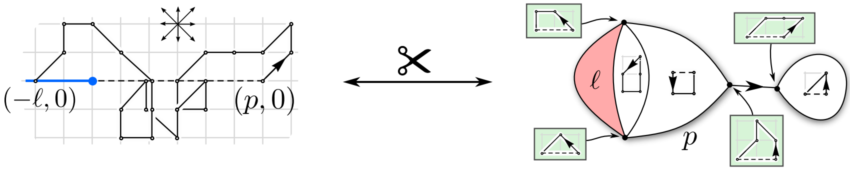

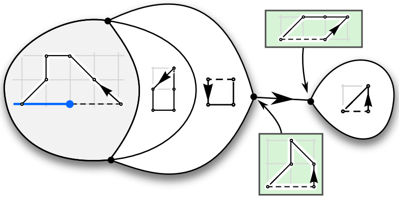

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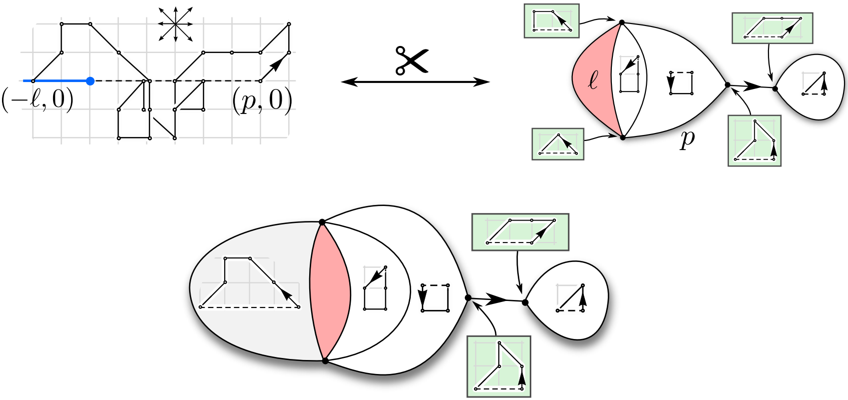

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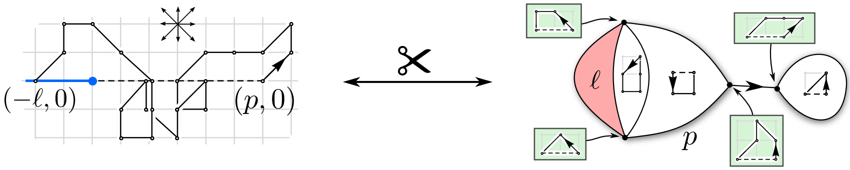
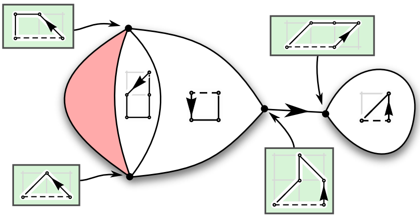
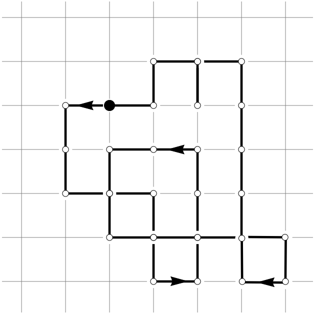




Byproduct: winding field of a random loop

- Consider a uniform loop of length 2ℓ on $\mathbb{Z}^2$.



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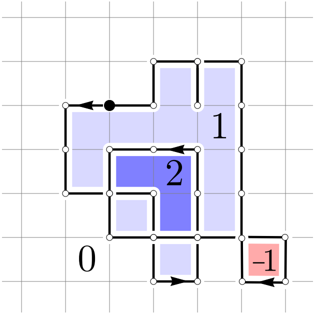

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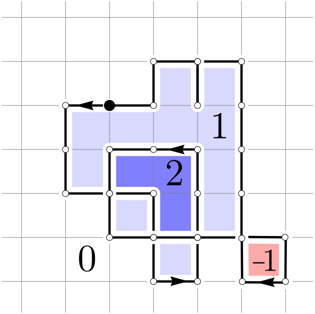

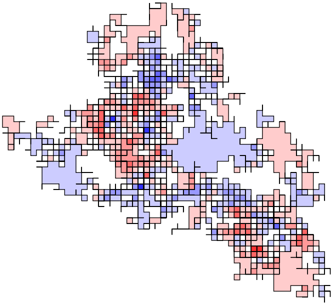

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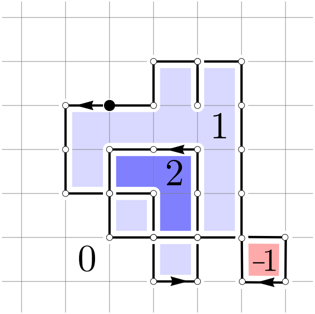

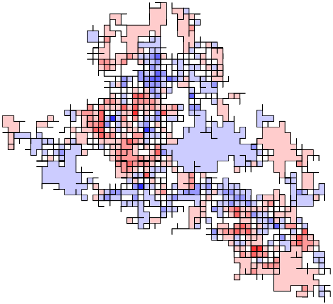

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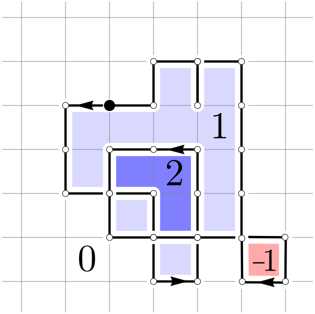

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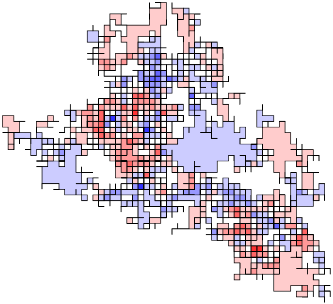
- It can be expressed explicitly as [TB, '17]

$$\frac{4^{2\ell}}{\binom{2\ell}{\ell}^2} \frac{\ell}{n} [k^{2\ell}] \frac{2q^{2n}}{1-q^{4n}} ,$$

where $q(k)$ is the nome of modulus k.



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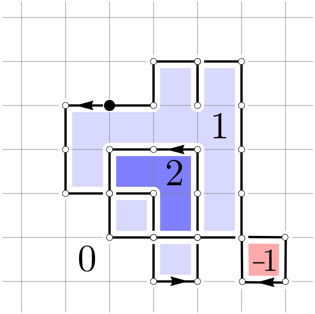
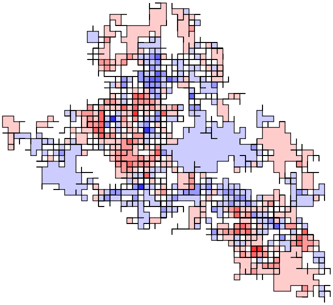
$$\frac{4^{2\ell}}{\binom{2\ell}{\ell}^2}\frac{\ell}{n}[k^{2\ell}]\frac{2q^{2n}}{1-q^{4n}} \sim \frac{\ell}{2\pi n^2},$$

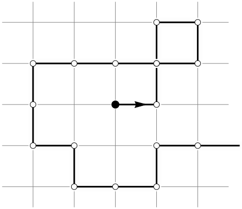
  where $q(k)$ is the nome of modulus k.
- Reproduces result of the 2d Brownian bridge as $\ell \to \infty$.
  [Garban, Trujillo-Ferreras,'06]





Winding angle of a simple random walk

- The winding angle Θ of random walks on $\mathbb{Z}^2$ were only known asymptotically: "hyperbolic secant laws" [Rudnick, Hu, '87] [Bélisle, '89] [Shi, '98].



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- An application:

Theorem (Discrete hyperbolic secant law [TB, '17])

The winding angle Θ around $(-\frac{1}{2}, -\frac{1}{2})$ of a simple random walk on $\mathbb{Z}^2$ shortly after a geometric random time with parameter k satisfies for $\alpha = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots,$

$$\mathbb{P}[\Theta \in (\alpha - \tfrac{\pi}{2}, \alpha + \tfrac{\pi}{2})] = c \operatorname{sech}(\tau \alpha), \quad c = \frac{\pi}{2kK(k)}, \quad \tau = \frac{K(\sqrt{1-k^2})}{K(k)}.$$