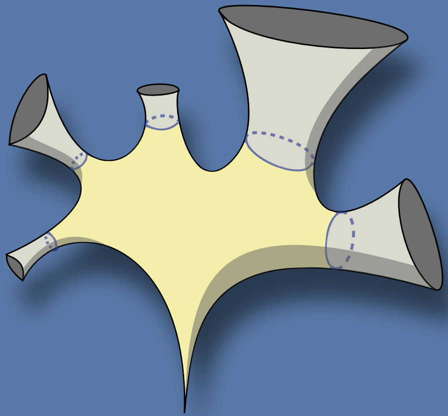
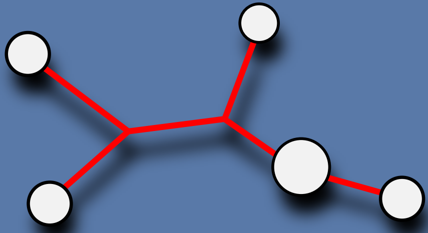
03-10-2023 CIRM workshop: Probability and Geometry in, on and of non-Euclidian spaces

The geometry of random genus-0 hyperbolic surfaces via trees

Timothy Budd





w.i.p. with T. Meeusen & B. Zonneveld
and with N. Curien

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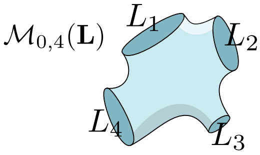
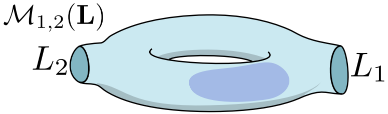
The partition function of hyperbolic surfaces: WP volumes

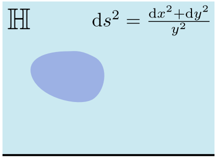
[Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

- Consider the Moduli space

$$\mathcal{M}_{g,n}(\mathbf{L}) = \left\{ \begin{array}{l} \text{genus-}g \text{ hyperbolic surface with } n \text{ geodesic} \\ \text{boundaries of lengths } \mathbf{L} = (L_1, \ldots, L_n) \end{array} \right\} / \mathrm{Isom}^+$$







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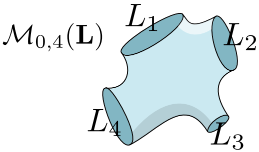

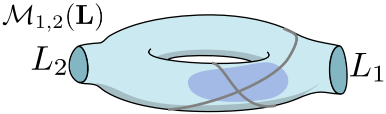

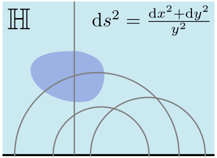

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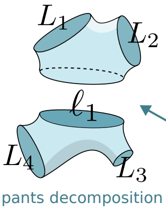

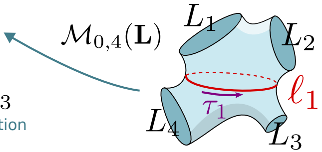

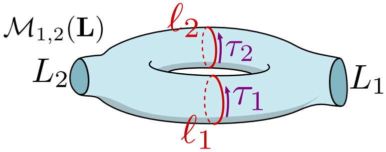

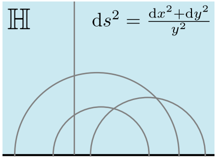

- Carries natural Weil-Petersson volume form μ_{WP}. In local Fenchel-Nielsen length & twist coordinates $\ell_1, \tau_1, \ldots, \ell_{3g-3+n}, \tau_{3g-3+n}$ for a pants decomposition:

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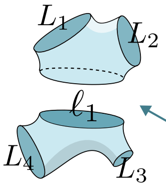

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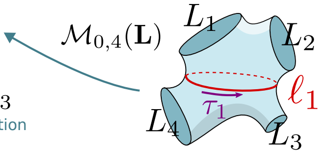

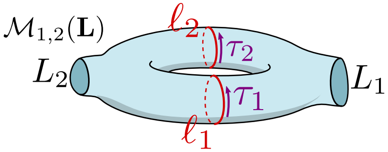

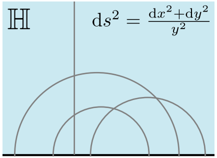

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- Weil-Petersson volume: $V_{g,n}(\mathbf{L}) := \int_{\mathcal{M}_{g,n}(\mathbf{L})} \mu_{\mathrm{WP}}$ is a polynomial in $L_1^2, \ldots, L_n^2, \pi^2$. [Mirzakhani, '07]

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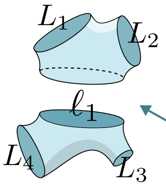

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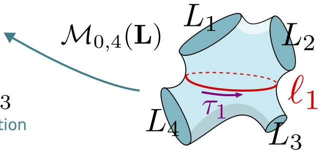

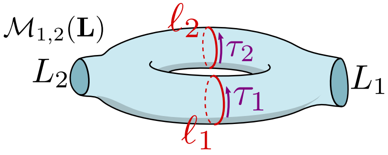

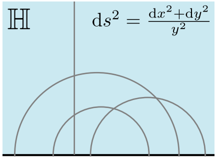

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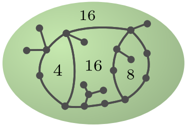
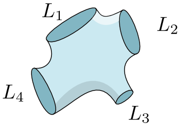
- Examples: $V_{0,3}(\mathbf{L}) = 1, \qquad V_{0,4}(\mathbf{L}) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2) + 2\pi^2,$
$V_{1,2}(\mathbf{L}) = \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2).$

(Bipartite) Maps on surfaces



Hyperbolic surfaces



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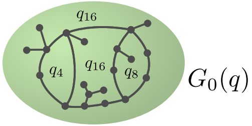

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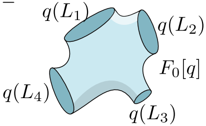

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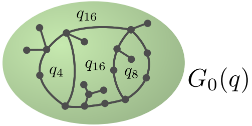

$e^{\sum_g G_g}$ is τ-function of 2-Toda hierarchy

[Kadomtsev, Petriashvili, Panharipande, Okounkov, Kazarian, ...]

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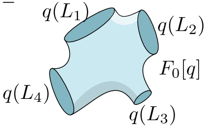

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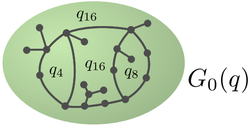

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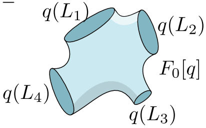

$e^{\sum_g F_g}$ is τ-function of KdV hierarchy

[Witten, '91][Kontsevich, '92][Kaufmann, Manin, Zagier, '96][Mirzakhani, '07]

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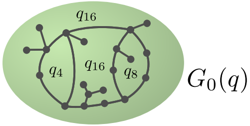

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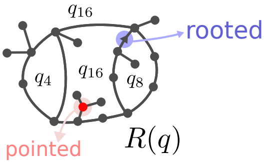

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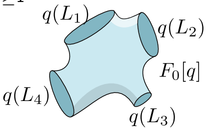

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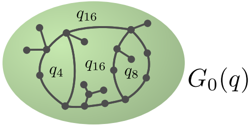

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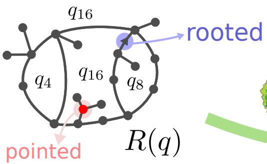

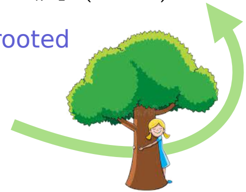

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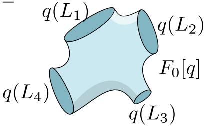

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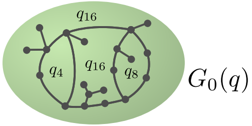

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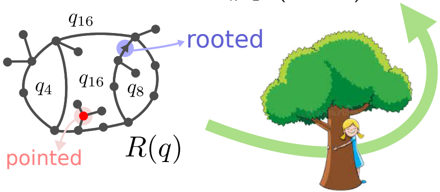

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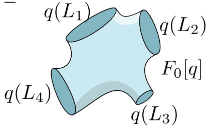

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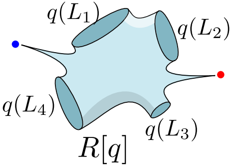

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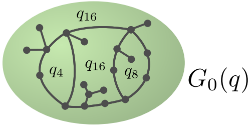

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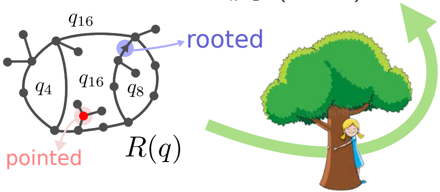

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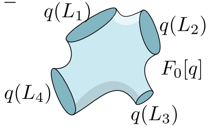

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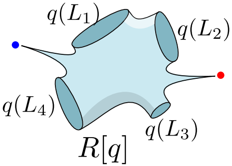
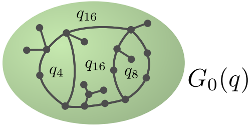
$$t_k = \frac{2}{k!} \int_0^{\infty} \left(\frac{L}{2}\right)^{2k} \mathrm{d}q(L) \qquad \gamma_k = \frac{(-1)^k \pi^{2k-2}}{(k-1)!} \mathbf{1}_{k\geq 2}$$



(Bipartite) Maps on surfaces

- genus-g generating function

$$G_g(q) = \sum_{n\geq 1} \frac{1}{n!} \sum_{d_1=0}^{\infty} q_{2d_1} \cdots \sum_{d_n=0}^{\infty} q_{2d_n} \# \left\{ \begin{array}{l} \text{genus-}g \text{ maps with} \\ \text{face degrees } 2d_1, \ldots, 2d_n \end{array} \right\}$$



- $g = 0$ determined by string eq. for $R(q) = \frac{\partial G_0}{\partial q_0 \partial q_1}$

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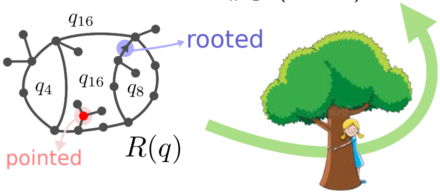

Hyperbolic surfaces

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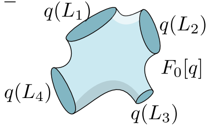

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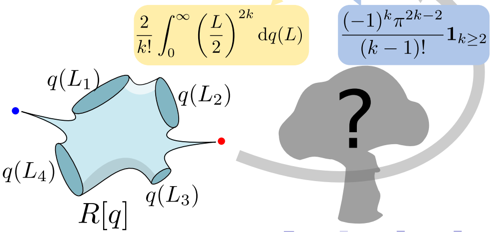

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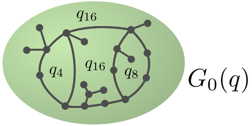

- $G_0(q) \xrightarrow[\text{interpretation}]{\text{probabilistic}}$ Boltzmann planar map $\mathfrak{m}$

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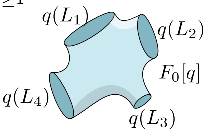

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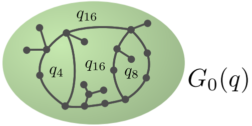

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- Scaling limit (if q sufficiently regular):

$$(\mathfrak{m}, n^{-\frac{1}{4}} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{\text{(d) GH}} \text{Brownian sphere}$$ [Le gall, Miermont]

- Random metric space
- Hausdorff dimension 4
- Topology of 2-sphere [Le Gall, Miermont, Marckert, Marzouk, ...]
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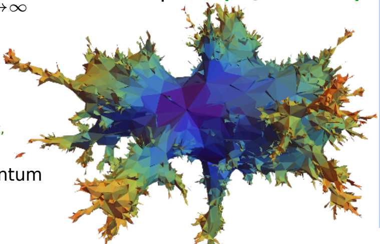

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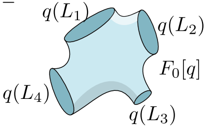

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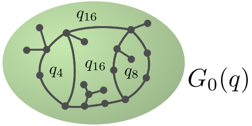

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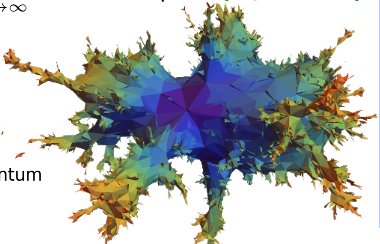

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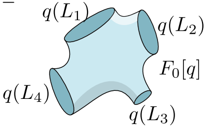

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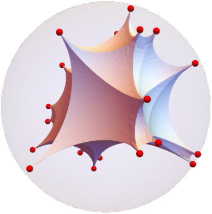
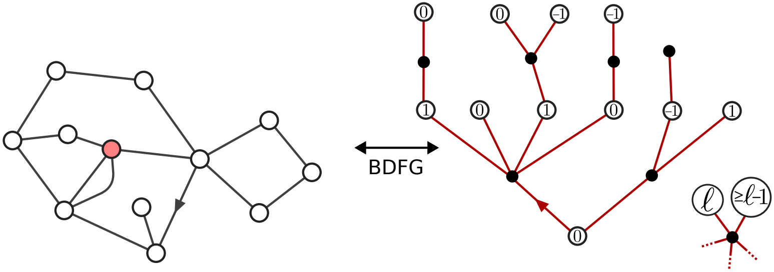


$\longrightarrow$
$n \to \infty$

Bouttier–Di Francesco–Guitter bijection [BDFG, '04]

$$\left\{ \begin{array}{c} \text{rooted bipartite planar maps} \\ \text{with marked vertex ("origin")} \end{array} \right\} \overset{\text{2-to-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{mobiles (bicolored plane trees} \\ \text{with labeled white vertices)} \end{array} \right\}$$



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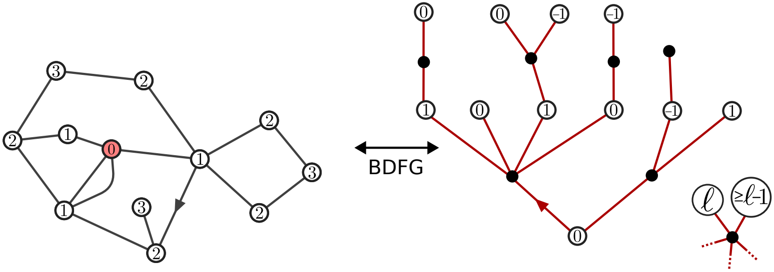

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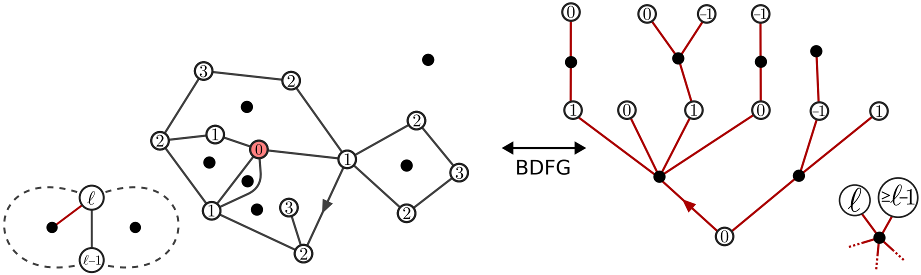

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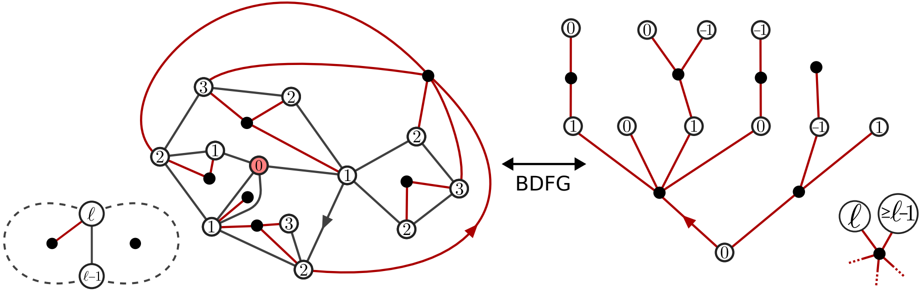

- Face of degree $2k$ $\longleftrightarrow$ Black vertex of degree k.

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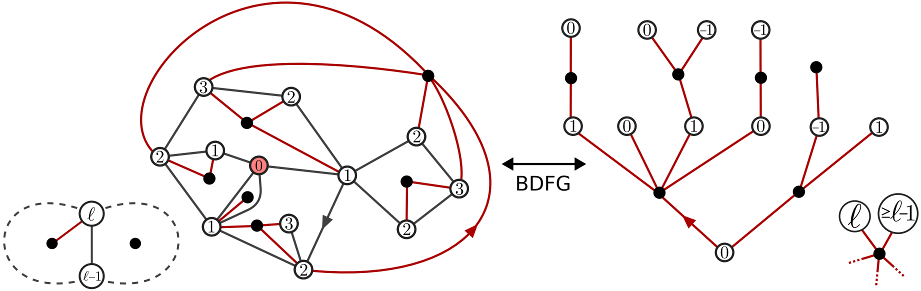

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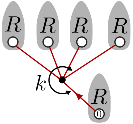
$$R = ⓪ + \sum_{k=1}^{\infty} q_{2k} \sum_{\text{labels}}$$  $$= 1 + \sum_{k=1}^{\infty} q_{2k} \binom{2k-1}{k} R^k,$$

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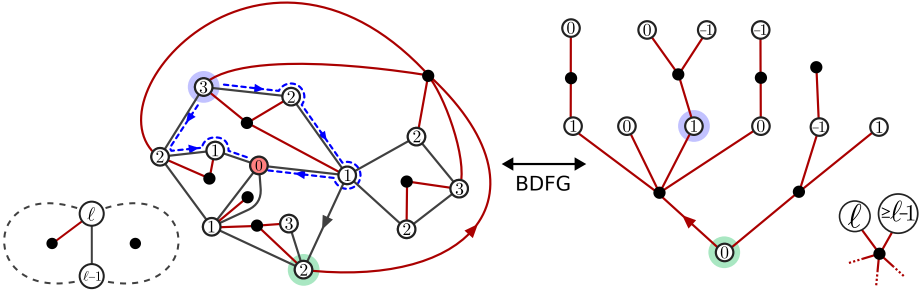

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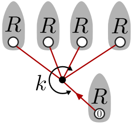
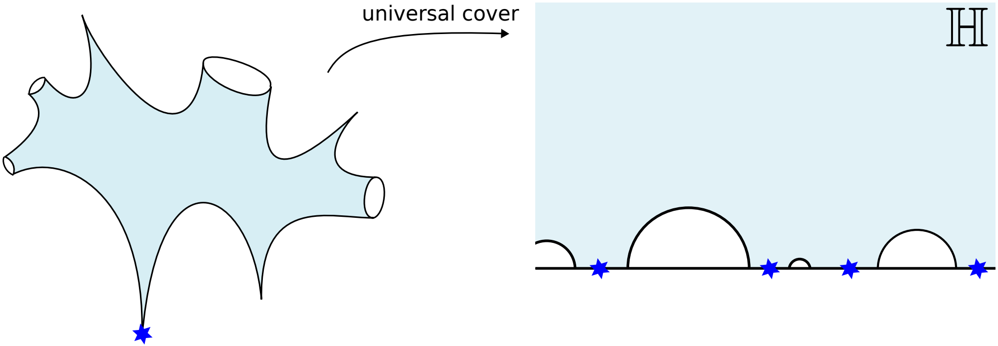
$$R = ⓪ + \sum_{k=1}^{\infty} q_{2k} \sum_{\text{labels}}$$  $$= 1 + \sum_{k=1}^{\infty} q_{2k} \binom{2k-1}{k} R^k,$$

- Vertex at distance $r > 0$ to origin $\longleftrightarrow$ White vertex with label $r - r_{\text{root}}$.

Tree in a hyperbolic surface?



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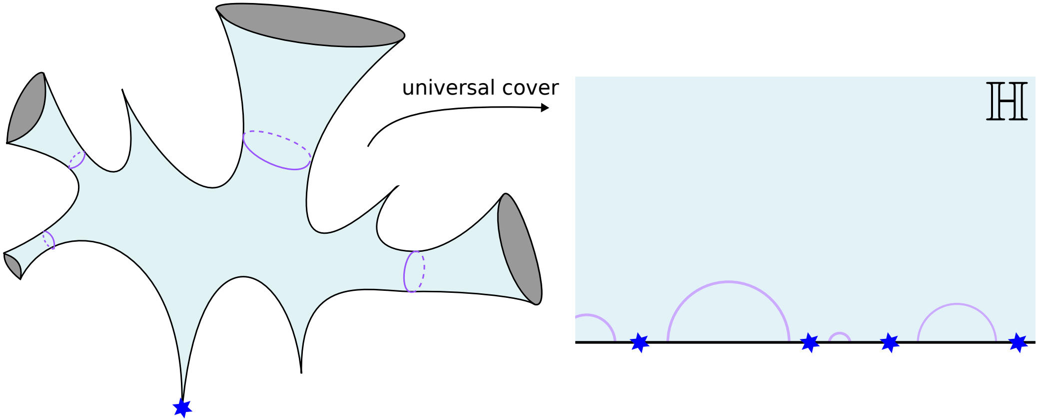

- Extend boundaries with hyperbolic funnels.

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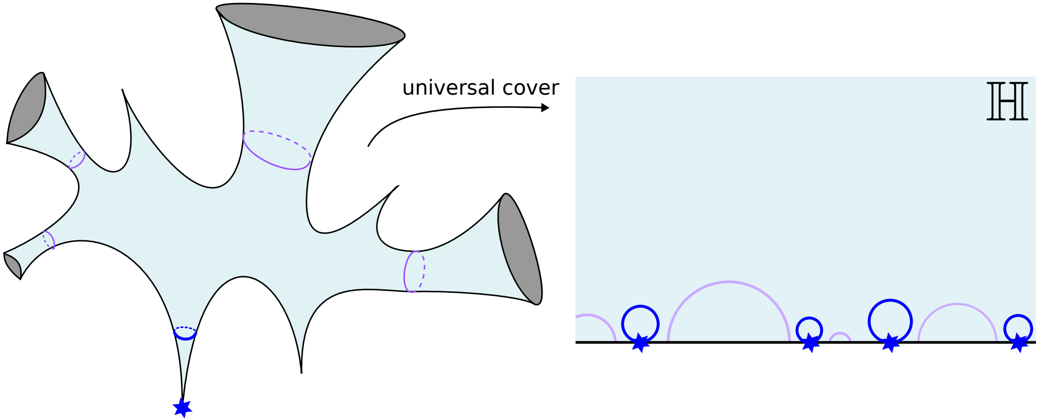

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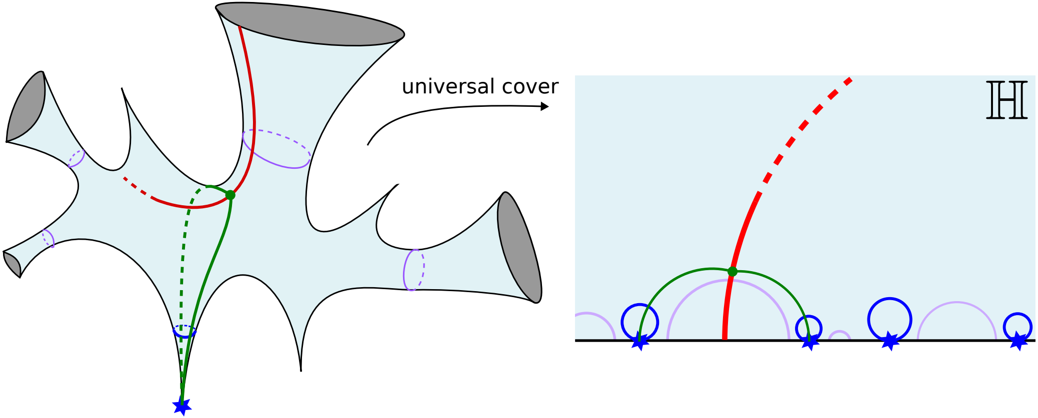

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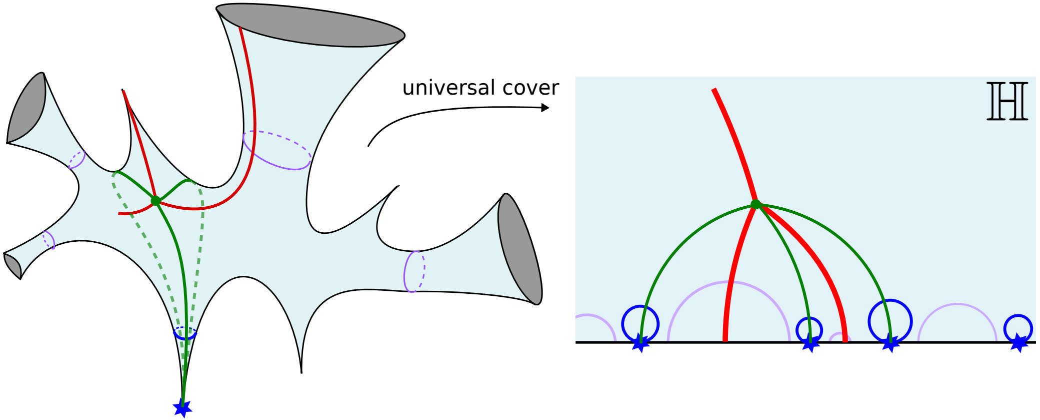

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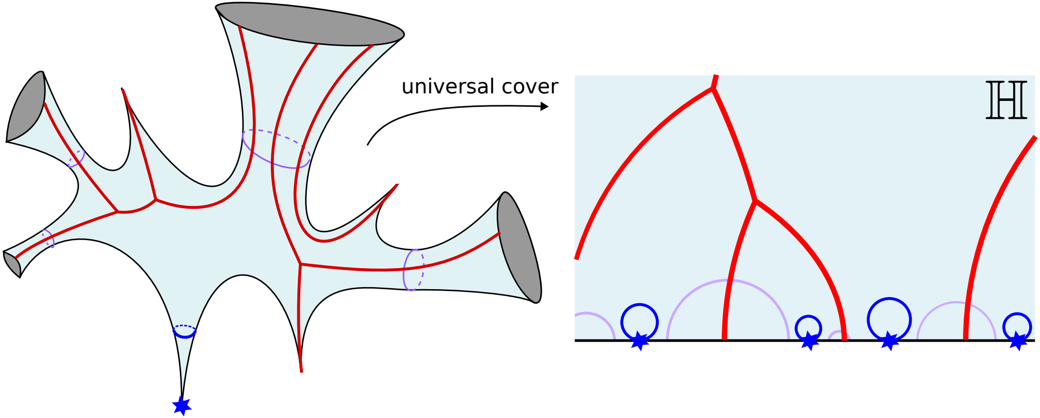

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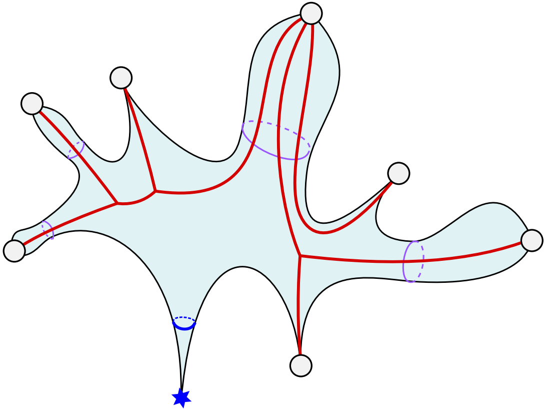

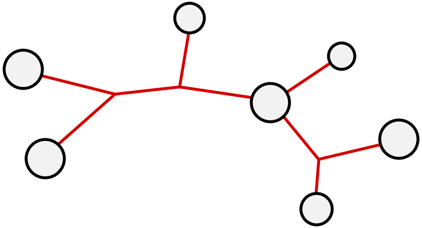

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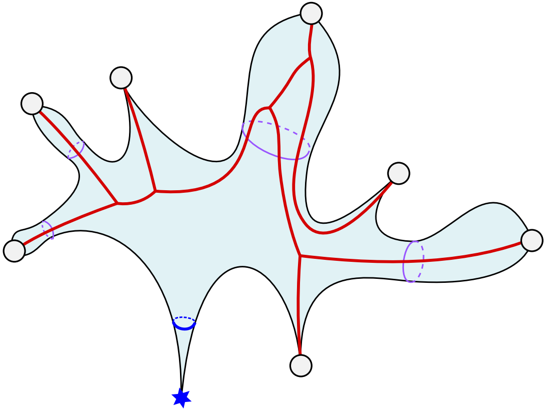

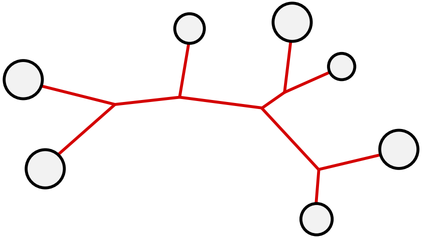

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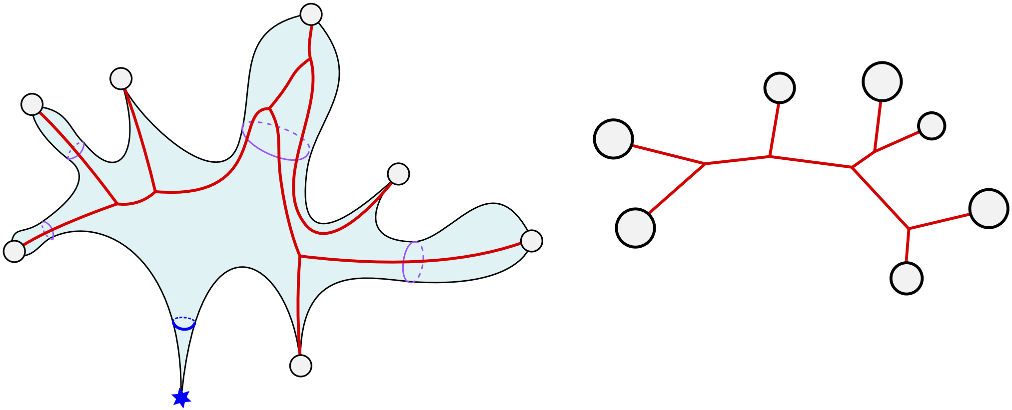

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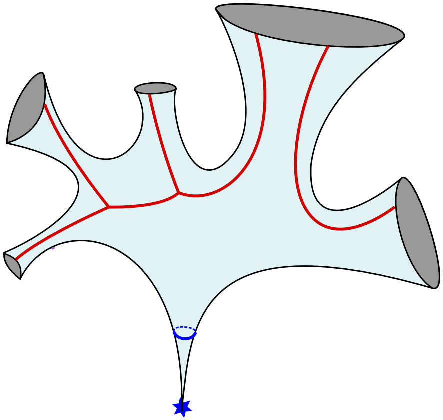
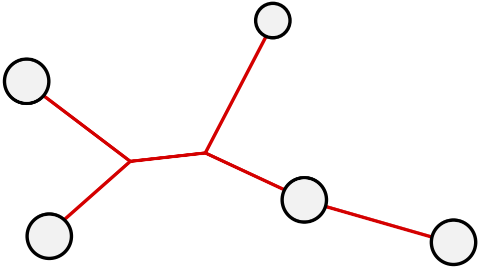
- Can we label the tree to make this a bijection?

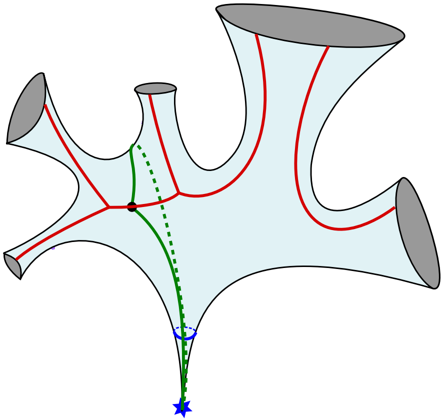
Labels: angles on half edges

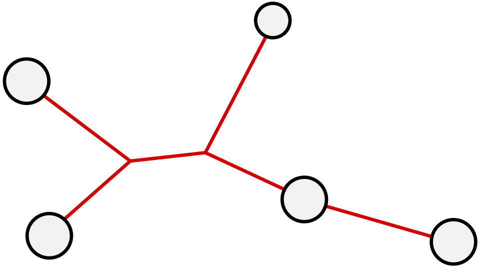




- The surface is canonically triangulated by

Labels: angles on half edges





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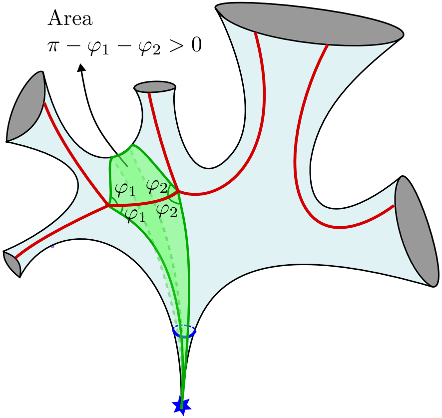

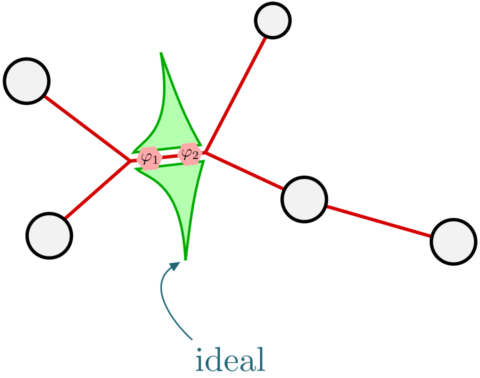

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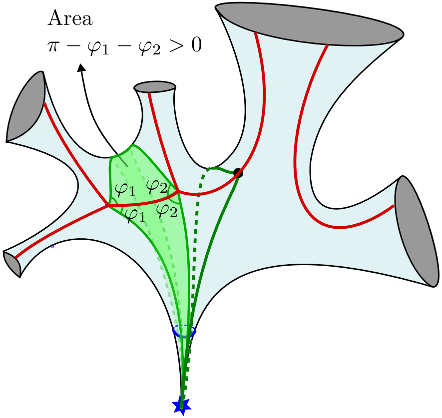

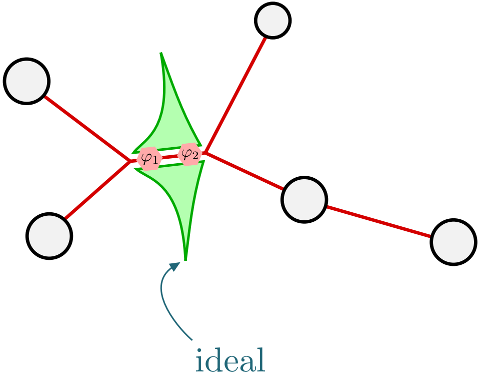

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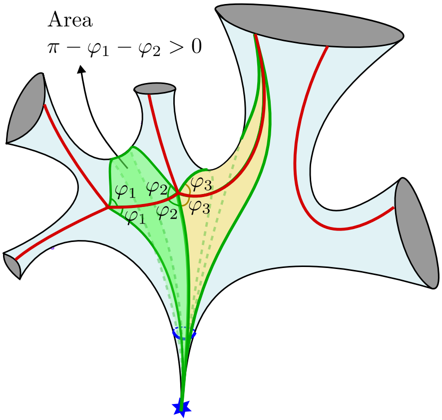

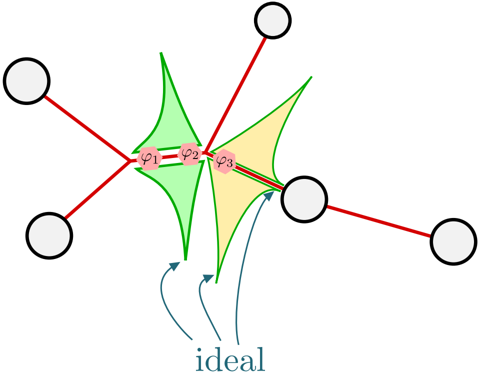

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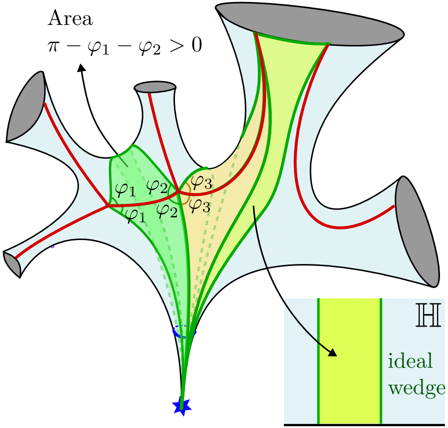

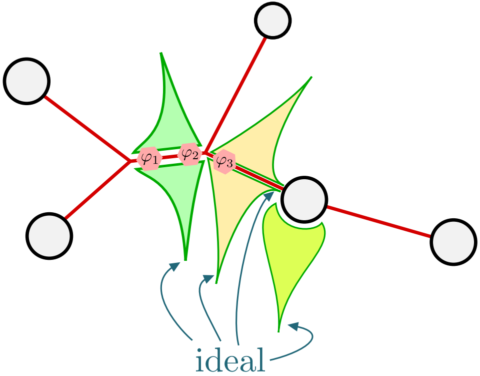

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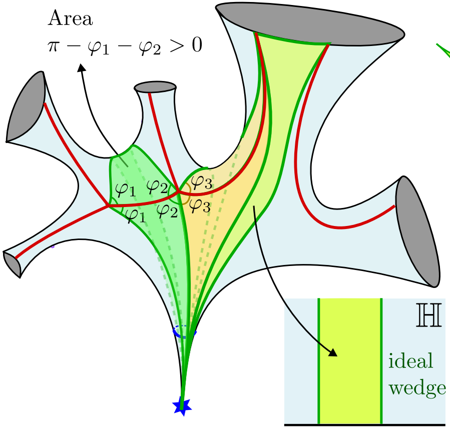

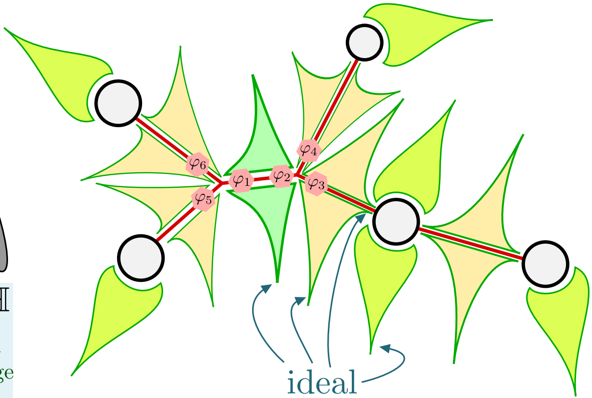

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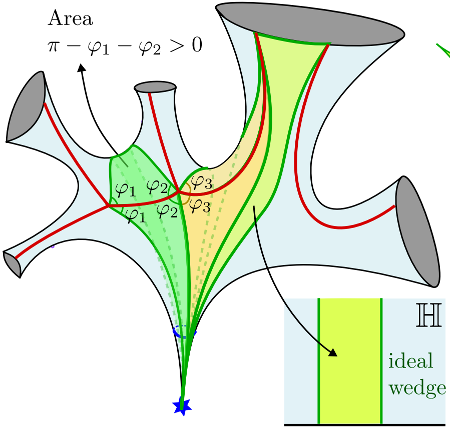
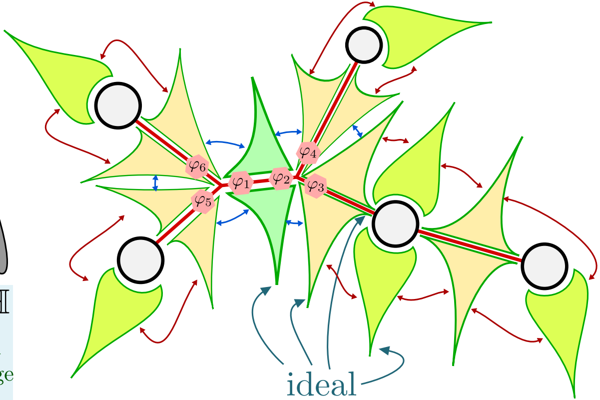
  - for each corner of white vertex: an ideal wedge.

Labels: angles on half edges





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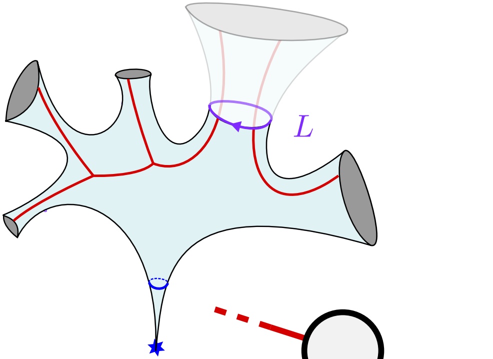
  - for each spine edge: pair of triangles with angles $\varphi_i, \varphi_j, 0$ (so $\varphi_i + \varphi_j < \pi$)
  - for each corner of white vertex: an ideal wedge.
- Gluing of triangles is unique, except for bi-infinite sides: need extra parameters for injectivity.

Labels: geometry around boundary



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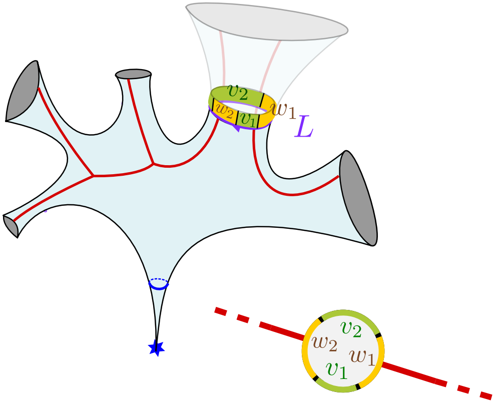

- Boundary of degree k partitions into $2k$ segments of lengths $v_1, \ldots, v_k$, $w_1, \ldots, w_k$.

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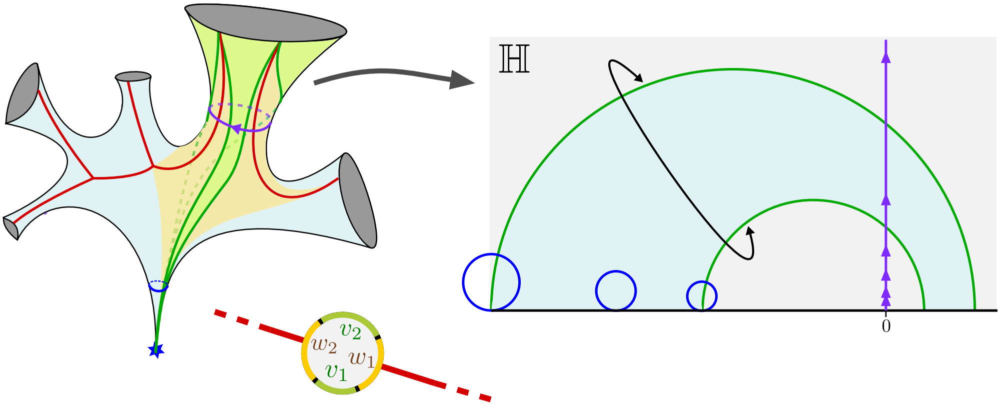

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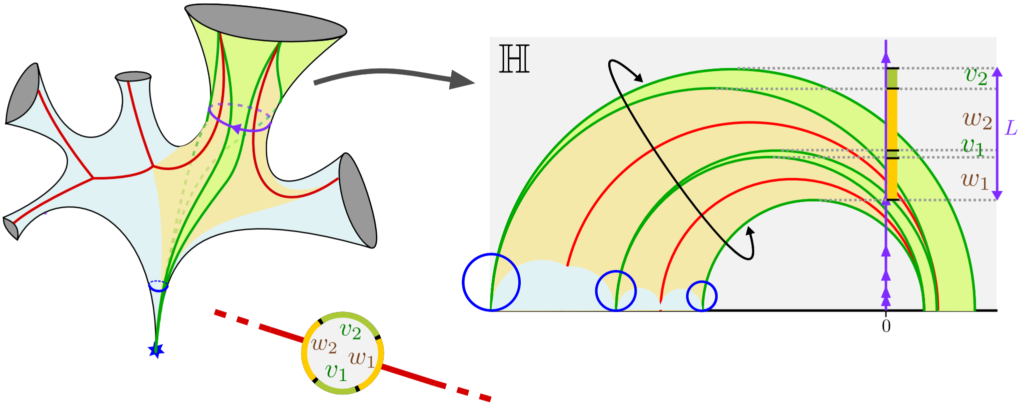
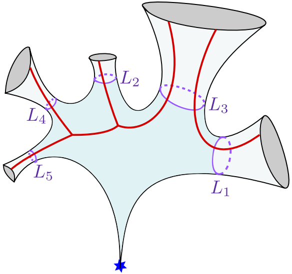
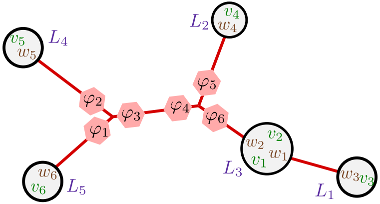


- Boundary of degree k partitions into $2k$ segments of lengths $v_1, \ldots, v_k$, $w_1, \ldots, w_k$.
- Uniquely determines gluing, so should label vertex by

$$\left\{ (v_i, w_i)_{i=1}^k : \sum_{i=1}^k v_i = \sum_{i=1}^k w_i = \frac{L}{2} \right\}.$$

Bijective result





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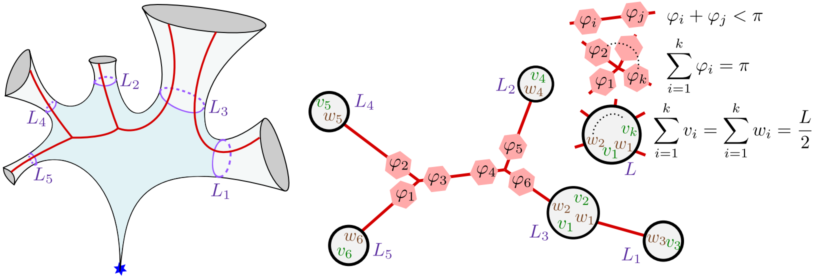

- For plane tree $\mathfrak{t}$ with n white vertices (deg ≥ 1) and red vertices (deg ≥ 3),

$$\mathcal{A}_{\mathfrak{t}}(L_1, \ldots, L_n) = \left\{ (\phi_i, v_i, w_i) : \phi_i > 0, v_i \geq 0, w_i > 0, \text{constraints above} \right\}.$$

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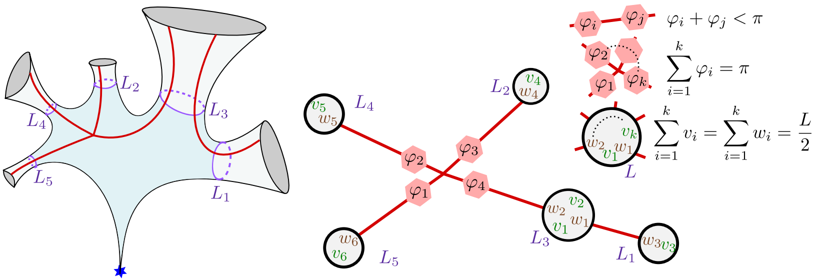

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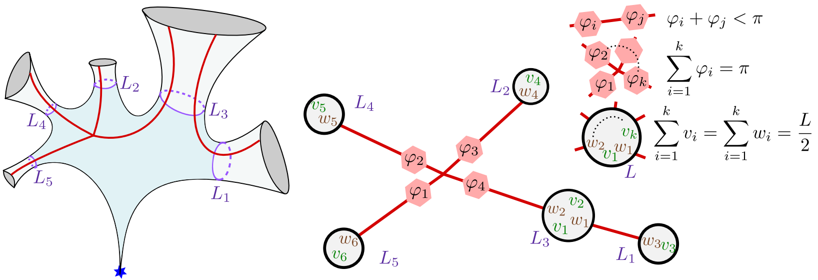

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Theorem (TB, Meeusen, Zonneveld, '23+)

This determines a bijection

$$\Phi : \mathcal{M}_{0,n+1}(0, \mathbf{L}) \longrightarrow \bigsqcup_{\mathfrak{t}} \mathcal{A}_{\mathfrak{t}}(\mathbf{L}).$$

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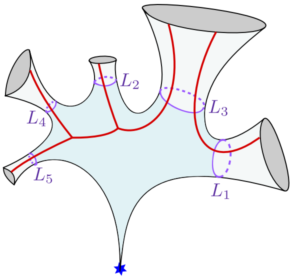

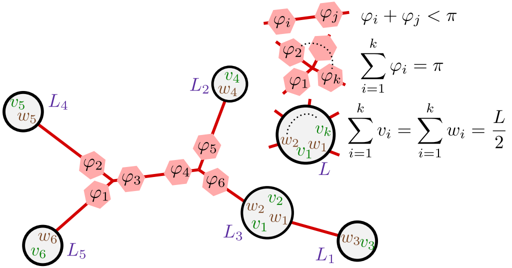

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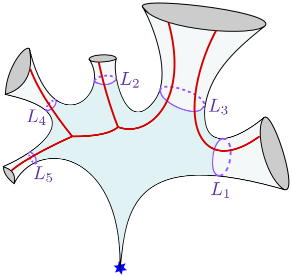

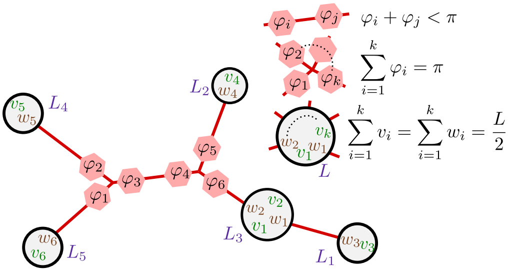

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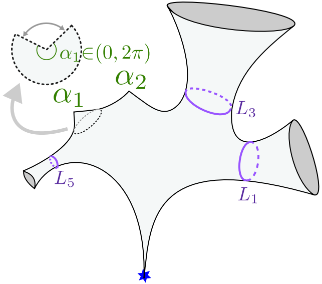
- Corollary: $V_{0,n+1}(0, \mathbf{L}) = \sum_{\mathfrak{t}} |\mathcal{A}_{\mathfrak{t}}(\mathbf{L})|$, and $|\mathcal{A}_{\mathfrak{t}}(\mathbf{L})| = \text{rational} \times \pi^{2\#\bullet} \prod_{\circ_i} L_i^{2(\deg \circ_i - 1)}$.

Remark: extension to surfaces with cone points



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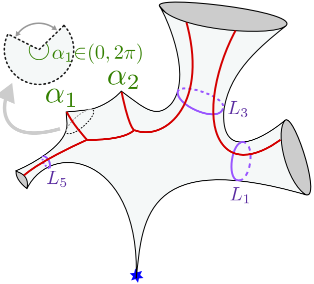

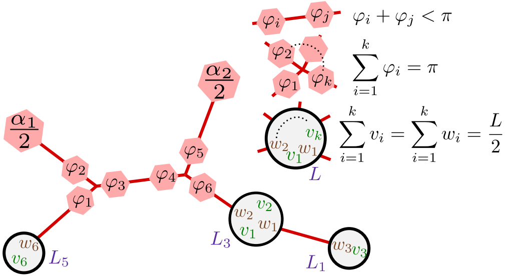

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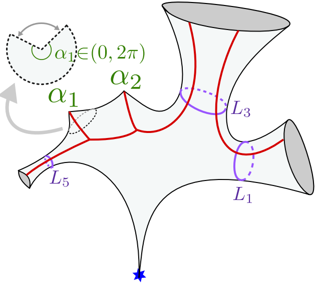

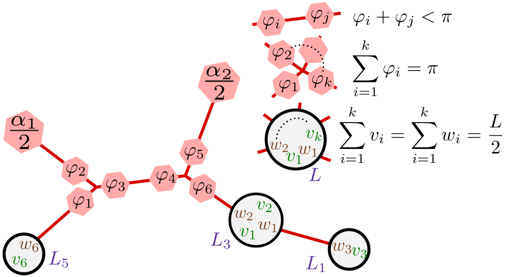

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[Mirzakhani, '07] [Tan, Wong, Zhang, '06]

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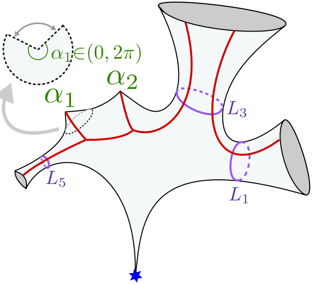

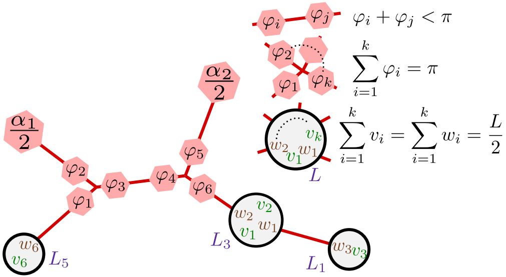

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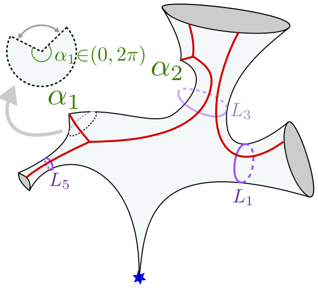

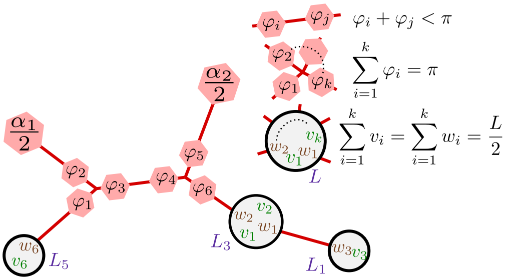

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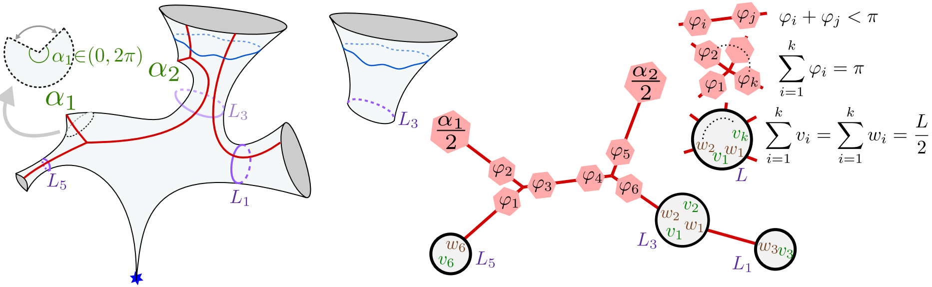

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$\mathrm{Vol}_{\mathrm{WP}}$ (moduli space of surfaces with cone points $\boldsymbol{\alpha}$ & ends of type $\mathbf{L}$)

non-polynomial

WP volume generating function

- Why does the generating function $R[q] = \sum_{n\geq 1} \frac{1}{n!} \int_0^\infty \mathrm{d}q(L_1)\cdots \mathrm{d}q(L_n) V^{\mathrm{WP}}_{0,n+2}(0,0,\mathbf{L})$ satisfy

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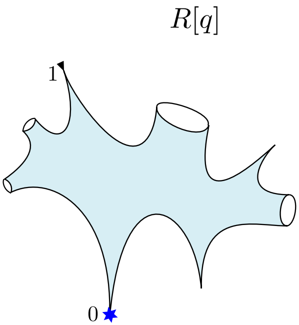

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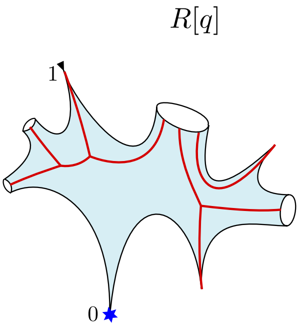

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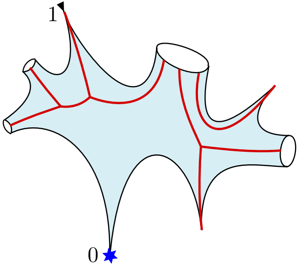
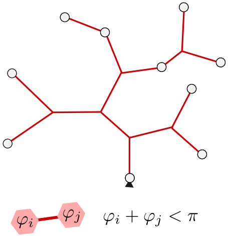
$$R[q] = \sum_{\text{trees } \mathfrak{t}} \left| \mathcal{A}_{\mathfrak{t}}(\mathbf{L}) \right|$$





$\varphi_i + \varphi_j < \pi$

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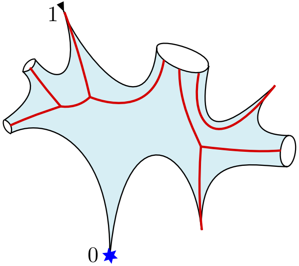

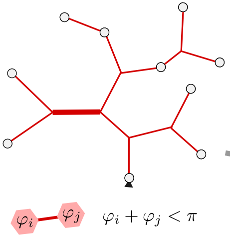

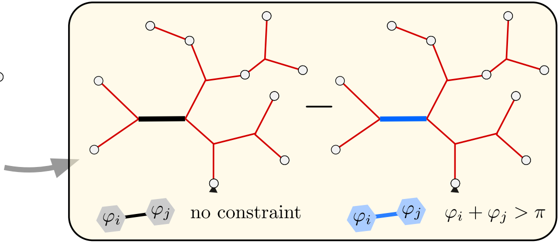

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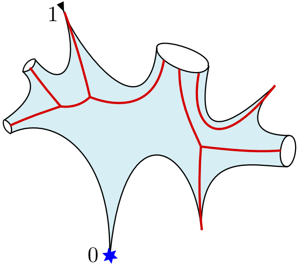

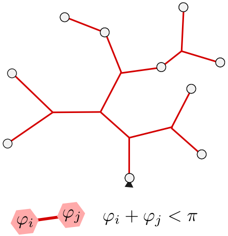

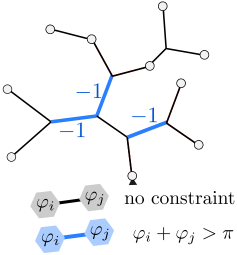

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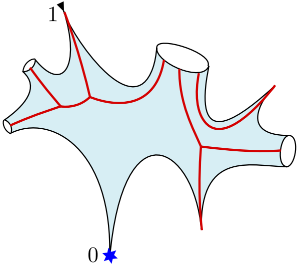
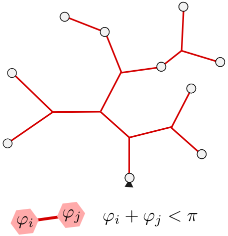
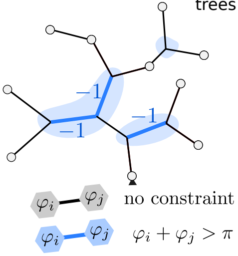
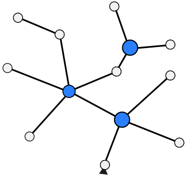




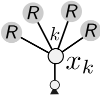
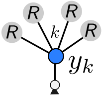




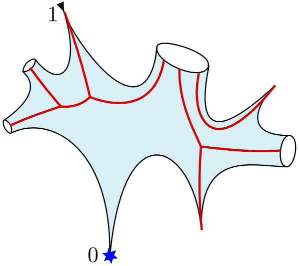
WP volume generating function

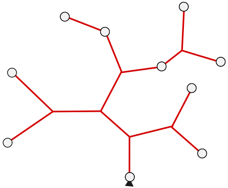
- Why does the generating function $R[q] = \sum_{n\geq 1} \frac{1}{n!} \int_0^\infty \mathrm{d}q(L_1)\cdots\mathrm{d}q(L_n) V^{\mathrm{WP}}_{0,n+2}(0,0,\mathbf{L})$ satisfy

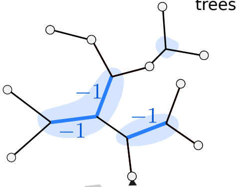
$$R = \sum_{k=0}^{\infty} \frac{2^{k-1}}{k!} t_k R^k + \sum_{k=2}^{\infty} \frac{2^{k-1}}{k!} \gamma_k R^k, \qquad t_k = \frac{2}{k!}\int_0^\infty \left(\tfrac{L}{2}\right)^{2k} \mathrm{d}q(L), \quad \gamma_k = \frac{(-1)^k \pi^{2k-2}}{(k-1)!} \; ?$$

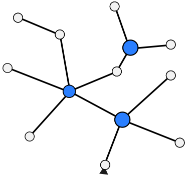
$= \sum_{k=0}^{\infty}$  $+ \sum_{k=2}^{\infty}$ 

$$R[q] \;=\; \sum_{\text{trees } \mathfrak{t}} \left|\mathcal{A}_{\mathfrak{t}}(\mathbf{L})\right| \;=\; \sum_{\substack{\text{trees } \mathfrak{t} \text{ with} \\ \text{blue/black edges}}} (-1)^{\#/} \left|\mathcal{A}^{>}_{\mathfrak{t}}(\mathbf{L})\right| \;=\; \sum_{\text{2-type trees}} \prod_{\circ} x_{\deg\circ - 1} \prod_{\bullet} y_{\deg\bullet - 1}$$









φ_i — φ_j (red): $\varphi_i + \varphi_j < \pi$

φ_i — φ_j (black): no constraint

φ_i — φ_j (blue): $\varphi_i + \varphi_j > \pi$

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- Why does the generating function $R[q] = \sum_{n\geq 1} \frac{1}{n!} \int_0^\infty \mathrm{d}q(L_1)\cdots \mathrm{d}q(L_n) V^{\mathrm{WP}}_{0,n+2}(0,0,\mathbf{L})$ satisfy

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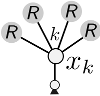
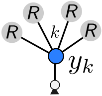
$= \sum_{k=0}^{\infty}$  $+ \sum_{k=2}^{\infty}$ 

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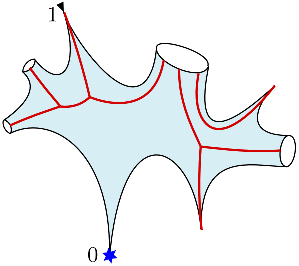
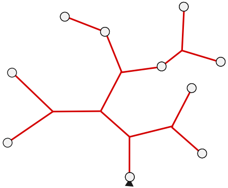
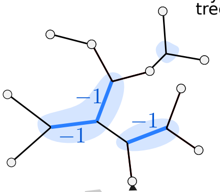
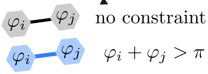
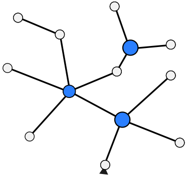




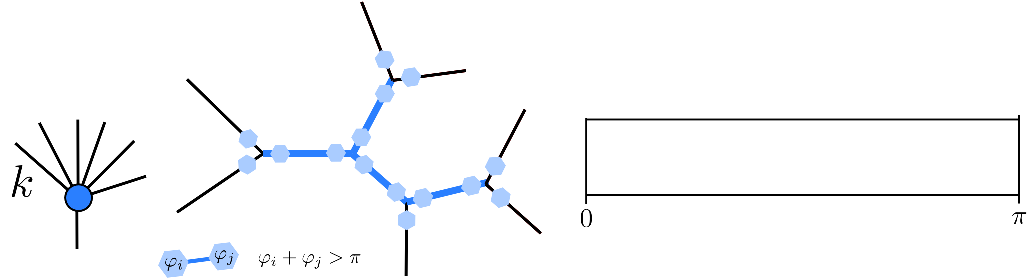






WP volume of blue vertices

- The reversed condition $\varphi_i + \varphi_j > \pi$ is simpler, because WP volume is independent of tree structure:



$$y_k = (-1)^k 2^{k-1} \sum_{\text{binary trees}} \int_{\mathcal{A}_t^{>}} \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{2k-2} =$$

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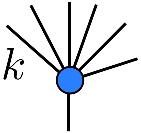

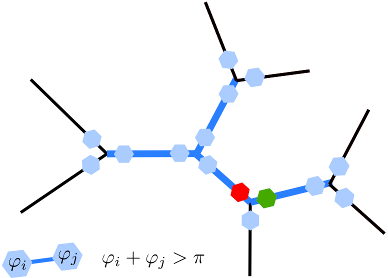

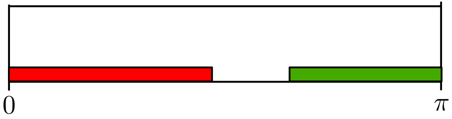

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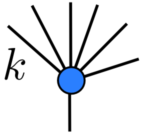

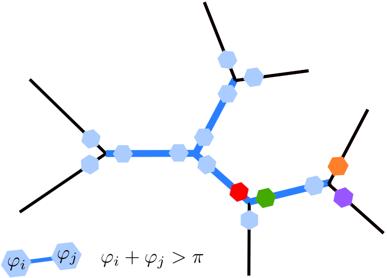

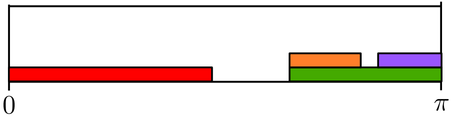

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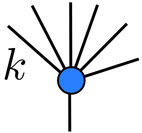

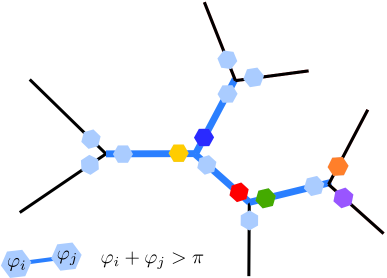

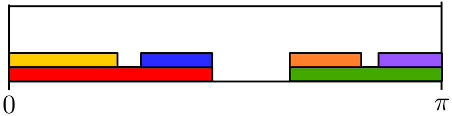

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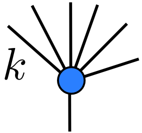

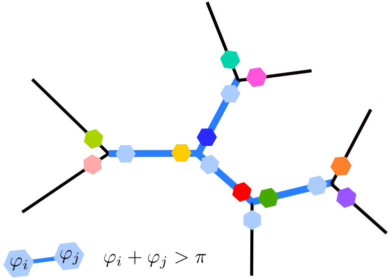

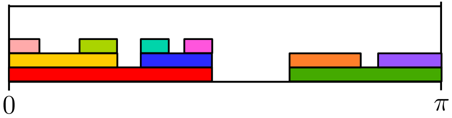

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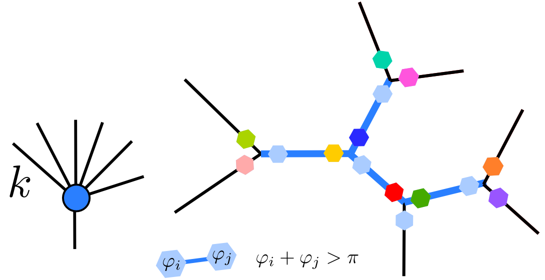

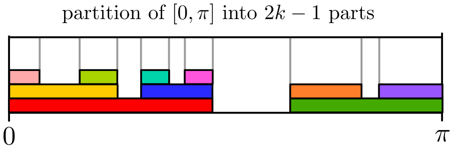

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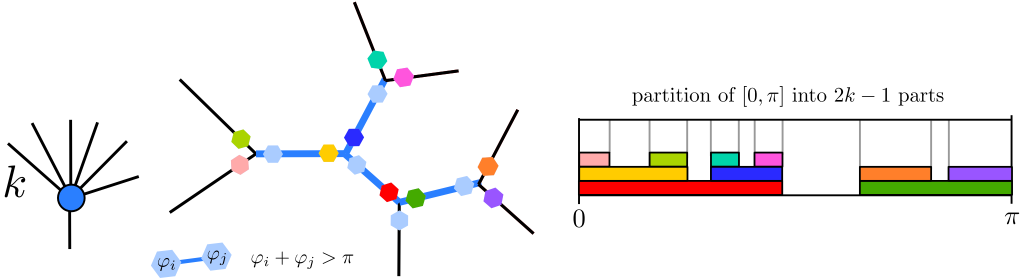
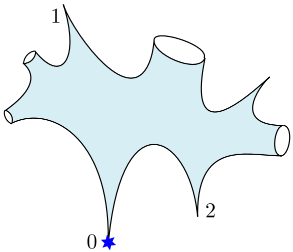


$$y_k = (-1)^k 2^{k-1} \sum_{\text{binary trees}} \int_{\mathcal{A}_t^>} d\varphi_1 \cdots d\varphi_{2k-2} = (-1)^k 2^{k-1} \mathrm{Cat}(k-1) \frac{\pi^{2k-2}}{(2k-2)!} = (-1)^k 2^{k-1} \frac{\pi^{2k-2}}{k!(k-1)!} = 2^{k-1} \frac{\gamma_k}{k!}$$

Not just volumes: geodesic distance control!

- Boltzmann hyperbolic sphere $X \in \bigcup_{n\geq 0} \mathcal{M}_{n+3}(0,0,0,\mathbf{L})$: $\mathbb{P}(X) \propto \mathrm{d}q(L_1)\cdots\mathrm{d}q(L_n)\mathrm{d}\mu_{\mathrm{WP}}$.



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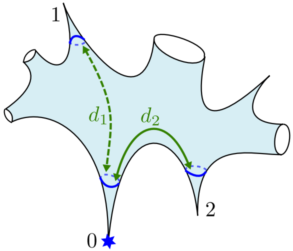

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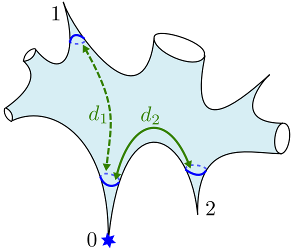

Theorem (TB, Meeusen, Zonneveld, '23+)

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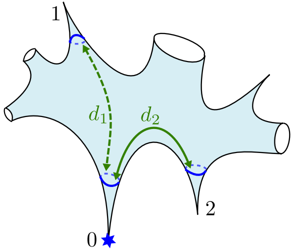

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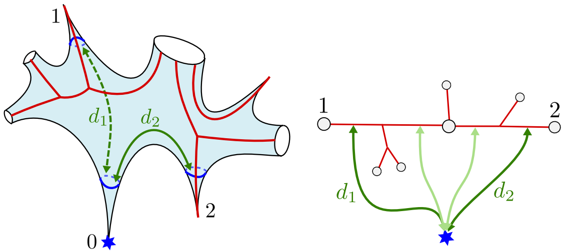

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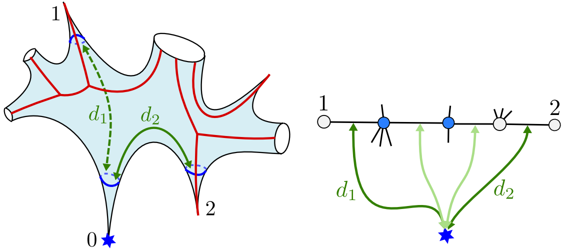

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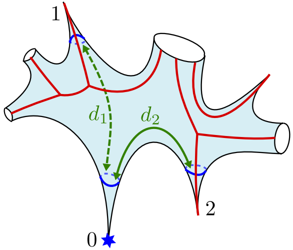

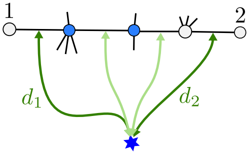

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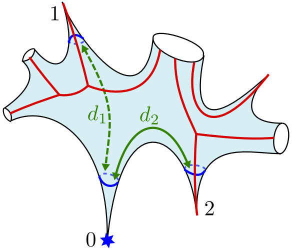

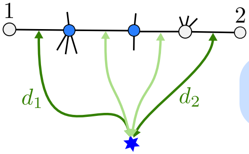

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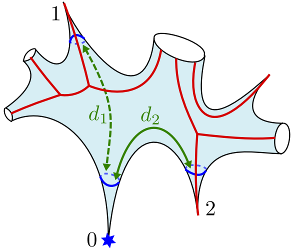

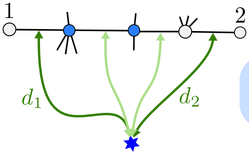

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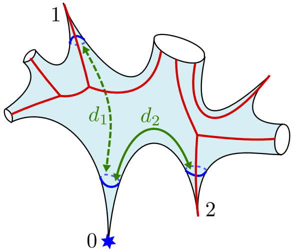
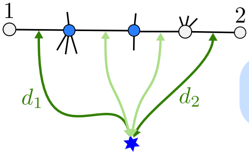
$$\mathbb{E}[e^{2u(d_1-d_2)}] = \frac{\sin 2\pi u}{2\pi}\frac{y'(0)}{y(u)}, \quad y(u) = [u^{\geq 0}]\frac{\sin 2\pi z}{\pi} - \int_0^\infty \mathrm{d}q(L)\frac{\cosh Lz}{z}, \quad z = \sqrt{u^2 + 2R}$$

- $(u^2, y(u))$ is curve for topological recursion of "tight" Weil–Petersson volumes [TB, Zonneveld, '23]

Not just volumes: geodesic distance control!

- Boltzmann hyperbolic sphere $X \in \bigcup_{n\geq 0} \mathcal{M}_{n+3}(0,0,0,\mathbf{L})$: $\mathbb{P}(X) \propto \mathrm{d}q(L_1)\cdots\mathrm{d}q(L_n)\mathrm{d}\mu_{\mathrm{WP}}$.





$$\mathbb{E}[e^{2u(d_1-d_2)}] \propto \frac{1}{1-X_{\bullet}(u)-X_{\circ}(u)}$$

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- $(u^2, y(u))$ is curve for topological recursion of "tight" Weil–Petersson volumes [TB, Zonneveld, '23]
- Singularity analysis: $d_1 - d_2 \approx n^{1/4}$ in Boltzmann hyperbolic sphere for n large. Same universality class as Boltzmann planar map?

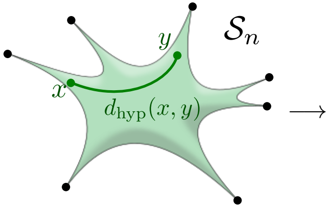
Geometry of sphere with many cusps

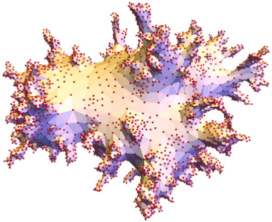
- In the case of only cusps, $q(L) = x\delta_0(L)$, this is indeed true:

Theorem (TB, Curien, '23+)

If $S_n \in \mathcal{M}_{0,n}(0)$ is sampled with probability density $\mu_{WP}/V_{0,n}(0)$, then we have the convergence in distribution of the random metric space in the Gromov–Prokhorov topology

$$\left(S_n, \frac{d_{\mathrm{hyp}}}{c\, n^{1/4}}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \textit{Brownian sphere}, \quad c = 2.339\ldots.$$





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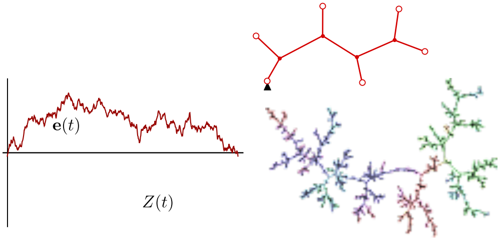
Proof ingredients: Le Gall's strategy [Le Gall, '11]

convergence of labeled tree to Brownian snake



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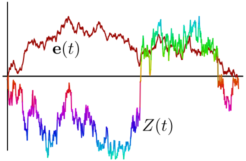

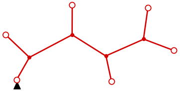

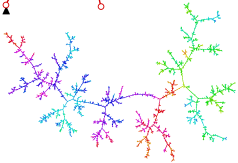

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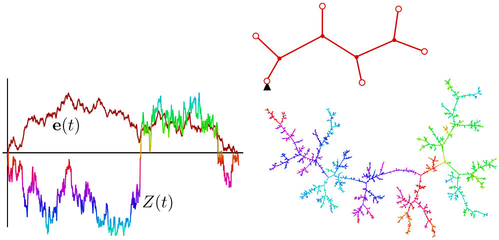

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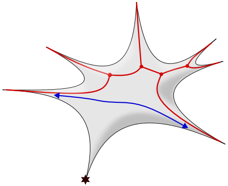
bound on distances between arbitrary cusps



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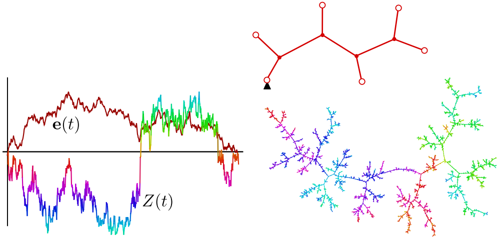

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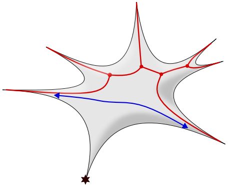

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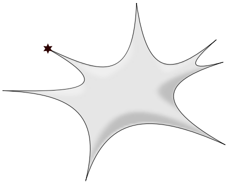
invariance under change of origin



Thanks for your attention!