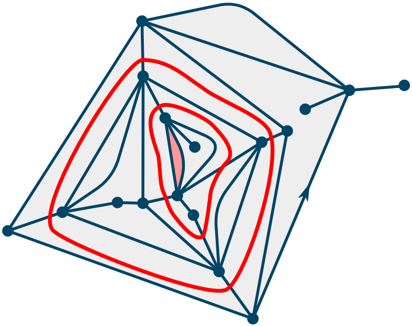
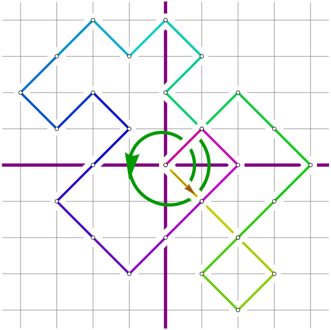
Workshop on Large Random Structures in Two Dimensions,
IHP, January 19th, 2017

On a connection between planar map combinatorics and lattice walks

Timothy Budd





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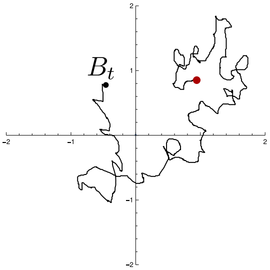
Introduction: Hyperbolic secant law

- The winding angle θ_t of 2d Brownian motion satisfies Spitzer's law [Spitzer '58]

$$\frac{2\theta_t}{\log(t)} \xrightarrow[t\to\infty]{(d)} \text{Cauchy}$$



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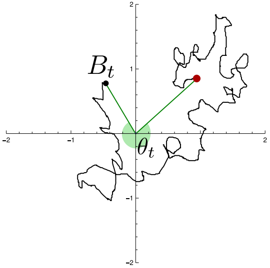

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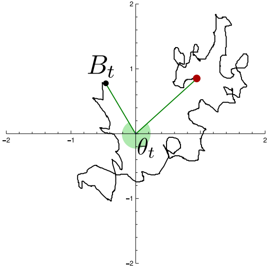

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- The winding angle θ_n of 2d random walk satisfies hyperbolic secant law [Rudnick, Hu '87] [Bélisle '89]

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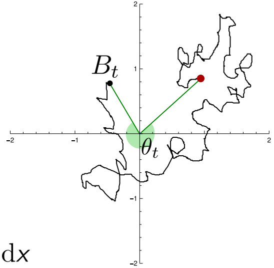

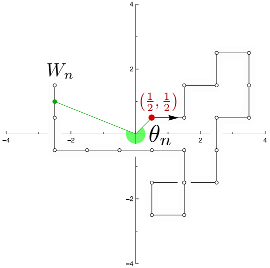

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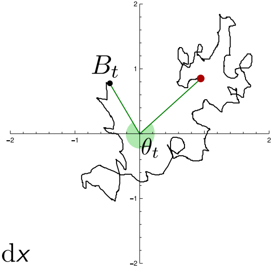

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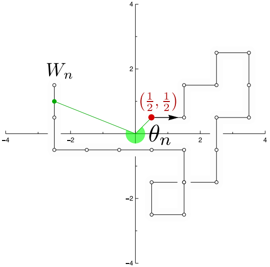
- Surprising discrete analogue for SRW started at $(\frac{1}{2},\frac{1}{2})$: if $n_p \geq 1$ is geometric with parameter p, then for $a,b\in\mathbb{Z}$:

$$\mathbb{P}\left[\frac{\theta_{n_p}}{\pi} \in (a,b)\right] = C_p \sum_{x=a+\frac{1}{2}}^{b-\frac{1}{2}} \operatorname{sech}(\pi x T_p)$$



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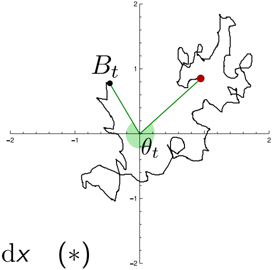

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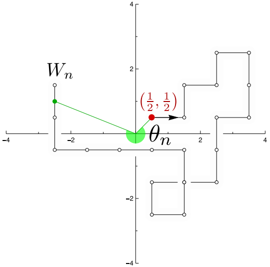

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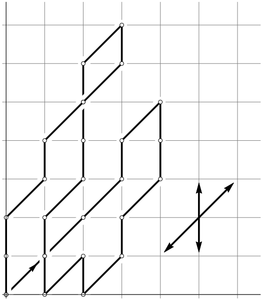
- $T_p \sim \frac{1}{\log(1-p)}$ as $p \to 1$. Reproduces $(*)$.

Introduction: Gessel numbers

- In 2001 Ira Gessel conjectured the number of walks with $2n$ steps $\in \{N, S, SW, NE\}$ in the quadrant starting and ending at 0 to be

$$16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = 2, 11, 85, 782, \ldots$$



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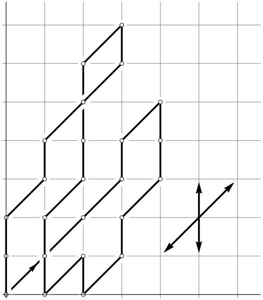

- Turned out to be a notoriously difficult problem, but by now we have...
  - ...a computer-aided proof. [Kauers, Koutschan, Zeilberger, '08]
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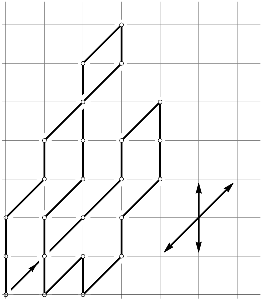

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- We will see that control of winding numbers provides an alternative route.

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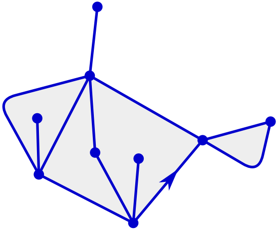
- Planar map = rooted planar graph embedded in $\mathbb{R}^2$ up to homeomorphisms.



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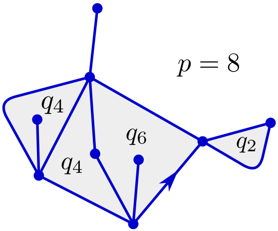

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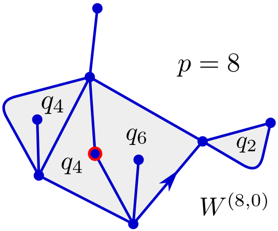

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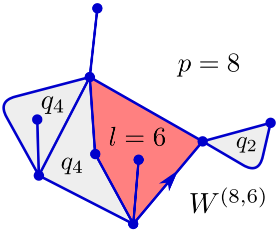
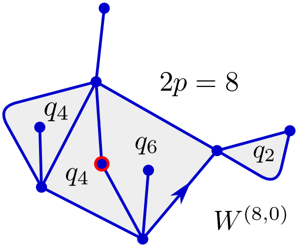


Relation between maps and walks?

- Classical result: for bipartite maps the GF with marked vertex takes a universal form (with $\rho_{\mathbf{q}}$ a formal power series in $q_2, q_4, \ldots$)

$$W^{(2p,0)} = \binom{2p}{p} \left(\frac{\rho_{\mathbf{q}}}{4}\right)^p$$



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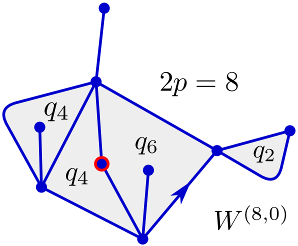

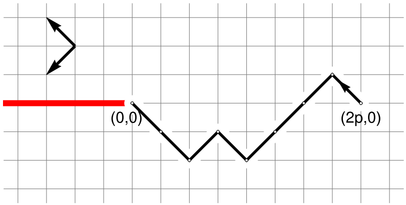

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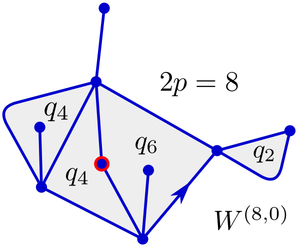

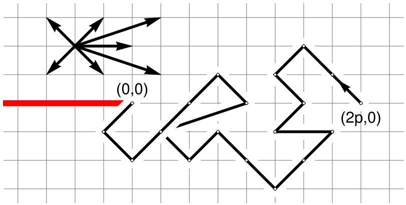

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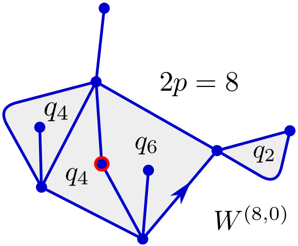
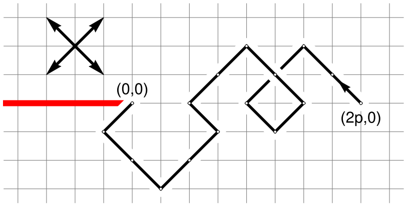
  - in particular simple diagonal walks $(\rho \to \frac{1-\sqrt{1-16t^2}}{8t^2} - 1)$.





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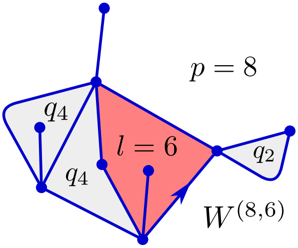

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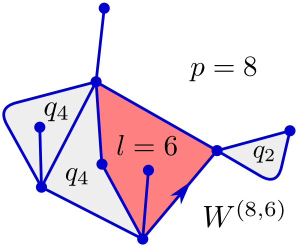

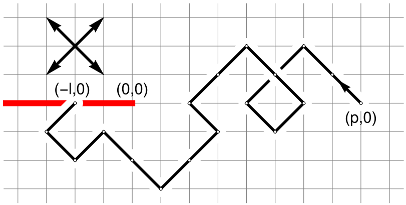

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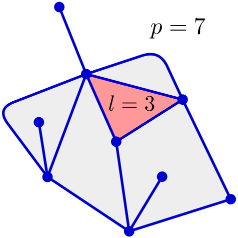

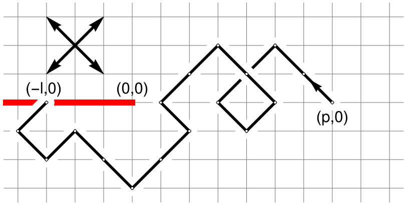

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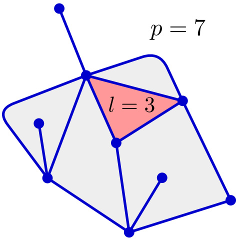

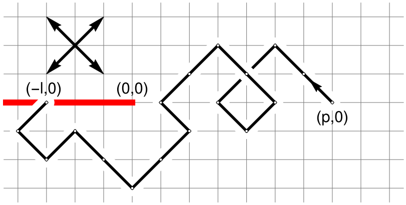

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- Up to factor of two (and $\rho \to \frac{1-\sqrt{1-16t^2}}{8t^2} - 1$) this also counts walks on slit plane ending at $(-l, 0)$. Coincidence?

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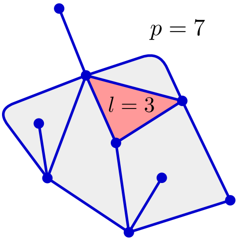

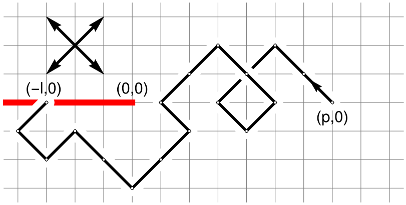

A bijective explanation

Proposition

For any step set $\mathfrak{S} \subset \{-1, 0, 1, 2, \ldots\} \times \{-1, 0, 1\}$, *there is a 2-to-1 map*

$$\Phi^{(p,l)} : \{\mathfrak{S}\text{-walks } (p, 0), \cdots, (-l, 0) \text{ on slit plane}\}$$

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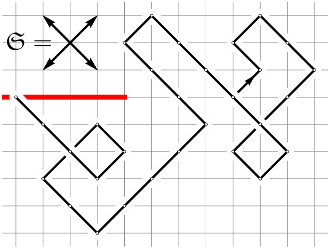

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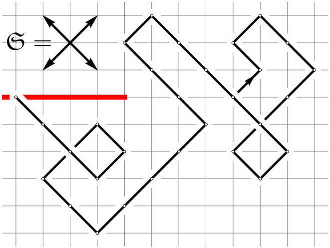
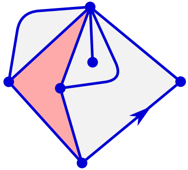
- A $\mathfrak{S}$-walk-decorated map is a rooted planar map with a marked face together with. . .





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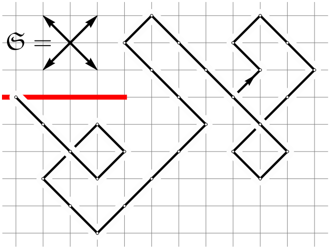
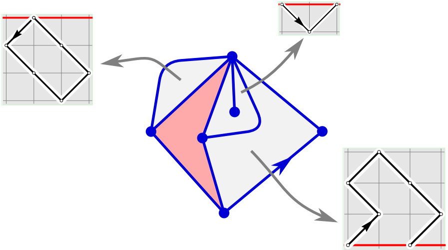
$$\longrightarrow \left\{ \begin{array}{l} \text{“}\mathfrak{S}\text{-walk-decorated maps” with root face degree } p \\ \text{and marked face degree } l \end{array} \right\}$$

- A $\mathfrak{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree k an excursion $(0,0),\ldots,(k-2,0)$ above or below x-axis.





A bijective explanation

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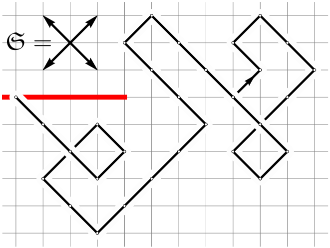

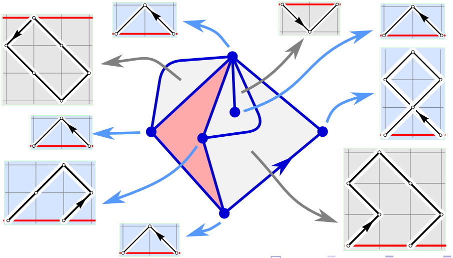

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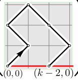
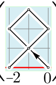
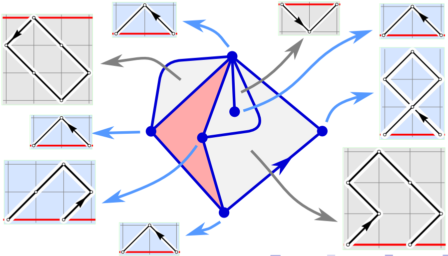
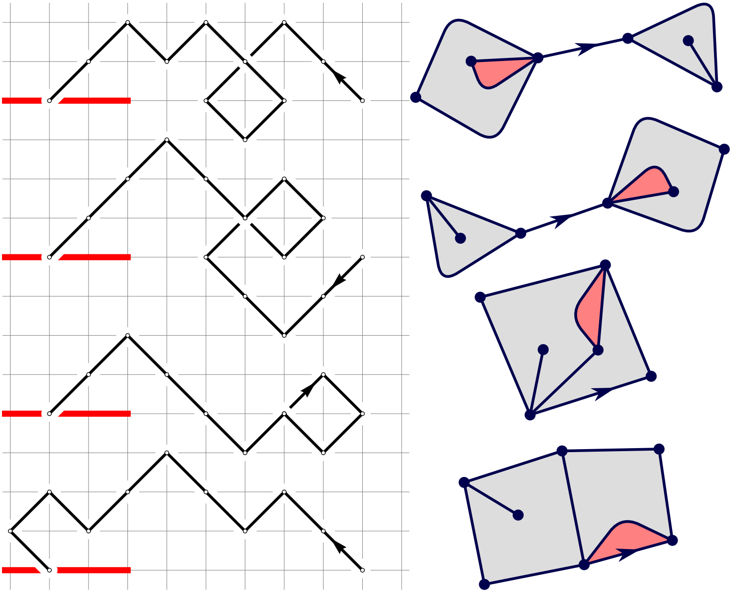
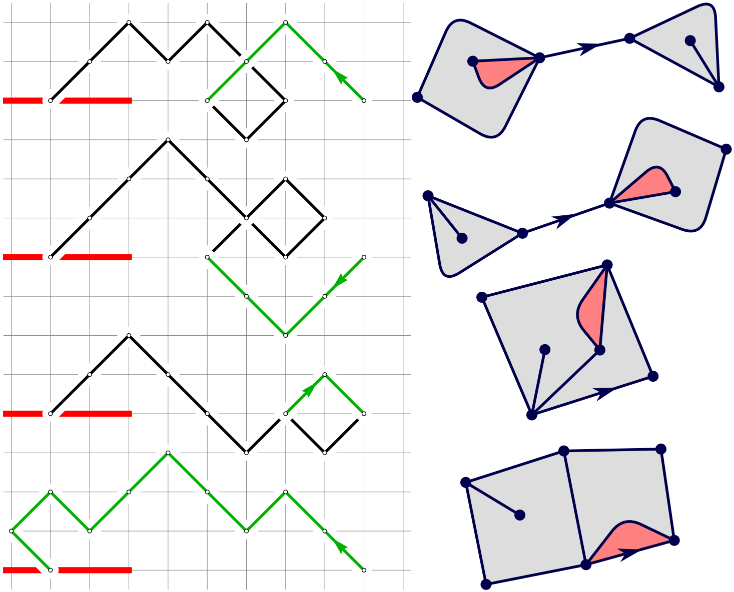
- Substituting in $2W^{(p,l)}(\{q_i\})$ the GF

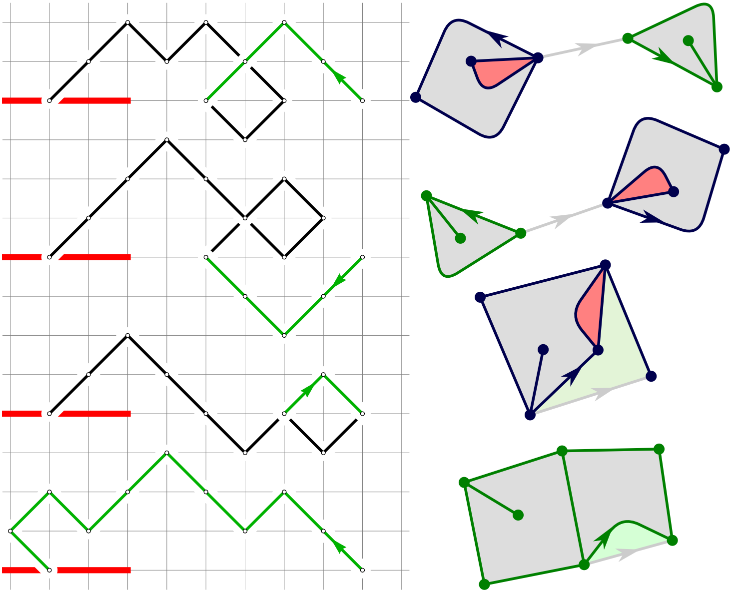
$q_k \to$ () $\cdot$ () $^{\frac{k-2}{2}}$

leads to $H^{(p,l)}(t)$ (up to $(\cdot)^{p+l}$).

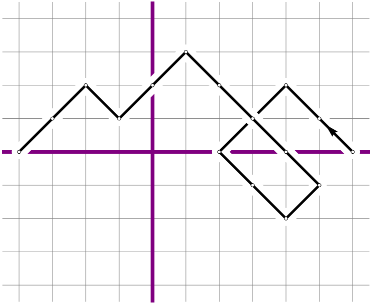


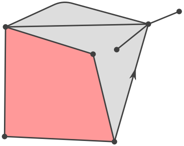






From walks to (rigid) loop-decorated maps





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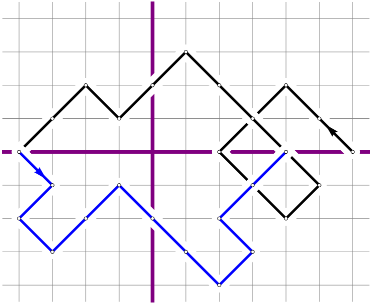

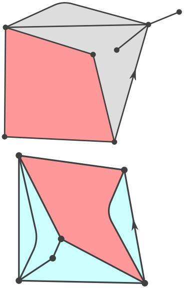

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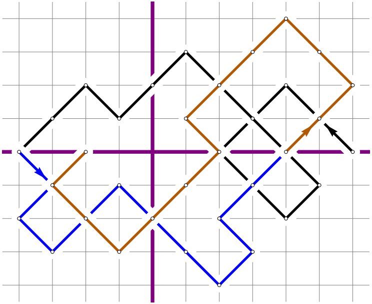

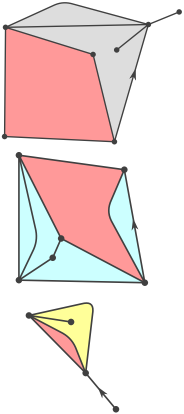

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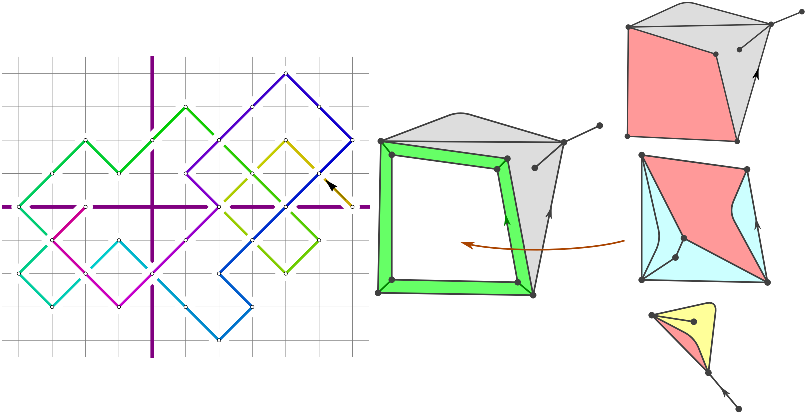

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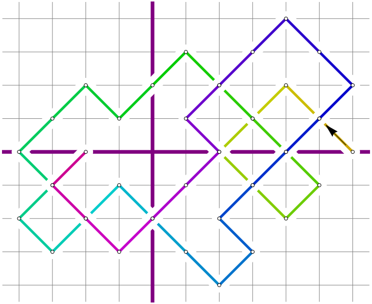

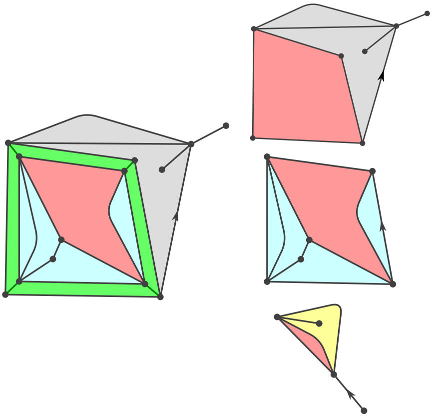

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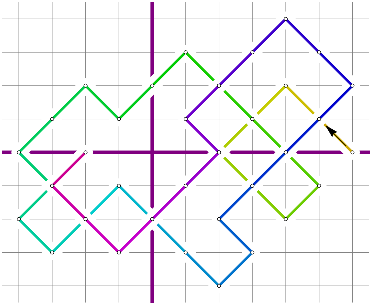

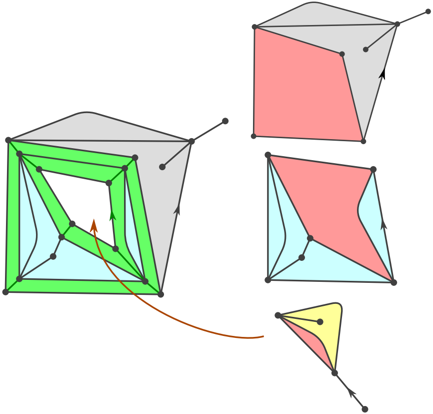

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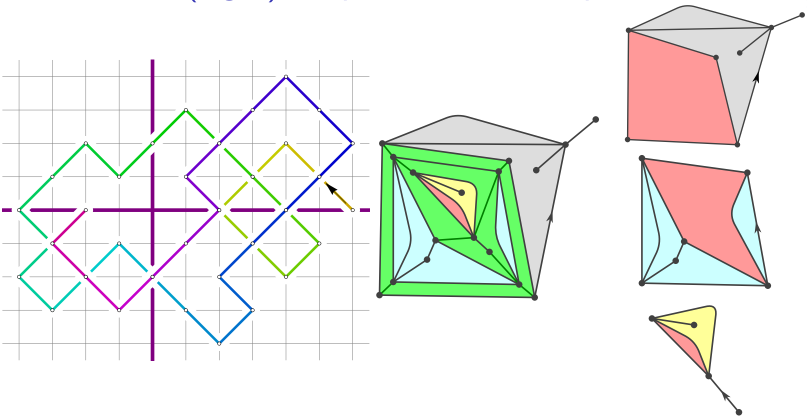

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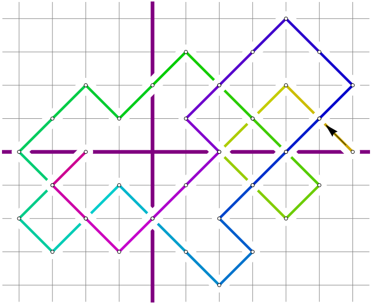

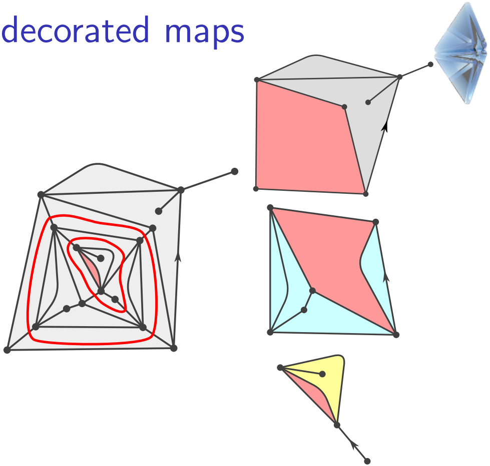

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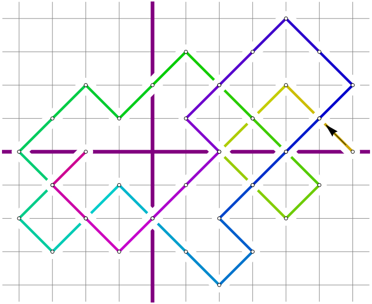

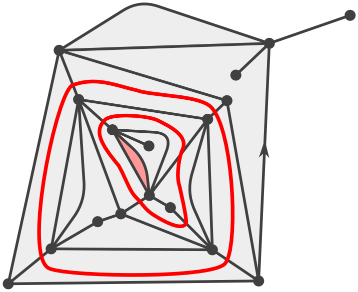

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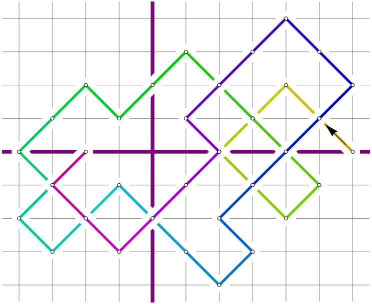

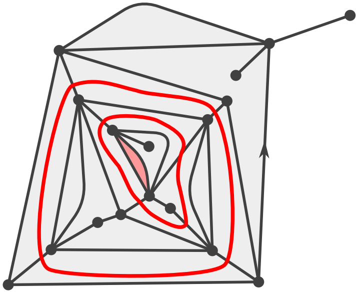

- Such walks from $(p,0)$ to $(\pm l,0)$ with winding angle θ_w have GF

$$\mathcal{G}_b^{(p,l)} := \sum_w t^{|w|} e^{ib\theta_w} = \sum_{N=1}^{\infty} \left(\frac{e^{ib\pi} + e^{-ib\pi}}{2} \right)^N \sum_{k_1,\ldots,k_{N-1}\geq 1} H^{(p,k_1)} H^{(k_1,k_2)} \ldots H^{(k_{N-1},l)}$$

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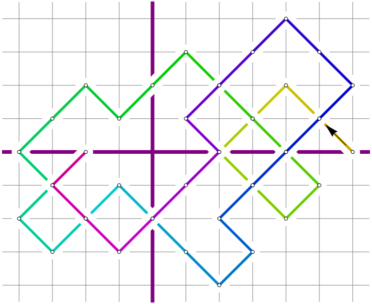

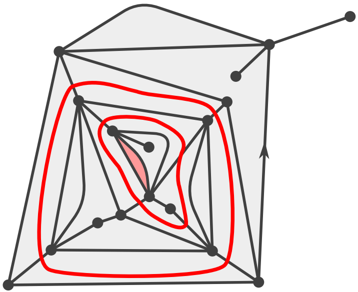

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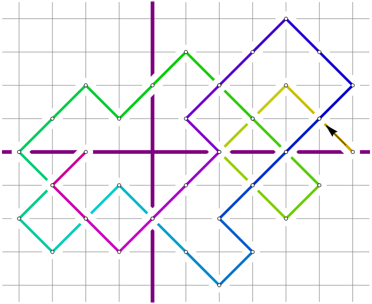

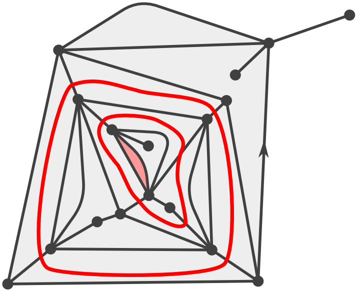

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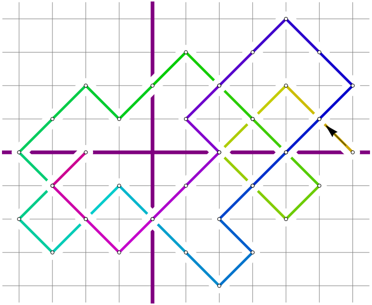

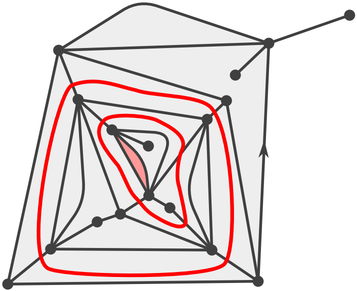

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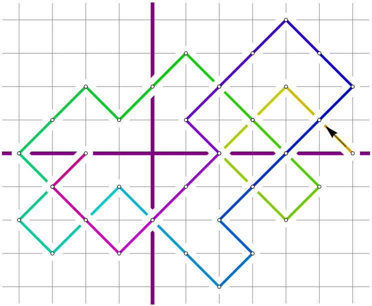

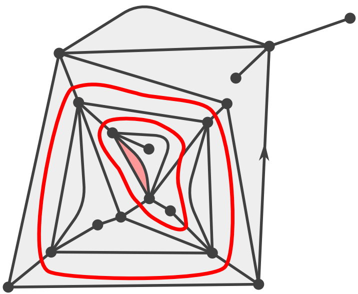

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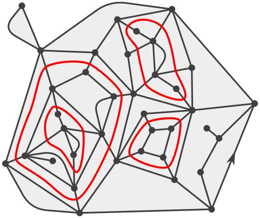
- But this also enumerates planar maps decorated with rigid loops carrying a weight $n := 2\cos(\pi b)$ each (and a redundant overall factor of n).

Planar maps coupled to a rigid $O(n)$ loop model

- Rigid $O(n)$ model: a planar map + disjoint loops, that intersect solely quadrangles through opposite sides. Enumerated with

$$\text{weight} \quad n^{\#\text{loops}} g^{\#\text{loop faces}} \prod_{\text{regular faces}} q_{\text{degree}}$$



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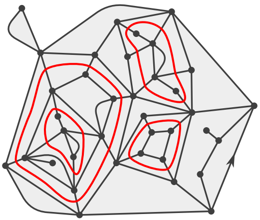

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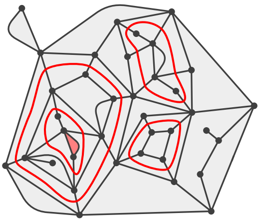

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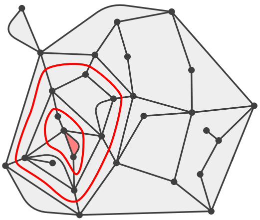

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- Recently in [Borot, Bouttier, Duplantier, '16] (in slightly different setting) exact statistics for the nesting of loops was obtained, i.e. distribution of # loops surrounding a marked vertex/face.
- Importantly: suppressing loops that do not surround mark affects GF's only through renormalization of **q**.

- Adapting GF from [Borot, Bouttier, Duplantier, '16], setting $n = 2\cos(\pi b)$ and computing a series representation:

$$\mathcal{G}_b(x_1, x_2; t) := \sum_{p,l \geq 1} x_1^p x_2^l \, \mathcal{G}_b^{(p,l)}$$

$$= 4 \sum_{m=1}^{\infty} \frac{2\cos(\pi b)}{q^m + q^{-m} - 2\cos(\pi b)} \frac{\cos(2\pi m\, v(x_2))\; x_1 \frac{\partial}{\partial x_1} \cos(2\pi m\, v(x_1))}{m(q^{-m} - q^m)}$$

where $q = q(4t)$ is elliptic nome of modulus $4t$ and

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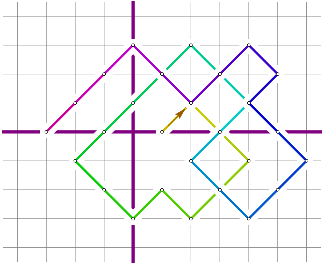
Proposition (Diagonalization of $\mathcal{H}$)

$\mathcal{H} = U^T \cdot \Lambda_q \cdot U$ in the sense of operators on $\ell^2(\mathbb{R})$ with

$$\Lambda_q = diag\left(\frac{2}{q^m + q^{-m}}\right)_{m\geq 1}, \quad U_{mp} = \sqrt{\frac{4p}{m(q^{-m} - q^m)}}\,[x^p]\cos(2\pi m\, v(x))$$

Application 1: hyperbolic secant law

- Recall $\sqrt{\frac{p}{l}}(\mathcal{H}^N)_{pl}$ enumerates walks $(p,0) \to (\pm l, 0)$ that alternate between half-axes N times.



Application 1: hyperbolic secant law

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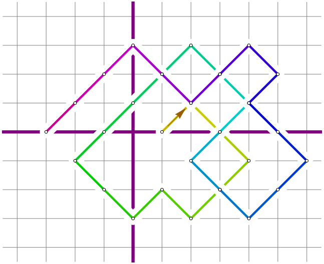

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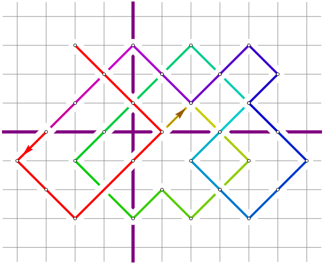

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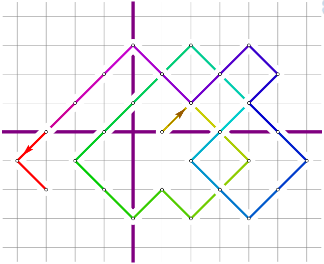

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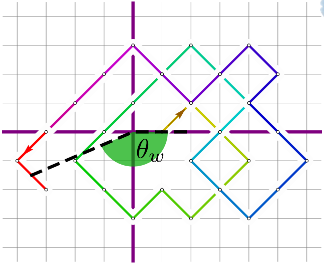

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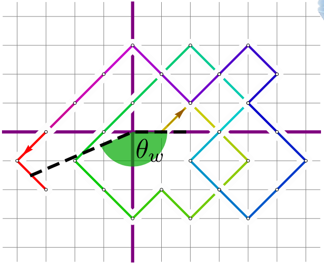

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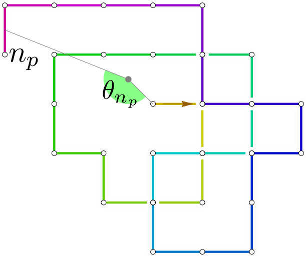

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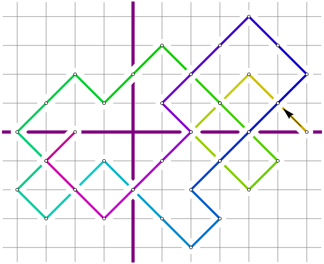
Theorem (Winding angle of SRW on $\mathbb{Z}^2$ around $(-\frac{1}{2}, \frac{1}{2})$)

If $n_p \geq 1$ is a geometric RV with parameter $0 < p < 1$ then

$$\mathbb{P}\left[k\pi < \theta_{n_p} < (k+1)\pi\right] = \frac{\mathrm{sech}(\pi(k+\frac{1}{2})T)}{\sum_{k\in\mathbb{Z}}\mathrm{sech}(\pi(k+\frac{1}{2})T)}, \quad T = \frac{K(\sqrt{1-p^2})}{K(p)}$$

Refinement: increase winding angle resolution

- Up to now: decomposed walk into sequence of walks on slit plane.



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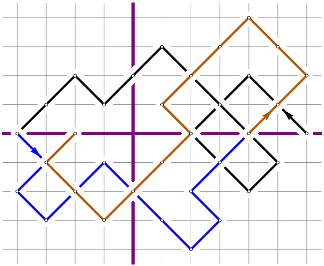

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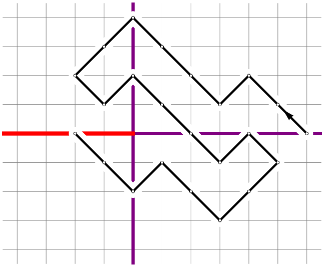

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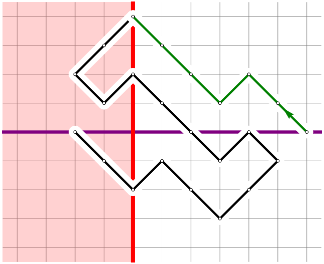

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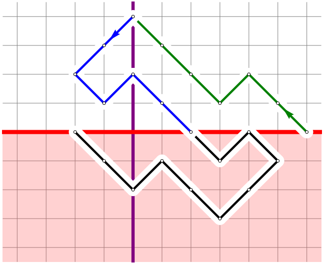

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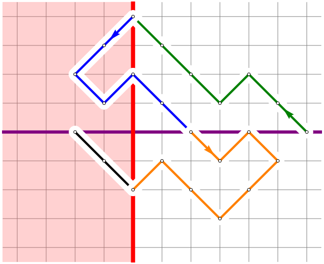

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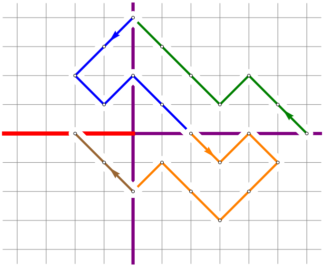
$$\mathcal{H} = \frac{1}{2}\mathcal{J}^2(I + \mathcal{H}), \quad \mathcal{J} = \sqrt{\frac{2\mathcal{H}}{I + \mathcal{H}}}$$



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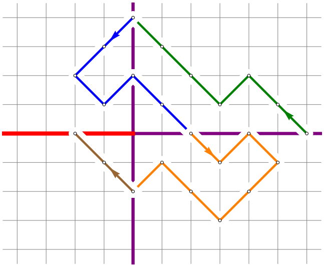


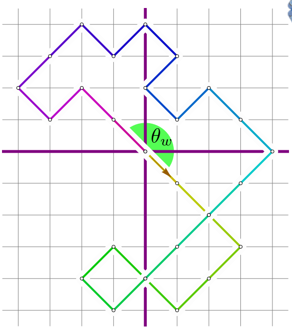
- Hence $\mathcal{J}$ has same eigenmodes as $\mathcal{H}$ but eigenvalues are $\frac{2}{q^{m/2}+q^{-m/2}}$ instead of $\frac{2}{q^m+q^{-m}}$. Such an operation $q \to \sqrt{q}$ on elliptic functions are well-known as "Landen transformations".

Winding angle of excursions

- Wish to enumerate excursions from origin by length and winding angle:

$$F(t,b) := \sum_{w} t^{|w|} e^{ib\,\theta_w}$$

$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \dots$$



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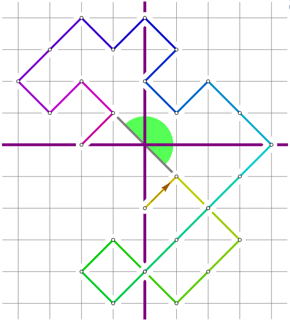

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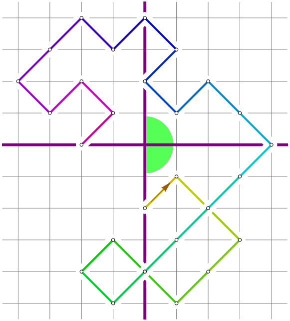

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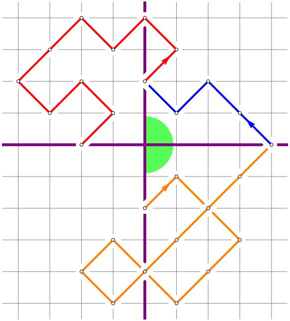

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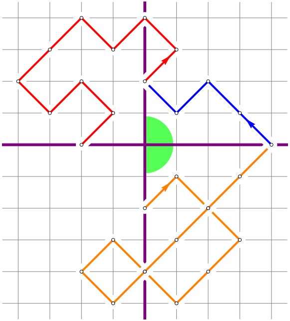

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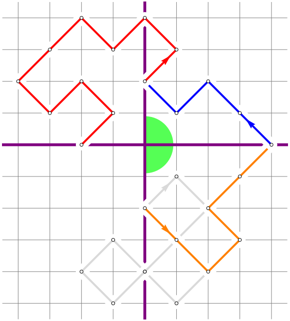

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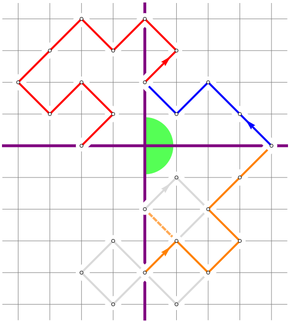

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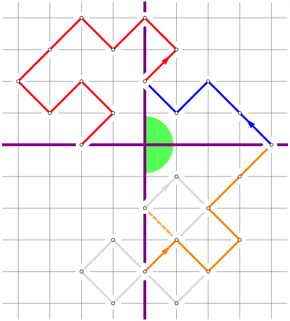

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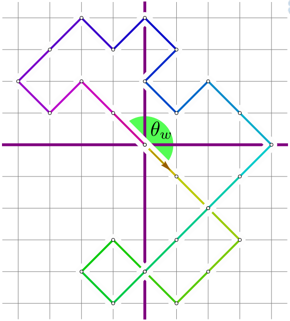

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Application 2: walks in cones

Theorem (Excursions in the $\frac{n\pi}{4}$-cone.)

For integers $m-n<p<m<n$ the GF for simple walks $(0,0),(1,0),\ldots,(0,0)$ with winding angle $\frac{p\pi}{2}$ staying strictly inside angular region $(\frac{p+m-n}{4}\pi, \frac{p+m}{4}\pi)$ is

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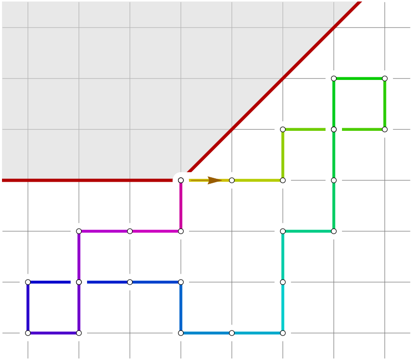

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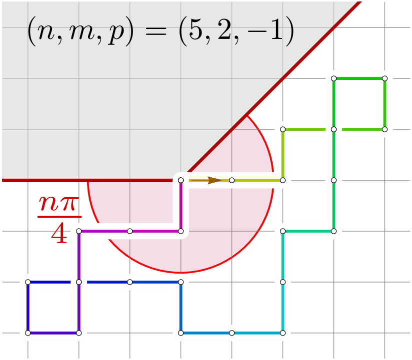

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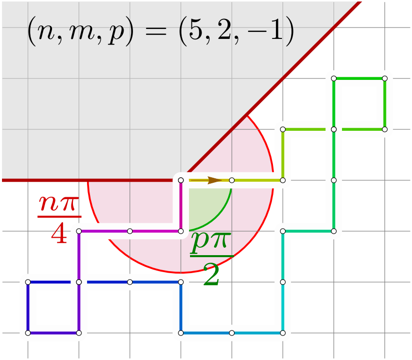

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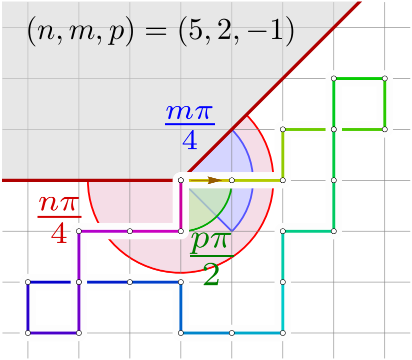

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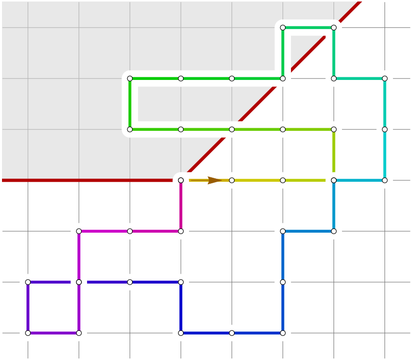

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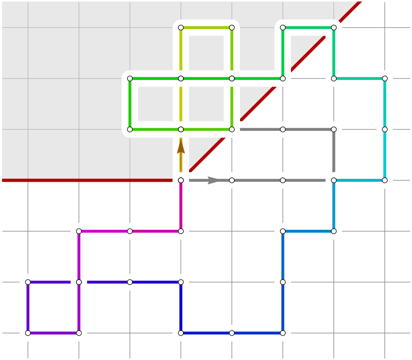

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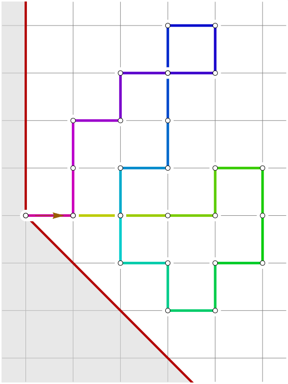
- Special case: $(n, m, p) = (3, 2, 0)$



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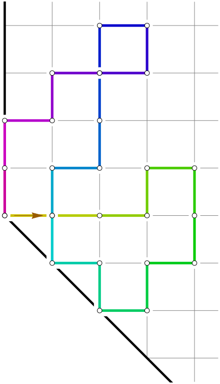

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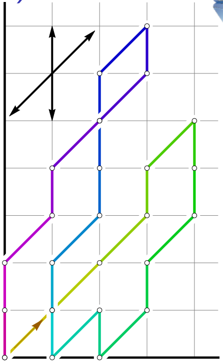

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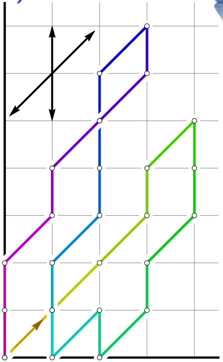
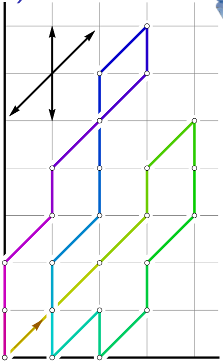
$$\begin{aligned}\frac{1}{t^2} F_{3,2,0}(t) &= \frac{1}{4t^2} F(t, \tfrac{4}{3}) \\ &= \frac{1}{2t^2}\left[\frac{\sqrt{3}\pi}{2K(4t)} \frac{\theta_1'(\frac{\pi}{3}, \sqrt{q})}{\theta_1(\frac{\pi}{3}, \sqrt{q})} - 1\right] \\ &= 1 + 2t^2 + 11t^4 + 85t^6 + \cdots\end{aligned}$$

which is [OEIS sequence A135404]



Application 2: walks in cones (Gessel case)



- Special case: $(n, m, p) = (3, 2, 0)$
- Gessel-type walks returning to origin in quadrant are enumerated by

$$\frac{1}{t^2} F_{3,2,0}(t) = \frac{1}{4t^2} F(t, \tfrac{4}{3})$$
$$= \frac{1}{2t^2}\left[\frac{\sqrt{3}\pi}{2K(4t)} \frac{\theta_1'(\frac{\pi}{3}, \sqrt{q})}{\theta_1(\frac{\pi}{3}, \sqrt{q})} - 1\right]$$
$$= 1 + 2t^2 + 11t^4 + 85t^6 + \cdots$$

which is [OEIS sequence A135404]

- But not obvious that this reproduces the known GF

$$\sum_{n=0}^{\infty} t^{2n}\, 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = \frac{1}{2t^2}\left[{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{6}; \tfrac{2}{3}; (4t)^2 \right) - 1 \right],$$

nor that it is algebraic [Bostan, Kauers, '09].

Further questions

- Which generating functions are algebraic?
- Other walks with small steps?
- Why are some of the generating functions biperiodic and other ones only quasi-biperiodic?
- Finally, here is an interpretation of the nome q as function of the elliptic modulus k. Why is it so simple?

$$q(k) = \lim_{n\to\infty} \mathbb{P}\left[\begin{array}{l}\text{SRW on } \mathbb{Z}^2 \text{ reaches winding angle } n\pi \\ \text{before geometric time with parameter } k\end{array}\right]^{1/n}$$